# Advancements on SEFE and Partitioned Book Embedding Problems 

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#### Abstract

In this work we investigate the complexity of some problems related to the Simultaneous Embedding with Fixed Edges (SEFE) of $k$ planar graphs and the Partitioned $k$-Page Book embedding (PBE- $k$ ) problems, which are known to be equivalent under certain conditions. While the computational complexity of SEFE for $k=2$ is still a central open question in Graph Drawing, the problem is $\mathcal{N \mathcal { P }}$-complete for $k \geq 3$ [Gassner et al., WG '06], even if the intersection graph is the same for each pair of graphs (sunflower intersection) [Schaefer, JGAA (2013)]. We improve on these results by proving that SEFE with $k \geq 3$ and sunflower intersection is $\mathcal{N} \mathcal{P}$-complete even when the intersection graph is a tree and all the input graphs are biconnected. Also, we prove $\mathcal{N} \mathcal{P}$-completeness for $k \geq 3$ of problem PBE- $k$ and of problem Partitioned T-Coherent $k$-Page Book Embedding (PTBE- $k$ ) - that is the generalization of PBE- $k$ in which the ordering of the vertices on the spine is constrained by a tree $T$ - even when two input graphs are biconnected. Further, we provide a linear-time algorithm for PTBE$k$ when $k-1$ pages are assigned a connected graph. Finally, we prove that the problem of maximizing the number of edges that are drawn the same in a SEFE of two graphs is $\mathcal{N} \mathcal{P}$-complete in several restricted settings (optimization version of SEFE, Open Problem 9, Chapter 11 of the Handbook of Graph Drawing and Visualization).


## 1 Introduction

Let $G_{1}, \ldots, G_{k}$ be $k$ graphs on the same set $V$ of vertices. A simultaneous embedding with fixed edges (SEFE) of $G_{1}, \ldots, G_{k}$ consists of $k$ planar drawings $\Gamma_{1}, \ldots, \Gamma_{k}$ of $G_{1}, \ldots, G_{k}$, respectively, such that each vertex $v \in V$ is mapped to the same point in every drawing $\Gamma_{i}$ and each edge that is common to more than one graph is represented by the same simple curve in the drawings of all such graphs. The SEFE problem is the problem of testing whether $k$ input graphs $G_{1}, \ldots, G_{k}$ admit a SEFE [14].

The possibility of drawing together a set of graphs gives the opportunity to represent at the same time a set of different binary relationships among the same objects, hence making this topic an fundamental tool in Information Visualization [15]. Motivated by such applications and by their theoretical appealing, simultaneous graph embeddings received wide research attention in the last few years. For an up-to-date survey, see [6].

Recently, a new major milestone to assert the importance of SEFE has been provided by Schaefer [28], who discussed its relationships with some other famous prob-
lems in Graph Drawing, proving that SEFE generalizes several of them. In particular, he showed a polynomial-time reduction to SEFE with $k=2$ from the clustered planarity testing problem [12[13], that can be arguably considered as one of the most important open problems in the field.

The SEFE problem has been proved $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$ by Gassner et al. [18]. On the other hand, if the embedding of the input graphs is fixed, SEFE becomes polynomialtime solvable for $k=3$, but remains $\mathcal{N} \mathcal{P}$-complete for $k \geq 14$ [2].

In Chapter 11 of the Handbook of Graph Drawing and Visualization [6], the SEFE problem with sunflower intersection (SUNFLOWER SEFE) is cited as an open question (Open Problem 7). In this setting, the intersection graph $G_{\cap}$ (that is, the graph composed of the edges that are common to at least two graphs) is such that, if an edge belongs to $G_{\cap}$, then it belongs to all the input graphs. Haeupler et al. [19] conjectured that SUNFLOWER SEFE is polynomial-time solvable. However, Schaefer [28] recently proved that this problem is $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$. The reduction is from the $\mathcal{N} \mathcal{P}$-complete [21] problem Partitioned T-Coherent $k$-Page Book embedDING (PTBE- $k$ ), defined [4] as follows. Given a set $V$ of vertices, a tree $T$ whose leaves are the elements of $V$, and a collection of edge-sets $E_{i} \subseteq V \times V$, for $i=1, \ldots, k$, is there a $k$-page book embedding such that the edges in $E_{i}$ are placed on the $i$-th page and the ordering of the elements of $V$ on the spine is represented by $T$ ? Note that, the $\mathcal{N P}$-completeness of PTBE- $k$ holds for $k$ unbounded [21], which implies that the $\mathcal{N} \mathcal{P}$-completeness of SUNFLOWER SEFE for $k \geq 3$ holds for instances in which the intersection graph is a spanning forest composed of an unbounded number of star graphs [28].

In this paper, we improve on this result by proving that SUNFLOwER SEFE is $\mathcal{N} \mathcal{P}$-complete with $k \geq 3$ even if $G_{\cap}$ consists of a single spanning tree and all the input graphs are biconnected. Note that, for $k=2$, having $G_{\cap}$ connected and all the input graphs biconnected suffices to have a polynomial-time algorithm for the problem [9].

Since SUNFLOWER SEFE when the intersection graph is connected has been proved equivalent to the PTBE- $k$ problem [4] (where the equivalence sets each graph $G_{i}$ equal to tree $T$ plus the edge-set $E_{i}$ ), our result implies the $\mathcal{N} \mathcal{P}$-completeness of PTBE- $k$ for $k \geq 3$, but with no guarantees on the biconnectivity of the input graphs. We prove for this problem even stronger results, namely that PTBE- $k$ remains $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$ even if two of the input graphs $G_{i}=T \cup E_{i}$ are biconnected or if $T$ is a star. This latter setting, in which the tree $T$ basically does not impose any constraint on the ordering of the vertices on the spine, is also known as Partitioned $k$-PAGE Book Embedding (PBE- $k$ ). Note that, for $k=2$, PBE- $k$ can be solved in linear time [20].

From the algorithmic point of view, we prove that PTBE- $k$ with $k \geq 2$ can be solved in linear time if $k-1$ of the input edge-sets $E_{i}$ induce connected graphs (a stronger condition than graph $G_{i}$ being biconnected), hence improving on a result by Hoske [21], that was based on all the $k$ input edge-sets having this property. Of course, relaxing this constraint on one of the $k$ input edge-sets becomes more relevant for small values of $k$; in particular, it contributes to extend the class of instances that can be solved in polynomial time also for $k=2$, that is the most studied setting both for PTBE- $k$ and for SEFE (note that every instance of SEFE with $k=2$ obviously has sunflower intersection).

| Problem | $\mathbf{G}_{\cap}$ | T-Coherent | k | Biconnected | $\mathcal{T}$-Biconnected | xity |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| SUNFLOWER | tree | NO | $k \geq 3$ | $k$ | - | NPC (Th 1] |
| PBE- $k$ | star | YES | $k \geq 3$ | - | - | NPC (Th. ${ }^{\text {4 }}$ ) |
| PBE-2 | star | YES | $k=2$ | - | - | $O(n)([20])$ |
| PTBE-3 | caterpillar | YES | $k=3$ | 2 | - | NPC (Th. 21 |
| PTBE- $k$ | tree | YES | $k \geq 2$ | $k-1$ | $k-1$ | $O(n)$ (Th. ${ }^{5}$ ) |
| PTBE-2 | tree | YES | $k=2$ | 2 | - | $O\left(n^{2}\right)([8])$ |
|  | binary trtree | YES | $k=2$ | - | - | $O\left(n^{2}\right)([21])$ |
|  |  | YES | $k=2$ | 1 | - | OPEN (Th. 6 |

Table 1: Complexity status for SUNFLOWER SEFE, PTBE- $k$, and PBE- $k$.

In fact, even if the complexity of SEFE and of PTBE- $k$ is still unknown for $k=2$, polynomial-time algorithms exist for instances in which: (i) one of $G_{1}$ and $G_{2}$ has a fixed embedding [3]; (ii) the intersection graph $G_{\cap}$ is biconnected [4[19], a star graph [4], or a subcubic graph [21|28]; (iii) each connected component of $G_{\cap}$ has a fixed embedding [7]; or (iv) $G_{1}$ and $G_{2}$ are biconnected and $G_{\cap}$ is connected [8].

For the setting $k=2$, we also prove that, given any instance of PTBE- $k$ (and hence of SEFE in which $G_{\cap}$ is connected), it is possible to construct an equivalent instance of the same problem in which one of the input graphs, say $G_{1}$, is biconnected and seriesparallel. This implies that it would be sufficient to find a polynomial-time algorithm for this seemingly restricted case in order to have a polynomial-time algorithm for the whole problem.

An updated summary of the results on SUNFLOWER SEFE and on PTBE- $k$ is presented in Table 1

Still in the setting $k=2$, we study the optimization version of SEFE, that we call MAX SEFE, which is cited as an open question by Haeupler et al. [19] and in Chapter 11 (Open Problem 9) of the Handbook of Graph Drawing and Visualization [6]. In this problem, one asks for drawings of $G_{1}$ and $G_{2}$ such that as many edges of $G_{\cap}$ as possible are drawn the same. We prove that MAX SEFE is $\mathcal{N} \mathcal{P}$-complete, even under some strong constraints. Namely, the problem is $\mathcal{N} \mathcal{P}$-complete if $G_{1}$ and $G_{2}$ are triconnected, and $G_{\cap}$ is composed of a triconnected component plus a set of isolated vertices. This implies that the problem is computationally hard both in the fixed and in the variable embedding case. In the latter case, however, we can prove that MAX SEFE is $\mathcal{N} \mathcal{P}$-complete even if $G_{\cap}$ has degree at most 2 . Observe that any of these constraints would be sufficient to obtain polynomial-time algorithms for the original decision problem.

In Sect. 2 we give some preliminary definitions. In Sect. 3 we deal with the sunflower intersection scenario; in Sect. 4 we focus on the PTBE- $k$ problem; while in Sect. 5 we study the MAx SEFE problem. Finally, in Sect. 6 we give concluding remarks and discuss some open problems.

## 2 Preliminaries

A drawing of a graph is a mapping of each vertex to a point of the plane and of each edge to a simple curve connecting its endpoints. A drawing is planar if the curves representing its edges do not cross except, possibly, at common endpoints. A graph is planar if it admits a planar drawing. A planar drawing $\Gamma$ determines a subdivision of the plane into connected regions, called faces, and a clockwise ordering of the edges incident to each vertex, called rotation scheme. The unique unbounded face is the outer face. Two drawings are equivalent if they have the same rotation schemes. A planar embedding is an equivalence class of planar drawings.

The SEFE problem can be studied both in terms of embeddings and in terms of drawings, since edges can be represented by arbitrary curves without geometric restrictions, and since Jünger and Schulz [22] proved that two graphs $G_{1}$ and $G_{2}$ with intersection graph $G_{\cap}$ have a SEFE if and only if there exists a planar embedding $\Gamma_{1}$ of $G_{1}$ and a planar embedding $\Gamma_{2}$ of $G_{2}$ inducing the same embedding of $G_{\cap}$. This condition extends to more than two graphs in the sunflower intersection setting.

A graph is connected if every pair of vertices is connected by a path. A $k$-connected graph $G$ is such that removing any $k-1$ vertices leaves $G$ connected; 3-connected and 2 -connected graphs are also called triconnected and biconnected, respectively. A tree is a graph with no cycle. A caterpillar is a tree such that the removal of all the leaves yields a path. A subgraph $H$ of a graph $G$ is spanning if for each vertex $v \in G$ there exists an edge of $H$ incident to $v$.

A series-parallel graph (SP-graph) is a graph with no $K_{4}$-minor. SP-graphs are inductively defined as follows. An edge $(u, v)$ is an SP-graph with poles $u$ and $v$. Denote by $u_{i}$ and $v_{i}$ the poles of an SP-graph graph $G_{i}$. A series composition of SP-graphs $G_{0}, \ldots, G_{k}$, with $k \geq 1$, is an SP-graph with poles $u=u_{0}$ and $v=v_{k}$, containing graphs $G_{i}$ as subgraphs, and such that $v_{i}=u_{i+1}$, for each $i=0,1, \ldots, k-1$. A parallel composition of SP-graphs $G_{0}, \ldots, G_{k}$, with $k \geq 1$, is an SP-graph with poles $u=u_{0}=u_{1}=\cdots=u_{k}$ and $v=v_{0}=v_{1}=\cdots=v_{k}$ and containing graphs $G_{i}$ as subgraphs.

The dual of a graph $G$ with respect to an embedding $\Gamma$ of $G$ is the graph $G^{\star}$ having a vertex $v_{f}$ for each face $f$ of $\Gamma$ and an edge $\left(v_{f_{1}}, v_{f_{2}}\right)$ if and only if faces $f_{1}$ and $f_{2}$ of $\Gamma$ have a common edge $e$ in $G$. We say that edge $\left(v_{f_{1}}, v_{f_{2}}\right)$ is the dual edge of $e$, and vice versa.

## 3 Sunflower SEFE

In this section we study the SUNFLOWER SEFE problem, that is the restriction of SEFE to instances in which the intersection graph $G_{\cap}$ is the same for each pair of graphs, that is, $G_{\cap}=G_{i} \cap G_{j}$ for each $1 \leq i<j \leq k$. We prove that SUNFLOWER SEFE is $\mathcal{N} \mathcal{P}$-complete with $k \geq 3$ even if $G_{\cap}$ is a spanning tree and all the input graphs are biconnected.

The proof is based on a polynomial-time reduction from the $\mathcal{N} \mathcal{P}$-complete [26] problem Betweenness, that takes as input a finite set $A$ of $n$ objects and a set $C$ of $m$ ordered triples of distinct elements of $A$, and asks whether a linear ordering $\mathcal{O}$ of the elements of $A$ exists such that for each triple $\langle\alpha, \beta, \gamma\rangle$ of $C$, we have either $\mathcal{O}=<\ldots, \alpha, \ldots, \beta, \ldots, \gamma, \ldots>$ or $\mathcal{O}=<\ldots, \gamma, \ldots, \beta, \ldots, \alpha, \ldots>$.

In order to simplify the proof, we first give in Lemma 1 an $\mathcal{N} \mathcal{P}$-completeness proof for a less restricted setting of SUNFLOWER SEFE and then describe how the produced instances can be modified in order to obtain equivalent instances with the desired properties.

A pseudo-tree is a connected graph containing only one cycle.
Lemma 1. SUNFLOWER SEFE with $k=3$ is $\mathcal{N} \mathcal{P}$-complete even if two of the input graphs are biconnected and the intersection graph $G_{\cap}$ is a spanning pseudo-tree.

Proof. The membership in $\mathcal{N P}$ has been proved in [18] by reducing SEFE to the Weak Realizability Problem [23|24].

The $\mathcal{N} \mathcal{P}$-hardness is proved by means of a polynomial-time reduction from problem Betweenness. Given an instance $\langle A, C\rangle$ of BETWEENNESS, we construct an instance $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ of SUNFLOWER SEFE that admits a SEFE if and only if $\langle A, C\rangle$ is a positive instance of BETWEENNESS, as follows.

Refer to Fig. 1 for an illustration of the construction of $G_{\cap}, G_{1}, G_{2}$, and $G_{3}$.


Fig. 1: Illustration of the composition of $G_{\cap}, G_{1}, G_{2}$, and $G_{3}$ in Lemma 1, focused on the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$ with $i=2$.

Graph $G_{\cap}$ contains a cycle $\mathcal{C}=u_{1}, v_{1}, u_{2}, v_{2}, \ldots, u_{m}, v_{m}, w_{m}, \ldots, w_{1}$ of $3 m$ vertices. Also, for each $i=1, \ldots, m, G_{\cap}$ contains a star $S_{i}$ with $n$ leaves centered at $u_{i}$ and a star $T_{i}$ with $n$ leaves centered at $v_{i}$. For each $i=1, \ldots, m$, the leaves of $S_{i}$ are labeled $x_{i}^{j}$ and the leaves of $T_{i}$ are labeled $y_{i}^{j}$, for $j=1, \ldots, n$. Graph $G_{1}$ contains all the edges of $G_{\cap}$ plus a set of edges $\left(y_{i}^{j}, x_{i+1}^{j}\right)$, for $i=1, \ldots, m$ and $j=1, \ldots, n$. Here and in the following, $i+1$ is computed modulo $m$. Graph $G_{2}$ contains all the edges of $G_{\cap}$ plus a set of edges $\left(x_{i}^{j}, y_{i}^{j}\right)$, for $i=1, \ldots, m$ and $j=1, \ldots, n$. Graph $G_{3}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. For each $i=1, \ldots, m$, consider the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$, and the corresponding vertices $x_{i}^{\alpha}, x_{i}^{\beta}$, and $x_{i}^{\gamma}$ of $S_{i}$; graph $G_{3}$ contains edges $\left(w_{i}, x_{i}^{\alpha}\right),\left(w_{i}, x_{i}^{\beta}\right),\left(w_{i}, x_{i}^{\gamma}\right),\left(x_{i}^{\alpha}, x_{i}^{\beta}\right)$, and $\left(x_{i}^{\beta}, x_{i}^{\gamma}\right)$.

First note that, by construction, $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ is an instance of SUNFLOWER SEFE, and graph $G_{\cap}$ is a spanning pseudo-tree. Also, one can easily verify that $G_{1}$ and $G_{2}$ are biconnected. In the following we prove that $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ is a positive instance if and only if $\langle A, C\rangle$ is a positive instance of BETWEENNESS.

Suppose that $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ is a positive instance, that is, $G_{1}, G_{2}$, and $G_{3}$ admit a SEFE $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$. Observe that, for each $i=1, \ldots, m$, the subgraph of $G_{1}$ induced by the vertices of $T_{i}$ and the vertices of $S_{i+1}$ is composed of a set of $n$ paths of length 3 between $v_{i}$ and $u_{i+1}$, where the $j$-th path contains internal vertices $y_{i}^{j}$ and $x_{i+1}^{j}$, for $i=1, \ldots, n$. Hence, in any SEFE of $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$, the ordering of the edges of $T_{i}$ around $v_{i}$ is reversed with respect to the ordering of the edges of $S_{i+1}$ around $u_{i+1}$, where the vertices of $T_{i}$ and $S_{i+1}$ are identified based on index $j$. Also observe that, for each $i=1, \ldots, m$, the subgraph of $G_{2}$ induced by the vertices of $S_{i}$ and the vertices of $T_{i}$ is composed of a set of $n$ paths of length 3 between $u_{i}$ and $v_{i}$, where the $j$-th path contains internal vertices $x_{i}^{j}$ and $y_{i}^{j}$, for $i=1, \ldots, n$. Hence, in any SEFE of $G_{1}, G_{2}$, and $G_{3}$, the ordering of the edges of $S_{i}$ around $u_{i}$ is the reverse of the ordering of the edges of $T_{i}$ around $v_{i}$, where the vertices of $S_{i}$ and $T_{i}$ are identified based on $j$. The two observations imply that, in any SEFE of $G_{1}, G_{2}$, and $G_{3}$, for each $i=1, \ldots, m$ the ordering of the edges of $S_{i}$ around $u_{i}$ is the same as the ordering of the edges of $S_{i+1}$ around $v_{i+1}$, where the vertices of $S_{i}$ and $S_{i+1}$ are identified based on $j$.

We construct a linear ordering $\mathcal{O}$ of the elements of $A$ from the ordering of the leaves of $S_{1}$ in $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$. Initialize $\mathcal{O}=\emptyset$; then, starting from the edge of $S_{1}$ clockwise following $\left(u_{1}, w_{1}\right)$ around $u_{1}$, consider all the leaves of $S_{1}$ in clockwise order. For each considered leaf $x_{1}^{j}$, append $j$ as the last element of $\mathcal{O}$. We prove that $\mathcal{O}$ is a solution of $\langle A, C\rangle$. For each $i=1, \ldots, m$, the subgraph of $G_{3}$ induced by vertices $w_{i}$, $u_{i}, x_{i}^{\alpha}, x_{i}^{\beta}$, and $x_{i}^{\gamma}$ is such that adding edge $\left(u_{i}, w_{i}\right)$ would make it triconnected. Hence, it admits two planar embeddings, which differ by a flip. Thus, in any SEFE of $G_{1}, G_{2}$, and $G_{3}$, edges $\left(u_{i}, x_{i}^{\alpha}\right),\left(u_{i}, x_{i}^{\beta}\right)$, and $\left(u_{i}, x_{i}^{\gamma}\right)$ appear either in this order or in the reverse order around $u_{i}$. Since for each triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ in $C$ there exists vertices $w_{i}$, $u_{i}, x_{i}^{\alpha}, x_{i}^{\beta}$, and $x_{i}^{\gamma}$ inducing a subgraph of $G_{3}$ with the above properties, and since the clockwise ordering of the leaves of $S_{i}$ is the same for every $i, \mathcal{O}$ is a solution of $\langle A, C\rangle$.

Suppose that $\langle A, C\rangle$ is a positive instance, that is, there exists an ordering $\mathcal{O}$ of the elements of $A$ in which for each triple $t_{i}$ of $C$, the three elements of $t_{i}$ appear in one of their two admissible orderings. We construct an embedding for $G_{1}, G_{2}$, and $G_{3}$. For each $i=1, \ldots, m$, the rotation schemes of $u_{i}$ and $v_{i}$ are constructed as follows. Initialize first $=v_{i-1}$ if $i>1$, otherwise first $=w_{1}$. Also, initialize last $=u_{i+1}$ if $i<m$, otherwise last $=w_{m}$. For each element $j$ of $\mathcal{O}$, place $\left(u_{i}, x_{i}^{j}\right)$ between ( $u_{i}$, first) and $\left(u_{i}, v_{i}\right)$ in the rotation scheme of $u_{i}$, and set first $=x_{i}^{j}$. Also, place $\left(v_{i}, x_{i}^{j}\right)$ between $\left(v_{i}\right.$, last $)$ and $\left(v_{i}, u_{i}\right)$ in the rotation scheme of $v_{i}$, and set last $=x_{i}^{j}$. Since all the vertices of $G_{1}$ and of $G_{2}$ different from $u_{i}$ and $v_{i}(i=1, \ldots, m)$ have degree 2 , the embeddings $\Gamma_{1}$ and $\Gamma_{2}$ of $G_{1}$ and $G_{2}$, are completely specified. To obtain the embedding $\Gamma_{3}$ of $G_{3}$, we have to specify the rotation scheme of $w_{i}$ and of the three leaves of $S_{i}$ adjacent to $w_{i}$, for $i=1, \ldots, m$. Consider a triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$. Initialize first $=w_{i-1}$, if $i>1$, and first $=u_{1}$ otherwise. Also, initialize last $=w_{i+1}$, if $i<m$, and last $=v_{m}$ otherwise. Recall that $\alpha, \beta$, and $\gamma$ appear in $\mathcal{O}$ either in this order or in the reverse one. In the former case, the rotation scheme of
$w_{i}$ is $\left(w_{i}\right.$, last $),\left(w_{i}, x_{i}^{\gamma}\right),\left(w_{i}, x_{i}^{\beta}\right),\left(w_{i}, x_{i}^{\alpha}\right),\left(w_{i}, f i r s t\right)$; the rotation scheme of $x_{i}^{\alpha}$ is $\left(x_{i}^{\alpha}, w_{i}\right),\left(x_{i}^{\alpha}, x_{i}^{\beta}\right),\left(x_{i}^{\alpha}, u_{i}\right)$; the rotation scheme of $x_{i}^{\beta}$ is $\left(x_{i}^{\beta}, x_{i}^{\alpha}\right),\left(x_{i}^{\beta}, w_{i}\right),\left(x_{i}^{\beta}, x_{i}^{\gamma}\right)$, $\left(x_{i}^{\beta}, u_{i}\right)$; and the rotation scheme of $x_{i}^{\gamma}$ is $\left(x_{i}^{\gamma}, x_{i}^{\beta}\right),\left(x_{i}^{\gamma}, w_{i}\right),\left(x_{i}^{\gamma}, u_{i}\right)$. In the latter case, the rotation scheme of $w_{i}$ is $\left(w_{i}\right.$, last $),\left(w_{i}, x_{i}^{\alpha}\right),\left(w_{i}, x_{i}^{\beta}\right),\left(w_{i}, x_{i}^{\gamma}\right),\left(w_{i}\right.$, first $)$; the rotation scheme of $x_{i}^{\alpha}$ is $\left(x_{i}^{\alpha}, x_{i}^{\beta}\right),\left(x_{i}^{\alpha}, w_{i}\right),\left(x_{i}^{\alpha}, u_{i}\right)$; the rotation scheme of $x_{i}^{\beta}$ is $\left(x_{i}^{\beta}, x_{i}^{\gamma}\right),\left(x_{i}^{\beta}, w_{i}\right),\left(x_{i}^{\beta}, x_{i}^{\alpha}\right),\left(x_{i}^{\beta}, u_{i}\right)$; and the rotation scheme of $x_{i}^{\gamma}$ is $\left(x_{i}^{\gamma}, w_{i}\right)$, $\left(x_{i}^{\gamma}, x_{i}^{\beta}\right),\left(x_{i}^{\gamma}, u_{i}\right)$. In order to prove that $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$ is a SEFE, we first observe that the embeddings of $G_{\cap}$ obtained by restricting $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ to the edges of $G_{\cap}$, respectively, coincide by construction. The planarity of $\Gamma_{1}$ and $\Gamma_{2}$ descends from the fact that the orderings of the edges incident to $u_{i}$ and $v_{i}$, for $i=1, \ldots, m$, is one the reverse of the other (where vertices are identified based on index $j$ ). The planarity of $\Gamma_{3}$ is due to the fact that, by construction, for each $i=1, \ldots, m$, the subgraph induced by $w_{i}, u_{i}$, $x_{i}^{\alpha}, x_{i}^{\beta}$, and $x_{i}^{\gamma}$ is planar in $\Gamma_{3}$. This concludes the proof of the theorem.

We are now ready to prove the main result of the section, by showing how to modify the reduction of Lemma 1 to obtain instances in which all graphs are biconnected and $G_{\cap}$ is a tree.

Theorem 1. SUNFLOWER SEFE is $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$ even if all the input graphs are biconnected and the intersection graph is a spanning tree.

Proof. The membership in $\mathcal{N P}$ has been proved in [18].
The $\mathcal{N} \mathcal{P}$-hardness is proved by means of a polynomial-time reduction from problem Betweenness. Given an instance $\langle A, C\rangle$ of BETWEENNESS, we first construct an instance $\left\langle G_{1}^{*}, G_{2}^{*}, G_{3}^{*}\right\rangle$ of SUNFLOWER SEFE that admits a SEFE if and only if $\langle A, C\rangle$ is a positive instance of BETWEENNESS by applying the reduction shown in Lemma 1 . We show how to modify $\left\langle G_{1}^{*}, G_{2}^{*}, G_{3}^{*}\right\rangle$ to obtain an equivalent instance $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ with the required properties.

Refer to Fig. 2 for an illustration of the construction of $G_{\cap}, G_{1}, G_{2}$, and $G_{3}$.
Graph $G_{\cap}$ is initialized to $G_{\cap}^{*}$. For $i=1, \ldots, m$, subdivide edge $\left(w_{u}, w_{i+1}\right)$ (where $w_{m+1}=v_{m}$ ) with two vertices $s_{i}$ and $t_{i}$, add a star with 3 leaves $\alpha_{i}, \beta_{i}$, and $\gamma_{i}$ with center $c_{i}$, and add an edge connecting $w_{i}$ to $c_{i}$. Graph $G_{1}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. As in $\left\langle G_{1}^{*}, G_{2}^{*}, G_{3}^{*}\right\rangle$, for $i=1, \ldots, m$, graph $G_{1}$ contains edges $\left(y_{i}^{j}, x_{i+1}^{j}\right)$, with $j=1, \ldots, n$, connecting the leaves of $T_{i}$ to the leaves of $S_{i+1}$. Additionally, for $i=1, \ldots, m, G_{1}$ contains edges $\left(w_{i}, \alpha_{i}\right),\left(\alpha_{i}, \beta_{i}\right),\left(\beta_{i}, \gamma_{i}\right)$, $\left(\gamma_{i}, w_{i}\right)$, and $\left(\beta_{i}, s_{i}\right)$. Here and in the following, $i+1$ is computed modulo $m$. Graph $G_{2}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. As in $\left\langle G_{1}^{*}, G_{2}^{*}, G_{3}^{*}\right\rangle$, for $i=1, \ldots, m$, graph $G_{2}$ contains edges $\left(x_{i}^{j}, y_{i}^{j}\right)$, with $j=1, \ldots, n$. Additionally, for $i=1, \ldots, m, G_{2}$ contains edges $\left(\alpha_{i}, t_{i}\right),\left(\beta_{i}, t_{i}\right)$, and $\left(\gamma_{i}, t_{i}\right)$. Graph $G_{3}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. For each $i=1, \ldots, m$, consider the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$, and the corresponding vertices $x_{i}^{\alpha}, x_{i}^{\beta}$, and $x_{i}^{\gamma}$ of $S_{i}$; graph $G_{3}$ contains edges $\left(\alpha_{i}, x_{i}^{\alpha}\right),\left(\beta_{i}, x_{i}^{\beta}\right),\left(\gamma_{i}, x_{i}^{\gamma}\right)$, and edges $\left(x_{i}^{j}, c_{i}\right)$, for every $j \notin\{\alpha, \beta, \gamma\}$. Also, for $i=1, \ldots, m$, graph $G_{3}$ contains edges $\left(y_{i}^{j}, t_{i}\right)$, with $j=1, \ldots, n$.

Observe that, graph $G_{\cap}$ is a pseudo-tree and graphs $G_{1}, G_{2}$, and $G_{3}$ are biconnected. We first prove that the constructed instance $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ of SUNFLOWER SEFE
is equivalent to instance $\langle A, C\rangle$ of Betweenness. Then, we show how to modify $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ in such a way that $G_{\cap}$ is a tree, without losing the biconnectivity of the input graphs.

Suppose that $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ is a positive instance, that is, $G_{1}, G_{2}$, and $G_{3}$ admit a SEFE $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$. Observe that, as proved in Lemma 1 for $\left\langle G_{1}^{*}, G_{2}^{*}, G_{3}^{*}\right\rangle$, in any SEFE of $G_{1}, G_{2}$, and $G_{3}$, for each $i=1, \ldots, m$, the ordering of the edges of $S_{i}$ around $u_{i}$ is the same as the ordering of the edges of $S_{i+1}$ around $v_{i+1}$, where the vertices of $S_{i}$ and $S_{i+1}$ are identified based on index $j$.

We construct a linear ordering $\mathcal{O}$ of the elements of $A$ from the ordering of the leaves of $S_{1}$ in $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$ as described in Lemma 1 .

We prove that $\mathcal{O}$ is a solution of $\langle A, C\rangle$. For each $i=1, \ldots, m$, the subgraph of $G_{1}$ induced by vertices $w_{i}, \alpha_{i}, \beta_{i}, \gamma_{i}$, and $c_{i}$ is a triconnected subgraph attached to the rest of the graph through the split pair $\left\{w_{i}, \beta_{i}\right\}$. Hence, in any planar embedding of $G_{1}$ (and hence also in $\Gamma_{1}$ ) the clockwise order of the edges around $c_{i}$ is either $\left(c_{i}, \alpha_{i}\right),\left(c_{i}, w_{i}\right)$, $\left(c_{i}, \gamma_{i}\right)$, and $\left(c_{i}, \beta_{i}\right)$, or $\left(c_{i}, \alpha_{i}\right),\left(c_{i}, \beta_{i}\right),\left(c_{i}, \gamma_{i}\right)$, and $\left(c_{i}, w_{i}\right)$. Also, the ordering of the edges of $G_{3}$ around $c_{i}$ in $\Gamma_{1}$ restricted to those belonging to $G_{\cap}$ is the same as in $\Gamma_{1}$. Further, for each $i=1, \ldots, m$, consider the subgraph of $G_{3}$ composed of the paths connecting $c_{i}$ and $u_{i}$, and containing a leaf of $S_{i}$. In any planar embedding of $G_{3}$ the ordering of the edges around $c_{i}$ is reversed with respect to the ordering of the edges around $u_{i}$, where the edges are identified based on the path they belong to. Hence, the ordering of the edges of $S_{i}$ around $u_{i}$ (and thus $\mathcal{O}$ ) is such that edges $\left(u_{i}, x_{i}^{\alpha}\right),\left(u_{i}, x_{i}^{\beta}\right)$, and $\left(u_{i}, x_{i}^{\gamma}\right)$ appear either in this order or in the reverse order. Since the clockwise ordering of the edges of $S_{i}$ around $u_{i}$ is the same for every $i, \mathcal{O}$ is a solution of $\langle A, C\rangle$.

Suppose that $\langle A, C\rangle$ is a positive instance, that is, there exists an ordering $\mathcal{O}$ of the elements of $A$ in which for each triple $t_{i}$ of $C$ the three elements of $t_{i}$ appear in one of their two admissible orderings. We construct embeddings $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ for $G_{1}, G_{2}$, and $G_{3}$, respectively. For each $i=1, \ldots, m$, the rotation schemes of $u_{i}$ and $v_{i}$ in $\Gamma_{1}$, in $\Gamma_{2}$, and in $\Gamma_{3}$ are constructed based on $\mathcal{O}$ as described in the proof of Lemma 1 Note that, in any SEFE of $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ all the vertices not belonging to the only cycle of $G_{\cap}$ lie on the same side with respect to it, as removing such a cycle from the union graph $G_{\cup}$ results in a connected graph. Hence, the rotation scheme of $w_{i}$ restricted to the edges of $G_{\cap}$ is determined in $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$. Also, the rotation scheme of $s_{i}$ is determined in $\Gamma_{1}$. Consider the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$. We set the rotation scheme of $c_{i}$ restricted to the edges of $G_{\cap}$ in $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ to be either $\left(c_{i}, w_{i}\right),\left(c_{i}, \gamma_{i}\right),\left(c_{i}, \beta_{i}\right)$, and $\left(c_{i}, \alpha_{i}\right)$, if $\alpha, \beta$, and $\gamma$ appear in this order in $\mathcal{O}$, or $\left(c_{i}, w_{i}\right),\left(c_{i}, \alpha_{i}\right),\left(c_{i}, \beta_{i}\right)$, and $\left(c_{i}, \gamma_{i}\right)$, if they appear in the reverse order in $\mathcal{O}$. Note that, given the rotation scheme of $c_{i}$ in $\Gamma_{1}$ and in $\Gamma_{2}$, the rotations schemes of $w_{i}, \alpha_{i}, \beta_{i}$, and $\gamma_{i}$ in $\Gamma_{1}$ and of $t_{i}$ in $\Gamma_{2}$ are univocally determined. Observe that $\Gamma_{1}$ and $\Gamma_{2}$ are planar by construction. We prove that $\Gamma_{3}$ can be completed to a planar drawing of $G_{3}$. In order to do that, we need to specify the rotation schemes of $c_{i}$ and $t_{i}$ in $\Gamma_{3}$. We set the rotation schemes of $c_{i}$ and of $t_{i}$ to be the reverse with respect to the rotation schemes of $u_{i}$ and of $v_{i}$, respectively, where edges are identified based on the path they belong to. As for $t_{i}$, this clearly does not introduce crossings in $\Gamma_{3}$, while for $c_{i}$ this is due to the fact that the ordering of the edges of $G_{\cap}$ incident to $c_{i}$ determined by the $i$-th triple is consistent with the rotation scheme of $u_{i}$, since this has been determined by $\mathcal{O}$.

In order to prove that $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$ is a SEFE, we observe that the embeddings of $G_{\cap}$ obtained by restricting $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ to the edges of $G_{\cap}$, respectively, coincide by construction.

Finally, in order to make $G_{\cap}$ a spanning tree, remove edge $\left(u_{1}, w_{1}\right)$ from $G_{\cap}$; add to $G_{\cap}$ two star graphs with 3 leaves, and add to $G_{\cap}$ an edge connecting $u_{1}$ to the center of the first star and an edge connecting $w_{1}$ to the center of the second star. Also, add edges to $G_{1}$, to $G_{2}$, and to $G_{3}$ among vertices of the two stars so that (i) all graphs remains biconnected, (ii) there exists an edge of $G_{1}$, an edge of $G_{2}$, and an edge of $G_{3}$ connecting a leaf of the first star to a leaf of the second star, and (iii) no edge is added to more than one graph. A suitable augmentation is shown in Fig. 2 .


Fig. 2: Illustration of the composition of $G_{\cap}, G_{1}, G_{2}$, and $G_{3}$ in Theorem 1 , focused on the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$ with $i=2$.

The above discussion proves the statement for $k=3$. To extend the theorem to any value of $k$ observe that, given an instance of SUNFLOWER SEFE with $k_{0} \geq 3$ biconnected graphs whose intersection graph $G_{\cap}$ is a tree, an equivalent instance with $k_{0}+1$ biconnected graphs whose intersection graph is a tree can be obtained by subdividing an edge of $G_{\cap}$ with a dummy vertex and by connecting it to all the leaves of $G_{\cap}$ with edges only belonging to the $\left(k_{0}+1\right)$-th graph.

## 4 PARTITIONED $k$-PAGE BOOK EMBEDDING

In this section we turn our attention to the problem of computing $k$-page book embeddings in which the assignment of the $k$ sets of edges to the $k$ pages is given as part of the input. We study this problem both in its original definition [20], called Partitioned $k$ Page Book embedding (PBE- $k$ ), and in a generalization of it, called Partitioned T-Coherent $k$-Page Book embedding (PTBE- $k$ ), in which the order of the vertices on the spine must satisfy an additional constraint, namely it must be represented by a tree $T$, also given as part of the input. Observe that, problem PTBE- $k$ in which $T$ is a star is exactly the same problem as PBE- $k$.

Problem PTBE- $k$ has been defined in [4] and proved equivalent to the case of SUNFLOWER SEFE in which the intersection graph $G_{\cap}$ is a spanning tree and all the edges not belonging to $G_{\cap}$ are incident to two leaves of such tre ${ }^{1}$ For this reason, in the following we will indifferently denote an instance $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ of PTBE$k$ by the corresponding instance $\left\langle G_{1}, \ldots, G_{k}\right\rangle$ of SUNFLOWER SEFE, where $G_{i}=$ $\left(V(T), E(T) \cup E_{i}\right)$, for each $i=1, \ldots, k$, and vice versa.

We remark that the instances of SUNFLOWER SEFE constructed in the reduction performed in Theorem 1 are such that the intersection graph $G_{\cap}$ is a spanning tree, but there exist edges not belonging to $G_{\cap}$ that are incident to internal vertices of such tree. In order to obtain equivalent instances of SUNFLOWER SEFE satisfying both properties, it would be possible to apply a procedure described in [4] that, for each edge $e \in \bigcup_{i=1}^{k} E_{i}$ incident to an internal vertex $v$ of $G_{\cap}$, adds a new leaf to $G_{\cap}$ attached to $v$ and replaces $v$ with this leaf as an endvertex of $e$. Hence, Theorem 1 implies that PTBE- $k$ is $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$. However, every time a new leaf is attached to an internal vertex, such a vertex becomes a cut-vertex for $k-1$ of the input graphs; thus, none of the $k$ graphs $G_{i}$ can be assumed to be biconnected after the whole procedure has been applied.

The relevance of this latter observation is motivated by the fact that the biconnectivity of the input graphs $G_{i}$, together with the "simplicity" of $T$, seems to be the key factor that allows for polynomial-time algorithms for the partitioned book embedding problems. Indeed, Hoske [21] proved that PBE- $k$ becomes solvable in linear-time if each graph $G_{i}$ is $T$-biconnected, that is, $E_{i}$ induces a connected graph. Notice that, $T$ biconnectivity is a stronger requirement than biconnectivity, since the former implies the latter, while the converse does not always hold. We observe that the algorithm by Hoske can be easily generalized from PBE- $k$ to PTBE- $k$ in which $T$ is not necessarily a star; hence, the same algorithmic result can be stated also for PTBE- $k$. Furthermore, to support the importance of the above mentioned key factors, we recall that PTBE- $k$ is polynomial-time solvable for $k=2$ if either both input graphs are biconnected [9], or $T=G_{\cap}$ is a star [20], or $T=G_{\cap}$ is a binary tree [21|28].

In this section we provide several results that considerably narrow the gap between the instances of the partitioned book embedding problems that can be solved in polynomial time and those that cannot (unless $P=N P$ ), by studying their complexity with respect to such factors. Namely, we prove that:

- PTBE- $k$ remains $\mathcal{N} \mathcal{P}$-complete for $k=3$ when $T$ is a caterpillar and 2 of the input graphs are biconnected (Theorem 2);
- PBE- $k$ (with no restriction on the biconnectivity) is $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$ (Theorem 4], which was known only for $k$ unbounded [28];
- PTBE- $k$ is linear-time solvable if $k-1$ of the input graphs are $T$-biconnected (Theorem 57;
- requiring one of the two graphs of an instance $\left\langle T, E_{1}, E_{2}\right\rangle$ of PTBE-2 to be biconnected (and even series-parallel) does not alter the computational complexity of the problem (Theorem6).

[^0]Due to the equivalence between PTBE- $k$ and SUNFLOWER SEFE in which $G_{\cap}$ is a spanning tree and all the edges not belonging to $G_{\cap}$ connect two of its leaves, in order to prove Theorem 2 it suffices to show that the instances produced in the reduction of Lemma 1 can be modified to obtain equivalent instances satisfying the above properties in which two of the input graphs are biconnected.

Theorem 2. PTBE-k is $\mathcal{N} \mathcal{P}$-complete for $k=3$ even if two of the input graphs are biconnected and $T=G_{\cap}$ is a caterpillar tree.

Proof. Consider an instance $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ obtained from the reduction described in Lemma 1. We describe how to obtain an equivalent instance satisfying the required properties.

Refer to Fig. 1 and to Fig. 3. First, for $i=1, \ldots, m$, replace the edges $\left(w_{i}, x_{i}^{\alpha}\right)$, $\left(w_{i}, x_{i}^{\beta}\right)$, and $\left(w_{i}, x_{i}^{\gamma}\right)$ of $G_{3}$ with length-2 paths composed of a black and of a green edge and such that the black edge is incident to $w_{i}$. Denote by $\Phi_{i}$ the star graph centered at $w_{i}$ induced by the newly inserted black edges. Second, for $i=1, \ldots, m$, subdivide edge $\left(w_{i}, w_{i+1}\right)$ of $G_{\cap}$ (where $w_{m+1}=v_{m}$ ) with a dummy vertex $d_{i}$, and add to $G_{\cap}$ a star graph $\Psi_{i}$ centered at $d_{i}$ and with 3 leaves. Observe that, at this stage of the construction, $G_{\cap}$ is a spanning pseudo-caterpillar.


Fig. 3: Illustration of how to modify the instance of SUNFLOWER SEFE so that: (i) the intersection graph $G_{\cap}$ is a spanning caterpillar and (ii) $G_{1}$ and $G_{2}$ are biconnected.

It is now possible to obtain an equivalent instance of SUNFLOWER SEFE where $G_{1}$ and $G_{2}$ are biconnected and $G_{\cap}$ remains a spanning pseudo-caterpillar, by only adding edges to $G_{1}$ and to $G_{2}$ among the leaves of $\Phi_{i}$ and $\Psi_{i}$, for $i=1, \ldots, m$.

Further, in order to make $G_{\cap}$ a spanning caterpillar, remove edge $\left(u_{1}, w_{1}\right)$ from $G_{\cap}$; add to $G_{\cap}$ two star graphs with 3 leaves, and add to $G_{\cap}$ an edge connecting $u_{1}$ to the center of the first star and an edge connecting $w_{1}$ to the center of the second star.

Finally, add edges to $G_{1}$, to $G_{2}$, and to $G_{3}$ among the leaves of the two stars so that (i) $G_{1}$ and $G_{2}$ are biconnected, (ii) there exists an edge of $G_{3}$ connecting a leaf of the
first star to a leaf of the second star, and (iii) no edge is added to more than one graph. A suitable augmentation is shown in Fig. 3 .

It is easy to observe that the constructed instance satisfies the required properties.
In the following we prove that dropping the requirement of biconnectivity of the graphs allows us to prove $\mathcal{N} \mathcal{P}$-completeness also for PBE- $k$ when $k$ is bounded by a constant, thus improving on the result of Hoske [21]. We first prove that the $\mathcal{N} \mathcal{P}$-completeness of PTBE- $k$ for $k \geq 3$ proved in Theorem 2 implies the $\mathcal{N} \mathcal{P}$-completeness of PBE- $k$ for $k \geq 4$. Then, in Theorem 4 we show that PBE- $k$ is $\mathcal{N} \mathcal{P}$-complete even for $k=3$. We recall that a linear-time algorithm for the problem is known when $k=2$ [20].

Theorem 3. PTBE- $k$ is polynomial-time reducible to $\operatorname{PBE}-(k+1)$.
Proof. Let $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ be an instance of PTBE- $k$. We construct an instance $\left\langle V^{*}, E_{1}^{*}, \ldots, E_{k}^{*}, E_{k+1}^{*}\right\rangle$ of $\operatorname{PBE}-(k+1)$ as follows.

Set $V^{*}=V(T)$ and $E_{k+1}^{*}=E(T)$. Then, for each $i=1, \ldots, k$, set $E_{i}^{*}=E_{i}$. Refer to Fig. 4


Fig. 4: Illustration of the proof of Theorem 3

We prove that $\left\langle V^{*}, E_{1}^{*}, \ldots, E_{k}^{*}, E_{k+1}^{*}\right\rangle$ is a positive instance of $\operatorname{PBE}-(k+1)$ if and only if $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ is a positive instance of PTBE- $k$.

Suppose that $\left\langle V^{*}, E_{1}^{*}, \ldots, E_{k}^{*}, E_{k+1}^{*}\right\rangle$ admits a partitioned $(k+1)$-page book embedding $\mathcal{O}^{*}$. Let $\mathcal{O}$ be the order obtained by restricting $\mathcal{O}^{*}$ to the leaves of $T$. We show that $\mathcal{O}$ is a partitioned $T$-coherent $k$-page book embedding of $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$.

For each $i=1, \ldots, k$, no two edges of $E_{i}$ alternate in $\mathcal{O}$, as otherwise the corresponding two edges of $E_{i}^{*}$ would alternate in $\mathcal{O}^{*}$, hence contradicting the hypothesis that $\mathcal{O}^{*}$ is a partitioned $(k+1)$-page book embedding. Also, we claim that order $\mathcal{O}$ is represented by $T$. Namely, place the vertices of $T$ on a horizontal line in the same order as they appear in $\mathcal{O}^{*}$; since $\mathcal{O}^{*}$ supports a crossing-free drawing of the edges of $E_{k+1}^{*}=E(T)$ on a single page and since $\mathcal{O}^{*}$ restricted to the leaves of $T$ coincides with $\mathcal{O}$, the claim follows.

Suppose that $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ admits a partitioned $T$-coherent $k$-page book embedding $\mathcal{O}$. We show how to construct a partitioned $(k+1)$-page book embedding $\mathcal{O}^{*}$ of $\left\langle V^{*}, E_{1}^{*}, \ldots, E_{k}^{*}, E_{k+1}^{*}\right\rangle$.

Initialize $\mathcal{O}^{*}=\mathcal{O}$. Root $T$ at an arbitrary internal vertex. Then, consider each internal vertex $w$ of $T$ according to a bottom-up traversal. Consider the subtree $T(w)$
of $T$ rooted at $w$ and consider the vertex $z$ of $T(w)$ appearing in $\mathcal{O}^{*}$ right before all the other vertices of $T(w)$. Place $w$ right before $z$ in $\mathcal{O}^{*}$.

We show that $\mathcal{O}^{*}$ is a partitioned $(k+1)$-page book embedding of $\left\langle V^{*}, E_{1}^{*}, \ldots, E_{k}^{*}, E_{k+1}^{*}\right\rangle$.
For each $i=1, \ldots, k$, no two edges of $E_{i}^{*}$ alternate in $\mathcal{O}^{*}$, as otherwise the corresponding two edges of $E_{i}$ would alternate in $\mathcal{O}$, hence contradicting the hypothesis that $\mathcal{O}$ is a partitioned $T$-coherent $k$-page book embedding. Also, the fact that no two edges of $E_{k+1}^{*}$ alternate in $\mathcal{O}^{*}$ descends from the fact that, for each vertex $w$ of $T$, all the vertices belonging to the subtree $T(w)$ of $T$ rooted at $w$ appear consecutively in $\mathcal{O}^{*}$. We prove this property by induction. In the base case $w$ is the parent of a set of leaves. In this case, the statement holds since $\mathcal{O}$ is represented by $T$. Inductively assume that, for all children $u_{i}$ of $w$, the vertices of $T\left(u_{i}\right)$ are consecutive in $\mathcal{O}^{*}$. Also, by construction, $w$ has been placed right before all vertices of $T(w)$. It follows that all vertices of $T(w)$ (including $w$ ) are consecutive in $\mathcal{O}^{*}$. This concludes the proof of the theorem.

As PBE- $k$ is a special case of PTBE- $k$, the problem belongs to $\mathcal{N} \mathcal{P}$. Hence, putting together the results of Theorem 3 and of Theorem 2, we obtain the following:

Corollary 1. PBE- $k$ is $\mathcal{N} \mathcal{P}$-complete for $k \geq 4$.
We strengthen this result by proving that the $\mathcal{N} \mathcal{P}$-hardness of PBE- $k$ holds even for $k=3$. As for Theorem 2 we describe the proof in terms of the corresponding SUNFLOWER SEFE problem, namely in the case in which $G_{\cap}$ is a star graph and all the edges not belonging to $G_{\cap}$ connect two of its leaves.

Theorem 4. PBE- $k$ is $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$.
Proof. We prove the statement for $k=3$, as for $k \geq 4$ it descends from Corollary 1 . The $\mathcal{N} \mathcal{P}$-hardness is shown by means of a polynomial-time reduction from problem BETWEENNESS. Given an instance $\langle A, C\rangle$ of BETWEENNESS, we construct an instance $\left\langle V, E_{1}, E_{2}, E_{3}\right\rangle$ of PBE-3 that admits a partitioned 3-page book embedding if and only if $\langle A, C\rangle$ is a positive instance of BETWEENNESS.

We describe instance $\left\langle V, E_{1}, E_{2}, E_{3}\right\rangle$ in terms of the corresponding instance $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ of SUnflower SEFE in which $G_{\cap}$ is a star. Refer to Fig. 5 .

Graph $G_{\cap}$ is initialized to a star graph with center $\phi$, a leaf $\omega$ and, for $i=0, \ldots, m$, leaves $a_{i}$ and $b_{i}$. Also, for $i=1, \ldots, m, G_{\cap}$ contains $n$ leaves $x_{i}^{1}, \ldots, x_{i}^{n}, n$ leaves $y_{i}^{1}, \ldots, y_{i}^{n}$, plus two additional leaves $x_{i}^{*}$ and $y_{i}^{*}$. Finally, $G_{\cap}$ contains $n$ leaves $y_{0}^{1}, \ldots, y_{0}^{n}$, plus an additional leaf $y_{0}^{*}$.

Graph $G_{1}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. For $i=1, \ldots, m$, graph $G_{1}$ contains an edge $\left(\omega, a_{i}\right)$. Also, for $i=1, \ldots, m$, graph $G_{1}$ contains edges $\left(x_{i}^{j}, y_{i-1}^{j}\right)$, with $j=1, \ldots n$, and edge $\left(x_{i}^{*}, y_{i-1}^{*}\right)$.

Graph $G_{2}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. For $i=0, \ldots, m-1$, graph $G_{2}$ contains an edge $\left(\omega, b_{i}\right)$. Also, for $i=1, \ldots, m$, graph $G_{2}$ contains edges $\left(x_{i}^{j}, y_{i}^{j}\right)$, with $j=1, \ldots n$, and edge $\left(x_{i}^{*}, y_{i}^{*}\right)$.

Graph $G_{3}$ contains all the edges of $G_{\cap}$ plus a set of edges defined as follows. Graph $G_{3}$ contains edges $\left(\omega, a_{o}\right)$ and $\left(\omega, b_{m}\right)$. Also, for each $i=0, \ldots, m$, graph $G_{3}$ contains edges $\left(a_{i}, b_{i}\right),\left(a_{i}, y_{i}^{*}\right),\left(b_{i}, y_{i}^{*}\right)$, and edges $\left(y_{i}^{*}, x_{i}^{j}\right)$, with $j=1, \ldots, n$. Finally, for $i=1$, dots, $m$, consider the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$, and the corresponding


Fig. 5: Illustration of the composition of $G_{\cap}, G_{1}, G_{2}$, and $G_{3}$ in Theorem 4 , focused on the $i$-th triple $t_{i}=\langle\alpha, \beta, \gamma\rangle$ of $C$.
vertices $x_{i}^{\alpha}, x_{i}^{\beta}$, and $x_{i}^{\gamma}$; graph $G_{3}$ contains edges $\left(a_{i}, x_{i}^{\alpha}\right),\left(a_{i}, x_{i}^{\beta}\right),\left(a_{i}, x_{i}^{\gamma}\right),\left(x_{i}^{\alpha}, x_{i}^{\beta}\right)$, and $\left(x_{i}^{\beta}, x_{i}^{\gamma}\right)$.

We prove that the constructed instance $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ of SUNFLOWER SEFE is equivalent to instance $\langle A, C\rangle$ of BETWEENNESS.

Suppose that $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ is a positive instance, that is, $G_{1}, G_{2}$, and $G_{3}$ admit a SEFE $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$. Observe that, vertices $\phi, \omega$, and vertices $a_{i}$ and $b_{i}$, with $i=$ $1, \ldots, m$, induce a wheel with central vertex $\phi$ in $G_{3}$. Hence, in any planar embedding of $G_{3}$, edges $(\omega, \phi),\left(a_{o}, \phi\right),\left(b_{o}, \phi\right), \ldots,\left(a_{m}, \phi\right)$, and $\left(b_{m}, \phi\right)$ appear in this order (or in the reverse order) around $\phi$. Also, since $y_{i}^{*}$ is adjacent in $G_{3}$ to both $a_{i}$ and $b_{i}$, for $i=0, \ldots, m$, edge $\left(y_{i}^{*}, \phi\right)$ appears between edges $\left(a_{i}, \phi\right)$ and $\left(b_{i}, \phi\right)$ around $\phi$ in any planar embedding of $G_{3}$. Hence, since all vertices $y_{i}^{j}$, with $j=1, \ldots, n$, are adjacent in $G_{3}$ to $y_{i}^{*}$, also edges $\left(y_{i}^{j}, \phi\right)$ appear between $\left(a_{i}, \phi\right)$ and $\left(b_{i}, \phi\right)$ around $\phi$ in any planar embedding of $G_{3}$. Furthermore, for $i=1, \ldots, m$, edges $\left(x_{i}^{j}, \phi\right)$, with $j=1 \ldots, n$, and edge $\left(x_{i}^{*}, \phi\right)$ appear between $\left(b_{i-1}, \phi\right)$ and $\left(a_{i}, \phi\right)$ around $\phi$ in $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$. This is due to the following two facts: (1) all vertices $x_{i}^{j}$ and vertex $x_{i}^{*}$ are adjacent in $G_{1}$ to a vertex $y_{i-1}$ such that edge $\left(y_{i-1}, \phi\right)$ appears between edges $\left(a_{i-1}, \phi\right)$ and $\left(b_{i_{1}}, \phi\right)$ around $\phi$, and in $G_{2}$ to a vertex $y_{i}$ such that edge $\left(y_{i}, \phi\right)$ appears between edges $\left(a_{i}, \phi\right)$ and $\left(b_{i}, \phi\right)$ around $\phi$; (2) there exists edges $\left(\omega, b_{i-1}\right)$ in $G_{2}$ and $\left(\omega, a_{i}\right)$ in $G_{1}$. Refer to Fig. 5 for a possible ordering of the edges around $\phi$ in a SEFE.

Observe that, due to the properties of the ordering of the edges of $G_{\cap}$ around $\phi$ discussed above, for $i=1, \ldots, m$, edge $\left(x_{i}^{*}, \phi\right)$ and edges $\left(x_{i}^{j}, \phi\right)$, with $j=1, \ldots, n$, behave similarly to the edges of the star graph $S_{i}$ used in Lemma 1 and edge ( $y_{i}^{*}, \phi$ ) and edges $\left(y_{i}^{j}, \phi\right)$, with $j=1, \ldots, n$, behave similarly to the edges of the star graph $T_{i}$ used in Lemma 1 Namely, in any SEFE of $G_{1}, G_{2}$, and $G_{3}$, for each $i=1, \ldots, m-1$, the ordering of the edges $\left(x_{i}^{j}, \phi\right)$, with $j=1, \ldots, n$, and edge $\left(x_{i}^{*}, \phi\right)$ around $\phi$ is the same as the ordering of the edges $\left(x_{i+1}^{j}, \phi\right)$, with $j=1, \ldots, n$, and edge $\left(x_{i+1}^{*}, \phi\right)$ around $\phi$, where the vertices are identified based on index $j$.

We construct a linear ordering $\mathcal{O}$ of the elements of $A$ from the ordering of the leaves of $x_{1}^{j}$, with $j=1, \ldots, n$, in $\left\langle\Gamma_{1}, \Gamma_{2}, \Gamma_{3}\right\rangle$ as described in Lemma 1

We prove that $\mathcal{O}$ is a solution of $\langle A, C\rangle$. For each $i=1, \ldots, m$, the subgraph of $G_{3}$ induced by vertices $\phi_{i}, x_{i}^{\alpha}, x_{i}^{\beta}, x_{i}^{\gamma}$, and $a_{i}$ is a triconnected subgraph attached to the rest of the graph through the split pair $\left\{\phi, a_{i}\right\}$. Hence, in any planar embedding of $G_{3}$ (and hence also in $\left.\Gamma_{3}\right)$ edges $\left(\phi, x_{i}^{\alpha}\right),\left(\phi, x_{i}^{\beta}\right),\left(\phi, x_{i}^{\gamma}\right)$ appear either in this order or in the reverse order around $\phi$. Since the ordering of the edges $\left(x_{i+1}^{j}, \phi\right)$, with $j=1, \ldots, n$, around $\phi$ is the same for every $i, \mathcal{O}$ is a solution of $\langle A, C\rangle$.

Suppose that $\langle A, C\rangle$ is a positive instance, that is, there exists an ordering $\mathcal{O}$ of the elements of $A$ in which for each triple $t_{i}$ of $C$ the three elements of $t_{i}$ appear in one of their two admissible orderings. In order to construct embeddings $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ for $G_{1}, G_{2}$, and $G_{3}$, respectively, we describe the order of the edges of $G_{\cap}$ around $\phi$. Initialize the rotation scheme of $\phi$ to $(\omega, \phi),\left(a_{o}, \phi\right),\left(y_{0}^{*}, \phi\right),\left(b_{o}, \phi\right)$, and, for $i=1, \ldots, m$, $\left(x_{i}^{*}, \phi\right),\left(a_{i}, \phi\right),\left(y_{i}^{*}, \phi\right)$, and $\left(b_{i}, \phi\right)$. Then, for $i=1, \ldots, m$, initialize first $_{i}=a_{i}$ and last $_{i}=x_{i}^{*}$. For each element $j$ of $\mathcal{O}$, place $\left(x_{i}^{j}, \phi\right)$ between $\left(\right.$ first $\left._{i}, \phi\right)$ and (last $\left.{ }_{i}, \phi\right)$ in the rotation scheme of $\phi$, and set first $_{i}=x_{i}^{j}$. Also, for $i=0, \ldots, m$, initialize first $_{i}=y_{i}^{*}$ and last ${ }_{i}=b_{i}$. For each element $j$ of $\mathcal{O}$, place $\left(y_{i}^{j}, \phi\right)$ between $\left(\right.$ first $\left._{i}, \phi\right)$ and $\left(\right.$ last $\left._{i}, \phi\right)$ in the rotation scheme of $\phi$, and set last ${ }_{i}=y_{i}^{j}$. Refer to Fig. 5 for an illustration of the construction of the rotation scheme of $\phi$.

The rest of the construction of $\Gamma_{1}, \Gamma_{2}$, and $\Gamma_{3}$ and the proof that such embeddings determine a SEFE of $\left\langle G_{1}, G_{2}, G_{3}\right\rangle$ works as in the proof of Lemma 1 . In particular, the fact that the rotation scheme of $\phi$ determines a planar embedding of the triconnected subgraphs of $G_{3}$ induced by vertices $\phi, a_{i}, x_{i}^{\alpha}, x_{i}^{\beta}, x_{i}^{\gamma}$, for $i=1, \ldots, m$, derives from the fact that $\mathcal{O}$ is a solution of instance $\langle A, C\rangle$ of BETWEENNESS. This concludes the proof of the theorem.

Although PTBE- $k$ has been shown $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$ even when two of the input graphs are biconnected in Theorem 2, we show that stronger conditions on the connectivity of the graphs allow for a polynomial-time solution of the problem. As observed before, the linear-time algorithm by Hoske [21] for PBE- $k$ when each graph is $T$-biconnected can be easily extended to solve PTBE- $k$ under the same conditions. In the following theorem we prove that for $k \geq 2$ this is true even if only $k-1$ graphs are $T$-biconnected.

At this aim, we describe an algorithm that we call ALGO- $(k-1)-T$-BICO to decide whether an instance $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ of PTBE- $k$ is positive in the case in which $k-1$ graphs $G_{i}$ are $T$-biconnected. In the description of the algorithm we assume, without loss of generality, that graphs $G_{1}, \ldots, G_{k-1}$ are $T$-biconnected.
STEP 1. For $i=1, \ldots, k-1$, we construct an auxiliary graph $H_{i}$ as follows. Initialize $H_{i}$ to $G_{i}$; remove from $H_{i}$ the internal vertices of $T$ and their incident edges; and add to $H_{i}$ a vertex $w_{i}$ and connect it to all vertices of $H_{i}$ (that is, to all leaves of $T$ ).
STEP 2. For $i=1, \ldots, k-1$, we construct a PQ-tree $\mathcal{T}_{i}$ representing all possible orders of the edges around $w_{i}$ in a planar embedding of $H_{i}$ by applying the planarity testing algorithm of Booth and Lueker [11]. Since, by construction, all vertices of $H_{i}$ different from $w_{i}$ are adjacent to $w_{i}$, the leaves of $\mathcal{T}_{i}$ are in one-to-one correspondence with the leaves of $T$. Hence, all PQ-trees $\mathcal{T}_{i}$ have the same leaves.

STEP 3. We intersect all PQ-trees $\mathcal{T}_{1}, \ldots, \mathcal{T}_{k-1}$ to obtain a PQ-tree $\mathcal{T}^{*}$ representing all the possible partitioned book embeddings of graphs $H_{i} \backslash w_{i}$, for $i=1, \ldots, k-1$. We remark that the procedure described so far is analogous to the one described in [21] to compute a PBE- $k$ of $k T$-biconnected graphs.
STEP 4. We intersect $\mathcal{T}^{*}$ with $T$ to obtain a PQ-tree $\mathcal{T}$ representing all the possible partitioned $T$-coherent book embeddings of instance $\left\langle T, E_{1}, \ldots, E_{k-1}\right\rangle$.
STEP 5. We construct a representative graph $G_{\mathcal{T}}$ from $\mathcal{T}$, as described in [16], composed of wheel graphs (that is, graphs consisting of a central vertex and of a cycle, called the rim of the wheel, such that the central vertex is connected to every vertex of the rim), edges connecting vertices of the rims of different wheels not creating simple cycles containing vertices belonging to more than one wheel, and vertices of degree 1 , which are in one-to-one correspondence with the leaves of $\mathcal{T}$, each connected to a vertex of the rim of some wheel.
STEP 6. We extend graph $G_{\mathcal{T}}$ by adding an edge between two degree- 1 vertices if and only if the two leaves of $T$ corresponding to such vertices are connected by an edge of $E_{k}$; hence obtaining graph $H$.
STEP 7. We return YES if $H$ is planar, otherwise we return NO.
In the following theorem we prove the correctness and the time complexity of ALGO- $(k-1)-T$-BICO,
Theorem 5. Let $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ be an instance of PTBE- $k$ with $k \geq 2$ in which $k-1$ graphs are T-biconnected. There exists an $O(k \cdot n)$-time algorithm to decide whether $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ admits a Partitioned T-Coherent $k$-Page Book embedding, where $n$ is the number of vertices of $T$.
Proof. The algorithm that decides PTBE- $k$ for $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ is ALGO- $(k-1)-T$ BICO.

We prove the correctness. First, observe that, as proved in [21], the PQ-tree $\mathcal{T}^{*}$ constructed at STEP 3 encodes all and only the partitioned $(k-1)$-page book embeddings of instance $\left\langle\mathcal{L}(T), E_{1}, \ldots, E_{k-1}\right\rangle$. Thus, intersecting $\mathcal{T}^{*}$ with tree $T$ yields a PQ-tree $\mathcal{T}$ (see STEP 4) encoding all and only the partitioned $T$-coherent $(k-1)$-page book embedding $\int^{2}$ ] of instance $\left\langle T, E_{1}, \ldots, E_{k-1}\right\rangle$.

Also, as proved in [16], there exists a one-to-one correspondence between the possible orderings of the leaves of $\mathcal{T}$ and the possible orderings obtained by restricting the order of the vertices in an Eulerian tour of the outer face in a planar embedding of $G_{\mathcal{T}}$ to the degree- 1 vertices.

Given a planar embedding $\Gamma$ of $H$ (see Fig. 6(a), we construct a partitioned $T$ coherent $k$-page book embedding $\mathcal{O}$ of $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$. We claim that $\Gamma$ can be modified in order to obtain a planar embedding $\Gamma^{\prime}$ of $H$ (see Fig. 6(c)) such that all the degree-1 vertices of $G_{\mathcal{T}}$ lie on the outer face of the embedding $\bar{\Gamma}_{\mathcal{T}}$ of $G_{\mathcal{T}}$ obtained by restricting $\Gamma^{\prime}$ to the vertices and edges of $G_{\mathcal{T}}$.

The claim implies that the order $\mathcal{O}$ of the degree-1 vertices in a Eulerian tour of the outer face of $\Gamma_{\mathcal{T}}$ is a partitioned $T$-coherent $k$-page book embedding of $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$ since (i) $\mathcal{O}$ is represented by $\mathcal{T}$ and (ii) no two edges of $E_{k}$ alternate in $\mathcal{O}$, given that $\Gamma^{\prime}$ is planar.

[^1]We prove the claim. First, we show that starting from $\Gamma$ we can obtain a planar drawing $\Gamma^{*}$ of $H$ such that every wheel of $G_{\mathcal{T}}$ is drawn canonically (see Fig. 6(b) , namely, with its central vertex lying in the interior of its rim. Consider any wheel $W$ of $G_{\mathcal{T}}$ with central vertex $\omega$ that is not drawn canonically in $\Gamma$. This implies that there exist two vertices $a$ and $b$ of the rim of $W$ such that all the vertices of $W$ different from $a, b$, and $\omega$ lie in the interior of cycle $\langle a, b, \omega\rangle$. Since, by construction of $G_{\mathcal{T}}$ and of $H$, vertex $\omega$ is not adjacent to any vertex not belonging to $W$, it is possible to reroute edge $(a, b)$ as a curve arbitrarily close to path $(a, \omega, b)$ so that cycle $\langle a, b, \omega\rangle$ does not enclose any vertex of $H$. Observe that, such an operation might determine a change in the rotation scheme of $a$ or $b$. Applying such a procedure to all non-canonically drawn wheels, eventually results in a planar drawing $\Gamma^{*}$ of $H$ such that all wheels of $G_{\mathcal{T}}$ are drawn canonically. Second, we show how to obtain $\Gamma^{\prime}$ starting from $\Gamma^{*}$ (see Fig. 6(c).). Consider any wheel $W$ of $G_{\mathcal{T}}$, with central vertex $\omega$. For each two adjacent vertices $a$ and $b$ of the rim of $W$, if there exist vertices of $H$ in the interior of cycle $\langle a, b, \omega\rangle$, then we reroute edge $(a, b)$ as a curve arbitrarily close to path $(a, \omega, b)$ so that cycle $\langle a, b, \omega\rangle$ does not enclose any vertex of $H$. Since $\omega$ is not connected to vertices of $H$ other than those belonging to the rim of $W$, this operation does not introduce any crossing. After this operation has been performed for every two adjacent edges of the rim of $W$, there exists no vertex of $H$ not belonging to $W$ in the interior of the rim of $W$, since $W$ is drawn canonically. This concludes the proof of the claim, since $G_{\mathcal{T}}$ does not contain any simple cycle containing vertices belonging to more than one wheel and no wheel of $G_{\mathcal{T}}$ contains in its interior vertices of $H$ not belonging to it.


Fig. 6: Illustration for the proof of Theorem 5 Edges of $G_{\mathcal{T}}$ are black solid curves. Edges of $E_{k}$ are blue dotted curves. Edges of $H$ which have been redrawn with respect to the previous drawing are red dashed curves. Central vertices of the wheels are white squares. Degree-1 vertices of $G_{\mathcal{T}}$ are white circles. (a) Planar drawing $\Gamma$ of $H$. (b) Planar drawing $\Gamma^{*}$ of $H$ in which every wheel of $G_{\mathcal{T}}$ is drawn canonically. (c) Planar drawing $\Gamma^{\prime}$ of $H$ in which all the degree-1 vertices of $G_{\mathcal{T}}$ lie in the outer face of $\Gamma^{\prime}$ restricted to $G_{\mathcal{T}}$.

Given a partitioned $T$-coherent $k$-page book embedding $\mathcal{O}$ of $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$, we construct a planar embedding $\Gamma$ of $H$. To obtain $\Gamma$, we first augment $G_{\mathcal{T}}$ to an auxiliary graph $U$ by adding a dummy edge between two degree- 1 vertices of $G_{\mathcal{T}}$ if and only
if the corresponding leaves of $T$ are either adjacent in $\mathcal{O}$ or appear as the first and last element in $\mathcal{O}$. Since $\mathcal{O}$ is a partitioned $T$-coherent $k$-page book embedding $\mathcal{O}$ of $\left\langle T, E_{1}, \ldots, E_{k}\right\rangle$, it is possible to find a planar embedding of $G_{\mathcal{T}}$ in which the degree-1 vertices appear along the Eulerian tour of the outer face in the same order as $\mathcal{O}$. Hence, graph $U$ is planar. Produce a planar drawing $\Gamma^{*}$ of $H$ whose outer face is the cycle composed of all the dummy edges. Since $\mathcal{O}$ is a partitioned $T$-coherent $k$-page book embedding, no two edges of $E_{k}$ alternate in $\mathcal{O}$. Hence they can be drawn in the outer face of $\Gamma^{*}$ without introducing crossings. Removing all dummy edges yields a planar embedding $\Gamma$ of $H$.

We prove the time complexity. STEP 1 and STEP 2 take $O(k \cdot n)$ time, since the time-complexity of constructing a PQ-tree on a ground set of $n$ elements is linear in the size of the ground set [10]11]. STEP 3 and STEP 4 take $O((k-2) \cdot n)$ and $O(n)$ time, respectively, since the intersection of two PQ-trees can be performed in amortized linear time in their size [10] and the size of the obtained PQ-tree stays linear in the size of the ground set. STEP 5 takes linear time in the size of $\mathcal{T}$, since it corresponds to replacing each Q-node with a wheel and each P-node with a cut vertex connecting the wheels [16]. Observe that, graph $G_{\mathcal{T}}$ has size linear in $n$, since each vertex of the rim of a wheel corresponds to exactly one edge of $\mathcal{T}$. STEP 6 takes $O\left(\left|E_{k}\right|\right)=O(n)$ time and produces a graph $H$ with $O(n)$ vertices. Finally, testing the planarity of $H$ takes linear time in the size of $H$ [11].

This concludes the proof of the theorem.

### 4.1 Partitioned T-Coherent 2-Page Book embedding

In this subsection we restrict our attention to instances $\left\langle T, E_{1}, E_{2}\right\rangle$ of PTBE- $k$ with $k=2$. We remark that this problem has been proved [4] equivalent to SEFE for $k=2$ when the intersection graph $G_{\cap}$ is connected. This problem was only known to be polynomial-time solvable if (i) $T$ is a star [20], (ii) $G_{1}=\left(V(T), E(T) \cup E_{1}\right)$ and $G_{2}=\left(V(T), E(T) \cup E_{2}\right)$ are biconnected [9], or (iii) $T$ is binary [21|28]. Theorem5] extends the class of polynomially-solvable instances by showing that PTBE-2 is lineartime solvable if either $G_{1}$ or $G_{2}$ is $T$-biconnected.

In the following we prove that, in order to find a polynomial-time algorithm for the general setting of PTBE-2, it suffices to focus on instances of PTBE-2 in which only one of the two graphs is biconnected (not $T$-biconnected) and series-parallel.

Theorem 6. Let $\left\langle T, E_{1}, E_{2}\right\rangle$ be an instance of PTBE-2. There exists an equivalent instance $\left\langle T^{*}, E_{1}^{*}, E_{2}^{*}\right\rangle$ of PTBE-2 such that one of the two graphs is biconnected and series-parallel.

Proof. We describe how to construct instance $\left\langle T^{*}, E_{1}^{*}, E_{2}^{*}\right\rangle$ starting from $\left\langle T, E_{1}, E_{2}\right\rangle$. Refer to Fig7.

Let $r$ be any internal vertex of $T$. Tree $T^{*}$ is constructed as follows. Initialize tree $T^{*}$ to the union of two copies $T^{\prime}$ and $T^{\prime \prime}$ of $T$. For each vertex $v \in T$, let $v^{\prime}$ and $v^{\prime \prime}$ be the two copies of $v$ in $T^{\prime}$ and in $T^{\prime \prime}$, respectively. Add a vertex $r^{*}$ to $T^{*}$ and edges $\left(r^{*}, r^{\prime}\right)$ and $\left(r^{*}, r^{\prime \prime}\right)$. Sets $E_{1}^{*}$ and $E_{2}^{*}$ are defined as follows. Set $E_{1}^{*}=\left\{\left(v_{i}^{\prime}, v_{j}^{\prime}\right)\right.$ : $\left.\left(v_{i}, v_{j}\right) \in E_{1}\right\} \cup\left\{\left(v_{i}^{\prime \prime}, v_{j}^{\prime \prime}\right):\left(v_{i}, v_{j}\right) \in E_{2}\right\}$. Also, set $E_{2}^{*}=\left\{\left(v_{i}^{\prime}, v_{i}^{\prime \prime}\right): v_{i} \in \mathcal{L}(T)\right\}$, where $\mathcal{L}(T)$ denotes the set of leaves of $T$.


Fig. 7: Illustration of the proof of Theorem 6

It is straightforward to observe that, by construction, the graph $G_{2}^{*}$ composed of $\mathcal{T}^{*}$ plus the edges in $E_{2}^{*}$ is biconnected and series-parallel. We prove that $\left\langle T^{*}, E_{1}^{*}, E_{2}^{*}\right\rangle$ is equivalent to $\left\langle T, E_{1}, E_{2}\right\rangle$.

Suppose that $\left\langle T, E_{1}, E_{2}\right\rangle$ admits a partitioned $T$-coherent 2-page book embedding $\mathcal{O}$. We construct an order $\mathcal{O}^{*}$ for $\left\langle T^{*}, E_{1}^{*}, E_{2}^{*}\right\rangle$ as follows. For each $i=1, \ldots,|\mathcal{L}(T)|$, consider the vertex $v_{j}$ at position $i$ in $\mathcal{O}$. Place vertices $v_{j}^{\prime}$ and $v_{j}^{\prime \prime}$ at positions $i$ and $2 \cdot|\mathcal{L}(T)|-i+1$ in $\mathcal{O}^{*}$, respectively.

We prove that $\mathcal{O}^{*}$ is a partitioned $T$-coherent 2-page book embedding of $\left\langle T^{*}, E_{1}^{*}, E_{2}^{*}\right\rangle$. First, we observe that $\mathcal{O}^{*}$ is represented by $T^{*}$, as (i) $T^{*}$ is composed of two copies of $T$ connected through $r^{*}$, (ii) $\mathcal{O}^{*}$ is composed of two suborders of which the first coincides with $\mathcal{O}$ and the second coincides with the reverse of $\mathcal{O}$, where each element $v_{j}$ of $\mathcal{O}$ is identified with elements $v_{j}^{\prime}$ and $v_{j}^{\prime \prime}$ of $\mathcal{O}^{*}$, and (iii) $\mathcal{O}$ is represented by $T$. Second, we prove that the endvertices of edges in $E_{1}^{*}$ and $E_{2}^{*}$ do not alternate in $\mathcal{O}^{*}$. As for the edges in $E_{2}^{*}$, we observe that for every two edges $\left(v_{i}^{\prime}, v_{i}^{\prime \prime}\right)$ and $\left(v_{j}^{\prime}, v_{j}^{\prime \prime}\right)$ with $i<j$, both vertices $v_{j}^{\prime}$ and $v_{j}^{\prime \prime}$ lie between $v_{i}^{\prime}$ and $v_{i}^{\prime \prime}$ in $\mathcal{O}^{*}$. As for the edges in $E_{1}^{*}$, we first observe that no alternation occurs between the endvertices of edges $\left(v_{i}^{\prime}, v_{j}^{\prime}\right)$ and $\left(v_{h}^{\prime \prime}, v_{k}^{\prime \prime}\right)$ as both $v_{i}^{\prime}$ and $v_{j}^{\prime}$ appear in $\mathcal{O}^{*}$ before $v_{h}^{\prime \prime}$ and $v_{k}^{\prime \prime}$, by construction. Also, no two edges $\left(v_{i}^{\prime}, v_{j}^{\prime}\right)$ and $\left(v_{h}^{\prime}, v_{k}^{\prime}\right)$ alternate in $\mathcal{O}^{*}$ as otherwise edges $\left(v_{i}, v_{j}\right)$ and $\left(v_{h}, v_{k}\right)$ would alternate in $\mathcal{O}$. For the same reason, no two edges $\left(v_{i}^{\prime \prime}, v_{j}^{\prime \prime}\right)$ and $\left(v_{h}^{\prime \prime}, v_{k}^{\prime \prime}\right)$ alternate in $\mathcal{O}^{*}$.

Suppose that $\left\langle T^{*}, E_{1}^{*}, E_{2}^{*}\right\rangle$ admits a partitioned $T$-coherent 2-page book embedding $\mathcal{O}^{*}$. We first observe that in $\mathcal{O}^{*}$ either all vertices $v_{i}^{\prime} \in T^{\prime}$ appear consecutively or all vertices $v_{i}^{\prime \prime} \in T^{\prime \prime}$ do, as $\mathcal{O}^{*}$ is represented by $T^{*}$ and $T^{*}$ consists of the two copies $T^{\prime}$ and $T^{\prime \prime}$ of $T$. Also, given a partitioned $T$-coherent 2-page book embedding $\mathcal{O}^{1}$, it is possible to obtain a new one $\mathcal{O}^{2}$ by performing a circular shift on the elements of $\mathcal{O}^{1}$, that is, by setting the first element of $\mathcal{O}^{1}$ as the last element of $\mathcal{O}^{2}$ and by setting the element at position $i$ in $\mathcal{O}^{1}$ as the element at position $i-1$ in $\mathcal{O}^{2}$, for each $i=2, \ldots,\left|\mathcal{O}^{1}\right|$. Hence, in the following, we will assume that $\mathcal{O}^{*}$ is such that all the vertices $v_{i}^{\prime} \in T^{\prime}$ appear before all the vertices $v_{j}^{\prime \prime} \in T^{\prime \prime}$.

We construct an order $\mathcal{O}$ for $\left\langle T, E_{1}, E_{2}\right\rangle$ as follows. For each $i=1, \ldots,\left|\mathcal{L}\left(T^{\prime}\right)\right|$, consider the vertex $v_{j}^{\prime}$ at position $i$ in $\mathcal{O}^{*}$ and place vertex $v_{j}$ at position $i$ in $\mathcal{O}$.

We prove that $\mathcal{O}$ is a partitioned $T$-coherent 2-page book embedding of $\left\langle T, E_{1}, E_{2}\right\rangle$. First, we observe that $\mathcal{O}$ is represented by $T$, as the suborder of $\mathcal{O}^{*}$ restricted to its first $|\mathcal{L}(T)|$ elements (that corresponds to a copy of $\mathcal{O}$ ) is represented by $T^{\prime}$ (that is a copy of $T$, where vertex $v_{i}^{\prime} \in T^{\prime}$ is identified with vertex $v_{i} \in T$ ). Second, we prove that the endvertices of edges in $E_{1}$ and $E_{2}$ do not alternate in $\mathcal{O}$. In order to prove that, first observe that the suborder $\mathcal{O}^{\prime}$ of $\mathcal{O}^{*}$ restricted to its first $|\mathcal{L}(T)|$ elements is the reverse of the suborder $\mathcal{O}^{\prime \prime}$ of $\mathcal{O}^{*}$ restricted to its last $|\mathcal{L}(T)|$ elements, where vertex $v_{i}^{\prime} \in T^{\prime}$ is identified with vertex $v_{i}^{\prime \prime} \in T^{\prime \prime}$. This is due to the fact that (i) for every $i=1, \ldots,|\mathcal{L}(T)|$, there exists edge $\left(v_{i}^{\prime}, v_{i}^{\prime \prime}\right)$ and (ii) all the vertices $v_{i}^{\prime} \in T^{\prime}$ appear before all the vertices $v_{j}^{\prime \prime} \in T^{\prime \prime}$. This implies that if the endvertices of two edges $\left(v_{i}, v_{j}\right)$ and $\left(v_{h}, v_{k}\right)$ belonging to $E_{1}^{\prime}$ (to $E_{2}$ ) alternate in $\mathcal{O}$, then the corresponding copies $v_{i}^{\prime}$, $v_{j}^{\prime}, v_{h}^{\prime}$, and $v_{k}^{\prime}$ (the corresponding copies $v_{i}^{\prime \prime}, v_{j}^{\prime \prime}, v_{h}^{\prime \prime}$, and $v_{k}^{\prime \prime}$ ) alternate in $\mathcal{O}^{*}$. However, this contradicts the fact that $\mathcal{O}^{*}$ is a partitioned $T$-coherent 2-page book embedding of $\left\langle T, E_{1}, E_{2}\right\rangle$, since edges $\left(v_{i}^{\prime}, v_{j}^{\prime}\right)$ and $\left(v_{h}^{\prime}, v_{k}^{\prime}\right)$ (edges $\left(v_{i}^{\prime \prime}, v_{j}^{\prime \prime}\right)$ and $\left(v_{h}^{\prime \prime}, v_{k}^{\prime \prime}\right)$ ) exist in $E_{1}^{*}$ by construction. This concludes the proof of the theorem.

## 5 Max SEFE

In this section we study the optimization version of the SEFE problem, in which two embeddings of the input graphs $G_{1}$ and $G_{2}$ are searched so that as many edges of $G_{\cap}$ as possible are drawn the same. We study the problem in its decision version and call it MAX SEFE. Namely, given a triple $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ composed of two planar graphs $G_{1}$ and $G_{2}$, and an integer $k^{*}$, the MAX SEFE problem asks whether $G_{1}$ and $G_{2}$ admit a simultaneous embedding $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ in which at most $k^{*}$ edges of $G_{\cap}$ have a different drawing in $\Gamma_{1}$ and in $\Gamma_{2}$. First, in Lemma 2, we state the membership of MAX SEFE to $\mathcal{N} \mathcal{P}$, which descends from the fact that SEFE belongs to $\mathcal{N} \mathcal{P}$. Then, in Theorem 7 we prove the $\mathcal{N} \mathcal{P}$-completeness in the general case. Finally, in Theorem 8 , we prove that the problem remains $\mathcal{N} \mathcal{P}$-complete even if stronger restrictions are imposed on the intersection graph $G_{\cap}$ of $G_{1}$ and $G_{2}$.

Lemma 2. MAX SEFE is in $\mathcal{N P}$.
Proof. The statement descends from the fact that the SEFE problem belongs to $\mathcal{N} \mathcal{P}$ [18]. Namely, let $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ be an instance of MAX SEFE. Non-deterministically construct in polynomial time all the sets of at most $k^{*}$ edges of $G_{\cap}$. Then, for each of the constructed sets, replace every edge in the set with a path of length 2 in one of the two graphs, say $G_{1}$, hence obtaining a graph $G_{1}^{\prime}$, and test whether a SEFE of $G_{1}^{\prime}$ and $G_{2}$ exists in polynomial time with a non-deterministic Turing machine [18]. If at least one of the performed tests succeeds, then $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ is a positive instance.

In order to prove that MAX SEFE is $\mathcal{N} \mathcal{P}$-complete, we show a reduction from a variant of the $\mathcal{N} \mathcal{P}$-complete problem Planar Steiner Tree (PST) 17], defined as follows: Given an instance $\langle G(V, E), S, k\rangle$ of PST, where $G(V, E)$ is a planar graph whose edges have weights $\omega: E \rightarrow \mathbb{N}, S \subset V$ is a set of terminals, and $k>0$ is an integer; does a tree $T^{*}\left(V^{*}, E^{*}\right)$ exist such that (1) $V^{*} \subseteq V$, (2) $E^{*} \subseteq E$, (3) $S \subseteq$
$V^{*}$, and (4) $\sum_{e \in E^{*}} \omega(e) \leq k$ ? The edge weights in $\omega$ are bounded by a polynomial function $p(n)$ (see [17]). In our variant, that we call Uniform Triconnected PST (UTPST), graph $G$ is a triconnected planar graph and all the edge weights are equal to 1 . We remark that a variant of PST in which all the edge weights are equal to 1 and in which $G$ is a subdivision of a triconnected planar graph (and no subdivision vertex is a terminal) is known to be $\mathcal{N} \mathcal{P}$-complete [1]. However, using this variant of the problemwould create multiple edges in our reduction. Actually, the presence of multiple edges might be handled by replacing them in the constructed instance with a set of length-2 paths. However, we think that an $\mathcal{N} \mathcal{P}$-completeness proof for the PST problem with $G$ triconnected and uniform edge weights may be of independent interest.

## Lemma 3. Uniform Triconnected PST is $\mathcal{N} \mathcal{P}$-complete.

Proof. The membership in $\mathcal{N P}$ follows from the fact that an instance of UTPST is also an instance of PST. The $\mathcal{N} \mathcal{P}$-hardness is proved by means of a polynomial-time


Fig. 8: (a) Gadget added inside a face to make $G^{*}$ triconnected. (b) Gadget replacing a vertex of degree greater than 3 to make $G_{\cap}$ subcubic.
reduction from PST. Let $\langle G, S, k\rangle$ be any instance of PST. We construct an equivalent instance $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$ of UTPST as follows. Initialize $G^{\prime}=G$. Let $w=\sum_{e \in G^{\prime}} w(e)$. Since the weights in $\omega$ are bounded by a polynomial function $p(n)$, the value of $w$ is also bounded by a polynomial function $n \cdot p(n)$. Augment $G^{\prime}$ to a triconnected planar graph by adding dummy edges and set $\omega\left(e_{d}\right)=w$ for each dummy edge $e_{d}$. Then, replace each edge $e$ in $G^{\prime}$ with a path $P(e)$ of $\omega(e)$ weight-1 edges. Further, for each face $f$ of the unique planar embedding of $G^{\prime}$, consider the vertices $v_{1}, \ldots, v_{h}$ of $f$ as they appear on the boundary of $f$. Add to $G^{\prime}$ a set $V_{f}$ of $h$ vertices $u_{1}, \ldots, u_{h}$ and, for $i=1, \ldots, h$, add to $G^{\prime}$ a weight- 1 edge $\left(u_{i}, v_{i}\right)$ and a weight- 1 edge $\left(u_{i}, u_{i+1}\right)$, where $h+1=1$ (see Fig. 8(a)). Note that, $G^{\prime}$ is triconnected. Finally, set $S^{\prime}=S$ and $k^{\prime}=k$. Since $w$ is bounded by a polynomial function, $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$ can be constructed in polynomial time.

We prove that $\langle G, S, k\rangle$ is a positive instance of PST if and only if $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$ is a positive instance of UTPST.

Suppose that $\langle G, S, k\rangle$ is a positive instance of PST. Starting from the solution $T$ of $\langle G, S, k\rangle$, we construct a solution $T^{\prime}$ of $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$ by replacing each edge $e$ of $T$ with path $P(e)$. By construction, $T^{\prime}$ is a tree, each terminal vertex in $S^{\prime}$ belongs to $T^{\prime}$, and $\sum_{e \in T^{\prime}} 1=\sum_{e \in T} \omega(e) \leq k=k^{\prime}$.

Suppose that $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$ is a positive instance of UTPST. Let $T^{\prime}$ be the solution of $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$. Assume that $T^{\prime}$ is the optimal solution of $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$, i.e., there exists no solution $T^{\sharp}$ of $\left\langle G^{\prime}, S^{\prime}, k^{\prime}\right\rangle$ such that $\sum_{e \in T^{\sharp}} \omega(e)<\sum_{e \in T^{\prime}} \omega(e)$. Observe that, if an edge of a path $P(e)$ belongs to $T^{\prime}$, then all the edges of $P(e)$ belong to $T^{\prime}$, as the internal vertices of $P(e)$ do not belong to $S^{\prime}$, by construction. Moreover, no edge of a path $P\left(e_{d}\right)$ such that $e_{d}$ is a dummy edge belongs to $T^{\prime}$, since the total weight of the edges of $P\left(e_{d}\right)$ is $w$. Finally, no edge incident to a vertex $u_{i} \in V_{f}$, for some face $f$, belongs to $T^{\prime}$, as $S^{\prime} \cap V_{f}=\emptyset$ and every path $v_{i}, u_{i}, \ldots, u_{l}, \ldots, u_{j}, v_{j}$ connecting two vertices $v_{i}$ and $v_{j}$ of $f$ and passing through vertices of $V_{f}$ is two units longer than path $v_{i}, \ldots, v_{l}, \ldots, v_{j}$ only passing through vertices of $f$. Hence, we construct a solution $T$ of $\langle G, S, k\rangle$ by replacing in $T^{\prime}$ all the edges of each path $P(e)$ with an edge $e$. By construction, $T$ is a tree, each terminal vertex in $S$ belongs to $T$, and $\sum_{e \in T} \omega(e)=\sum_{e \in T^{\prime}} 1 \leq k^{\prime}=k$. This concludes the proof of the lemma.

Then, based on the previous lemma, we prove the main result of this section.

## Theorem 7. Max SEFE is $\mathcal{N} \mathcal{P}$-complete.

Proof. The membership in $\mathcal{N P}$ follows from Lemma 2
The $\mathcal{N} \mathcal{P}$-hardness is proved by means of a polynomial-time reduction from problem UTPST. Let $\langle G, S, k\rangle$ be an instance of UTPST. We construct an instance $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ of MAx SEFE as follows (refer to Fig. 9).

Since $G$ is triconnected, it admits a unique planar embedding $\Gamma_{G}$, up to a flip. We now construct $G_{\cap}, G_{1}$, and $G_{2}$. Initialize $G_{\cap}=G_{1} \cap G_{2}$ as the dual of $G$ with respect to $\Gamma_{G}$. Since $G$ is triconnected, its dual is triconnected. Consider a terminal vertex $s^{*} \in S$, the set $E_{G}\left(s^{*}\right)$ of the edges incident to $s^{*}$ in $G$, and the face $f_{s^{*}}$ of $G_{\cap}$ composed of the edges that are dual to the edges in $E_{G}\left(s^{*}\right)$. Let $v^{*}$ be any vertex incident to $f_{s^{*}}$, and let $v_{1}^{*}$ and $v_{2}^{*}$ be the neighbors of $v^{*}$ on $f_{s^{*}}$. Subdivide edges $\left(v^{*}, v_{1}^{*}\right)$ and $\left(v^{*}, v_{2}^{*}\right)$ with dummy vertices $u_{1}^{*}$ and $u_{2}^{*}$, respectively. Add to $G_{\cap}$ vertex $s^{*}$ and edges $\left(s^{*}, u_{1}^{*}\right)$, $\left(s^{*}, u_{2}^{*}\right)$, and $\left(s^{*}, v^{*}\right)$. Since $v^{*}$ has at least a neighbor not incident to $f_{s^{*}}$, vertices $u_{1}^{*}$ and $u_{2}^{*}$ do not create a separation pair. Hence, $G_{\cap}$ remains triconnected. See Fig. 9(a).

Graph $G_{1}$ contains all the vertices and edges of $G_{\cap}$ plus a set of vertices and edges defined as follows. For each terminal $s \in S$, consider the set $E_{G}(s)$ of edges incident to $s$ in $G$ and the face $f_{s}$ of $G_{\cap}$ composed of the edges dual to the edges in $E_{G}(s)$. Add to $G_{1}$ vertex $s$ and an edge $\left(s, v_{i}\right)$ for each vertex $v_{i}$ incident to $f_{s}$, without introducing multiple edges. Note that, graph $G_{1}$ is triconnected. Hence, the rotation scheme of each vertex is the one induced by the unique planar embedding of $G_{1}$. See Fig. 9(b).

Graph $G_{2}$ contains all the vertices and edges of $G \cap$ plus a set of vertices and edges defined as follows. Rename the terminal vertices in $S$ as $x_{1}, \ldots, x_{|S|}$, in such a way that $s^{*}=x_{1}$. For $i=1, \ldots,|S|-1$, add edge $\left(x_{i}, x_{i+1}\right)$ to $G_{2}$. The rotation scheme of the vertices of $G_{2}$ different from $x_{1}, \ldots, x_{|S|}$ is induced by the embedding of $G_{\cap}$. The rotation scheme of vertices $x_{2}, \ldots, x_{|S|}$ is unique, as they have degree less or equal to 2 . Finally, the rotation scheme of $s^{*}$ is obtained by extending the rotation scheme induced by the planar embedding of $G_{\cap}$, in such a way that edges $\left(s^{*}, v^{*}\right)$ and $\left(s^{*}, x_{2}\right)$ are not consecutive. In order to obtain an instance of MAX SEFE in which both graphs are triconnected, we can augment $G_{2}$ to triconnected by only adding edges among vertices $\left\{u_{1}^{*}, u_{2}^{*}\right\} \cup\left\{x_{1}, \ldots, x_{|S|}\right\}$. See Fig. 9(b) Finally, set $k^{*}=k$.


Fig. 9: Illustration for the proof of Theorem 7 Black lines are edges of $G_{\cap}$; grey lines are edges of $G$; dashed red and solid blue lines are edges of $G_{1}$ and $G_{2}$, respectively; green edges compose the Steiner tree $T$; white squares and white circles are terminal vertices and non-terminal vertices of $G$, respectively. (a) $G_{\cap}, G$ and $T$; (b) $G_{1} \cup G_{2}$; (c) a drawing of $G \cap$ where 4 edges have two different drawings; and (d) a solution $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ of $\left\langle G_{1}, G_{2}, 4\right\rangle$.

We show that $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ admits a solution if and only if $\langle G, S, k\rangle$ does.
Suppose that $\langle G, S, k\rangle$ admits a solution $T$. Construct a planar drawing $\Gamma_{1}$ of $G_{1}$. The drawing $\Gamma_{2}$ of $G_{2}$ is constructed as follows. The edges of $G_{\cap}$ that are not dual to edges of $T$ are drawn in $\Gamma_{2}$ with the same curve as in $\Gamma_{1}$. Observe that, in the current drawing $\Gamma_{2}$ all the terminal vertices in $S$ lie inside the same face $f$ (see Fig. 9(c)). Hence, all the remaining edges of $G_{2}$ can be drawn [27] inside $f$ without intersections, as the subgraph of $G_{2}$ induced by the vertices incident to $f$ and by the vertices of $S$ is planar (see Fig. 9(d). Since the only edges of $G_{\cap}$ that have a different drawing in $\Gamma_{1}$ and $\Gamma_{2}$ are those that are dual to edges of $T,\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is a solution for $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$.

Suppose that $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ admits a solution $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ and assume that $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is optimal (that is, there exists no solution with fewer edges of $G_{\cap}$ not drawn the same). Consider the graph $T$ composed of the dual edges of the edges of $G_{\cap}$ that are not drawn the same. We claim that $T$ has at least one edge incident to each terminal in $S$ and that $T$ is connected. The claim implies that $T$ is a solution to the instance $\langle G, S, k\rangle$ of UTPST, since $T$ has at most $k$ edges and since $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is optimal.

Suppose for a contradiction that there exist two connected components $T_{1}$ and $T_{2}$ of $T$ (possibly composed of a single vertex). Consider the edges of $G$ incident to vertices of $T_{1}$ and not belonging to $T_{1}$, and consider the face $f_{1}$ composed of their dual edges. Note that, $f_{1}$ is a cycle of $G_{\cap}$. By definition of $T$, all the edges incident to $f_{1}$ have the same drawing in $\Gamma_{1}$ and in $\Gamma_{2}$. Finally, there exists at least one vertex of $S$ that lies inside $f_{1}$ and at least one that lies outside $f_{1}$. Since all the vertices in $S$ belong to a connected subgraph of $G_{2}$ not containing any vertex incident to $f_{1}$, there exist two terminal vertices $s^{\prime}$ and $s^{\prime \prime}$ such that $s^{\prime}$ lies inside $f_{1}, s^{\prime \prime}$ lies outside $f_{1}$, and edge $\left(s^{\prime}, s^{\prime \prime}\right)$ belongs to $G_{2}$. This implies that $\left(s^{\prime}, s^{\prime \prime}\right)$ crosses an edge incident to $f_{1}$ in $\Gamma_{2}$, a contradiction. This concludes the proof of the theorem.

We note from Theorem 7 that MAX SEFE is $\mathcal{N} \mathcal{P}$-complete even if the two input graphs $G_{1}$ and $G_{2}$ are triconnected, and if the intersection graph $G_{\cap}$ is composed of a triconnected component and of a set of isolated vertices (those corresponding to terminal vertices). We remark that, under these conditions, the original SEFE problem is polynomial-time solvable (actually, it is polynomial-time solvable even if only one of the input graphs has a unique embedding [3]). Further, it is possible to transform the constructed instances so that all the vertices of $G_{\cap}$ have degree at most 3 , by replacing each vertex $v$ of degree $d(v)>3$ in $G_{\cap}$ with a gadget as in Fig. 8(b). Such a gadget is composed of a cycle of $2 d(v)$ vertices and of an internal grid with degree-3 vertices whose size depends on $d(v)$. Edges incident to $v$ are assigned to non-adjacent vertices of the cycle, in the order defined by the rotation scheme of $v$. Hence, the MAX SEFE problem remains $\mathcal{N} \mathcal{P}$-complete even for instances in which $G_{\cap}$ is subcubic, that is another sufficient condition to make SEFE polynomial-time solvable [28].

In the following we go farther in this direction and prove that MAX SEFE remains $\mathcal{N} \mathcal{P}$-complete even if the degree of the vertices in $G_{\cap}$ is at most 2 . The proof is based on a reduction from the $\mathcal{N} \mathcal{P}$-complete problem MAX 2-XORSAT [25], which takes as input (i) a set of Boolean variables $B=\left\{x_{1}, \ldots, x_{l}\right\}$, (ii) a 2 -XorSat formula $F=$ $\bigwedge_{x_{i}, x_{j} \in B}\left(l_{i} \oplus l_{j}\right)$, where $l_{i}$ is either $x_{i}$ or $\overline{x_{i}}$ and $l_{j}$ is either $x_{j}$ or $\overline{x_{j}}$, and (iii) an integer $k>0$, and asks whether there exists a truth assignment $A$ for the variables in $B$ such that at most $k$ of the clauses in $F$ are not satisfied by $A$.
 input graphs $G_{1}$ and $G_{2}$ is composed of a set of cycles of length 3 .

Proof. The membership in $\mathcal{N P}$ follows from Lemma 2
The $\mathcal{N} \mathcal{P}$-hardness is proved by means of a polynomial-time reduction from problem MAX 2-XorSat. Let $\langle B, F, k\rangle$ be an instance of MAX 2-XORSAT. We construct an instance $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ of MAX SEFE as follows. Refer to Fig. 10(a).

Graph $G_{1}$ is composed of a cycle $C$ with $2 l$ vertices $v_{1}, v_{2}, \ldots, v_{l}, u_{l}, u_{l-1}, \ldots, u_{1}$. Also, for each variable $x_{i} \in B$, with $i=1, \ldots, l, G_{1}$ contains a set of vertices and edges defined as follows. First, $G_{1}$ contains a 4-cycle $V_{i}=\left(a_{i}, b_{i}, c_{i}, d_{i}\right)$, that we call variable gadget, connected to $C$ through edge $\left(a_{i}, v_{i}\right)$. Further, for each clause $\left(l_{i} \oplus l_{j}\right) \in F$ (or $\left.\left(l_{j} \oplus l_{i}\right) \in F\right)$ such that $l_{i} \in\left\{x_{i}, \overline{x_{i}}\right\}, G_{1}$ contains (i) a 3-cycle $V_{i, j}=\left(a_{i, j}, b_{i, j}, c_{i, j}\right)$, that we call clause-variable gadget, (ii) an edge $\left(b_{i, j}, w\right)$, where either $w=b_{i}$, if $l_{i}=x_{i}$, or $w=d_{i}$, if $l_{i}=\overline{x_{i}}$, and (iii) an edge $\left(a_{i, j}, c_{i, h}\right)$, where $\left(l_{i} \oplus l_{h}\right)\left(\right.$ or $\left.\left(l_{h} \oplus l_{i}\right)\right)$ is the last considered clause to which $l_{i}$ participates; if $\left(l_{i} \oplus l_{j}\right)$ (or $\left.\left(l_{j} \oplus l_{i}\right)\right)$ is the first


Fig. 10: (a) Illustration of the construction of instance $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ of MAX SEFE. (b) Illustration of the two cases in which $l_{i}$ evaluates to true in $A$.
considered clause containing $l_{i}$, then $c_{i, h}=c_{i}$. When the last clause $\left(l_{i} \oplus l_{q}\right)$ (or $\left(l_{q} \oplus l_{i}\right)$ ) has been considered, an edge $\left(c_{i, q}, u_{i}\right)$ is added to $G_{1}$. Note that, the subgraph $G_{1}^{i}$ of $G_{1}$ induced by the vertices of the variable gadget $V_{i}$ and of all the clause-variable gadgets $V_{i, j}$ to which $l_{i}$ participates would result in a subdivision of a triconnected planar graph by adding edge $\left(c_{i, q}, a_{i}\right)$, and hence it has a unique planar embedding (up to a flip). Graph $G_{2}$ is composed as follows. For each clause $\left(l_{i} \oplus l_{j}\right) \in F$, with $l_{i} \in\left\{x_{i}, \overline{x_{i}}\right\}$ and $l_{j} \in\left\{x_{j}, \overline{x_{j}}\right\}$, graph $G_{2}$ contains a triconnected graph $G_{2}^{i, j}$, that we call clause gadget, composed of all the vertices and edges of the clause-variable gadgets $V_{i, j}$ and $V_{j, i}$, plus three edges $\left(a_{i, j}, a_{j, i}\right),\left(b_{i, j}, b_{j, i}\right)$, and $\left(c_{i, j}, c_{j, i}\right)$. Finally, set $k^{*}=k$.

Note that, with this construction, graph $G_{\cap}$ is composed of a set of $2|F|$ cycles of length 3, namely the two clause-variable gadgets $V_{i, j}$ and $V_{j, i}$ for each clause $\left(l_{i} \oplus l_{j}\right)$.

We show that $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ admits a solution if and only if $\langle B, F, k\rangle$ does.
Suppose that $\langle B, F, k\rangle$ admits a solution, that is, an assignment $A$ of truth values for the variables of $B$ not satisfying at most $k$ clauses of $F$. We construct a solution $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ of $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$. First, we construct $\Gamma_{1}$. Let the face composed only of the edges of $C$ be the outer face. For each variable $x_{i}$, with $i=1, \ldots, l$, if $x_{i}$ is true in $A$, then the rotation scheme of $a_{i}$ in $\Gamma_{1}$ is $\left(a_{i}, v_{i}\right),\left(a_{i}, b_{i}\right),\left(a_{i}, d_{i}\right)$ (as in Fig. 10(a). Otherwise, $x_{i}$ is false in $A$, and the rotation scheme of $a_{i}$ is the reverse (as for $a_{j}$ in Fig. 10(a). Since $G_{1}^{i}$ has a unique planar embedding, the rotation scheme of all its vertices is univocally determined. Second, we construct $\Gamma_{2}$. Consider each clause $\left(l_{i} \oplus l_{j}\right) \in F$, with $l_{i} \in\left\{x_{i}, \overline{x_{i}}\right\}$ and $l_{j} \in\left\{x_{j}, \overline{x_{j}}\right\}$. If $l_{i}$ evaluates to $\operatorname{true}$ in $A$, then the embedding of $G_{2}^{i, j}$ is such that the rotation scheme of $a_{i, j}$ in $\Gamma_{2}$ is $\left(a_{i, j}, b_{i, j}\right),\left(a_{i, j}, c_{i, j}\right)$, $\left(a_{i, j}, a_{j, i}\right)$ (as in Fig. 10(a). Otherwise, $l_{i}$ is false in $A$ and the rotation scheme of $a_{i, j}$ is the reverse (as for $a_{j, i}$ in Fig. $10(\mathrm{a})$. Since $G_{2}^{i, j}$ is triconnected, this determines the rotation scheme of all its vertices. To obtain $\Gamma_{2}$, compose the embeddings of all the clause gadgets in such a way that each clause gadget lies on the outer face of all the others.

We prove that $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is a solution of the MAX SEFE instance, namely that at most $k^{*}$ edges of $G_{\cap}$ have a different drawing in $\Gamma_{1}$ and in $\Gamma_{2}$. Since $G_{\cap}$ is composed of 3 -cycles, this corresponds to saying that at most $k^{*}$ of such 3 -cycles have a different embedding in $\Gamma_{1}$ and in $\Gamma_{2}$ (where the embedding of a 3 -cycle is defined by the clockwise order of the vertices on its boundary). In fact, a 3-cycle with a different embedding in $\Gamma_{1}$ and in $\Gamma_{2}$ can always be realized by drawing only one of its edges with a different curve in the two drawings. By this observation and by the fact that at most $k=k^{*}$ clauses are not satisifed by $A$, the following claim is sufficient to prove the statement.

Claim 1 For each clause $\left(l_{i} \oplus l_{j}\right) \in F$, if $\left(l_{i} \oplus l_{j}\right)$ is satisifed by $A$, then both $V_{i, j}$ and $V_{j, i}$ have the same embedding in $\Gamma_{1}$ and in $\Gamma_{2}$, while if $\left(l_{i} \oplus l_{j}\right)$ is not satisifed by $A$, then exactly one of them has the same embedding in $\Gamma_{1}$ and in $\Gamma_{2}$.

Proof. Consider a clause $\left(l_{i} \oplus l_{j}\right) \in F$, where $l_{i} \in\left\{x_{i}, \overline{x_{i}}\right\}$ and $l_{j} \in\left\{x_{j}, \overline{x_{j}}\right\}$. First, we prove that $V_{i, j}$ has the same embedding in $\Gamma_{1}$ and in $\Gamma_{2}$, independently of whether ( $l_{i} \oplus l_{j}$ ) is satisfied or not. Namely, the flip of $G_{1}^{i}$ selected in the construction of $\Gamma_{1}$ is such that the rotation scheme of $a_{i, j}$ in $\Gamma_{1}$ is $\left(a_{i, j}, b_{i, j}\right),\left(a_{i, j}, c_{i, j}\right),\left(a_{i, j}, c_{x}\right)$ if and only if $l_{i}$ evaluates to true in $A$ (where $c_{x}=c_{i}$ if $\left(l_{i} \oplus l_{j}\right)$ is the first considered clause involving either $x_{i}$ or $\overline{x_{i}}$ in the construction of $G_{1}$, otherwise $c_{x}=c_{i, h}$ where $\left(l_{i} \oplus l_{h}\right)$ (or $\left(l_{h} \oplus l_{i}\right)$ ) is the clause involving either $x_{i}$ or $\overline{x_{i}}$ considered before $\left(l_{i} \oplus l_{j}\right)$ in the construction of $G_{1}$ ). This can be easily verified by considering the flip of $G_{1}^{i}$ in $\Gamma_{1}$ in the two cases in which $l_{i}$ evaluates to true in $A$, namely when either $x_{i}=$ true and $l_{i}=x_{i}$ or when $x_{i}=$ false and $l_{i}=\overline{x_{i}}$, that are depicted in Fig. 10(b) Recall that, by construction, the rotation scheme of $a_{i, j}$ in $\Gamma_{2}$ is $\left(a_{i, j}, b_{i, j}\right),\left(a_{i, j}, c_{i, j}\right)$, and $\left(a_{i, j}, a_{j, i}\right)$ if and only if $l_{i}$ evaluates to true in $A$. Since $c_{x}$ lies outside $V_{i, j}$ in $\Gamma_{1}$ and $a_{j, i}$ lies outside $V_{i, j}$ in $\Gamma_{2}$, the embedding of $V_{i, j}$ is determined by the evaluation of $l_{i}$ in $A$ in the same way in $\Gamma_{1}$ as in $\Gamma_{2}$.

Hence, it remains to prove that, if $\left(l_{i} \oplus l_{j}\right)$ is satisifed by $A$, then also $V_{j, i}$ has the same embedding in $\Gamma_{1}$ and in $\Gamma_{2}$. Suppose that $l_{j}$ evaluates to false in $A$. By construction, the flip of $G_{1}^{j}$ selected in the construction of $\Gamma_{1}$ is such that the rotation scheme of $a_{j, i}$ in $\Gamma_{1}$ is $\left(a_{j, i}, c_{j, i}\right),\left(a_{j, i}, b_{j, i}\right),\left(a_{j, i}, c_{x}\right)$ (where $c_{x}$ is defined as above). This can be easily verified by considering the flip of $G_{1}^{i}$ in $\Gamma_{1}$ in the two cases in which $l_{j}$ evaluates to false in $A$, namely when either $x_{j}=\mathrm{false}$ and $l_{j}=x_{j}$ or when $x_{j}=$ true and $l_{j}=\overline{x_{j}}$. Further, since $\left(l_{i} \oplus l_{j}\right)$ is satisifed by $A$ and $l_{j}$ evaluates to false, $l_{i}$ evaluates to true. Hence, by construction, the rotation scheme of $a_{i, j}$ in $\Gamma_{2}$ is $\left(a_{i, j}, b_{i, j}\right),\left(a_{i, j}, c_{i, j}\right),\left(a_{i, j}, a_{j, i}\right)$. Since $G_{2}^{i, j}$ is triconnected, the rotation scheme of $a_{j, i}$ in $\Gamma_{2}$ is $\left(a_{j, i}, c_{j, i}\right),\left(a_{j, i}, b_{j, i}\right),\left(a_{j, i}, a_{i, j}\right)$. Since $c_{x}$ lies outside $V_{j, i}$ in $\Gamma_{1}$ and $a_{i, j}$ lies outside $V_{j, i}$ in $\Gamma_{2}$, the embedding of $V_{j, i}$ is the same in $\Gamma_{1}$ and in $\Gamma_{2}$ when $l_{j}$ evaluates to false in $A$.

The fact that the embedding of $V_{j, i}$ be the same in $\Gamma_{1}$ and in $\Gamma_{2}$ when $l_{j}$ evaluates to true in $A$ (and hence $l_{i}$ evaluates to false) can be proved analogously.

Suppose that $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ admits a solution $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$. Assume that $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is optimal, that is, there exists no solution of $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ with fewer edges of $G_{\cap}$ drawn differently. We construct a truth assignment $A$ that is a solution of $\langle B, F, k\rangle$, as follows. For each variable $x_{i}$, with $i=1, \ldots, l$, assign true to $x_{i}$ if the rotation scheme of $a_{i}$ in $\Gamma_{1}$ is $\left(a_{i}, v_{i}\right),\left(a_{i}, b_{i}\right),\left(a_{i}, d_{i}\right)$. Otherwise, assign false to $x_{i}$.

We prove that $A$ is a solution of the MAX 2-XORSAT instance, namely that at most $k$ clauses of $B$ are not satisfied by $A$. Since $\left\langle\Gamma_{1}, \Gamma_{2}\right\rangle$ is optimal, for any 3 -cycle $V_{i, j}$ of $G_{\cap}$, at most one edge has a different drawing in $\Gamma_{1}$ and in $\Gamma_{2}$. Also, for any clause ( $l_{i} \oplus l_{j}$ ), at most one of $V_{i, j}$ and $V_{j, i}$ has an edge drawn differently in $\Gamma_{1}$ and in $\Gamma_{2}$, as otherwise one could flip $G_{2}^{i, j}$ in $\Gamma_{2}$ (that is, revert the rotation scheme of all its vertices) and draw all the edges of $V_{i, j}$ and $V_{j, i}$ with the same curves as in $\Gamma_{1}$. Since $k=k^{*}$, the following claim is sufficient to prove the statement.

Claim 2 For each clause gadget $G_{2}^{i, j}$ such that $V_{i, j}$ and $V_{j, i}$ have the same drawing in $\Gamma_{1}$ and in $\Gamma_{2}$, the corresponding clause $\left(l_{i} \oplus l_{j}\right)$ is satisfied by $A$.

Proof. Consider a clause gadget $G_{2}^{i, j}$ and the drawing of the corresponding clausevariable gadgets $V_{i, j}$ and $V_{j, i}$ in $\Gamma_{2}$. Note that, since $G_{2}^{i, j}$ is triconnected, if the rotation scheme of $a_{i, j}$ is $\left(a_{i, j}, b_{i, j}\right),\left(a_{i, j}, c_{i, j}\right),\left(a_{i, j}, a_{j, i}\right)$, then the rotation scheme of $a_{j, i}$ is $\left(a_{j, i}, c_{j, i}\right),\left(a_{j, i}, b_{j, i}\right),\left(a_{j, i}, a_{i, j}\right)$. Otherwise, both the rotation schemes are reversed. Also, consider the clause-variable gadget $V_{i, j}$ corresponding to any clause $\left(l_{i} \oplus l_{j}\right)$ or $\left(l_{j} \oplus l_{i}\right)$ involving a variable $x_{i}$. Note that, if the rotation scheme of $a_{i, j}$ in $\Gamma_{1}$ is $\left(a_{i, j}, b_{i, j}\right),\left(a_{i, j}, c_{i, j}\right),\left(a_{i, j}, c_{x}\right)$ (where $c_{x}$ is defined as in the proof of Claim 1], then either edge $\left(b_{i, j}, b_{i}\right)$ exists in $G_{1}$ and the rotation scheme of $a_{i}$ is $\left(a_{i}, v_{i}\right),\left(a_{i}, b_{i}\right)$, $\left(a_{i}, d_{i}\right)$, or edge $\left(b_{i, j}, d_{i}\right)$ exists in $G_{1}$ and the rotation scheme of $a_{i}$ is $\left(a_{i}, v_{i}\right),\left(a_{i}, d_{i}\right)$, $\left(a_{i}, b_{i}\right)$. In both cases, literal $l_{i}$ evaluates to true in $A$. In fact, in the former case $l_{i}=x_{i}$ and $x_{i}$ is true in $A$, while in the latter case $l_{i}=\overline{x_{i}}$ and $x_{i}$ is false in $A$, by the construction of $\left\langle G_{1}, G_{2}, k^{*}\right\rangle$ and by the assignment chosen for $A$. Analogously, if the rotation scheme of $a_{i, j}$ is the opposite, then $l_{i}$ evaluates to false in $A$.

Consider any clause gadget $G_{2}^{i, j}$ such that $V_{i, j}$ and $V_{j, i}$ have the same drawing in $\Gamma_{1}$ and in $\Gamma_{2}$. By combining the observations on the relationships among the rotation schemes of the vertices belonging to the clause gadget $G_{2}^{i, j}$, to the clause-variable gadgets $V_{i, j}$ and $V_{j, i}$, and to the variable gadgets $V_{i}$ and $V_{j}$, it is possible to conclude that $l_{i}$ evaluates to true in $A$ if and only if $l_{j}$ evaluates to false in $A$, that is, $\left(l_{i} \oplus l_{j}\right)$ is satisfied by $A$.

This concludes the proof of the theorem.

## 6 Conclusions

In this paper we proved several results concerning the computational complexity of some problems related to the SEFE and the Partitioned T-Coherent $k$-Page Book embedding problems. We showed that the version of SEFE in which all graphs share the same intersection graph $G_{\cap}$ (SUNFLOWER SEFE) is $\mathcal{N} \mathcal{P}$-complete for $k \geq 3$ even when $G_{\cap}$ is a tree and all the input graphs are biconnected. This improves on the result by Schaefer [28] who proved $\mathcal{N} \mathcal{P}$-completeness when $G_{\cap}$ is a forest of stars and two of the input graphs consist of disjoint biconnected components. Further, we prove $\mathcal{N} \mathcal{P}$-completeness of problem PTBE- $k$ for $k \geq 3$ when $T$ is a caterpillar and two of the input graphs are biconnected, and of problem PBE- $k$ for $k \geq 3$. These results improve on the previously known $\mathcal{N} \mathcal{P}$-completeness for $k$ unbounded by Hoske [21]. Also, we provided a linear-time algorithm to decide PTBE- $k$ for $k \geq 2$ when $k-1$ of the input graphs are $T$-biconnected. Most notably, this result enlarges the set of instances of

PTBE-2, and hence of the long-standing open problem SEFE when $G_{\cap}$ is connected, for which a polynomially-time algorithm is known. For this problem, we also proved that all the instances can be encoded by equivalent instances in which one of the two graphs is biconnected and series-parallel. It is also known that the biconnectivity of both the input graphs suffices to make the problem polynomial-time solvable [7]. On one hand, our results push PTBE-2 closer to the boundary of polynomiality. On the other hand, since we proved that for $k \geq 3$ the biconnectivity of all the input graphs does not avoid $\mathcal{N} \mathcal{P}$-completeness, it is natural to wonder whether dropping the biconnectivy condition on one of the two graphs in the case $k=2$ would make it possible to simulate the degrees of freedom that are given by the fact of having more graphs.

Moreover, we considered the optimization version MAX SEFE of SEFE with $k=$ 2 , in which one wants to draw as many common edges as possible with the same curve in the drawings of the two input graphs. We showed $\mathcal{N} \mathcal{P}$-completeness of this problem even under strong restrictions on the embedding of the input graphs and on the degree of the intersection graph that are sufficient to obtain polynomial-time algorithms for the original decision version of the problem.

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[^0]:    ${ }^{1}$ Although [4] proves the equivalence for $k=2$, the result can be naturally extended to any $k$.

[^1]:    ${ }^{2}$ This is the extension of the algorithm by Hoske to instances of PTBE- $k$ mentioned before.

