Quantum Algorithms for Finding Constant-sized Sub-hypergraphs

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Abstract

We develop a general framework to construct quantum algorithms that detect if a 3-uniform hypergraph given as input contains a sub-hypergraph isomorphic to a prespecified constant-sized hypergraph. This framework is based on the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez [SODA'13], and extends the methodology designed by Lee, Magniez and Santha [SODA'13] for similar problems over graphs. As applications, we obtain a quantum algorithm for finding a 4-clique in a 3-uniform hypergraph on n vertices with query complexity $O(n^{1.883})$, and a quantum algorithm for determining if a ternary operator over a set of size n is associative with query complexity $O(n^{2.113})$.

1 Introduction

Quantum query complexity is a model of quantum computation, in which the cost of computing a function is measured by the number of queries that are made to the input given as a black-box. In this model, it was exhibited in the early stage of quantum computing research that there exist quantum algorithms superior to the classical counterparts, such as Deutsch and Jozsa's algorithm, Simon and Shor's period finding algorithms, and Grover's search algorithm. Extensive studies following them have invented a lot of powerful upper bound (i.e., algorithmic) techniques such as variations/generalizations of Grover's search algorithm or quantum walks. Although these techniques give tight bounds for many problems, there are still quite a few cases for which no tight bounds are known. Intensively studied problems among them are the k-distinctness problem [1, 4, 5] and the triangle finding problem [3, 7, 10, 13, 15].

A recent breakthrough is the concept of learning graph introduced by Belovs [3]. This concept enables one to easily find a special form of feasible solutions to the minimization form (i.e., the dual form) of the general adversary bound [8, 16], and makes possible to detour the need of solving a semidefinite program of exponential size to find a non-trivial upper bound. Indeed, Belovs [3] improved the long-standing $O(n^{13/10})$ upper bound [15] of the triangle finding problem to $O(n^{35/27})$. His idea was generalized by Lee, Magniez and Santha [12] and Zhu [22] to obtain a quantum algorithm that finds a constant-sized subgraph with complexity $o(n^{2-2/k})$, improving the previous best bound $O(n^{2-2/k})$ [15], where k is the size of the subgraph. Subsequently, Lee, Magniez and Santha [13] constructed a triangle finding algorithm with quantum query complexity $O(n^{9/7})$. This bound was later shown by Belovs and Rosmanis [6] to be the best possible bound attained by the family of quantum algorithms whose complexities depend only on the index set of 1-certificates. Ref. [13] also gave a framework of quantum algorithms for finding a constant-sized subgraph, based on which they showed that associativity testing (testing if a binary operator over a domain of size n is associative) has quantum query complexity $O(n^{10/7})$.

Recently, Jeffery, Kothari and Magniez [10] cast the idea of the above triangle finding algorithms into the framework of quantum walks (called nested quantum walks) by recursively performing the quantum walk algorithm given by Magniez, Nayak, Roland and Santha [14] (which extended two seminal works for quantum walk algorithms, Szegedy's algorithm [17] based on Markov chain and Ambainis' algorithm [1] for *k*-element distinctness). Indeed, they presented two quantum-walk-based triangle finding algorithms of complexities $\tilde{O}(n^{35/27})$ and $\tilde{O}(n^{9/7})$, respectively. The nested quantum walk framework was further employed in [5] (but in a different way from [10]) to obtain $\tilde{O}(n^{5/7})$ complexity for the 3-distinctness problem. This achieves the best known upper bound (up to poly-logarithmic factors), which was first obtained with the learning-graph-based approach [4].

The triangle finding problem also plays a central role in several areas beside query complexity, and it has been recently discovered that faster algorithms for (weighted versions of) triangle finding would lead to faster algorithms for matrix multiplication [11, 19], the 3SUM problem [18], and for Max-2SAT [20, 21]. In particular, Max-2SAT over n variables is reducible to finding a triangle with maximum weight over $O(2^{n/3})$ vertices; in this context, although the final goal is a *time-efficient* classical or quantum algorithm that finds a triangle with maximum weight, studying triangle finding in the query complexity model is a first step toward this goal.

Our results. Along this line of research, this paper studies the problem of finding a 4-clique (i.e., the complete 3-uniform hypergraph with 4 vertices) in a 3-uniform hypergraph, a natural generalization of finding a triangle in an ordinary graph (i.e., a 2-uniform hypergraph). Our initial motivation comes from the complexity-theoretic importance of the problem. Indeed, while it is now well-known that Max-3SAT over *n* variables is reducible to finding a 4-clique with maximum weight in a 3-uniform hypergraph of $O(2^{n/4})$ vertices (the reduction is similar to the reduction from Max-2SAT to triangle finding mentioned above; we refer to [21] for details), no efficient classical algorithm for 4-clique finding has been discovered so far. Constructing query-efficient algorithms for this problem can be seen as a first step to investigate the possibility of faster (in the time complexity setting) classical or quantum algorithms for Max-3SAT.

Concretely, and more generally, this paper gives a framework based on quantum walks for finding any constant-sized sub-hypergraph in a 3-uniform hypergraph (Theorem 5). This is an extension of the learning-graph-based algorithm in [13] to the hypergraph case in terms of a nested quantum walk [10]. We illustrate this methodology by constructing a quantum algorithm that finds a 4-clique in a 3-uniform hypergraph¹ with query complexity $\tilde{O}(n^{241/128}) = O(n^{1.883})$, while naïve Grover search over the $\binom{n}{4}$ combinations of vertices only gives $O(n^2)$. As another application, we also construct a quantum algorithm that determines if a ternary operator is associative using $\tilde{O}(n^{169/80}) = O(n^{2.113})$ queries, while naïve Grover search needs $O(n^{2.5})$ queries.

In the course of designing the quantum walk framework, we introduce several new technical ideas (outlined below) for analyzing nested quantum walks to cope with difficulties that do not arise in the 2-uniform case (i.e., ordinary graphs), such as the fact that the size of the random subset taken in an inner walk may vary depending on the random subsets taken in outer walks. We believe that these ideas may be applicable to various problems beyond sub-hypergraph finding.

Our framework is another demonstration of the power of the concept of nested quantum walks, and of its wide applicability. In particular, we crucially rely on the high-level description and analysis made possible by the nested quantum walk formalism to overcome all the technical difficulties that arise when considering 3-uniform hypergraphs.

Technical contribution. Roughly speaking, the subgraph finding algorithm by Lee, Magniez and San-

¹We stress that, while this quantum algorithm can also be used to find with the same complexity a 4-clique of maximal weight, this does not currently lead to a better algorithm for Max-3SAT since our algorithm is only query-efficient.

tha [13] works as follows. First, for each vertex i in the subgraph H that we want to find, a random subset V_i of vertices of the input graph is taken. This subset V_i represents a set of candidates for the vertex i. Next, for each edge (i, j) in the subgraph H, a random subset of pairs in $V_i \times V_j$ is taken, representing a set of candidates for the edge (i, j). The most effective feature of their algorithm is to introduce a parameter for each ordered pair (V_i, V_j) that controls the average degree of a vertex in the bipartite graph between V_i and V_j . To make the algorithm efficient, it is crucial to keep the degree of every vertex in V_i almost equal to the value specified by the parameter. For this, they carefully devise a procedure for taking pairs from $V_i \times V_j$.

Our basic idea is similar in that we first, for each vertex i in the sub-hypergraph H that we want to find, take a random subset V_i of vertices in the input 3-uniform hypergraph as a set of candidates for the vertex i and then, for each hyperedge $\{i, j, k\}$ of H, take a random subset of triples in $V_i \times V_j \times V_k$. One may think that the remaining task is to fit the pair-taking procedure into the hypergraph case. It, however, turns out to be technically very complicated to generalize the pair-taking procedure from [13] to an efficient triple-taking procedure. Instead we cast the idea into the nested quantum walk of Jeffery, Kothari and Magniez [10] and employ probabilistic arguments. More concretely, we introduce a parameter that specifies the number e_{ijk} of triples to be taken from $V_i \times V_j \times V_k$ for each hyperedge $\{i, j, k\}$ of H. We then argue that, for randomly chosen e_{ijk} triples, the degree of each vertex sharply concentrates around its average, where the degree means the number of triples including the vertex (in this sense, the parameters e_{ijk} play essentially the same role as those of "average degrees" used in [10], but introducing e_{ijk} gives a neat formulation of the algorithm and this is effective particularly in handling such complicated cases as hypergraphs). This makes it substantially easier to analyze the complexity of all involved quantum walks, and enables us to completely analyze the complexity of our approach. Unfortunately, it turns out that this approach (taking the sets V_i first, and then e_{ijk} triples from each $V_i \times V_j \times V_k$) does not lead to any improvement over the naïve $O(n^2)$ -query quantum algorithm.

Our key idea is to introduce, for each unordered pair $\{i, j\}$ of vertices in H, a parameter f_{ij} , and modify the approach as follows. After randomly choosing V_i, V_j, V_k , we take three random subsets $F_{ij} \subseteq V_i \times V_j$, $F_{jk} \subseteq V_j \times V_k$, and $F_{ik} \subseteq V_i \times V_k$ of size f_{ij}, f_{jk} and f_{ik} , respectively. We then randomly choose e_{ijk} triples from the set $\Gamma_{ijk} = \{(u, v, w) \mid (u, v) \in F_{ij}, (u, w) \in F_{ik} \text{ and } (v, w) \in F_{jk}\}$. The difficulty here is that the size of Γ_{ijk} varies depending on the sets F_{ij}, F_{jk}, F_{ik} . Another problem is that, after taking many quantum-walks (i.e., performing the update operation many times), the distribution of the set of pairs can change. To overcome these difficulties, we carefully define the "marked states" (i.e., "absorbing states") of each level of the nested quantum walk: besides requiring, as usual, that the set (of the form V_i, F_{ij} or Γ_{ijk}) associated to a marked state should contain a part (i.e., a vertex, a pair of vertices or a triple of vertices) of a copy of H, we also require that this set should *satisfy certain regularity conditions*. We then show that the associated sets almost always satisfy the regularity conditions, by using concentration theorems for hypergeometric distributions. This regularity enables us to effectively bound the complexity of our new approach, giving in particular the claimed $\tilde{O}(n^{241/128})$ -query upper bound when H is a 4-clique.

2 Preliminaries

For any $k \ge 2$, an undirected k-uniform hypergraph is a pair (V, E), where V is a finite set (the set of vertices), and E is a set of unordered k-tuples of elements in V (the set of hyperedges). An undirected 2-uniform hypergraph is simply an undirected graph.

In this paper, we use the standard quantum query complexity model formulated in Ref. [2]. We deal with (undirected) 3-uniform hypergraphs G = (V, E) as input, and the operation of the black-box is given as the unitary mapping $|\{u, v, w\}, b\rangle \mapsto |\{u, v, w\}, b \oplus \chi(\{u, v, w\})\rangle$ for $b \in \{0, 1\}$, where the triple $\{u, v, w\}$ is the query to the black-box and $\chi(\{u, v, w\})$ is the answer on whether the triple is a hyperedge of G, namely, $\chi(\{u, v, w\}) = 1$ if $\{u, v, w\} \in E$ and $\chi(\{u, v, w\}) = 0$ otherwise.

Our algorithmic framework is based on the concept of the nested quantum walk introduced by Jeffery, Kothari and Magniez [10]. In the nested quantum walk, for each positive integer t, the walk at level t checks whether the current state is marked or not by invoking the walk at level t + 1, and this is iterated recursively until some fixed level m. The data structure of the walk at level t is defined so that it includes the initial state of the walk at level t + 1, which means that the setup cost of the walk at level $t \ge 2$ is zero. Jeffery, Kothari and Magniez have shown (in Section 4.1 of [10]) that the overall complexity of such a walk is

$$\tilde{O}\left(\mathsf{S} + \sum_{t=1}^{m} \left(\prod_{r=1}^{t} \frac{1}{\sqrt{\varepsilon_r}}\right) \frac{1}{\sqrt{\delta_t}} \mathsf{U}_t\right)$$

if the checking cost at level m is zero, which will be our case. Here S denotes the setup cost of the whole nested walk, U_t denotes the cost of updating the database of the walk at level t, δ_t denotes the spectral gap of the walk at level t, and ε_r denotes the fraction of marked states for the walk at level r. As in most quantum walk papers, we only consider quantum walks on the Johnson graphs, where the Johnson graph J(N, K) is a graph such that each vertex is a subset with size K of a set with size N and two vertices corresponding to subsets S and S' are adjacent if and only if $|S\Delta S'| = 2$ (we denote by $S\Delta S'$ the symmetric difference between S and S'). If the walk at level t is on J(N, K), then its spectral gap δ_t is known to be $\Omega(1/K)$.

Consider the update operation of the walk at any level. The update cost may vary depending on the states of the walk we want to update. Assume without loss of generality that the update operation is of the form $U = \sum_i |i\rangle \langle i| \otimes U_i$, where each U_i can be implemented using q_i queries, and the quantum state to be updated is of the form $|s\rangle = \sum_i \alpha_i |i\rangle |s_i\rangle$. Then the following lemma, used in [10], shows that if the magnitude of the states $|i\rangle |s_i\rangle$ that cost much to update (i.e., such that q_i is large) is small enough, we can approximate the update operator U with good precision by replacing all U_i acting on such costly states with the identity operator.

Lemma 1 ([10]) Let $U = \sum_{i} |i\rangle \langle i| \otimes U_{i}$ be a controlled unitary operator and let q_{i} be the query complexity of exactly implementing U_{i} . For any fixed integer T, define \tilde{U} as $\sum_{i:q_{i} \leq T} |i\rangle \langle i| \otimes U_{i} + \sum_{i:q_{i} > T} |i\rangle \langle i| \otimes \mathbb{I}$, where \mathbb{I} is the identity operator on the space on which U_{i} acts. Then, for any quantum state $|s\rangle = \sum_{i} \alpha_{i} |i\rangle |s_{i}\rangle$, the inequality $|\langle s|\tilde{U}U|s\rangle| \geq 1 - \epsilon_{T}$ holds whenever $\epsilon_{T} \geq \sum_{i:q_{i} > T} |\alpha_{i}|^{2}$.

In the analysis of this paper, hypergeometric distributions will appear many times. Let HG(n, m, r) denote the hypergeometric distribution whose random variable X is defined by

$$\Pr[X=j] = \frac{\binom{m}{j}\binom{n-m}{r-j}}{\binom{n}{r}}$$

We first state below several tail bounds of hypergeometric distributions (the proof can be easily obtained from Theorem 2.10 in [9]).

Lemma 2 When X has a hypergeometric distribution with expectation μ , the following hold (where $\exp(x)$ denotes e^x):

- (1) For any $0 < \delta \le 1$, $\Pr[X \ge (1+\delta)\mu] \le \exp(-\frac{\mu\delta^2}{3})$.
- (2) For any $0 < \delta < 1$, $\Pr[X \le (1 \delta)\mu] \le \exp(-\frac{\mu\delta^2}{2})$.
- (3) For any $\delta > 2e 1$, $\Pr[X > (1 + \delta)\mu] < \left(\frac{1}{2}\right)^{(1+\delta)\mu}$.

3 Statement of our main result

In this section, we state our main result (an upper bound on the query complexity of finding a constant-sized sub-hypergraph in a 3-uniform hypergraph) in terms of loading schedules, which generalizes the concept of loading schedules for graphs introduced, in the learning graph framework, by Lee, Magniez and Santha [13], and used in the framework of nested quantum walks by Jeffery, Kothari and Magniez [10].

Let *H* be a 3-uniform hypergraph with κ vertices. We identify the set of vertices of *H* with the set $\Sigma_1 = \{1, \ldots, \kappa\}$. We identify the set of hyperedges of *H* with the set $\Sigma_3 \subseteq \{\{1, 2, 3\}, \{1, 2, 4\}, \ldots, \{\kappa - 2, \kappa - 1, \kappa\}\}$. We identify the set of (unordered) pairs of vertices included in at least one hyperedge of *H* with the set $\Sigma_2 = \{\{i, j\} \mid \{i, j, k\} \in \Sigma_3 \text{ for some } k\}$. A loading schedule for *H* is defined as follows.

Definition 3 A loading schedule for H of length m is a list $S = (s_1, \ldots, s_m)$ of m elements such that the following three properties hold for all $t \in \{1, \ldots, m\}$: (i) $s_t \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$; (ii) if $s_t = \{i, j\}$, then there exist $t_1, t_2 \in \{1, \ldots, t-1\}$ such that $s_{t_1} = i$ and $s_{t_2} = j$; (iii) if $s_t = \{i, j, k\}$, then there exist $t_1, t_2, t_3 \in \{1, \ldots, t-1\}$ such that $s_{t_1} = \{i, j\}$, $s_{t_2} = \{i, k\}$ and $s_{t_3} = \{j, k\}$. A loading schedule S is valid if no element of $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ appears more than once and, for any $\{i, j, k\} \in \Sigma_3$, there exists an index $t \in \{1, \ldots, m\}$ such that $s_t = \{i, j, k\}$.

We now introduce the concept of parameters associated to a loading schedule. Formally, these parameters are functions of the variable n representing the number of vertices of the input 3-uniform hypergraphs G = (V, E). We will nevertheless, in a slight abuse of notation, consider that n is fixed, and define them as integers (implicitly depending on n).

Definition 4 Let $S = (s_1, ..., s_m)$ be a loading schedule for H of length m. A set of parameters for S is a set of m integers defined as follows: for each $t \in \{1, ..., m\}$,

- if $s_t = i$, then the associated parameter is denoted by r_i and satisfies $r_i \in \{1, ..., n\}$;
- if $s_t = \{i, j\}$, then the associated parameter is denoted by f_{ij} and satisfies $f_{ij} \in \{1, \dots, r_i r_j\}$;
- if $s_t = \{i, j, k\}$, then the associated parameter is denoted by e_{ijk} and satisfies $e_{ijk} \in \{1, \dots, r_i r_j r_k\}$.

The set of parameters is admissible if $r_i \ge 1$, $e_{ijk} \ge 1$, $\frac{r_i r_j}{f_{ij}} \ge 1$, $\frac{f_{ij} f_{ik} f_{jk}/(r_i r_j r_k)}{e_{ijk}} \ge 1$, and the terms $\frac{n}{r_i}$, $\frac{f_{ij}}{r_i}$, $\frac{f_{ij}}{r_i}$, $\frac{f_{ij} f_{ik}}{r_i r_j r_k}$ are larger than n^{γ} for some constant $\gamma > 0$.

Now we state the main result in terms of loading schedules.

Theorem 5 Let *H* be any constant-sized 3-uniform hypergraph. Let $S = (s_1, ..., s_m)$ be a valid loading schedule for *H* with an admissible set of parameters. There exists a quantum algorithm that, given as input a 3-uniform hypergraph *G* with *n* vertices, finds a sub-hypergraph of *G* isomorphic to *H* (and returns "no" if there are no such sub-hypergraphs) with probability at least some constant, and has query complexity

$$\tilde{O}\left(\mathsf{S} + \sum_{t=1}^{m} \left(\prod_{r=1}^{t} \frac{1}{\sqrt{\varepsilon_r}}\right) \frac{1}{\sqrt{\delta_t}} \mathsf{U}_{\mathsf{t}}\right),\,$$

where S, U_t , δ_t and ε_r are evaluated as follows:

- $S = \sum_{\{i,j,k\}\in\Sigma_3} e_{ijk};$
- for $t \in \{1, \ldots, m\}$, (i) if $s_t = \{i\}$, then $\delta_t = \Omega(\frac{1}{r_i})$, $\varepsilon_t = \Omega(\frac{r_i}{n})$ and $U_t = \tilde{O}\left(1 + \sum_{\{j,k\} \in \Sigma_3} \frac{e_{ijk}}{r_i}\right)$; (ii) if $s_t = \{i, j\}$, then $\delta_t = \Omega(\frac{1}{f_{ij}})$, $\varepsilon_t = \Omega(\frac{f_{ij}}{r_i r_j})$ and $U_t = \tilde{O}\left(1 + \sum_{k:\{i,j,k\} \in \Sigma_3} \frac{e_{ijk}}{f_{ij}}\right)$; (iii) if $s_t = \{i, j, k\}$, then $\delta_t = \Omega(\frac{1}{e_{ijk}})$, $\varepsilon_t = \Omega(\frac{e_{ijk} r_i r_j r_k}{f_{ij} f_{ik} f_{jk}})$ and $U_t = O(1)$.

4 **Proof of Theorem 5**

In this section, we prove Theorem 5 by constructing an algorithm based on the concept of *m*-level nested quantum walks, in which the walk at level *t* will correspond to the element s_t of the loading schedule for each $t \in \{1, \ldots, m\}$. For convenience, we will write $M_{ijk} = 11 \frac{f_{ij} f_{ik} f_{jk}}{r_i r_i r_k}$ for each $\{i, j, k\} \in \Sigma_3$.

4.1 Definition of the walks

At level $t \in \{1, ..., m\}$, the quantum walk will differ according to the nature of s_t , so there are three cases to consider.

Case 1 [$s_t = i$]: The quantum walk will be over the Johnson graph $J(n, r_i)$. The space of the quantum walk will then be $\Omega_t = \{T \subseteq \{1, \ldots, n\} \mid |T| = r_i\}$. A state of this walk is an element $R_t \in \Omega_t$.

Case 2 $[s_t = \{i, j\}]$: The quantum walk will be over $J(r_i r_j, f_{ij})$. The space of the quantum walk will then be $\Omega_t = \{T \subseteq \{1, \ldots, r_i r_j\} \mid |T| = f_{ij}\}$. A state of this walk is an element $R_t \in \Omega_t$.

Case 3 $[s_t = \{i, j, k\}]$: The quantum walk will be over $J(M_{ijk}, e_{ijk})$. The space of the quantum walk will then be $\Omega_t = \{T \subseteq \{1, \dots, M_{ijk}\} \mid |T| = e_{ijk}\}$. A state of this walk is an element $R_t \in \Omega_t$.

4.2 Definition of the data structures of the walks

Let us fix an arbitrary ordering on the set $V \times V \times V$ of triples of vertices. For any set $\Gamma \subseteq V \times V \times V$ and any $R \subseteq \{1, \ldots, |V|^3\}$, define the set $Y(R, \Gamma)$ consisting of at most |R| triples of vertices which are taken from Γ by the process below.

- Construct a list Λ of all the triples in V × V × V as follows: first, all the triples in Γ are listed in increasing order and, then, all the triples in (V × V × V)\Γ are listed in increasing order.
- For any $a \in \{1, \dots, |V|^3\}$, let $\Lambda[a]$ denote the *a*-th triple of the list.
- Define $\mathsf{Y}(R,\Gamma) = \{\Lambda[a] \mid a \in R\} \cap \Gamma$.

The following lemma will be useful later in this section.

Lemma 6 Let Γ and Γ' be two subsets of $V \times V \times V$. Let p and r be any parameters such that $1 \leq r \leq p \leq |V|^3$. There exists a permutation π of $\{1, \ldots, p\}$ such that, if R is a subset of $\{1, \ldots, p\}$ of size r taken uniformly at random, then

$$\Pr_{R}\left[|\mathsf{Y}(R,\Gamma)\Delta\mathsf{Y}(\pi(R),\Gamma')| \leq \frac{22r|\Gamma\Delta\Gamma'|}{p} + 100\log n\right] \geq 1 - 2\left(\frac{1}{2}\right)^{\frac{11r|\Gamma\Delta\Gamma'|}{p} + 50\log n}$$

Proof. Let Λ and Λ' be the lists obtained when using the construction for Γ and Γ' , respectively. Let us write

$$\Lambda_1 = \{\Lambda[a] \mid 1 \le a \le p\} \cap \Gamma,$$

$$\Lambda'_1 = \{\Lambda'[a] \mid 1 \le a \le p\} \cap \Gamma'.$$

We can show the following inequality.

Claim 7 $|\Lambda_1 \Delta \Lambda'_1| \leq 2|\Gamma \Delta \Gamma'|.$

Proof. Λ_1 contains precisely the $|\Lambda_1 \cap (\Gamma \cap \Gamma')|$ smallest (with respect to the increasing order) elements of $\Gamma \cap \Gamma'$, while the other $|\Lambda_1 \cap (\Gamma \setminus \Gamma')|$ elements of Λ_1 are in $\Gamma \setminus \Gamma'$. Similarly, Λ'_1 contains precisely the $|\Lambda'_1 \cap (\Gamma \cap \Gamma')|$ smallest elements of $\Gamma \cap \Gamma'$, while the other $|\Lambda'_1 \cap (\Gamma' \setminus \Gamma)|$ elements of Λ'_1 are in $\Gamma' \setminus \Gamma$. We can write

$$\begin{split} \left| \Lambda_1 \Delta \Lambda_1' \right| &= \left| \left| \Lambda_1' \cap (\Gamma \cap \Gamma') \right| - \left| \Lambda_1 \cap (\Gamma \cap \Gamma') \right| \right| + \left| \Lambda_1 \cap (\Gamma \setminus \Gamma') \right| + \left| \Lambda_1' \cap (\Gamma' \setminus \Gamma) \right| \\ &\leq \left| \left| \Lambda_1' \cap (\Gamma \cap \Gamma') \right| - \left| \Lambda_1 \cap (\Gamma \cap \Gamma') \right| \right| + \left| \Gamma \setminus \Gamma' \right| + \left| \Gamma' \setminus \Gamma \right|. \end{split}$$

We have to consider two cases.

Case 1: $|\Lambda_1| = |\Lambda'_1| = p$ Assume, without loss of generality, that $|\Lambda_1 \cap (\Gamma \cap \Gamma')| \le |\Lambda'_1 \cap (\Gamma \cap \Gamma')|$. We have

without loss of generality, that $|X_1| + (1 + 1)| \le |X_1| + (1 + 1)|$. We have

 $|\Lambda_1 \cap (\Gamma \cap \Gamma')| = p - |\Lambda_1 \cap (\Gamma \backslash \Gamma')| \ge p - |\Gamma \backslash \Gamma'|$

and $|\Lambda'_1 \cap (\Gamma \cap \Gamma')| \leq p$. Thus

$$\left| |\Lambda'_1 \cap (\Gamma \cap \Gamma')| - |\Lambda_1 \cap (\Gamma \cap \Gamma')| \right| \le p - (p - |\Gamma \setminus \Gamma'|) = |\Gamma \setminus \Gamma'|,$$

which gives $|\Lambda_1 \Delta \Lambda'_1| \leq 2|\Gamma \Delta \Gamma'|$, as claimed.

Case 2:
$$\min(|\Lambda_1|, |\Lambda'_1|) < p$$

By symmetry, it suffices to show only the case where $|\Lambda'_1| \leq |\Lambda_1|$. Since $|\Lambda'_1| < p$, we have $\Lambda'_1 = \Gamma'$. This implies that $|\Lambda_1 \setminus \Lambda'_1| = |\Lambda_1 \setminus \Gamma'| \leq |\Gamma \setminus \Gamma'|$. Since $|\Lambda'_1| \leq |\Lambda_1|$, we have $|\Lambda'_1 \setminus \Lambda_1| \leq |\Lambda_1 \setminus \Lambda'_1| \leq |\Gamma \setminus \Gamma'|$. Hence, $|\Lambda_1 \Delta \Lambda'_1| \leq 2|\Gamma \setminus \Gamma'| \leq 2|\Gamma \Delta \Gamma'|$ also holds in this case.

For any $a \in \{1, \ldots, \min(p, |\Gamma|)\}$ such that $\Lambda[a]$ is in Λ'_1 , we set $\pi(a) = a'$, where a' is the (unique) index in $\{1, \ldots, \min(p, |\Gamma'|)\}$ such that $\Lambda[a] = \Lambda'[a']$. For all other $a \in \{1, \ldots, p\}$, we set $\pi(a)$ arbitrarily such that π becomes a permutation of $\{1, \ldots, p\}$.

Let R be any subset of $\{1, \ldots, p\}$. Define the subsets $S_R, S'_R \subseteq R$ as follows:

$$S_R = \left\{ a \in R \mid \Lambda[a] \in \Lambda_1 \backslash \Lambda'_1 \right\},$$

$$S'_R = \left\{ b \in \pi(R) \mid \Lambda'[b] \in \Lambda'_1 \backslash \Lambda_1 \right\}.$$

From the definition of π , we know that for any element $a \in R \cap \{1, \ldots, |\Gamma|\}$ such that $a \notin S_R$ we have $\Lambda[a] = \Lambda'[\pi(a)]$, which means that this element is not in $\Upsilon(R, \Gamma) \setminus \Upsilon(\pi(R), \Gamma')$. This implies that $\Upsilon(R, \Gamma) \setminus \Upsilon(\pi(R), \Gamma') \subseteq \{\Lambda[a] \mid a \in S_R\}$. Similarly, we have $\Upsilon(\pi(R), \Gamma') \setminus \Upsilon(R, \Gamma) \subseteq \{\Lambda'[a] \mid a \in S'_R\}$, which gives the inequality

$$|\mathsf{Y}(R,\Gamma)\Delta\mathsf{Y}(\pi(R),\Gamma')| \le |S_R| + |S_R'|.$$

Recall that R is taken uniformly at random from $\{1, \ldots, p\}$ so that |R| = r. Thus, $|S_R|$ has hypergeometric distribution $HG(p, |\Lambda_1 \setminus \Lambda'_1|, r)$ and its expectation is $\mu = \frac{r|\Lambda_1 \setminus \Lambda'_1|}{p}$. Taking $\delta = \frac{1}{\mu} (11 \frac{r|\Gamma \Delta \Gamma'|}{p} + 50 \log n) - 1$, we have

$$\Pr\left[|S_R| \ge 11 \frac{r |\Gamma \Delta \Gamma'|}{p} + 50 \log n\right] = \Pr\left[|S_R| \ge (1+\delta)\mu\right].$$

Note that by Claim 7, $1 + \delta = \frac{1}{\mu} (11 \frac{r |\Gamma \Delta \Gamma'|}{p} + 50 \log n) > 11 \cdot \frac{|\Gamma \Delta \Gamma'|}{|\Lambda_1 \Delta \Lambda'_1|} \ge 11/2$ and hence $\delta \ge 9/2 > 2e - 1$. By Lemma 2(3), we have

$$\Pr\left[|S_R| \ge 11 \frac{r|\Gamma \Delta \Gamma'|}{p} + 50 \log n\right] < (1/2)^{(1+\delta)\mu} = (1/2)^{11 \frac{r|\Gamma \Delta \Gamma'|}{p} + 50 \log n}$$

Similarly, $\Pr\left[|S_R'| \ge 11 \frac{r|\Gamma \Delta \Gamma'|}{p} + 50 \log n\right] < (1/2)^{11 \frac{r|\Gamma \Delta \Gamma'|}{p} + 50 \log n}$. Therefore, $\Pr\left[|\mathbf{Y}(R, \Gamma) \Delta \mathbf{Y}(\pi(R), \Gamma')| \ge 22 \frac{r|\Gamma \Delta \Gamma'|}{p} + 100 \log n\right]$ $\le \Pr\left[|S_R| + |S_R'| \ge 22 \frac{r|\Gamma \Delta \Gamma'|}{p} + 100 \log n\right]$ $< 2(1/2)^{11 \frac{r|\Gamma \Delta \Gamma'|}{p} + 50 \log n}$.

This completes the proof of Lemma 6.

Suppose that the states of the walks at levels $1, \ldots, m$ are R_1, \ldots, R_m , respectively. Assume that the set of vertices of G is $V = \{v_1, \ldots, v_n\}$. We first interpret the states R_1, \ldots, R_m as sets of vertices, sets of pairs of vertices or sets of triples of vertices in V, as follows. For each $t \in \{1, \ldots, m\}$, there are three cases to consider.

Case 1 $[s_t = i]$: In this case, $R_t = \{a_1, \ldots, a_{r_i}\} \subseteq \{1, \ldots, n\}$. We associate to R_t the set $V_i = \{v_{a_1}, \ldots, v_{a_{r_i}}\}$. For further reference, we will rename the vertices in this set as $V_i = \{v_1^i, \ldots, v_{r_i}^i\}$.

Case 2 $[s_t = \{i, j\}$ with i < j]: We know that, in this case, there exist $t_1, t_2 \in \{1, \ldots, t-1\}$ such that $s_{t_1} = i$ and $s_{t_2} = j$. The state R_t represents a set $\{(a_1, b_1), \ldots, (a_{f_{ij}}, b_{f_{ij}})\}$ of f_{ij} pairs in $R_{t_1} \times R_{t_2}$. We associate to it the set $F_{ij} = \{(v_{a_1}^i, v_{b_1}^j), \ldots, (v_{a_{f_{ij}}}^i, v_{b_{f_{ij}}}^j)\}$ of pairs of vertices.

Case 3 $[s_t = \{i, j, k\}$ with i < j < k]: We know that there exist $t_1, t_2, t_3 \in \{1, \ldots, t-1\}$ such that $s_{t_1} = \{i, j\}, s_{t_2} = \{i, k\}$ and $s_{t_3} = \{j, k\}$, and R_t is a subset of $\{1, \ldots, M_{ijk}\}$ with $|R_t| = e_{ijk}$. Let us define the set

$$\Gamma_{ijk} = \{(u, v, w) \in V_i \times V_j \times V_k \mid (u, v) \in F_{ij}, (u, w) \in F_{ik} \text{ and } (v, w) \in F_{jk} \}.$$

We associate to R_t the set $E_{ijk} = Y(R_t, \Gamma_{ijk})$.

We are now ready to define the data structures involved in the walks. When the states of the walks at levels $1, \ldots, (m-1)$ are R_1, \ldots, R_{m-1} , respectively, and the state of the most inner walk is R_m , the data structure associated with the most inner walk is denoted by $D(R_1, \ldots, R_m)$ and defined as:

$$D(R_1, \dots, R_m) = \left\{ (\{u, v, w\}, \chi(\{u, v, w\})) \mid (u, v, w) \in \bigcup_{\{i, j, k\} \in \Sigma_3 : \ i < j < k} E_{ijk} \right\}.$$

The data structure associated with the walk at level t, for each $t \in \{1, ..., m-1\}$, is defined as:

$$\sum_{R_{t+1}\in\Omega_{t+1}}\cdots\sum_{R_m\in\Omega_m}|R_{t+1}\rangle\cdots|R_m\rangle|D(R_1,\ldots,R_m)\rangle.$$

Here and hereafter we omit normalization factors.

4.3 Marked states of the walks

For any $t \in \{1, ..., m-1\}$, the purpose of the walk at level t + 1 is to check if the state of the walk t is marked (for the most inner walk, the state can be checked without running another walk, since all the information necessary is already in the database). In this subsection we define the set of marked states for each walk.

Assume that the hypergraph G contains a (without loss of generality, unique) sub-hypergraph isomorphic to H. Let $\{u_1, \ldots, u_\kappa\}$ denote the vertex set of this sub-hypergraph. For the most outer walk, $s_1 = j$ for some $j \in \{1, \ldots, \kappa\}$ and we say that R_1 is marked if and only if $u_j \in V_j$. Consider a state R_t of the walk at level t > 1, and suppose that the states R_1, \ldots, R_{t-1} are all marked. We have again three cases to consider.

Case 1 [$s_t = i$]: R_t corresponds to V_i . We say that R_t is marked if and only if $u_i \in V_i$.

Case 2 [$s_t = \{i, j\}$ with i < j]: R_t corresponds to F_{ij} , and we say that R_t is marked if and only if the following four conditions hold:

- (a) $(u_i, u_j) \in F_{ij};$
- (b) for all $u \in V_i$, $\frac{f_{ij}}{2r_i} \le |\{v \in V_j \mid (u, v) \in F_{ij}\}| \le 2\frac{f_{ij}}{r_i};$
- (c) for all $v \in V_j$, $\frac{f_{ij}}{2r_j} \le |\{u \in V_i \mid (u, v) \in F_{ij}\}| \le 2\frac{f_{ij}}{r_j};$
- (d) for any k such that there exists $t_1 \in \{1, \ldots, t-1\}$ for which $s_{t_1} = \{i, k\}$, and any $(v, w) \in V_j \times V_k$, $|\{u \in V_i \mid (u, v) \in F_{ij} \text{ and } (u, w) \in F_{ik}\}| \leq 11 \frac{f_{ij} f_{ik}}{r_i r_j r_k}$.

Case 3 [$s_t = \{i, j, k\}$ with i < j < k]: R_t corresponds to E_{ijk} , and we say that R_t is marked if and only if $(u_i, u_j, u_k) \in E_{ijk}$.

The next subsection will use the following lemma.

Lemma 8 Assume that, for Case 2, R_t is taken uniformly at random from Ω_t (i.e., F_{ij} corresponds to a set of f_{ij} pairs taken uniformly at random from $V_i \times V_j$). Then,

$$\Pr[Conditions\ (b), (c), (d)\ hold\ for\ F_{ij}] \ge 1 - 2r_i e^{-\frac{f_{ij}}{8r_i}} - 2r_j e^{-\frac{f_{ij}}{8r_j}} - r_j r_k \kappa 2^{-11\frac{f_{ij}f_{ik}}{r_i r_j r_k}}$$

Proof. For each $u \in V_i$, the quantity $|\{v \in V_j \mid (u, v) \in F_{ij}\}|$ is a random variable with hypergeometric distribution $HG(r_ir_j, r_j, f_{ij})$. Its expectation is f_{ij}/r_i , and by Lemma 2(1-2) it holds that

$$\Pr\left[|\{v \in V_j \mid (u, v) \in F_{ij}\}| > 2\frac{f_{ij}}{r_i}\right] \le \exp(-\frac{1}{3} \times \frac{f_{ij}}{r_i}) \le \exp(-\frac{1}{8} \times \frac{f_{ij}}{r_i}) \le \exp(-\frac{$$

A similar statement holds for the degree of each $v \in V_j$, and thus, from the union bound, we obtain

$$\Pr[\text{Condition (b) or (c) does not hold for } F_{ij}] \le 2r_i \exp(-\frac{f_{ij}}{8r_i}) + 2r_j \exp(-\frac{f_{ij}}{8r_j})$$

For any k such that there exists $t_1 \in \{1, \ldots, t-1\}$ for which $s_{t_1} = \{i, k\}$, consider any $(v, w) \in V_j \times V_k$. Let us write $S = \{u \in V_i \mid (u, w) \in F_{ik}\}$. Since R_{t_1} is marked, we know that $|S| \leq 2f_{ik}/r_k$. The quantity $|\{u \in V_i \mid (u, v) \in F_{ij} \text{ and } (u, w) \in F_{ik}\}|$ has hypergeometric distribution $HG(r_ir_j, |S|, f_{ij})$. Applying Lemma 2(3) with $\delta = \frac{11f_{ik}/r_k}{|S|} - 1 > 2e - 1$, we obtain

$$\Pr\left[|\{u \in V_i \mid (u, v) \in F_{ij} \text{ and } (u, w) \in F_{ik}\}| \ge 11 \frac{f_{ij} f_{ik}}{r_i r_j r_k}\right] \le 2^{-11 \frac{f_{ij} f_{ik}}{r_i r_j r_k}}$$

Using the union bound (note in particular that there are at most κ possibilities for k), we conclude that

Pr [Condition (d) does not hold for
$$F_{ij}$$
] $\leq r_j r_k \kappa 2^{-11 \frac{f_{ij} f_{ik}}{r_i r_j r_k}}$.

The statement of the lemma then follows from the union bound.

4.4 Analysis of the algorithm

Our nested quantum walk algorithm finds a marked state in the most inner walk and thus a sub-hypergraph isomorphic to H, with high probability, since, as will be shown below, the ideal nested quantum walks can be approximated with high accuracy. As explained in Section 2, the overall query complexity of the walk is

$$\tilde{O}\left(\mathsf{S} + \sum_{t=1}^{m} \left(\prod_{r=1}^{t} \frac{1}{\sqrt{\varepsilon_r}}\right) \frac{1}{\sqrt{\delta_t}} \mathsf{U}_t\right).$$

We will show below that the values of the terms S, U_t, δ_t and ε_t are as claimed in the statement of Theorem 5.

We first make the following simple observation: when computing U_t and ε_t , we can assume that the state R_{t-1} of the immediately outer walk is marked (and thus, by applying this argument recursively, that the states R_1, \ldots, R_{t-1} of all the outer walks are marked). Indeed, remember that the purpose of the walk at level t is to check if the state R_{t-1} is marked. We first evaluate its complexity under the assumption that R_{t-1} is marked, giving some upper bound T on the complexity. Now, since the checking procedure in our framework has one-sided error, in the case where R_{t-1} is not marked the checking procedure may not terminate after T queries, but we can stop it after T queries anyway and simply output that R_{t-1} is not marked.

The setup cost S for the algorithm is the number of queries needed to construct the superposition

$$\sum_{R_1 \in \Omega_1} \cdots \sum_{R_m \in \Omega_m} |R_1\rangle \cdots |R_m\rangle |D(R_1, \dots, R_m)\rangle.$$

This value is at most $\sum_{\{i,j,k\}\in\Sigma_3} e_{ijk}$. We next evaluate δ_t and ε_t . The analysis is again divided into three cases.

Case 1 [$s_t = i$]: Since the quantum walk is over $J(n, r_i)$ by the definition in Section 4.1, we have $\delta_t = \Omega(\frac{1}{r_i})$ and $\varepsilon_t = \Omega(\frac{r_i}{r_i})$.

Case 2 $[s_t = \{i, j\}$ with i < j]: Since the quantum walk is over $J(r_i r_j, f_{ij})$, we have $\delta_t = \Omega(\frac{1}{f_{ij}})$. The fraction of states F_{ij} such that $(u_i, u_j) \in F_{ij}$ is $\Omega(\frac{f_{ij}}{r_i r_j})$. While all those states may not be marked, Lemma 8 implies that the fraction of those states that are not marked is exponentially small when the set of parameters is admissible. Thus $\varepsilon_t = \Omega(\frac{f_{ij}}{r_i r_i})$.

Case 3 $[s_t = \{i, j, k\}$ with i < j < k]: In this case $\delta_t = \Omega(\frac{1}{e_{ijk}})$. Since all the states R_1, \ldots, R_{t-1} of the outer walks are assumed to be marked, by item (d) of the definition of the marked states in Section 4.3, we can upper-bound $|\Gamma_{ijk}| = \sum_{(v,w)\in F_{jk}} |\{u \in V_i \mid (u,v) \in F_{ij} \text{ and } (u,w) \in F_{ik}\}| \text{ by } |F_{jk}| \frac{11f_{ij}f_{ik}}{r_i r_j r_k} = M_{ijk}$. Thus, we have $\varepsilon_t = \Omega(\frac{e_{ijk}}{M_{ijk}})$.

Finally, we evaluate the cost U_t , which is the cost of transforming the quantum state

$$\sum_{R_{t+1}\in\Omega_{t+1}}\cdots\sum_{R_m\in\Omega_m}|R_{t+1}\rangle\cdots|R_m\rangle|D(R_1,\ldots,R_{t-1},R_t,R_{t+1},\ldots,R_m)\rangle$$

to the quantum state

$$\sum_{R_{t+1}\in\Omega_{t+1}}\cdots\sum_{R_m\in\Omega_m}|R_{t+1}\rangle\cdots|R_m\rangle|D(R_1,\ldots,R_{t-1},R'_t,R_{t+1},\ldots,R_m)\rangle,$$

for any two states R_t and R'_t adjacent in the corresponding Johnson graph. We again divide the analysis into three cases.

Case 1 $[s_t = i]$: In this case R_t and R'_t are two subsets of $\{1, \ldots, n\}$, both of size r_i , differing by exactly one element. The corresponding subsets V_i and V'_i also differ by exactly one element: let us write $V'_i = (V_i \setminus \{u\}) \cup \{u'\}$. For any $\{i, j, k\} \in \Sigma_3$, there exist some $t_1, t_2, t_3 \in \{t + 1, \ldots, m\}$ such that $s_{t_1} = \{i, j\}, s_{t_2} = \{i, k\}$ and $s_{t_3} = \{i, j, k\}$. There also exist some $t_4, t_5, t_6 \in \{1, \ldots, m\}$ such that $s_{t_4} = j, s_{t_5} = k$ and $s_{t_6} = \{j, k\}$. Note that t_4, t_5, t_6 can be smaller than t, but we will first assume that they are all larger than t (the other cases, which are actually easier to analyze, are discussed at the end of the analysis). A state R_{t_4} defines a set V_j of r_j vertices and, for any $R_{t_1} \in \Omega_{t_1}$, the state (R_t, R_{t_1}, R_{t_4}) defines a set of f_{ij} pairs in $V_i \times V_j$, as described in Section 4.3. In the same way, for any $R'_{t_1} \in \Omega_{t_1}$, the state $(R'_t, R'_{t_1}, R_{t_4})$ defines a set of f_{ij} pairs in $V'_i \times V_j$. There exists a permutation π_1 of the elements of Ω_{t_1} such that, for any $R_{t_1} \in \Omega_{t_1}$, the set F'_{ij} defined by (R'_t, R_{t_1}, R_{t_4}) and the set F'_{ij} defined by $(R'_t, \pi_1(R_{t_1}), R_{t_4})$ are related in the following way:

$$F'_{ij} = (F_{ij} \setminus \{(u,v) \in \{u\} \times V_j \mid (u,v) \in F_{ij}\}) \cup \{(u',v) \in \{u'\} \times V_j \mid (u,v) \in F_{ij}\},\$$

which means that each pair of the form (u, v) in F_{ij} is replaced by the pair (u', v) in F'_{ij} , while the other pairs are the same in F_{ij} and in F'_{ij} .

Similarly, there exists a permutation π_2 of the elements of Ω_{t_2} such that, for any $R_{t_2} \in \Omega_{t_2}$, the set F_{ik} defined by (R_t, R_{t_2}, R_{t_5}) and the set F'_{ik} defined by $(R'_t, \pi_2(R_{t_2}), R_{t_5})$ are related in the following way:

$$F'_{ik} = (F_{ik} \setminus \{(u, w) \in \{u\} \times V_k \mid (u, w) \in F_{ik}\}) \cup \{(u', w) \in \{u'\} \times V_k \mid (u, w) \in F_{ik}\}$$

The states $(R_t, R_{t_1}, R_{t_2}, R_{t_4}, R_{t_5}, R_{t_6})$ define sets $V_i, F_{ij}, F_{ik}, V_j, V_k, F_{jk}, \Gamma_{ijk}$, while the states $(R'_t, \pi_1(R_{t_1}), \pi_2(R_{t_2}), R_{t_4}, R_{t_5}, R_{t_6})$ define sets $V'_i, F'_{ij}, F'_{ik}, V_j, V_k, F_{jk}, \Gamma'_{ijk}$. Given any state R_{t_3} , let $E_{ijk}(R_t, R_{t_1}, R_{t_2}, R_{t_3}, R_{t_4}, R_{t_5}, R_{t_6})$ denote the set of hyperedges to be queried associated with Γ_{ijk} and R_{t_3} , and $E_{ijk}(R'_t, \pi_1(R_{t_1}), \pi_2(R_{t_2}), R_{t_3}, R_{t_4}, R_{t_5}, R_{t_6})$ denote the set of hyperedges to be queried associated associated with Γ'_{ijk} and R_{t_3} . By Lemmas 1 and 6, the mapping

$$\begin{split} |R_{t_1}\rangle |R_{t_2}\rangle |R_{t_4}\rangle |R_{t_5}\rangle |R_{t_6}\rangle &\sum_{R_{t_3}\in\Omega_{t_3}} |R_{t_3}\rangle |E_{ijk}(R_t, R_{t_1}, R_{t_2}, R_{t_3}, R_{t_4}, R_{t_5}, R_{t_6})\rangle \mapsto \\ |\pi_1(R_{t_1})\rangle |\pi_2(R_{t_2})\rangle |R_{t_4}\rangle |R_{t_5}\rangle |R_{t_6}\rangle &\sum_{R_{t_3}\in\Omega_{t_3}} |R_{t_3}\rangle |E_{ijk}(R_t', \pi_1(R_{t_1}), \pi_2(R_{t_2}), R_{t_3}, R_{t_4}, R_{t_5}, R_{t_6})\rangle \end{split}$$

can be approximated within inverse polynomial precision² using $\tilde{O}\left(\frac{e_{ijk}|\Gamma_{ijk}\Delta\Gamma'_{ijk}|}{M_{ijk}} + \log n\right) = \tilde{O}\left(\frac{e_{ijk}|\Gamma_{ijk}\Delta\Gamma'_{ijk}|}{M_{ijk}} + 1\right)$ queries (here Lemma 1 is used with $T = 22\frac{e_{ijk}|\Gamma_{ijk}\Delta\Gamma'_{ijk}|}{M_{ijk}} + 100\log n$, and then ϵ_T can be set to $2\left(\frac{1}{2}\right)^{11\frac{e_{ijk}|\Gamma_{ijk}\Delta\Gamma'_{ijk}|}{M_{ijk}} + 50\log n}$ by Lemma 6 with $p = M_{ijk}$ and $r = e_{ijk}$). We will use the following lemma.

 $^{^{2}}$ Note that a better estimation of the accuracy of the approximation can be obtained, but in this proof approximation within inverse polynomial will be enough for our purpose. In consequence, while stronger tail bounds can be proved, the statements of Lemmas 9 and 10 will be enough for our purpose.

Lemma 9 When R_{t_1}, R_{t_2} and R_{t_6} are taken uniformly at random,

$$\Pr\left[|\Gamma_{ijk}\Delta\Gamma'_{ijk}| \ge 44 \times \frac{f_{ij}f_{ik}f_{jk}}{r_i^2 r_j r_k}\right] = O\left(\frac{1}{n^{100}}\right).$$

Proof. Let us write

$$A = \{(u, v, w) \in \{u\} \times V_j \times V_k \mid (u, v) \in F_{ij}, (u, w) \in F_{ik} \text{ and } (v, w) \in F_{jk}\},\$$

$$B = \{(u', v, w) \in \{u'\} \times V_j \times V_k \mid (u', v) \in F'_{ij}, (u', w) \in F'_{ik} \text{ and } (v, w) \in F_{jk}\},\$$

and note that

$$\Gamma_{ijk}\Delta\Gamma'_{ijk}| = |A| + |B|.$$

Consider the set $C = \{v \in V_j \mid (u, v) \in F_{ij}\}$. When R_{t_1} is taken uniformly at random from $\Omega_{t_1} = \{T \subseteq \{1, \ldots, r_i r_j\} \mid |T| = f_{ij}\}$ (recall that $s_{t_1} = \{i, j\}$), the quantity |C| has hypergeometric distribution $HG(r_i r_j, r_j, f_{ij})$. By Lemma 2(1), we have

$$\Pr\left[|C| \ge 2\frac{f_{ij}}{r_i}\right] \le \exp(-\frac{1}{3} \times \frac{f_{ij}}{r_i}).$$

Let us fix C and, for any $v \in C$, write

$$C(v) = \{ w \in V_k \mid (v, w) \in F_{jk} \}.$$

When R_{t_6} is taken uniformly at random from $\Omega_{t_6} = \{T \subseteq \{1, \ldots, r_j r_k\} \mid |T| = f_{jk}\}$ (recall that $s_{t_6} = \{j, k\}$), the quantity |C(v)| has hypergeometric distribution $HG(r_j r_k, r_k, f_{jk})$. By Lemma 2(1), we have

$$\Pr\left[|C(v)| \ge 2\frac{f_{jk}}{r_j}\right] \le \exp\left(-\frac{1}{3} \times \frac{f_{jk}}{r_j}\right).$$
(1)

Let us fix C(v) and write

$$C'(v) = \{ w \in C(v) \mid (u, w) \in F_{ik} \}.$$

When R_{t_2} is taken uniformly at random from $\Omega_{t_2} = \{T \subseteq \{1, \ldots, r_i r_k\} \mid |T| = f_{ik}\}$ (recall that $s_{t_2} = \{i, k\}$), the quantity |C'(v)| has hypergeometric distribution $HG(r_i r_k, |C(v)|, f_{ik})$. Under the hypothesis $|C(v)| \leq 2\frac{f_{jk}}{r_j}$, we can apply Lemma 2(3) with $\delta = \frac{11f_{jk}/r_j}{|C(v)|} - 1 > 2e - 1$ to evaluate the size of C'(v). The union bound then gives

$$\Pr\left[|C'(v)| \le 11 \frac{f_{jk} f_{ik}}{r_i r_j r_k}\right] \ge 1 - 2^{-11 \frac{f_{jk} f_{ik}}{r_i r_j r_k}} - \exp(-\frac{f_{jk}}{3r_j}).$$

Finally, note that $|A| = \sum_{v \in C} |C'(v)|$. Thus the union bound gives

$$\Pr\left[|A| \le 22 \frac{f_{ij} f_{ik} f_{jk}}{r_i^2 r_j r_k}\right] \ge 1 - \exp(-\frac{f_{ij}}{3r_i}) - 2 \frac{f_{ij}}{r_i} \left(2^{-11 \frac{f_{jk} f_{ik}}{r_i r_j r_k}} + \exp(-\frac{f_{jk}}{3r_j})\right)$$

Similarly, we have

$$\Pr\left[|B| \le 22 \frac{f_{ij} f_{ik} f_{jk}}{r_i^2 r_j r_k}\right] \ge 1 - \exp(-\frac{f_{ij}}{3r_i}) - 2 \frac{f_{ij}}{r_i} \left(2^{-11 \frac{f_{jk} f_{ik}}{r_i r_j r_k}} + \exp(-\frac{f_{jk}}{3r_j})\right),$$

and thus

$$\Pr\left[|\Gamma_{ijk}\Delta\Gamma'_{ijk}| \ge 44 \times \frac{f_{ij}f_{ik}f_{jk}}{r_i^2 r_j r_k}\right] \le 2\exp(-\frac{f_{ij}}{3r_i}) + 4\frac{f_{ij}}{r_i}\left(2^{-11\frac{f_{jk}f_{ik}}{r_i r_j r_k}} + \exp(-\frac{f_{jk}}{3r_j})\right),$$

which is exponentially small since this set of parameters is admissible.

Lemmas 1 and 9 then show that the mapping

$$R_{t_4}\rangle|R_{t_5}\rangle \sum_{R_{t_1}\in\Omega_{t_1}}\sum_{R_{t_2}\in\Omega_{t_2}}\sum_{R_{t_6}\in\Omega_{t_6}}\sum_{R_{t_3}\in\Omega_{t_3}}|R_{t_1}\rangle|R_{t_2}\rangle|R_{t_6}\rangle|R_{t_3}\rangle|E_{ijk}(R_t,R_{t_1},\cdots,R_{t_6})\rangle \mapsto |R_{t_4}\rangle|R_{t_5}\rangle \sum_{R_{t_1}\in\Omega_{t_1}}\sum_{R_{t_2}\in\Omega_{t_2}}\sum_{R_{t_6}\in\Omega_{t_6}}\sum_{R_{t_3}\in\Omega_{t_3}}|R_{t_1}\rangle|R_{t_2}\rangle|R_{t_6}\rangle|R_{t_3}\rangle|E_{ijk}(R_t',R_{t_1},\cdots,R_{t_6})\rangle$$

can be approximated within inverse polynomial precision using $O(e_{ijk}/r_i + 1)$ queries. This argument is true for all $\{i, j, k\} \in \Sigma_3$, so the update cost is

$$\mathsf{U}_t = \tilde{O} \Bigg(1 + \sum_{\{j,k\} \text{ such that } \{i,j,k\} \in \Sigma_3} \frac{e_{ijk}}{r_i} \Bigg).$$

Let us finally consider the case where t_4, t_5, t_6 are not all larger than t. Whenever t_6 is larger than t, exactly the same analysis as above holds. When t_6 is smaller than t (which implies that t_4 and t_5 are also smaller than t), remember that we only need to do the analysis of the update cost under the condition that R_{t_6} is marked. This means that we can assume that, for any $v \in V_j$, we have $|\{w \in V_k \mid (v, w) \in F_{jk}\}| \le 2f_{jk}/r_j$. This property can be used instead of Inequality (1) in the proof of Lemma 9, and the analysis then becomes the same as above.

Case 2 $[s_t = \{i, j\}$ with i < j]: R_t and R'_t correspond to two subsets F_{ij} and F'_{ij} that also differ by exactly one element: let us write $F'_{ij} = (F_{ij} \setminus \{(u, v)\}) \cup \{(u', v')\}$. For any $\{i, j, k\} \in \Sigma_3$, there exist some $t_1, t_2 \in \{1, \ldots, t-1\}$ such that $s_{t_1} = i$, $s_{t_2} = j$ and some $t_3 \in \{t + 1, \ldots, m\}$ such that $s_{t_3} = \{i, j, k\}$. There also exist some $t_4, t_5, t_6 \in \{1, \ldots, m\}$ such that $s_{t_4} = k$, $s_{t_5} = \{i, k\}$ and $s_{t_6} = \{j, k\}$. Note that t_4, t_5, t_6 can be smaller than t, but we will first assume that they are all larger than t (the other cases, which are actually easier to analyze, are discussed at the end of the analysis). The states $(R_t, R_{t_1}, R_{t_2}, R_{t_4}, R_{t_5}, R_{t_6})$ define sets $F_{ij}, V_i, V_j, V_k, F_{ik}, F_{jk}, \Gamma_{ijk}$, while the states $(R'_t, R_{t_1}, R_{t_2}, R_{t_4}, R_{t_5}, R_{t_6})$ define sets $F_{ij}, V_i, V_j, V_k, F_{ik}, F_{jk}, \Gamma_{ijk}$, while $E_{ijk}(R_t, R_{t_1}, \dots, R_{t_6})$ denote the set of hyperedges to be queried associated with Γ'_{ijk} and R_{t_3} .

By Lemmas 1 and 6, we know that the mapping

$$\begin{split} |R_{t_4}\rangle |R_{t_5}\rangle |R_{t_6}\rangle \sum_{R_{t_3}\in\Omega_{t_3}} |R_{t_3}\rangle |E_{ijk}(R_t, R_{t_1}, R_{t_2}, R_{t_3}, R_{t_4}, R_{t_5}, R_{t_6})\rangle \mapsto \\ |R_{t_4}\rangle |R_{t_5}\rangle |R_{t_6}\rangle \sum_{R_{t_3}\in\Omega_{t_3}} |R_{t_3}\rangle |E_{ijk}(R_t', R_{t_1}, R_{t_2}, R_{t_3}, R_{t_4}, R_{t_5}, R_{t_6})\rangle \end{split}$$

can be approximated within inverse polynomial precision using

$$\tilde{O}\left(\frac{e_{ijk}|\Gamma_{ijk}\Delta\Gamma'_{ijk}|}{M_{ijk}} + \log n\right) = \tilde{O}\left(\frac{e_{ijk}|\Gamma_{ijk}\Delta\Gamma'_{ijk}|}{M_{ijk}} + 1\right)$$

queries. We now prove the following lemma.

Lemma 10 When R_{t_5} and R_{t_6} are taken uniformly at random,

$$\Pr\left[|\Gamma_{ijk}\Delta\Gamma'_{ijk}| \ge 22 \times \frac{f_{ik}f_{jk}}{r_i r_j r_k}\right] = O\left(\frac{1}{n^{100}}\right).$$

Proof. Let us write

$$A = \{ w \in V_k \mid (u, w) \in F_{ik} \text{ and } (v, w) \in F_{jk} \},\$$

$$B = \{ w \in V_k \mid (u', w) \in F_{ik} \text{ and } (v', w) \in F_{jk} \},\$$

and note that

$$|\Gamma_{ijk}\Delta\Gamma'_{ijk}| = |A| + |B|.$$

Let us write $A_1 = \{w \in V_k \mid (u, w) \in F_{ik}\}$. When R_{t_5} is taken uniformly at random from $\Omega_{t_5} = \{T \subseteq \{1, \ldots, r_i r_k\} \mid |T| = f_{ik}\}$ (recall that $s_{t_5} = \{i, k\}$), the quantity $|A_1|$ has hypergeometric distribution $HG(r_i r_k, r_k, f_{ik})$. By Lemma 2(1), we have

$$\Pr\left[|A_1| \ge 2\frac{f_{ik}}{r_i}\right] \le \exp(-\frac{1}{3} \times \frac{f_{ik}}{r_i}).$$
⁽²⁾

Once A_1 is fixed, the quantity $|\{w \in A_1 \mid (v, w) \in F_{jk}\}|$ has hypergeometric distribution $HG(r_jr_k, |A_1|, f_{jk})$ with expectation $\frac{|A_1|f_{jk}}{r_jr_k}$. Under the assumption $|A_1| \leq 2\frac{f_{ik}}{r_i}$, we can apply Lemma 2(3) with $\delta = \frac{11f_{ik}/r_i}{|A_1|} - 1 > 2e - 1$. The union bound then gives

$$\Pr\left[|A| \le 11 \frac{f_{ik} f_{jk}}{r_i r_j r_k}\right] \ge 1 - 2^{-11 \frac{f_{ik} f_{jk}}{r_i r_j r_k}} - \exp(-\frac{f_{ik}}{3r_i}).$$

Similarly we obtain

$$\Pr\left[|B| \le 11 \frac{f_{ik} f_{jk}}{r_i r_j r_k}\right] \ge 1 - 2^{-11 \frac{f_{ik} f_{jk}}{r_i r_j r_k}} - \exp(-\frac{f_{ik}}{3r_i}),$$

and thus

$$\Pr\left[|\Gamma_{ijk}\Delta\Gamma'_{ijk}| \geq 22 \frac{f_{ik}f_{jk}}{r_i r_j r_k}\right] \leq 2 \times \left(2^{-11 \frac{f_{ik}f_{jk}}{r_i r_j r_k}} + \exp(-\frac{f_{ik}}{3r_i})\right),$$

which is exponentially small since this set of parameters is admissible.

Using Lemma 1 and Lemma 10, the mapping

$$\begin{split} R_{t_4} & \sum_{R_{t_5} \in \Omega_{t_5}} \sum_{R_{t_6} \in \Omega_{t_6}} \sum_{R_{t_3} \in \Omega_{t_3}} |R_{t_5}\rangle |R_{t_6}\rangle |R_{t_3}\rangle |E_{ijk}(R_t, R_{t_1}, \cdots, R_{t_6})\rangle \mapsto \\ & |R_{t_4}\rangle \sum_{R_{t_5} \in \Omega_{t_5}} \sum_{R_{t_6} \in \Omega_{t_6}} \sum_{R_{t_3} \in \Omega_{t_3}} |R_{t_5}\rangle |R_{t_6}\rangle |R_{t_3}\rangle |E_{ijk}(R_t', R_{t_1}, \cdots, R_{t_6})\rangle \end{split}$$

can be approximated within inverse polynomial precision using $\tilde{O}(e_{ijk}/f_{ij}+1)$ queries. This argument is true for all $\{i, j, k\} \in \Sigma_3$, so the update cost is

$$\mathsf{U}_t = \tilde{O}\left(1 + \sum_{k \text{ such that } \{i,j,k\} \in \Sigma_3} \frac{e_{ijk}}{f_{ij}}\right).$$

Let us finally consider the case where t_4, t_5, t_6 are not all larger than t. Whenever both t_5 and t_6 are larger than t, exactly the same analysis as above holds. When $t_5 < t < t_6$, remember that we only need to do the analysis of the update cost under the condition that R_{t_5} is marked. This means that we can assume that, for any $u \in V_i$, we have $|\{w \in V_k | (u, w) \in F_{ik}\}| \le 2f_{ik}/r_i$. This property can be used instead of Inequality (2) in the proof of Lemma 10, and the analysis then becomes the same as above. When $t_6 < t < t_5$, the same argument holds by inverting the roles of $\{i, k\}$ and $\{j, k\}$ in the proof of Lemma 10. When $t_5, t_6 < t$, the fact that R_{t_5} and R_{t_6} are marked (more precisely, item (d) in the definition of marked states of Section 4.3) implies that for any $(u_1, v_1) \in V_i \times V_j$, $|\{w \in V_k | (u_1, w) \in F_{ik} \text{ and } (v_1, w) \in F_{jk}\}| \le 11 \frac{f_{ik} f_{jk}}{r_i r_j r_k}$, which immediately implies that $|\Gamma_{ijk} \Delta \Gamma'_{ijk}| \le 22 \frac{f_{ik} f_{jk}}{r_i r_j r_k}$.

Case 3 [$s_t = \{i, j, k\}$ with i < j < k]: R_t and R'_t are two subsets of $\{1, \ldots, M_{ijk}\}$, both of size e_{ijk} , differing by exactly one element. The corresponding E_{ijk} and E'_{ijk} are subsets of the same Γ_{ijk} , and have symmetric difference $|E_{ijk}\Delta E'_{ijk}| \le 2$, so $U_t \le 2$.

Now the proof of Theorem 5 is completed.

5 Applications: 4-clique detection and ternary associativity testing

In this section we describe how to use our method to construct efficient quantum algorithms for 4-clique detection and ternary associativity testing.

First, by applying Theorem 5 to the case where H is a 4-clique, and optimizing both the loading schedule and the parameters, we obtain the following result.

Theorem 11 There exists a quantum algorithm that detects if a 3-uniform hypergraph on n vertices has a 4-clique, with high probability, using $\tilde{O}(n^{241/128}) = O(n^{1.883})$ queries.

Proof. We use Theorem 5. Among the 1680384 possible valid loading schedules, we found, by numerical search, that the best schedule is

 $(1, 2, 3, 4, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}).$

The complexity of the algorithm for this schedule is minimized by the following values of parameters:

$r_1 = n^{1/2},$	$r_2 = n^{3/4},$	$r_3 = n^{7/8},$	$r_4 = n^{3/4},$
$f_{12} = n^{5/4},$	$f_{13} = n^{5/4},$	$f_{14} = n^{147/128},$	
$f_{23} = n^{193/128},$	$f_{24} = n^{83/64},$	$f_{34} = n^{181/128},$	
$e_{123} = n^{241/128},$	$e_{124} = n^{217/128},$	$e_{134} = n^{211/128},$	$e_{234} = n^{193/128}$

It is easy to check that this set of parameters is admissible. This gives query complexity $\tilde{O}(n^{241/128})$.

Next, we consider ternary associativity testing. Let X be a finite set with |X| = n. A ternary operator \mathcal{F} from $X \times X \times X$ to X is said to be *associative* if $\mathcal{F}(\mathcal{F}(a, b, c), d, e) = \mathcal{F}(a, \mathcal{F}(b, c, d), e) = \mathcal{F}(a, b, \mathcal{F}(c, d, e))$ holds for every 5-tuple $(a, b, c, d, e) \in X^5$. The function \mathcal{F} is given as a black-box: when we make a query (a, b, c) to \mathcal{F} , the answer $\mathcal{F}(a, b, c)$ is returned. We can show that that the property " \mathcal{F} is not associative" has a certificate corresponding to a sub-hypergraph of seven vertices in a 3-uniform directed hypergraph with each edge weighted by an element in X. By applying Theorem 5 with adaptations to directed hypergraphs with non-binary hyperedge weights, we obtain the following result.

Theorem 12 There exists a quantum algorithm that determines if \mathcal{F} is associative with high probability using $\tilde{O}(n^{169/80}) = \tilde{O}(n^{2.1125})$ queries.

Proof. To apply Theorem 5, we basically follow the approach of Ref. [13] for the (binary) associativity testing. If \mathcal{F} is not associative, there is a 5-tuple $(a_1, a_2, a_3, a_4, a_5) \in X^5$ such that (i) $\mathcal{F}(\mathcal{F}(a_1, a_2, a_3), a_4, a_5) \neq \mathcal{F}(a_1, \mathcal{F}(a_2, a_3, a_4), a_5)$ or (ii) $\mathcal{F}(a_1, \mathcal{F}(a_2, a_3, a_4), a_5) \neq \mathcal{F}(a_1, a_2, \mathcal{F}(a_3, a_4, a_5))$. Thus, it suffices to check case (i) and case (ii) individually.

We consider only case (i) since case (ii) is similarly analyzed and needs the same query complexity as the algorithm for case (i). A certificate to case (i) is given by a 7-tuple $(a_1, a_2, \ldots, a_7) \in X^7$ such that $\mathcal{F}(a_1, a_2, a_3) = a_6, \mathcal{F}(a_2, a_3, a_4) = a_7$ and $\mathcal{F}(a_6, a_4, a_5) \neq \mathcal{F}(a_1, a_7, a_5)$. Let H be a directed hypergraph on seven vertices with directed hyperedges (1, 2, 3), (2, 3, 4), (6, 4, 5), (1, 7, 5). Then, finding a certificate to case (i) can be reduced to finding a sub-hypergraph isomorphic to H in an n-vertex directed hypergraph with each hyperedge weighted with an element in X, to which we will apply Theorem 5. Note that, although the proof of Theorem 5 assumes the given hypergraph is undirected and each hyperedge is weighted with binary values, we can easily adapt the algorithm to handle directed hypergraphs with non-binary hyperedge weight: (1) to deal with directed hyperedges of H we simply replace a query to the black-box on an unordered triple by a query on the corresponding ordered triple (for instance, for $(u, v, w) \in V_4 \times V_5 \times V_6$ we will query $\chi((w, u, v))$ instead of $\chi(\{u, v, w\})$); (2) since the quantum walk actually does not use the property that hyperedges have binary weight, it works without modification for the case of non-binary hyperedge weights as well. Note also that the resulting algorithm searches H over X^7 , so we do not need to consider separately the case of detecting vertex contractions of H as in Ref. [13].

By numerical search, we found the following schedule:

 $(1,3,4,6,2,5,7,\{1,2\},\{1,3\},\{1,5\},\{1,7\},\{2,3\},\{2,4\},\{3,4\},\{4,5\},\{4,6\},\\ \{5,6\},\{5,7\},\{1,2,3\},\{1,5,7\},\{2,3,4\},\{4,5,6\}).$

The complexity of the algorithm for this schedule is minimized by the following values of parameters: $r_1 = n^{3/4}, r_2 = n, r_3 = n, r_4 = n^{7/8}, r_5 = n^{1/2}, r_6 = n, r_7 = n; f_{12} = n^{7/4}, f_{13} = n^{7/4}, f_{15} = n^{5/4}, f_{17} = n^{7/4}, f_{23} = n^{23/16}, f_{24} = n^{29/16}, f_{34} = n^{15/8}, f_{45} = n^{11/8}, f_{46} = n^{15/8}, f_{56} = n^{3/2}, f_{57} = n^{3/2}; e_{123} = n^{169/80}, e_{157} = n^{169/80}, e_{234} = n^{169/80} \text{ and } e_{456} = 1.$ It is easy to check that this set of parameters is admissible. This gives query complexity $\tilde{O}(n^{169/80})$.

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