# Quantum Algorithms for Finding Constant-sized Sub-hypergraphs 

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#### Abstract

We develop a general framework to construct quantum algorithms that detect if a 3 -uniform hypergraph given as input contains a sub-hypergraph isomorphic to a prespecified constant-sized hypergraph. This framework is based on the concept of nested quantum walks recently proposed by Jeffery, Kothari and Magniez [SODA'13], and extends the methodology designed by Lee, Magniez and Santha [SODA' 13] for similar problems over graphs. As applications, we obtain a quantum algorithm for finding a 4 -clique in a 3 -uniform hypergraph on $n$ vertices with query complexity $O\left(n^{1.883}\right)$, and a quantum algorithm for determining if a ternary operator over a set of size $n$ is associative with query complexity $O\left(n^{2.113}\right)$.


## 1 Introduction

Quantum query complexity is a model of quantum computation, in which the cost of computing a function is measured by the number of queries that are made to the input given as a black-box. In this model, it was exhibited in the early stage of quantum computing research that there exist quantum algorithms superior to the classical counterparts, such as Deutsch and Jozsa's algorithm, Simon and Shor's period finding algorithms, and Grover's search algorithm. Extensive studies following them have invented a lot of powerful upper bound (i.e., algorithmic) techniques such as variations/generalizations of Grover's search algorithm or quantum walks. Although these techniques give tight bounds for many problems, there are still quite a few cases for which no tight bounds are known. Intensively studied problems among them are the $k$-distinctness problem [1, 4, 5] and the triangle finding problem [3, 7, 10, 13, 15].

A recent breakthrough is the concept of learning graph introduced by Belovs [3]. This concept enables one to easily find a special form of feasible solutions to the minimization form (i.e., the dual form) of the general adversary bound [8, 16], and makes possible to detour the need of solving a semidefinite program of exponential size to find a non-trivial upper bound. Indeed, Belovs [3] improved the long-standing $O\left(n^{13 / 10}\right)$ upper bound [15] of the triangle finding problem to $O\left(n^{35 / 27}\right)$. His idea was generalized by Lee, Magniez and Santha [12] and Zhu [22] to obtain a quantum algorithm that finds a constant-sized subgraph with complexity $o\left(n^{2-2 / k}\right)$, improving the previous best bound $O\left(n^{2-2 / k}\right)$ [15], where $k$ is the size of the subgraph. Subsequently, Lee, Magniez and Santha [13] constructed a triangle finding algorithm with quantum query complexity $O\left(n^{9 / 7}\right)$. This bound was later shown by Belovs and Rosmanis [6] to be the best possible bound attained by the family of quantum algorithms whose complexities depend only on the index set of 1-certificates. Ref. [13] also gave a framework of quantum algorithms for finding a constant-sized subgraph,
based on which they showed that associativity testing (testing if a binary operator over a domain of size $n$ is associative) has quantum query complexity $O\left(n^{10 / 7}\right)$.

Recently, Jeffery, Kothari and Magniez [10] cast the idea of the above triangle finding algorithms into the framework of quantum walks (called nested quantum walks) by recursively performing the quantum walk algorithm given by Magniez, Nayak, Roland and Santha [14] (which extended two seminal works for quantum walk algorithms, Szegedy's algorithm [17] based on Markov chain and Ambainis' algorithm [1] for $k$-element distinctness). Indeed, they presented two quantum-walk-based triangle finding algorithms of complexities $\tilde{O}\left(n^{35 / 27}\right)$ and $\tilde{O}\left(n^{9 / 7}\right)$, respectively. The nested quantum walk framework was further employed in [5] (but in a different way from [10]) to obtain $\tilde{O}\left(n^{5 / 7}\right)$ complexity for the 3-distinctness problem. This achieves the best known upper bound (up to poly-logarithmic factors), which was first obtained with the learning-graph-based approach [4].

The triangle finding problem also plays a central role in several areas beside query complexity, and it has been recently discovered that faster algorithms for (weighted versions of) triangle finding would lead to faster algorithms for matrix multiplication [11, 19], the 3SUM problem [18], and for Max-2SAT [20, 21]. In particular, Max-2SAT over $n$ variables is reducible to finding a triangle with maximum weight over $O\left(2^{n / 3}\right)$ vertices; in this context, although the final goal is a time-efficient classical or quantum algorithm that finds a triangle with maximum weight, studying triangle finding in the query complexity model is a first step toward this goal.

Our results. Along this line of research, this paper studies the problem of finding a 4-clique (i.e., the complete 3 -uniform hypergraph with 4 vertices) in a 3 -uniform hypergraph, a natural generalization of finding a triangle in an ordinary graph (i.e., a 2 -uniform hypergraph). Our initial motivation comes from the complexity-theoretic importance of the problem. Indeed, while it is now well-known that Max-3SAT over $n$ variables is reducible to finding a 4 -clique with maximum weight in a 3 -uniform hypergraph of $O\left(2^{n / 4}\right)$ vertices (the reduction is similar to the reduction from Max-2SAT to triangle finding mentioned above; we refer to [21] for details), no efficient classical algorithm for 4-clique finding has been discovered so far. Constructing query-efficient algorithms for this problem can be seen as a first step to investigate the possibility of faster (in the time complexity setting) classical or quantum algorithms for Max-3SAT.

Concretely, and more generally, this paper gives a framework based on quantum walks for finding any constant-sized sub-hypergraph in a 3-uniform hypergraph (Theorem 5). This is an extension of the learning-graph-based algorithm in [13] to the hypergraph case in terms of a nested quantum walk [10]. We illustrate this methodology by constructing a quantum algorithm that finds a 4 -clique in a 3 -uniform hypergraph 1 with query complexity $\tilde{O}\left(n^{241 / 128}\right)=O\left(n^{1.883}\right)$, while naïve Grover search over the $\binom{n}{4}$ combinations of vertices only gives $O\left(n^{2}\right)$. As another application, we also construct a quantum algorithm that determines if a ternary operator is associative using $\tilde{O}\left(n^{169 / 80}\right)=O\left(n^{2.113}\right)$ queries, while naïve Grover search needs $O\left(n^{2.5}\right)$ queries.

In the course of designing the quantum walk framework, we introduce several new technical ideas (outlined below) for analyzing nested quantum walks to cope with difficulties that do not arise in the 2 -uniform case (i.e., ordinary graphs), such as the fact that the size of the random subset taken in an inner walk may vary depending on the random subsets taken in outer walks. We believe that these ideas may be applicable to various problems beyond sub-hypergraph finding.

Our framework is another demonstration of the power of the concept of nested quantum walks, and of its wide applicability. In particular, we crucially rely on the high-level description and analysis made possible by the nested quantum walk formalism to overcome all the technical difficulties that arise when considering 3 -uniform hypergraphs.

Technical contribution. Roughly speaking, the subgraph finding algorithm by Lee, Magniez and San-

[^0]tha [13] works as follows. First, for each vertex $i$ in the subgraph $H$ that we want to find, a random subset $V_{i}$ of vertices of the input graph is taken. This subset $V_{i}$ represents a set of candidates for the vertex $i$. Next, for each edge $(i, j)$ in the subgraph $H$, a random subset of pairs in $V_{i} \times V_{j}$ is taken, representing a set of candidates for the edge $(i, j)$. The most effective feature of their algorithm is to introduce a parameter for each ordered pair $\left(V_{i}, V_{j}\right)$ that controls the average degree of a vertex in the bipartite graph between $V_{i}$ and $V_{j}$. To make the algorithm efficient, it is crucial to keep the degree of every vertex in $V_{i}$ almost equal to the value specified by the parameter. For this, they carefully devise a procedure for taking pairs from $V_{i} \times V_{j}$.

Our basic idea is similar in that we first, for each vertex $i$ in the sub-hypergraph $H$ that we want to find, take a random subset $V_{i}$ of vertices in the input 3-uniform hypergraph as a set of candidates for the vertex $i$ and then, for each hyperedge $\{i, j, k\}$ of $H$, take a random subset of triples in $V_{i} \times V_{j} \times V_{k}$. One may think that the remaining task is to fit the pair-taking procedure into the hypergraph case. It, however, turns out to be technically very complicated to generalize the pair-taking procedure from [13] to an efficient triple-taking procedure. Instead we cast the idea into the nested quantum walk of Jeffery, Kothari and Magniez [10] and employ probabilistic arguments. More concretely, we introduce a parameter that specifies the number $e_{i j k}$ of triples to be taken from $V_{i} \times V_{j} \times V_{k}$ for each hyperedge $\{i, j, k\}$ of $H$. We then argue that, for randomly chosen $e_{i j k}$ triples, the degree of each vertex sharply concentrates around its average, where the degree means the number of triples including the vertex (in this sense, the parameters $e_{i j k}$ play essentially the same role as those of "average degrees" used in [10], but introducing $e_{i j k}$ gives a neat formulation of the algorithm and this is effective particularly in handling such complicated cases as hypergraphs). This makes it substantially easier to analyze the complexity of all involved quantum walks, and enables us to completely analyze the complexity of our approach. Unfortunately, it turns out that this approach (taking the sets $V_{i}$ first, and then $e_{i j k}$ triples from each $V_{i} \times V_{j} \times V_{k}$ ) does not lead to any improvement over the naïve $O\left(n^{2}\right)$-query quantum algorithm.

Our key idea is to introduce, for each unordered pair $\{i, j\}$ of vertices in $H$, a parameter $f_{i j}$, and modify the approach as follows. After randomly choosing $V_{i}, V_{j}, V_{k}$, we take three random subsets $F_{i j} \subseteq V_{i} \times V_{j}$, $F_{j k} \subseteq V_{j} \times V_{k}$, and $F_{i k} \subseteq V_{i} \times V_{k}$ of size $f_{i j}, f_{j k}$ and $f_{i k}$, respectively. We then randomly choose $e_{i j k}$ triples from the set $\Gamma_{i j k}=\left\{(u, v, w) \mid(u, v) \in F_{i j},(u, w) \in F_{i k}\right.$ and $\left.(v, w) \in F_{j k}\right\}$. The difficulty here is that the size of $\Gamma_{i j k}$ varies depending on the sets $F_{i j}, F_{j k}, F_{i k}$. Another problem is that, after taking many quantum-walks (i.e., performing the update operation many times), the distribution of the set of pairs can change. To overcome these difficulties, we carefully define the "marked states" (i.e., "absorbing states") of each level of the nested quantum walk: besides requiring, as usual, that the set (of the form $V_{i}, F_{i j}$ or $\Gamma_{i j k}$ ) associated to a marked state should contain a part (i.e., a vertex, a pair of vertices or a triple of vertices) of a copy of $H$, we also require that this set should satisfy certain regularity conditions. We then show that the associated sets almost always satisfy the regularity conditions, by using concentration theorems for hypergeometric distributions. This regularity enables us to effectively bound the complexity of our new approach, giving in particular the claimed $\tilde{O}\left(n^{241 / 128}\right)$-query upper bound when $H$ is a 4 -clique.

## 2 Preliminaries

For any $k \geq 2$, an undirected $k$-uniform hypergraph is a pair $(V, E)$, where $V$ is a finite set (the set of vertices), and $E$ is a set of unordered $k$-tuples of elements in $V$ (the set of hyperedges). An undirected 2 -uniform hypergraph is simply an undirected graph.

In this paper, we use the standard quantum query complexity model formulated in Ref. [2]. We deal with (undirected) 3-uniform hypergraphs $G=(V, E)$ as input, and the operation of the black-box is given as the unitary mapping $|\{u, v, w\}, b\rangle \mapsto|\{u, v, w\}, b \oplus \chi(\{u, v, w\})\rangle$ for $b \in\{0,1\}$, where the triple $\{u, v, w\}$ is the query to the black-box and $\chi(\{u, v, w\})$ is the answer on whether the triple is a hyperedge of $G$, namely, $\chi(\{u, v, w\})=1$ if $\{u, v, w\} \in E$ and $\chi(\{u, v, w\})=0$ otherwise.

Our algorithmic framework is based on the concept of the nested quantum walk introduced by Jeffery, Kothari and Magniez [10]. In the nested quantum walk, for each positive integer $t$, the walk at level $t$ checks whether the current state is marked or not by invoking the walk at level $t+1$, and this is iterated recursively until some fixed level $m$. The data structure of the walk at level $t$ is defined so that it includes the initial state of the walk at level $t+1$, which means that the setup cost of the walk at level $t \geq 2$ is zero. Jeffery, Kothari and Magniez have shown (in Section 4.1 of [10]) that the overall complexity of such a walk is

$$
\tilde{O}\left(\mathrm{~S}+\sum_{t=1}^{m}\left(\prod_{r=1}^{t} \frac{1}{\sqrt{\varepsilon_{r}}}\right) \frac{1}{\sqrt{\delta_{t}}} \mathrm{U}_{t}\right)
$$

if the checking cost at level $m$ is zero, which will be our case. Here $S$ denotes the setup cost of the whole nested walk, $\mathrm{U}_{\mathrm{t}}$ denotes the cost of updating the database of the walk at level $t, \delta_{t}$ denotes the spectral gap of the walk at level $t$, and $\varepsilon_{r}$ denotes the fraction of marked states for the walk at level $r$. As in most quantum walk papers, we only consider quantum walks on the Johnson graphs, where the Johnson graph $J(N, K)$ is a graph such that each vertex is a subset with size $K$ of a set with size $N$ and two vertices corresponding to subsets $S$ and $S^{\prime}$ are adjacent if and only if $\left|S \Delta S^{\prime}\right|=2$ (we denote by $S \Delta S^{\prime}$ the symmetric difference between $S$ and $S^{\prime}$ ). If the walk at level $t$ is on $J(N, K)$, then its spectral gap $\delta_{t}$ is known to be $\Omega(1 / K)$.

Consider the update operation of the walk at any level. The update cost may vary depending on the states of the walk we want to update. Assume without loss of generality that the update operation is of the form $U=\sum_{i}|i\rangle\langle i| \otimes U_{i}$, where each $U_{i}$ can be implemented using $q_{i}$ queries, and the quantum state to be updated is of the form $|s\rangle=\sum_{i} \alpha_{i}|i\rangle\left|s_{i}\right\rangle$. Then the following lemma, used in [10], shows that if the magnitude of the states $|i\rangle\left|s_{i}\right\rangle$ that cost much to update (i.e., such that $q_{i}$ is large) is small enough, we can approximate the update operator $U$ with good precision by replacing all $U_{i}$ acting on such costly states with the identity operator.

Lemma $1([\mathbf{1 0}])$ Let $U=\sum_{i}|i\rangle\langle i| \otimes U_{i}$ be a controlled unitary operator and let $q_{i}$ be the query complexity of exactly implementing $U_{i}$. For any fixed integer $T$, define $\tilde{U}$ as $\sum_{i: q_{i} \leq T}|i\rangle\langle i| \otimes U_{i}+\sum_{i: q_{i}>T}|i\rangle\langle i| \otimes$ $\mathbb{I}$, where $\mathbb{I}$ is the identity operator on the space on which $U_{i}$ acts. Then, for any quantum state $|s\rangle=$ $\sum_{i} \alpha_{i}|i\rangle\left|s_{i}\right\rangle$, the inequality $\left.|\langle s| \tilde{U} U| s\right\rangle \mid \geq 1-\epsilon_{T}$ holds whenever $\epsilon_{T} \geq \sum_{i: q_{i}>T}\left|\alpha_{i}\right|^{2}$.

In the analysis of this paper, hypergeometric distributions will appear many times. Let $H G(n, m, r)$ denote the hypergeometric distribution whose random variable $X$ is defined by

$$
\operatorname{Pr}[X=j]=\frac{\binom{m}{j}\binom{n-m}{r-j}}{\binom{n}{r}} .
$$

We first state below several tail bounds of hypergeometric distributions (the proof can be easily obtained from Theorem 2.10 in [9]).

Lemma 2 When $X$ has a hypergeometric distribution with expectation $\mu$, the following hold (where $\exp (x)$ denotes $e^{x}$ ):
(1) For any $0<\delta \leq 1, \operatorname{Pr}[X \geq(1+\delta) \mu] \leq \exp \left(-\frac{\mu \delta^{2}}{3}\right)$.
(2) For any $0<\delta<1, \operatorname{Pr}[X \leq(1-\delta) \mu] \leq \exp \left(-\frac{\mu \delta^{2}}{2}\right)$.
(3) For any $\delta>2 e-1, \operatorname{Pr}[X>(1+\delta) \mu]<\left(\frac{1}{2}\right)^{(1+\delta) \mu}$.

## 3 Statement of our main result

In this section, we state our main result (an upper bound on the query complexity of finding a constant-sized sub-hypergraph in a 3-uniform hypergraph) in terms of loading schedules, which generalizes the concept of loading schedules for graphs introduced, in the learning graph framework, by Lee, Magniez and Santha [13], and used in the framework of nested quantum walks by Jeffery, Kothari and Magniez [10].

Let $H$ be a 3-uniform hypergraph with $\kappa$ vertices. We identify the set of vertices of $H$ with the set $\Sigma_{1}=\{1, \ldots, \kappa\}$. We identify the set of hyperedges of $H$ with the set $\Sigma_{3} \subseteq\{\{1,2,3\},\{1,2,4\}, \ldots,\{\kappa-$ $2, \kappa-1, \kappa\}\}$. We identify the set of (unordered) pairs of vertices included in at least one hyperedge of $H$ with the set $\Sigma_{2}=\left\{\{i, j\} \mid\{i, j, k\} \in \Sigma_{3}\right.$ for some $\left.k\right\}$. A loading schedule for $H$ is defined as follows.

Definition 3 A loading schedule for $H$ of length $m$ is a list $S=\left(s_{1}, \ldots, s_{m}\right)$ of $m$ elements such that the following three properties hold for all $t \in\{1, \ldots, m\}$ : (i) $s_{t} \in \Sigma_{1} \cup \Sigma_{2} \cup \Sigma_{3}$; (ii) if $s_{t}=\{i, j\}$, then there exist $t_{1}, t_{2} \in\{1, \ldots, t-1\}$ such that $s_{t_{1}}=i$ and $s_{t_{2}}=j$; (iii) if $s_{t}=\{i, j, k\}$, then there exist $t_{1}, t_{2}, t_{3} \in\{1, \ldots, t-1\}$ such that $s_{t_{1}}=\{i, j\}, s_{t_{2}}=\{i, k\}$ and $s_{t_{3}}=\{j, k\}$. A loading schedule $S$ is valid if no element of $\Sigma_{1} \cup \Sigma_{2} \cup \Sigma_{3}$ appears more than once and, for any $\{i, j, k\} \in \Sigma_{3}$, there exists an index $t \in\{1, \ldots, m\}$ such that $s_{t}=\{i, j, k\}$.

We now introduce the concept of parameters associated to a loading schedule. Formally, these parameters are functions of the variable $n$ representing the number of vertices of the input 3-uniform hypergraphs $G=(V, E)$. We will nevertheless, in a slight abuse of notation, consider that $n$ is fixed, and define them as integers (implicitly depending on $n$ ).
Definition 4 Let $S=\left(s_{1}, \ldots, s_{m}\right)$ be a loading schedule for $H$ of length $m$. $A$ set of parameters for $S$ is a set of $m$ integers defined as follows: for each $t \in\{1, \ldots, m\}$,

- if $s_{t}=i$, then the associated parameter is denoted by $r_{i}$ and satisfies $r_{i} \in\{1, \ldots, n\}$;
- if $s_{t}=\{i, j\}$, then the associated parameter is denoted by $f_{i j}$ and satisfies $f_{i j} \in\left\{1, \ldots, r_{i} r_{j}\right\}$;
- if $s_{t}=\{i, j, k\}$, then the associated parameter is denoted by $e_{i j k}$ and satisfies $e_{i j k} \in\left\{1, \ldots, r_{i} r_{j} r_{k}\right\}$. The set of parameters is admissible if $r_{i} \geq 1, e_{i j k} \geq 1, \frac{r_{i} r_{j}}{f_{i j}} \geq 1, \frac{f_{i j} f_{i k} f_{j k} /\left(r_{i} r_{j} r_{k}\right)}{e_{i j k}} \geq 1$, and the terms $\frac{n}{r_{i}}$, $\frac{f_{i j}}{r_{i}}, \frac{f_{i j}}{r_{j}}, \frac{f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}$ are larger than $n^{\gamma}$ for some constant $\gamma>0$.

Now we state the main result in terms of loading schedules.
Theorem 5 Let $H$ be any constant-sized 3-uniform hypergraph. Let $S=\left(s_{1}, \ldots, s_{m}\right)$ be a valid loading schedule for $H$ with an admissible set of parameters. There exists a quantum algorithm that, given as input a 3-uniform hypergraph $G$ with $n$ vertices, finds a sub-hypergraph of $G$ isomorphic to $H$ (and returns "no" if there are no such sub-hypergraphs) with probability at least some constant, and has query complexity

$$
\tilde{O}\left(S+\sum_{t=1}^{m}\left(\prod_{r=1}^{t} \frac{1}{\sqrt{\varepsilon_{r}}}\right) \frac{1}{\sqrt{\delta_{t}}} U_{t}\right)
$$

where $\mathrm{S}, \mathrm{U}_{t}, \delta_{t}$ and $\varepsilon_{r}$ are evaluated as follows:

- $\mathrm{S}=\sum_{\{i, j, k\} \in \Sigma_{3}} e_{i j k}$;
- for $t \in\{1, \ldots, m\}$, (i) if $s_{t}=\{i\}$, then $\delta_{t}=\Omega\left(\frac{1}{r_{i}}\right), \varepsilon_{t}=\Omega\left(\frac{r_{i}}{n}\right)$ and $\mathrm{U}_{t}=$ $\tilde{O}\left(1+\sum_{\{j, k\}:\{i, j, k\} \in \Sigma_{3}} \frac{e_{i j k}}{r_{i}}\right)$; (ii) if $s_{t}=\{i, j\}$, then $\delta_{t}=\Omega\left(\frac{1}{f_{i j}}\right), \varepsilon_{t}=\Omega\left(\frac{f_{i j}}{r_{i} r_{j}}\right)$ and $\mathrm{U}_{t}=$ $\tilde{O}\left(1+\sum_{k:\{i, j, k\} \in \Sigma_{3}} \frac{e_{i j k}}{f_{i j}}\right)$; (iii) if $s_{t}=\{i, j, k\}$, then $\delta_{t}=\Omega\left(\frac{1}{e_{i j k}}\right), \varepsilon_{t}=\Omega\left(\frac{e_{i j k} r_{i} r_{j} r_{k}}{f_{i j} f_{k} f_{j k}}\right)$ and $\mathrm{U}_{t}=O(1)$.


## 4 Proof of Theorem 5

In this section, we prove Theorem 5 by constructing an algorithm based on the concept of $m$-level nested quantum walks, in which the walk at level $t$ will correspond to the element $s_{t}$ of the loading schedule for each $t \in\{1, \ldots, m\}$. For convenience, we will write $M_{i j k}=11 \frac{f_{i j} f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}$ for each $\{i, j, k\} \in \Sigma_{3}$.

### 4.1 Definition of the walks

At level $t \in\{1, \ldots, m\}$, the quantum walk will differ according to the nature of $s_{t}$, so there are three cases to consider.

Case $1\left[s_{t}=i\right]$ : The quantum walk will be over the Johnson graph $J\left(n, r_{i}\right)$. The space of the quantum walk will then be $\Omega_{t}=\left\{T \subseteq\{1, \ldots, n\}| | T \mid=r_{i}\right\}$. A state of this walk is an element $R_{t} \in \Omega_{t}$.
Case $2\left[s_{\boldsymbol{t}}=\{i, j\}\right]$ : The quantum walk will be over $J\left(r_{i} r_{j}, f_{i j}\right)$. The space of the quantum walk will then be $\Omega_{t}=\left\{T \subseteq\left\{1, \ldots, r_{i} r_{j}\right\}| | T \mid=f_{i j}\right\}$. A state of this walk is an element $R_{t} \in \Omega_{t}$.
Case 3 [ $s_{t}=\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ ]: The quantum walk will be over $J\left(M_{i j k}, e_{i j k}\right)$. The space of the quantum walk will then be $\Omega_{t}=\left\{T \subseteq\left\{1, \ldots, M_{i j k}\right\}| | T \mid=e_{i j k}\right\}$. A state of this walk is an element $R_{t} \in \Omega_{t}$.

### 4.2 Definition of the data structures of the walks

Let us fix an arbitrary ordering on the set $V \times V \times V$ of triples of vertices. For any set $\Gamma \subseteq V \times V \times V$ and any $R \subseteq\left\{1, \ldots,|V|^{3}\right\}$, define the set $\mathrm{Y}(R, \Gamma)$ consisting of at most $|R|$ triples of vertices which are taken from $\Gamma$ by the process below.

- Construct a list $\Lambda$ of all the triples in $V \times V \times V$ as follows: first, all the triples in $\Gamma$ are listed in increasing order and, then, all the triples in $(V \times V \times V) \backslash \Gamma$ are listed in increasing order.
- For any $a \in\left\{1, \ldots,|V|^{3}\right\}$, let $\Lambda[a]$ denote the $a$-th triple of the list.
- Define $\mathrm{Y}(R, \Gamma)=\{\Lambda[a] \mid a \in R\} \cap \Gamma$.

The following lemma will be useful later in this section.
Lemma 6 Let $\Gamma$ and $\Gamma^{\prime}$ be two subsets of $V \times V \times V$. Let $p$ and $r$ be any parameters such that $1 \leq r \leq$ $p \leq|V|^{3}$. There exists a permutation $\pi$ of $\{1, \ldots, p\}$ such that, if $R$ is a subset of $\{1, \ldots, p\}$ of size $r$ taken uniformly at random, then

$$
\operatorname{Pr}_{R}\left[\left|\mathrm{Y}(R, \Gamma) \Delta \mathrm{Y}\left(\pi(R), \Gamma^{\prime}\right)\right| \leq \frac{22 r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+100 \log n\right] \geq 1-2\left(\frac{1}{2}\right)^{\frac{11 r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n} .
$$

Proof. Let $\Lambda$ and $\Lambda^{\prime}$ be the lists obtained when using the construction for $\Gamma$ and $\Gamma^{\prime}$, respectively. Let us write

$$
\begin{aligned}
& \Lambda_{1}=\{\Lambda[a] \mid 1 \leq a \leq p\} \cap \Gamma, \\
& \Lambda_{1}^{\prime}=\left\{\Lambda^{\prime}[a] \mid 1 \leq a \leq p\right\} \cap \Gamma^{\prime} .
\end{aligned}
$$

We can show the following inequality.
Claim $7\left|\Lambda_{1} \Delta \Lambda_{1}^{\prime}\right| \leq 2\left|\Gamma \Delta \Gamma^{\prime}\right|$.

Proof. $\Lambda_{1}$ contains precisely the $\left|\Lambda_{1} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right|$ smallest (with respect to the increasing order) elements of $\Gamma \cap \Gamma^{\prime}$, while the other $\left|\Lambda_{1} \cap\left(\Gamma \backslash \Gamma^{\prime}\right)\right|$ elements of $\Lambda_{1}$ are in $\Gamma \backslash \Gamma^{\prime}$. Similarly, $\Lambda_{1}^{\prime}$ contains precisely the $\left|\Lambda_{1}^{\prime} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right|$ smallest elements of $\Gamma \cap \Gamma^{\prime}$, while the other $\left|\Lambda_{1}^{\prime} \cap\left(\Gamma^{\prime} \backslash \Gamma\right)\right|$ elements of $\Lambda_{1}^{\prime}$ are in $\Gamma^{\prime} \backslash \Gamma$. We can write

We have to consider two cases.
Case 1: $\left|\Lambda_{1}\right|=\left|\Lambda_{1}^{\prime}\right|=p$
Assume, without loss of generality, that $\left|\Lambda_{1} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right| \leq\left|\Lambda_{1}^{\prime} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right|$. We have

$$
\left|\Lambda_{1} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right|=p-\left|\Lambda_{1} \cap\left(\Gamma \backslash \Gamma^{\prime}\right)\right| \geq p-\left|\Gamma \backslash \Gamma^{\prime}\right|
$$

and $\left|\Lambda_{1}^{\prime} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right| \leq p$. Thus

$$
\left|\left|\Lambda_{1}^{\prime} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right|-\left|\Lambda_{1} \cap\left(\Gamma \cap \Gamma^{\prime}\right)\right|\right| \leq p-\left(p-\left|\Gamma \backslash \Gamma^{\prime}\right|\right)=\left|\Gamma \backslash \Gamma^{\prime}\right|,
$$

which gives $\left|\Lambda_{1} \Delta \Lambda_{1}^{\prime}\right| \leq 2\left|\Gamma \Delta \Gamma^{\prime}\right|$, as claimed.
Case 2: $\min \left(\left|\Lambda_{1}\right|,\left|\Lambda_{1}^{\prime}\right|\right)<p$
By symmetry, it suffices to show only the case where $\left|\Lambda_{1}^{\prime}\right| \leq\left|\Lambda_{1}\right|$. Since $\left|\Lambda_{1}^{\prime}\right|<p$, we have $\Lambda_{1}^{\prime}=\Gamma^{\prime}$. This implies that $\left|\Lambda_{1} \backslash \Lambda_{1}^{\prime}\right|=\left|\Lambda_{1} \backslash \Gamma^{\prime}\right| \leq\left|\Gamma \backslash \Gamma^{\prime}\right|$. Since $\left|\Lambda_{1}^{\prime}\right| \leq\left|\Lambda_{1}\right|$, we have $\left|\Lambda_{1}^{\prime} \backslash \Lambda_{1}\right| \leq\left|\Lambda_{1} \backslash \Lambda_{1}^{\prime}\right| \leq\left|\Gamma \backslash \Gamma^{\prime}\right|$. Hence, $\left|\Lambda_{1} \Delta \Lambda_{1}^{\prime}\right| \leq 2\left|\Gamma \backslash \Gamma^{\prime}\right| \leq 2\left|\Gamma \Delta \Gamma^{\prime}\right|$ also holds in this case.

For any $a \in\{1, \ldots, \min (p,|\Gamma|)\}$ such that $\Lambda[a]$ is in $\Lambda_{1}^{\prime}$, we set $\pi(a)=a^{\prime}$, where $a^{\prime}$ is (he (unique) index in $\left\{1, \ldots, \min \left(p,\left|\Gamma^{\prime}\right|\right)\right\}$ such that $\Lambda[a]=\Lambda^{\prime}\left[a^{\prime}\right]$. For all other $a \in\{1, \ldots, p\}$, we set $\pi(a)$ arbitrarily such that $\pi$ becomes a permutation of $\{1, \ldots, p\}$.

Let $R$ be any subset of $\{1, \ldots, p\}$. Define the subsets $S_{R}, S_{R}^{\prime} \subseteq R$ as follows:

$$
\begin{aligned}
& S_{R}=\left\{a \in R \mid \Lambda[a] \in \Lambda_{1} \backslash \Lambda_{1}^{\prime}\right\}, \\
& S_{R}^{\prime}=\left\{b \in \pi(R) \mid \Lambda^{\prime}[b] \in \Lambda_{1}^{\prime} \backslash \Lambda_{1}\right\} .
\end{aligned}
$$

From the definition of $\pi$, we know that for any element $a \in R \cap\{1, \ldots,|\Gamma|\}$ such that $a \notin S_{R}$ we have $\Lambda[a]=\Lambda^{\prime}[\pi(a)]$, which means that this element is not in $\mathrm{Y}(R, \Gamma) \backslash \mathrm{Y}\left(\pi(R), \Gamma^{\prime}\right)$. This implies that $\mathrm{Y}(R, \Gamma) \backslash \mathrm{Y}\left(\pi(R), \Gamma^{\prime}\right) \subseteq\left\{\Lambda[a] \mid a \in S_{R}\right\}$. Similarly, we have $\mathrm{Y}\left(\pi(R), \Gamma^{\prime}\right) \backslash \mathrm{Y}(R, \Gamma) \subseteq\left\{\Lambda^{\prime}[a] \mid a \in S_{R}^{\prime}\right\}$, which gives the inequality

$$
\left|\mathrm{Y}(R, \Gamma) \Delta \mathrm{Y}\left(\pi(R), \Gamma^{\prime}\right)\right| \leq\left|S_{R}\right|+\left|S_{R}^{\prime}\right| .
$$

Recall that $R$ is taken uniformly at random from $\{1, \ldots, p\}$ so that $|R|=r$. Thus, $\left|S_{R}\right|$ has hypergeometric distribution $H G\left(p,\left|\Lambda_{1} \backslash \Lambda_{1}^{\prime}\right|, r\right)$ and its expectation is $\mu=\frac{r\left|\Lambda_{1} \backslash \Lambda_{1}^{\prime}\right|}{p}$. Taking $\delta=\frac{1}{\mu}\left(11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n\right)-1$, we have

$$
\operatorname{Pr}\left[\left|S_{R}\right| \geq 11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n\right]=\operatorname{Pr}\left[\left|S_{R}\right| \geq(1+\delta) \mu\right] .
$$

Note that by Claim $71+\delta=\frac{1}{\mu}\left(11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n\right)>11 \cdot \frac{\left|\Gamma \Delta \Gamma^{\prime}\right|}{\left|\Lambda_{1} \Delta \Lambda_{1}^{\prime}\right|} \geq 11 / 2$ and hence $\delta \geq 9 / 2>2 e-1$. By Lemma 2(3), we have

$$
\operatorname{Pr}\left[\left|S_{R}\right| \geq 11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n\right]<(1 / 2)^{(1+\delta) \mu}=(1 / 2)^{11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n}
$$

Similarly, $\operatorname{Pr}\left[\left|S_{R}^{\prime}\right| \geq 11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n\right]<(1 / 2)^{11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n}$. Therefore,

$$
\begin{aligned}
\operatorname{Pr} & {\left[\left|\mathrm{Y}(R, \Gamma) \Delta \mathrm{Y}\left(\pi(R), \Gamma^{\prime}\right)\right| \geq 22 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+100 \log n\right] } \\
& \leq \operatorname{Pr}\left[\left|S_{R}\right|+\left|S_{R}^{\prime}\right| \geq 22 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+100 \log n\right] \\
& <2(1 / 2)^{11 \frac{r\left|\Gamma \Delta \Gamma^{\prime}\right|}{p}+50 \log n .}
\end{aligned}
$$

This completes the proof of Lemma6
Suppose that the states of the walks at levels $1, \ldots, m$ are $R_{1}, \ldots, R_{m}$, respectively. Assume that the set of vertices of $G$ is $V=\left\{v_{1}, \ldots, v_{n}\right\}$. We first interpret the states $R_{1}, \ldots, R_{m}$ as sets of vertices, sets of pairs of vertices or sets of triples of vertices in $V$, as follows. For each $t \in\{1, \ldots, m\}$, there are three cases to consider.

Case $1\left[s_{t}=i\right]$ : In this case, $R_{t}=\left\{a_{1}, \ldots, a_{r_{i}}\right\} \subseteq\{1, \ldots, n\}$. We associate to $R_{t}$ the set $V_{i}=$ $\left\{v_{a_{1}}, \ldots, v_{a_{r_{i}}}\right\}$. For further reference, we will rename the vertices in this set as $V_{i}=\left\{v_{1}^{i}, \ldots, v_{r_{i}}^{i}\right\}$.
Case $2\left[s_{t}=\{\boldsymbol{i}, \boldsymbol{j}\}\right.$ with $\boldsymbol{i}<j$ ]: We know that, in this case, there exist $t_{1}, t_{2} \in\{1, \ldots, t-1\}$ such that $s_{t_{1}}=i$ and $s_{t_{2}}=j$. The state $R_{t}$ represents a set $\left\{\left(a_{1}, b_{1}\right), \ldots,\left(a_{f_{i j}}, b_{f_{i j}}\right)\right\}$ of $f_{i j}$ pairs in $R_{t_{1}} \times R_{t_{2}}$. We associate to it the set $F_{i j}=\left\{\left(v_{a_{1}}^{i}, v_{b_{1}}^{j}\right), \ldots,\left(v_{a_{f_{i j}}}^{i}, v_{f_{f_{i j}}}^{j}\right)\right\}$ of pairs of vertices.
Case 3 [ $s_{t}=\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ with $\boldsymbol{i}<\boldsymbol{j}<\boldsymbol{k}$ ]: We know that there exist $t_{1}, t_{2}, t_{3} \in\{1, \ldots, t-1\}$ such that $s_{t_{1}}=\{i, j\}, s_{t_{2}}=\{i, k\}$ and $s_{t_{3}}=\{j, k\}$, and $R_{t}$ is a subset of $\left\{1, \ldots, M_{i j k}\right\}$ with $\left|R_{t}\right|=e_{i j k}$. Let us define the set

$$
\Gamma_{i j k}=\left\{(u, v, w) \in V_{i} \times V_{j} \times V_{k} \mid(u, v) \in F_{i j},(u, w) \in F_{i k} \text { and }(v, w) \in F_{j k}\right\} .
$$

We associate to $R_{t}$ the set $E_{i j k}=\mathrm{Y}\left(R_{t}, \Gamma_{i j k}\right)$.
We are now ready to define the data structures involved in the walks. When the states of the walks at levels $1, \ldots,(m-1)$ are $R_{1}, \ldots, R_{m-1}$, respectively, and the state of the most inner walk is $R_{m}$, the data structure associated with the most inner walk is denoted by $D\left(R_{1}, \ldots, R_{m}\right)$ and defined as:

$$
D\left(R_{1}, \ldots, R_{m}\right)=\left\{(\{u, v, w\}, \chi(\{u, v, w\})) \mid(u, v, w) \in \bigcup_{\{i, j, k\} \in \Sigma_{3}: i<j<k} E_{i j k}\right\} .
$$

The data structure associated with the walk at level $t$, for each $t \in\{1, \ldots, m-1\}$, is defined as:

$$
\sum_{R_{t+1} \in \Omega_{t+1}} \cdots \sum_{R_{m} \in \Omega_{m}}\left|R_{t+1}\right\rangle \cdots\left|R_{m}\right\rangle\left|D\left(R_{1}, \ldots, R_{m}\right)\right\rangle .
$$

Here and hereafter we omit normalization factors.

### 4.3 Marked states of the walks

For any $t \in\{1, \ldots, m-1\}$, the purpose of the walk at level $t+1$ is to check if the state of the walk $t$ is marked (for the most inner walk, the state can be checked without running another walk, since all the information necessary is already in the database). In this subsection we define the set of marked states for each walk.

Assume that the hypergraph $G$ contains a (without loss of generality, unique) sub-hypergraph isomorphic to $H$. Let $\left\{u_{1}, \ldots, u_{\kappa}\right\}$ denote the vertex set of this sub-hypergraph. For the most outer walk, $s_{1}=j$ for some $j \in\{1, \ldots, \kappa\}$ and we say that $R_{1}$ is marked if and only if $u_{j} \in V_{j}$. Consider a state $R_{t}$ of the walk at level $t>1$, and suppose that the states $R_{1}, \ldots, R_{t-1}$ are all marked. We have again three cases to consider.

Case $1\left[s_{\boldsymbol{t}}=i\right]: R_{t}$ corresponds to $V_{i}$. We say that $R_{t}$ is marked if and only if $u_{i} \in V_{i}$.
Case $2\left[s_{t}=\{i, j\}\right.$ with $\left.i<j\right]: R_{t}$ corresponds to $F_{i j}$, and we say that $R_{t}$ is marked if and only if the following four conditions hold:
(a) $\left(u_{i}, u_{j}\right) \in F_{i j}$;
(b) for all $u \in V_{i}, \frac{f_{i j}}{2 r_{i}} \leq\left|\left\{v \in V_{j} \mid(u, v) \in F_{i j}\right\}\right| \leq 2 \frac{f_{i j}}{r_{i}}$;
(c) for all $v \in V_{j}, \frac{f_{i j}}{2 r_{j}} \leq\left|\left\{u \in V_{i} \mid(u, v) \in F_{i j}\right\}\right| \leq 2 \frac{f_{i j}}{r_{j}}$;
(d) for any $k$ such that there exists $t_{1} \in\{1, \ldots, t-1\}$ for which $s_{t_{1}}=\{i, k\}$, and any $(v, w) \in V_{j} \times V_{k}$, $\mid\left\{u \in V_{i} \mid(u, v) \in F_{i j}\right.$ and $\left.(u, w) \in F_{i k}\right\} \left\lvert\, \leq 11 \frac{f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}\right.$.

Case 3 [ $s_{t}=\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ with $\boldsymbol{i}<\boldsymbol{j}<\boldsymbol{k}$ ]: $R_{t}$ corresponds to $E_{i j k}$, and we say that $R_{t}$ is marked if and only if $\left(u_{i}, u_{j}, u_{k}\right) \in E_{i j k}$.

The next subsection will use the following lemma.
Lemma 8 Assume that, for Case 2, $R_{t}$ is taken uniformly at random from $\Omega_{t}$ (i.e., $F_{i j}$ corresponds to a set of $f_{i j}$ pairs taken uniformly at random from $V_{i} \times V_{j}$ ). Then,

$$
\operatorname{Pr}\left[\text { Conditions }(b),(c),(d) \text { hold for } F_{i j}\right] \geq 1-2 r_{i} e^{-\frac{f_{i j}}{8 r_{i}}}-2 r_{j} e^{-\frac{f_{i j}}{8 r_{j}}}-r_{j} r_{k} \kappa 2^{-11 \frac{f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}} .
$$

Proof. For each $u \in V_{i}$, the quantity $\left|\left\{v \in V_{j} \mid(u, v) \in F_{i j}\right\}\right|$ is a random variable with hypergeometric distribution $H G\left(r_{i} r_{j}, r_{j}, f_{i j}\right)$. Its expectation is $f_{i j} / r_{i}$, and by Lemma2(1-2) it holds that

$$
\begin{aligned}
\operatorname{Pr}\left[\left|\left\{v \in V_{j} \mid(u, v) \in F_{i j}\right\}\right|>2 \frac{f_{i j}}{r_{i}}\right] & \leq \exp \left(-\frac{1}{3} \times \frac{f_{i j}}{r_{i}}\right) \leq \exp \left(-\frac{1}{8} \times \frac{f_{i j}}{r_{i}}\right), \\
\operatorname{Pr}\left[\left|\left\{v \in V_{j} \mid(u, v) \in F_{i j}\right\}\right|<\frac{f_{i j}}{2 r_{i}}\right] & \leq \exp \left(-\frac{1}{8} \times \frac{f_{i j}}{r_{i}}\right) .
\end{aligned}
$$

A similar statement holds for the degree of each $v \in V_{j}$, and thus, from the union bound, we obtain

$$
\operatorname{Pr}\left[\text { Condition (b) or (c) does not hold for } F_{i j}\right] \leq 2 r_{i} \exp \left(-\frac{f_{i j}}{8 r_{i}}\right)+2 r_{j} \exp \left(-\frac{f_{i j}}{8 r_{j}}\right) .
$$

For any $k$ such that there exists $t_{1} \in\{1, \ldots, t-1\}$ for which $s_{t_{1}}=\{i, k\}$, consider any $(v, w) \in V_{j} \times V_{k}$. Let us write $S=\left\{u \in V_{i} \mid(u, w) \in F_{i k}\right\}$. Since $R_{t_{1}}$ is marked, we know that $|S| \leq 2 f_{i k} / r_{k}$. The quantity $\mid\left\{u \in V_{i} \mid(u, v) \in F_{i j}\right.$ and $\left.(u, w) \in F_{i k}\right\} \mid$ has hypergeometric distribution $\operatorname{HG}\left(r_{i} r_{j},|S|, f_{i j}\right)$. Applying Lemma 2(3) with $\delta=\frac{11 f_{i k} / r_{k}}{|S|}-1>2 e-1$, we obtain

$$
\operatorname{Pr}\left[\mid\left\{u \in V_{i} \mid(u, v) \in F_{i j} \text { and }(u, w) \in F_{i k}\right\} \left\lvert\, \geq 11 \frac{f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}\right.\right] \leq 2^{-11 \frac{f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}} .
$$

Using the union bound (note in particular that there are at most $\kappa$ possibilities for $k$ ), we conclude that

$$
\operatorname{Pr}\left[\text { Condition (d) does not hold for } F_{i j}\right] \leq r_{j} r_{k} \kappa 2^{-11 \frac{f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}} .
$$

The statement of the lemma then follows from the union bound.

### 4.4 Analysis of the algorithm

Our nested quantum walk algorithm finds a marked state in the most inner walk and thus a sub-hypergraph isomorphic to $H$, with high probability, since, as will be shown below, the ideal nested quantum walks can be approximated with high accuracy. As explained in Section 2 the overall query complexity of the walk is

$$
\tilde{O}\left(\mathrm{~S}+\sum_{t=1}^{m}\left(\prod_{r=1}^{t} \frac{1}{\sqrt{\varepsilon_{r}}}\right) \frac{1}{\sqrt{\delta_{t}}} \mathrm{U}_{t}\right) .
$$

We will show below that the values of the terms $\mathrm{S}, \mathrm{U}_{\mathrm{t}}, \delta_{t}$ and $\varepsilon_{t}$ are as claimed in the statement of Theorem 5
We first make the following simple observation: when computing $\mathrm{U}_{t}$ and $\varepsilon_{t}$, we can assume that the state $R_{t-1}$ of the immediately outer walk is marked (and thus, by applying this argument recursively, that the states $R_{1}, \ldots, R_{t-1}$ of all the outer walks are marked). Indeed, remember that the purpose of the walk at level $t$ is to check if the state $R_{t-1}$ is marked. We first evaluate its complexity under the assumption that $R_{t-1}$ is marked, giving some upper bound $T$ on the complexity. Now, since the checking procedure in our framework has one-sided error, in the case where $R_{t-1}$ is not marked the checking procedure may not terminate after $T$ queries, but we can stop it after $T$ queries anyway and simply output that $R_{t-1}$ is not marked.

The setup cost $S$ for the algorithm is the number of queries needed to construct the superposition

$$
\sum_{R_{1} \in \Omega_{1}} \cdots \sum_{R_{m} \in \Omega_{m}}\left|R_{1}\right\rangle \cdots\left|R_{m}\right\rangle\left|D\left(R_{1}, \ldots, R_{m}\right)\right\rangle .
$$

This value is at most $\sum_{\{i, j, k\} \in \Sigma_{3}} e_{i j k}$.
We next evaluate $\delta_{t}$ and $\varepsilon_{t}$. The analysis is again divided into three cases.
Case $1\left[s_{t}=i\right]$ : Since the quantum walk is over $J\left(n, r_{i}\right)$ by the definition in Section 4.1 we have $\delta_{t}=\Omega\left(\frac{1}{r_{i}}\right)$ and $\varepsilon_{t}=\Omega\left(\frac{r_{i}}{n}\right)$.
Case $2\left[s_{\boldsymbol{t}}=\{i, j\}\right.$ with $\left.i<j\right]$ : Since the quantum walk is over $J\left(r_{i} r_{j}, f_{i j}\right)$, we have $\delta_{t}=\Omega\left(\frac{1}{f_{i j}}\right)$. The fraction of states $F_{i j}$ such that $\left(u_{i}, u_{j}\right) \in F_{i j}$ is $\Omega\left(\frac{f_{i j}}{r_{i} r_{j}}\right)$. While all those states may not be marked, Lemma 8 implies that the fraction of those states that are not marked is exponentially small when the set of parameters is admissible. Thus $\varepsilon_{t}=\Omega\left(\frac{f_{i j}}{r_{i} r_{j}}\right)$.
Case 3 [ $s_{\boldsymbol{t}}=\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ with $\left.\boldsymbol{i}<\boldsymbol{j}<\boldsymbol{k}\right]$ : In this case $\delta_{t}=\Omega\left(\frac{1}{e_{i j k}}\right)$. Since all the states $R_{1}, \ldots, R_{t-1}$ of the outer walks are assumed to be marked, by item (d) of the definition of the marked states in Section4.3, we can upper-bound $\left|\Gamma_{i j k}\right|=\sum_{(v, w) \in F_{j k}} \mid\left\{u \in V_{i} \mid(u, v) \in F_{i j}\right.$ and $\left.(u, w) \in F_{i k}\right\} \mid$ by $\left|F_{j k}\right| \frac{11 f_{i j} f_{i k}}{r_{i} r_{j} r_{k}}=M_{i j k}$. Thus, we have $\varepsilon_{t}=\Omega\left(\frac{e_{i j k}}{M_{i j k}}\right)$.

Finally, we evaluate the cost $\mathrm{U}_{t}$, which is the cost of transforming the quantum state

$$
\sum_{R_{t+1} \in \Omega_{t+1}} \cdots \sum_{R_{m} \in \Omega_{m}}\left|R_{t+1}\right\rangle \cdots\left|R_{m}\right\rangle\left|D\left(R_{1}, \ldots, R_{t-1}, R_{t}, R_{t+1}, \ldots, R_{m}\right)\right\rangle,
$$

to the quantum state

$$
\sum_{R_{t+1} \in \Omega_{t+1}} \cdots \sum_{R_{m} \in \Omega_{m}}\left|R_{t+1}\right\rangle \cdots\left|R_{m}\right\rangle\left|D\left(R_{1}, \ldots, R_{t-1}, R_{t}^{\prime}, R_{t+1}, \ldots, R_{m}\right)\right\rangle,
$$

for any two states $R_{t}$ and $R_{t}^{\prime}$ adjacent in the corresponding Johnson graph. We again divide the analysis into three cases.

Case $1\left[s_{t}=i\right]$ : In this case $R_{t}$ and $R_{t}^{\prime}$ are two subsets of $\{1, \ldots, n\}$, both of size $r_{i}$, differing by exactly one element. The corresponding subsets $V_{i}$ and $V_{i}^{\prime}$ also differ by exactly one element: let us write $V_{i}^{\prime}=\left(V_{i} \backslash\{u\}\right) \cup\left\{u^{\prime}\right\}$. For any $\{i, j, k\} \in \Sigma_{3}$, there exist some $t_{1}, t_{2}, t_{3} \in\{t+1, \ldots, m\}$ such that $s_{t_{1}}=\{i, j\}, s_{t_{2}}=\{i, k\}$ and $s_{t_{3}}=\{i, j, k\}$. There also exist some $t_{4}, t_{5}, t_{6} \in\{1, \ldots, m\}$ such that $s_{t_{4}}=j, s_{t_{5}}=k$ and $s_{t_{6}}=\{j, k\}$. Note that $t_{4}, t_{5}, t_{6}$ can be smaller than $t$, but we will first assume that they are all larger than $t$ (the other cases, which are actually easier to analyze, are discussed at the end of the analysis). A state $R_{t_{4}}$ defines a set $V_{j}$ of $r_{j}$ vertices and, for any $R_{t_{1}} \in \Omega_{t_{1}}$, the state ( $R_{t}, R_{t_{1}}, R_{t_{4}}$ ) defines a set of $f_{i j}$ pairs in $V_{i} \times V_{j}$, as described in Section4.3. In the same way, for any $R_{t_{1}}^{\prime} \in \Omega_{t_{1}}$, the state ( $R_{t}^{\prime}, R_{t_{1}}^{\prime}, R_{t_{4}}$ ) defines a set of $f_{i j}$ pairs in $V_{i}^{\prime} \times V_{j}$. There exists a permutation $\pi_{1}$ of the elements of $\Omega_{t_{1}}$ such that, for any $R_{t_{1}} \in \Omega_{t_{1}}$, the set $F_{i j}$ defined by $\left(R_{t}, R_{t_{1}}, R_{t_{4}}\right)$ and the set $F_{i j}^{\prime}$ defined by $\left(R_{t}^{\prime}, \pi_{1}\left(R_{t_{1}}\right), R_{t_{4}}\right)$ are related in the following way:

$$
F_{i j}^{\prime}=\left(F_{i j} \backslash\left\{(u, v) \in\{u\} \times V_{j} \mid(u, v) \in F_{i j}\right\}\right) \cup\left\{\left(u^{\prime}, v\right) \in\left\{u^{\prime}\right\} \times V_{j} \mid(u, v) \in F_{i j}\right\},
$$

which means that each pair of the form $(u, v)$ in $F_{i j}$ is replaced by the pair $\left(u^{\prime}, v\right)$ in $F_{i j}^{\prime}$, while the other pairs are the same in $F_{i j}$ and in $F_{i j}^{\prime}$.

Similarly, there exists a permutation $\pi_{2}$ of the elements of $\Omega_{t_{2}}$ such that, for any $R_{t_{2}} \in \Omega_{t_{2}}$, the set $F_{i k}$ defined by ( $R_{t}, R_{t_{2}}, R_{t_{5}}$ ) and the set $F_{i k}^{\prime}$ defined by $\left(R_{t}^{\prime}, \pi_{2}\left(R_{t_{2}}\right), R_{t_{5}}\right)$ are related in the following way:

$$
F_{i k}^{\prime}=\left(F_{i k} \backslash\left\{(u, w) \in\{u\} \times V_{k} \mid(u, w) \in F_{i k}\right\}\right) \cup\left\{\left(u^{\prime}, w\right) \in\left\{u^{\prime}\right\} \times V_{k} \mid(u, w) \in F_{i k}\right\} .
$$

The states $\left(R_{t}, R_{t_{1}}, R_{t_{2}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)$ define sets $V_{i}, F_{i j}, F_{i k}, V_{j}, V_{k}, F_{j k}, \Gamma_{i j k}$, while the states $\left(R_{t}^{\prime}, \pi_{1}\left(R_{t_{1}}\right), \pi_{2}\left(R_{t_{2}}\right), R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)$ define sets $V_{i}^{\prime}, F_{i j}^{\prime}, F_{i k}^{\prime}, V_{j}, V_{k}, F_{j k}, \Gamma_{i j k}^{\prime}$. Given any state $R_{t_{3}}$, let $E_{i j k}\left(R_{t}, R_{t_{1}}, R_{t_{2}}, R_{t_{3}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)$ denote the set of hyperedges to be queried associated with $\Gamma_{i j k}$ and $R_{t_{3}}$, and $E_{i j k}\left(R_{t}^{\prime}, \pi_{1}\left(R_{t_{1}}\right), \pi_{2}\left(R_{t_{2}}\right), R_{t_{3}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)$ denote the set of hyperedges to be queried associated with $\Gamma_{i j k}^{\prime}$ and $R_{t_{3}}$. By Lemmas 1 and 6 , the mapping

$$
\begin{aligned}
& \left|R_{t_{1}}\right\rangle\left|R_{t_{2}}\right\rangle\left|R_{t_{4}}\right\rangle\left|R_{t_{5}}\right\rangle\left|R_{t_{6}}\right\rangle \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}, R_{t_{1}}, R_{t_{2}}, R_{t_{3}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)\right\rangle \mapsto \\
& \left|\pi_{1}\left(R_{t_{1}}\right)\right\rangle\left|\pi_{2}\left(R_{t_{2}}\right)\right\rangle\left|R_{t_{4}}\right\rangle\left|R_{t_{5}}\right\rangle\left|R_{t_{6}}\right\rangle \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}^{\prime}, \pi_{1}\left(R_{t_{1}}\right), \pi_{2}\left(R_{t_{2}}\right), R_{t_{3}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)\right\rangle
\end{aligned}
$$

can be approximated within inverse polynomial precision ${ }^{2}$ using $\tilde{O}\left(\frac{e_{i j k}\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|}{M_{i j k}}+\log n\right)=$ $\tilde{O}\left(\frac{e_{i j k}\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|}{M_{i j k}}+1\right)$ queries (here Lemma 1 is used with $T=22 \frac{e_{i j k}\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|}{M_{i j k}}+100 \log n$, and then $\epsilon_{T}$ can be set to $2\left(\frac{1}{2}\right)^{11 \frac{e_{i j k}\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|}{M_{i j k}}+50 \log n}$ by Lemma 6 with $p=M_{i j k}$ and $r=e_{i j k}$ ).

We will use the following lemma.

[^1]Lemma 9 When $R_{t_{1}}, R_{t_{2}}$ and $R_{t_{6}}$ are taken uniformly at random,

$$
\operatorname{Pr}\left[\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right| \geq 44 \times \frac{f_{i j} f_{i k} f_{j k}}{r_{i}^{2} r_{j} r_{k}}\right]=O\left(\frac{1}{n^{100}}\right)
$$

Proof. Let us write

$$
\begin{aligned}
& A=\left\{(u, v, w) \in\{u\} \times V_{j} \times V_{k} \mid(u, v) \in F_{i j},(u, w) \in F_{i k} \text { and }(v, w) \in F_{j k}\right\} \\
& B=\left\{\left(u^{\prime}, v, w\right) \in\left\{u^{\prime}\right\} \times V_{j} \times V_{k} \mid\left(u^{\prime}, v\right) \in F_{i j}^{\prime},\left(u^{\prime}, w\right) \in F_{i k}^{\prime} \text { and }(v, w) \in F_{j k}\right\},
\end{aligned}
$$

and note that

$$
\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|=|A|+|B| .
$$

Consider the set $C=\left\{v \in V_{j} \mid(u, v) \in F_{i j}\right\}$. When $R_{t_{1}}$ is taken uniformly at random from $\Omega_{t_{1}}=$ $\left\{T \subseteq\left\{1, \ldots, r_{i} r_{j}\right\}\left||T|=f_{i j}\right\}\right.$ (recall that $s_{t_{1}}=\{i, j\}$ ), the quantity $|C|$ has hypergeometric distribution $H G\left(r_{i} r_{j}, r_{j}, f_{i j}\right)$. By Lemma (1), we have

$$
\operatorname{Pr}\left[|C| \geq 2 \frac{f_{i j}}{r_{i}}\right] \leq \exp \left(-\frac{1}{3} \times \frac{f_{i j}}{r_{i}}\right)
$$

Let us fix $C$ and, for any $v \in C$, write

$$
C(v)=\left\{w \in V_{k} \mid(v, w) \in F_{j k}\right\}
$$

When $R_{t_{6}}$ is taken uniformly at random from $\Omega_{t_{6}}=\left\{T \subseteq\left\{1, \ldots, r_{j} r_{k}\right\}| | T \mid=f_{j k}\right\}$ (recall that $s_{t_{6}}=$ $\{j, k\}$ ), the quantity $|C(v)|$ has hypergeometric distribution $H G\left(r_{j} r_{k}, r_{k}, f_{j k}\right)$. By Lemman (1), we have

$$
\begin{equation*}
\operatorname{Pr}\left[|C(v)| \geq 2 \frac{f_{j k}}{r_{j}}\right] \leq \exp \left(-\frac{1}{3} \times \frac{f_{j k}}{r_{j}}\right) \tag{1}
\end{equation*}
$$

Let us fix $C(v)$ and write

$$
C^{\prime}(v)=\left\{w \in C(v) \mid(u, w) \in F_{i k}\right\} .
$$

When $R_{t_{2}}$ is taken uniformly at random from $\Omega_{t_{2}}=\left\{T \subseteq\left\{1, \ldots, r_{i} r_{k}\right\}| | T \mid=f_{i k}\right\}$ (recall that $s_{t_{2}}=$ $\{i, k\}$ ), the quantity $\left|C^{\prime}(v)\right|$ has hypergeometric distribution $H G\left(r_{i} r_{k},|C(v)|, f_{i k}\right)$. Under the hypothesis $|C(v)| \leq 2 \frac{f_{j k}}{r_{j}}$, we can apply Lemma【2(3) with $\delta=\frac{11 f_{j k} / r_{j}}{\mid C(v)}-1>2 e-1$ to evaluate the size of $C^{\prime}(v)$. The union bound then gives

$$
\operatorname{Pr}\left[\left|C^{\prime}(v)\right| \leq 11 \frac{f_{j k} f_{i k}}{r_{i} r_{j} r_{k}}\right] \geq 1-2^{-11 \frac{f_{j k} f_{i k}}{r_{i} r_{j} r_{k}}}-\exp \left(-\frac{f_{j k}}{3 r_{j}}\right) .
$$

Finally, note that $|A|=\sum_{v \in C}\left|C^{\prime}(v)\right|$. Thus the union bound gives

$$
\operatorname{Pr}\left[|A| \leq 22 \frac{f_{i j} f_{i k} f_{j k}}{r_{i}^{2} r_{j} r_{k}}\right] \geq 1-\exp \left(-\frac{f_{i j}}{3 r_{i}}\right)-2 \frac{f_{i j}}{r_{i}}\left(2^{-1 \frac{f_{j k} f_{i k}}{r_{i} r_{j} r_{k}}}+\exp \left(-\frac{f_{j k}}{3 r_{j}}\right)\right)
$$

Similarly, we have

$$
\operatorname{Pr}\left[|B| \leq 22 \frac{f_{i j} f_{i k} f_{j k}}{r_{i}^{2} r_{j} r_{k}}\right] \geq 1-\exp \left(-\frac{f_{i j}}{3 r_{i}}\right)-2 \frac{f_{i j}}{r_{i}}\left(2^{-11 \frac{f_{j k} f_{i k}}{r_{i} r_{j} r_{k}}}+\exp \left(-\frac{f_{j k}}{3 r_{j}}\right)\right)
$$

and thus

$$
\operatorname{Pr}\left[\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right| \geq 44 \times \frac{f_{i j} f_{i k} f_{j k}}{r_{i}^{2} r_{j} r_{k}}\right] \leq 2 \exp \left(-\frac{f_{i j}}{3 r_{i}}\right)+4 \frac{f_{i j}}{r_{i}}\left(2^{-11 \frac{f_{j j} f_{i k}}{r_{i} r_{j} r_{k}}}+\exp \left(-\frac{f_{j k}}{3 r_{j}}\right)\right),
$$

which is exponentially small since this set of parameters is admissible.
Lemmas 1 and 9 then show that the mapping

$$
\begin{array}{r}
\left|R_{t_{4}}\right\rangle\left|R_{t_{5}}\right\rangle \sum_{R_{t_{1}} \in \Omega_{t_{1}}} \sum_{R_{t_{2}} \in \Omega_{t_{2}}} \sum_{R_{t_{6}} \in \Omega_{t_{6}}} \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{1}}\right\rangle\left|R_{t_{2}}\right\rangle\left|R_{t_{6}}\right\rangle\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}, R_{t_{1}}, \cdots, R_{t_{6}}\right)\right\rangle \mapsto \\
\left|R_{t_{4}}\right\rangle\left|R_{t_{5}}\right\rangle \sum_{R_{t_{1}} \in \Omega_{t_{1}}} \sum_{R_{t_{2}} \in \Omega_{t_{2}}} \sum_{R_{t_{6}} \in \Omega_{t_{6}}} \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{1}}\right\rangle\left|R_{t_{2}}\right\rangle\left|R_{t_{6}}\right\rangle\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}^{\prime}, R_{t_{1}}, \cdots, R_{t_{6}}\right)\right\rangle
\end{array}
$$

can be approximated within inverse polynomial precision using $\tilde{O}\left(e_{i j k} / r_{i}+1\right)$ queries. This argument is true for all $\{i, j, k\} \in \Sigma_{3}$, so the update cost is

$$
\mathbf{U}_{t}=\tilde{O}\left(1+\sum_{\{j, k\} \text { such that }\{i, j, k\} \in \Sigma_{3}} \frac{e_{i j k}}{r_{i}}\right) .
$$

Let us finally consider the case where $t_{4}, t_{5}, t_{6}$ are not all larger than $t$. Whenever $t_{6}$ is larger than $t$, exactly the same analysis as above holds. When $t_{6}$ is smaller than $t$ (which implies that $t_{4}$ and $t_{5}$ are also smaller than $t$ ), remember that we only need to do the analysis of the update cost under the condition that $R_{t_{6}}$ is marked. This means that we can assume that, for any $v \in V_{j}$, we have $\left|\left\{w \in V_{k} \mid(v, w) \in F_{j k}\right\}\right| \leq$ $2 f_{j k} / r_{j}$. This property can be used instead of Inequality (1) in the proof of Lemma 9 , and the analysis then becomes the same as above.

Case $2\left[s_{t}=\{i, j\}\right.$ with $\left.i<j\right]: R_{t}$ and $R_{t}^{\prime}$ correspond to two subsets $F_{i j}$ and $F_{i j}^{\prime}$ that also differ by exactly one element: let us write $F_{i j}^{\prime}=\left(F_{i j} \backslash\{(u, v)\}\right) \cup\left\{\left(u^{\prime}, v^{\prime}\right)\right\}$. For any $\{i, j, k\} \in \Sigma_{3}$, there exist some $t_{1}, t_{2} \in\{1, \ldots, t-1\}$ such that $s_{t_{1}}=i, s_{t_{2}}=j$ and some $t_{3} \in\{t+1, \ldots, m\}$ such that $s_{t_{3}}=\{i, j, k\}$. There also exist some $t_{4}, t_{5}, t_{6} \in\{1, \ldots, m\}$ such that $s_{t_{4}}=k, s_{t_{5}}=$ $\{i, k\}$ and $s_{t_{6}}=\{j, k\}$. Note that $t_{4}, t_{5}, t_{6}$ can be smaller than $t$, but we will first assume that they are all larger than $t$ (the other cases, which are actually easier to analyze, are discussed at the end of the analysis). The states $\left(R_{t}, R_{t_{1}}, R_{t_{2}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)$ define sets $F_{i j}, V_{i}, V_{j}, V_{k}, F_{i k}, F_{j k}, \Gamma_{i j k}$, while the states $\left(R_{t}^{\prime}, R_{t_{1}}, R_{t_{2}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)$ define sets $F_{i j}^{\prime}, V_{i}, V_{j}, V_{k}, F_{i k}, F_{j k}, \Gamma_{i j k}^{\prime}$. Given any state $R_{t_{3}}$, let $E_{i j k}\left(R_{t}, R_{t_{1}}, \ldots, R_{t_{6}}\right)$ denote the set of hyperedges to be queried associated with $\Gamma_{i j k}$ and $R_{t_{3}}$, and $E_{i j k}\left(R_{t}^{\prime}, R_{t_{1}}, \ldots, R_{t_{6}}\right)$ denote the set of hyperedges to be queried associated with $\Gamma_{i j k}^{\prime}$ and $R_{t_{3}}$.

By Lemmas 1 and 6 we know that the mapping

$$
\begin{aligned}
& \left|R_{t_{4}}\right\rangle\left|R_{t_{5}}\right\rangle\left|R_{t_{6}}\right\rangle \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}, R_{t_{1}}, R_{t_{2}}, R_{t_{3}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)\right\rangle \mapsto \\
& \left|R_{t_{4}}\right\rangle\left|R_{t_{5}}\right\rangle\left|R_{t_{6}}\right\rangle \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}^{\prime}, R_{t_{1}}, R_{t_{2}}, R_{t_{3}}, R_{t_{4}}, R_{t_{5}}, R_{t_{6}}\right)\right\rangle
\end{aligned}
$$

can be approximated within inverse polynomial precision using

$$
\tilde{O}\left(\frac{e_{i j k}\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|}{M_{i j k}}+\log n\right)=\tilde{O}\left(\frac{e_{i j k}\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|}{M_{i j k}}+1\right)
$$

queries. We now prove the following lemma.

Lemma 10 When $R_{t_{5}}$ and $R_{t_{6}}$ are taken uniformly at random,

$$
\operatorname{Pr}\left[\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right| \geq 22 \times \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}\right]=O\left(\frac{1}{n^{100}}\right)
$$

Proof. Let us write

$$
\begin{aligned}
& A=\left\{w \in V_{k} \mid(u, w) \in F_{i k} \text { and }(v, w) \in F_{j k}\right\}, \\
& B=\left\{w \in V_{k} \mid\left(u^{\prime}, w\right) \in F_{i k} \text { and }\left(v^{\prime}, w\right) \in F_{j k}\right\},
\end{aligned}
$$

and note that

$$
\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right|=|A|+|B| .
$$

Let us write $A_{1}=\left\{w \in V_{k} \mid(u, w) \in F_{i k}\right\}$. When $R_{t_{5}}$ is taken uniformly at random from $\Omega_{t_{5}}=\{T \subseteq$ $\left\{1, \ldots, r_{i} r_{k}\right\}\left||T|=f_{i k}\right\}$ (recall that $s_{t_{5}}=\{i, k\}$ ), the quantity $\left|A_{1}\right|$ has hypergeometric distribution $H G\left(r_{i} r_{k}, r_{k}, f_{i k}\right)$. By Lemma2(1), we have

$$
\begin{equation*}
\operatorname{Pr}\left[\left|A_{1}\right| \geq 2 \frac{f_{i k}}{r_{i}}\right] \leq \exp \left(-\frac{1}{3} \times \frac{f_{i k}}{r_{i}}\right) . \tag{2}
\end{equation*}
$$

Once $A_{1}$ is fixed, the quantity $\left|\left\{w \in A_{1} \mid(v, w) \in F_{j k}\right\}\right|$ has hypergeometric distribution $H G\left(r_{j} r_{k},\left|A_{1}\right|, f_{j k}\right)$ with expectation $\frac{\left|A_{1}\right| f_{j k}}{r_{j} r_{k}}$. Under the assumption $\left|A_{1}\right| \leq 2 \frac{f_{i k}}{r_{i}}$, we can apply Lemma 2(3) with $\delta=\frac{11 f_{i k} / r_{i}}{\left|A_{1}\right|}-1>2 e-1$. The union bound then gives

$$
\operatorname{Pr}\left[|A| \leq 11 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}\right] \geq 1-2^{-11 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}}-\exp \left(-\frac{f_{i k}}{3 r_{i}}\right) .
$$

Similarly we obtain

$$
\operatorname{Pr}\left[|B| \leq 11 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}\right] \geq 1-2^{-11 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}}-\exp \left(-\frac{f_{i k}}{3 r_{i}}\right),
$$

and thus

$$
\operatorname{Pr}\left[\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right| \geq 22 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}\right] \leq 2 \times\left(2^{-11 \frac{f_{i l} f_{j k}}{r_{i} r_{j} r_{k}}}+\exp \left(-\frac{f_{i k}}{3 r_{i}}\right)\right),
$$

which is exponentially small since this set of parameters is admissible.
Using Lemma 1 and Lemma 10 the mapping

$$
\begin{aligned}
& \left|R_{t_{4}}\right\rangle \sum_{R_{t_{5}} \in \Omega_{t_{5}}} \sum_{R_{t_{6}} \in \Omega_{t_{6}}} \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{5}}\right\rangle\left|R_{t_{6}}\right\rangle\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}, R_{t_{1}}, \cdots, R_{t_{6}}\right)\right\rangle \mapsto \\
& \quad\left|R_{t_{4}}\right\rangle \sum_{R_{t_{5}} \in \Omega_{t_{5}}} \sum_{R_{t_{6}} \in \Omega_{t_{6}}} \sum_{R_{t_{3}} \in \Omega_{t_{3}}}\left|R_{t_{5}}\right\rangle\left|R_{t_{6}}\right\rangle\left|R_{t_{3}}\right\rangle\left|E_{i j k}\left(R_{t}^{\prime}, R_{t_{1}}, \cdots, R_{t_{6}}\right)\right\rangle
\end{aligned}
$$

can be approximated within inverse polynomial precision using $\tilde{O}\left(e_{i j k} / f_{i j}+1\right)$ queries. This argument is true for all $\{i, j, k\} \in \Sigma_{3}$, so the update cost is

$$
\mathrm{U}_{t}=\tilde{O}\left(1+\sum_{k \text { such that }\{i, j, k\} \in \Sigma_{3}} \frac{e_{i j k}}{f_{i j}}\right) .
$$

Let us finally consider the case where $t_{4}, t_{5}, t_{6}$ are not all larger than $t$. Whenever both $t_{5}$ and $t_{6}$ are larger than $t$, exactly the same analysis as above holds. When $t_{5}<t<t_{6}$, remember that we only need to do the analysis of the update cost under the condition that $R_{t_{5}}$ is marked. This means that we can assume that, for any $u \in V_{i}$, we have $\left|\left\{w \in V_{k} \mid(u, w) \in F_{i k}\right\}\right| \leq 2 f_{i k} / r_{i}$. This property can be used instead of Inequality (2) in the proof of Lemma 10, and the analysis then becomes the same as above. When $t_{6}<t<t_{5}$, the same argument holds by inverting the roles of $\{i, k\}$ and $\{j, k\}$ in the proof of Lemma 10 When $t_{5}, t_{6}<t$, the fact that $R_{t_{5}}$ and $R_{t_{6}}$ are marked (more precisely, item (d) in the definition of marked states of Section 4.3) implies that for any $\left(u_{1}, v_{1}\right) \in V_{i} \times V_{j}, \mid\left\{w \in V_{k} \mid\left(u_{1}, w\right) \in F_{i k}\right.$ and $\left.\left(v_{1}, w\right) \in F_{j k}\right\} \left\lvert\, \leq 11 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}\right.$, which immediately implies that $\left|\Gamma_{i j k} \Delta \Gamma_{i j k}^{\prime}\right| \leq 22 \frac{f_{i k} f_{j k}}{r_{i} r_{j} r_{k}}$.
Case 3 [ $s_{t}=\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ with $\left.i<j<k\right]: R_{t}$ and $R_{t}^{\prime}$ are two subsets of $\left\{1, \ldots, M_{i j k}\right\}$, both of size $e_{i j k}$, differing by exactly one element. The corresponding $E_{i j k}$ and $E_{i j k}^{\prime}$ are subsets of the same $\Gamma_{i j k}$, and have symmetric difference $\left|E_{i j k} \Delta E_{i j k}^{\prime}\right| \leq 2$, so $\mathrm{U}_{\mathrm{t}} \leq 2$.

Now the proof of Theorem 5 is completed.

## 5 Applications: 4-clique detection and ternary associativity testing

In this section we describe how to use our method to construct efficient quantum algorithms for 4-clique detection and ternary associativity testing.

First, by applying Theorem 5 to the case where $H$ is a 4 -clique, and optimizing both the loading schedule and the parameters, we obtain the following result.

Theorem 11 There exists a quantum algorithm that detects if a 3-uniform hypergraph on $n$ vertices has a 4-clique, with high probability, using $\tilde{O}\left(n^{241 / 128}\right)=O\left(n^{1.883}\right)$ queries.

Proof. We use Theorem [5] Among the 1680384 possible valid loading schedules, we found, by numerical search, that the best schedule is

$$
(1,2,3,4,\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}) .
$$

The complexity of the algorithm for this schedule is minimized by the following values of parameters:

$$
\begin{array}{llll}
r_{1}=n^{1 / 2}, & r_{2}=n^{3 / 4}, & r_{3}=n^{7 / 8}, & r_{4}=n^{3 / 4}, \\
f_{12}=n^{5 / 4}, & f_{13}=n^{5 / 4}, & f_{14}=n^{147 / 128}, & \\
f_{23}=n^{193 / 128}, & f_{24}=n^{83 / 64}, & f_{34}=n^{181 / 128}, & \\
e_{123}=n^{241 / 128}, & e_{124}=n^{217 / 128}, & e_{134}=n^{211 / 128}, & e_{234}=n^{193 / 128} .
\end{array}
$$

It is easy to check that this set of parameters is admissible. This gives query complexity $\tilde{O}\left(n^{241 / 128}\right)$.
Next, we consider ternary associativity testing. Let $X$ be a finite set with $|X|=n$. A ternary operator $\mathcal{F}$ from $X \times X \times X$ to $X$ is said to be associative if $\mathcal{F}(\mathcal{F}(a, b, c), d, e)=\mathcal{F}(a, \mathcal{F}(b, c, d), e)=$ $\mathcal{F}(a, b, \mathcal{F}(c, d, e))$ holds for every 5 -tuple $(a, b, c, d, e) \in X^{5}$. The function $\mathcal{F}$ is given as a black-box: when we make a query $(a, b, c)$ to $\mathcal{F}$, the answer $\mathcal{F}(a, b, c)$ is returned. We can show that that the property " $\mathcal{F}$ is not associative" has a certificate corresponding to a sub-hypergraph of seven vertices in a 3-uniform directed hypergraph with each edge weighted by an element in $X$. By applying Theorem 5 with adaptations to directed hypergraphs with non-binary hyperedge weights, we obtain the following result.

Theorem 12 There exists a quantum algorithm that determines if $\mathcal{F}$ is associative with high probability using $\tilde{O}\left(n^{169 / 80}\right)=\tilde{O}\left(n^{2.1125}\right)$ queries.

Proof. To apply Theorem [5] we basically follow the approach of Ref. [13] for the (binary) associativity testing. If $\mathcal{F}$ is not associative, there is a 5-tuple ( $a_{1}, a_{2}, a_{3}, a_{4}, a_{5}$ ) $\in X^{5}$ such that (i) $\mathcal{F}\left(\mathcal{F}\left(a_{1}, a_{2}, a_{3}\right), a_{4}, a_{5}\right) \neq \mathcal{F}\left(a_{1}, \mathcal{F}\left(a_{2}, a_{3}, a_{4}\right), a_{5}\right)$ or (ii) $\mathcal{F}\left(a_{1}, \mathcal{F}\left(a_{2}, a_{3}, a_{4}\right), a_{5}\right) \neq$ $\mathcal{F}\left(a_{1}, a_{2}, \mathcal{F}\left(a_{3}, a_{4}, a_{5}\right)\right)$. Thus, it suffices to check case (i) and case (ii) individually.

We consider only case (i) since case (ii) is similarly analyzed and needs the same query complexity as the algorithm for case (i). A certificate to case (i) is given by a 7 -tuple $\left(a_{1}, a_{2}, \ldots, a_{7}\right) \in X^{7}$ such that $\mathcal{F}\left(a_{1}, a_{2}, a_{3}\right)=a_{6}, \mathcal{F}\left(a_{2}, a_{3}, a_{4}\right)=a_{7}$ and $\mathcal{F}\left(a_{6}, a_{4}, a_{5}\right) \neq \mathcal{F}\left(a_{1}, a_{7}, a_{5}\right)$. Let $H$ be a directed hypergraph on seven vertices with directed hyperedges $(1,2,3),(2,3,4),(6,4,5),(1,7,5)$. Then, finding a certificate to case (i) can be reduced to finding a sub-hypergraph isomorphic to $H$ in an $n$-vertex directed hypergraph with each hyperedge weighted with an element in $X$, to which we will apply Theorem[5, Note that, although the proof of Theorem 5 assumes the given hypergraph is undirected and each hyperedge is weighted with binary values, we can easily adapt the algorithm to handle directed hypergraphs with non-binary hyperedge weight: (1) to deal with directed hyperedges of $H$ we simply replace a query to the black-box on an unordered triple by a query on the corresponding ordered triple (for instance, for $(u, v, w) \in V_{4} \times V_{5} \times V_{6}$ we will query $\chi((w, u, v))$ instead of $\chi(\{u, v, w\}))$; (2) since the quantum walk actually does not use the property that hyperedges have binary weight, it works without modification for the case of non-binary hyperedge weights as well. Note also that the resulting algorithm searches $H$ over $X^{7}$, so we do not need to consider separately the case of detecting vertex contractions of $H$ as in Ref. [13].

By numerical search, we found the following schedule:

$$
\begin{gathered}
(1,3,4,6,2,5,7,\{1,2\},\{1,3\},\{1,5\},\{1,7\},\{2,3\},\{2,4\},\{3,4\},\{4,5\},\{4,6\}, \\
\{5,6\},\{5,7\},\{1,2,3\},\{1,5,7\},\{2,3,4\},\{4,5,6\}) .
\end{gathered}
$$

The complexity of the algorithm for this schedule is minimized by the following values of parameters: $r_{1}=n^{3 / 4}, r_{2}=n, r_{3}=n, r_{4}=n^{7 / 8}, r_{5}=n^{1 / 2}, r_{6}=n, r_{7}=n ; f_{12}=n^{7 / 4}, f_{13}=n^{7 / 4}, f_{15}=n^{5 / 4}$, $f_{17}=n^{7 / 4}, f_{23}=n^{23 / 16}, f_{24}=n^{29 / 16}, f_{34}=n^{15 / 8}, f_{45}=n^{11 / 8}, f_{46}=n^{15 / 8}, f_{56}=n^{3 / 2}, f_{57}=n^{3 / 2}$; $e_{123}=n^{169 / 80}, e_{157}=n^{169 / 80}, e_{234}=n^{169 / 80}$ and $e_{456}=1$. It is easy to check that this set of parameters is admissible. This gives query complexity $\tilde{O}\left(n^{169 / 80}\right)$.

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## References

[1] Andris Ambainis. Quantum walk algorithm for element distinctness. SIAM J. Comput. 37(1): 210-239, 2007.
[2] Robert Beals, Harry Buhrman, Richard Cleve, Michele Mosca and Ronald de Wolf. Quantum lower bounds by polynomials. J. ACM 48(4): 778-797, 2001.
[3] Aleksandrs Belovs. Span programs for functions with constant-sized 1-certificates: extended abstract. In Proceedings of STOC, pages 77-84, 2012.
[4] Aleksandrs Belovs. Learning-graph-based quantum algorithm for $k$-distinctness. In Proceedings of FOCS, pages 207-216, 2012.
[5] Aleksandrs Belovs, Andrew M. Childs, Stacey Jeffery, Robin Kothari and Frédéric Magniez. Timeefficient quantum walks for 3-distinctness. In Proceedings of ICALP, Part I, pages 105-122, 2013.
[6] Aleksandrs Belovs and Ansis Rosmanis. On the power of non-adaptive learning graphs. In Proceedings of $C C C$, pages 44-55, 2013.
[7] Harry Buhrman, Christoph Dürr, Mark Heiligman, Peter Høyer, Frédéric Magniez, Miklos Santha and Ronald de Wolf. Quantum algorithms for element distinctness. SIAM J. Comput. 34(6): 1324-1330, 2005.
[8] Peter Høyer, Troy Lee and Robert Špalek. Negative weights make adversaries stronger. In Proceedings of STOC, pages 526-535, 2007.
[9] Svante Janson, Tomasz Łuczak, and Andrzej Ruciński. Random Graphs. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley \& Sons, 2000.
[10] Stacey Jeffery, Robin Kothari and Frédéric Magniez. Nested quantum walks with quantum data structures. In Proceedings of SODA, pages 1474-1485, 2013.
[11] François Le Gall. Improved output-sensitive quantum algorithms for Boolean matrix multiplication. In Proceedings of SODA, pages 1464-1476, 2012.
[12] Troy Lee, Frédéric Magniez and Miklos Santha. Learning graph based quantum query algorithms for finding constant-size subgraphs. Chicago J. Theor. Comput. Sci. Article 10, 2012.
[13] Troy Lee, Frédéric Magniez and Miklos Santha. Improved quantum query algorithms for triangle finding and associativity testing. In Proceedings of SODA, pages 1486-1502, 2013.
[14] Frédéric Magniez, Ashwin Nayak, Jérémie Roland and Miklos Santha. Search via quantum walk. SIAM J. Comput. 40(1): 142-164, 2011.
[15] Frédéric Magniez, Miklos Santha and Mario Szegedy. Quantum algorithms for the triangle problem. SIAM J. Comput. 37(2): 413-424, 2007.
[16] Ben Reichardt. Span programs and quantum query complexity: The general adversary bound is nearly tight for every Boolean function. In Proceedings of FOCS, pages 544-551, 2009.
[17] Mario Szegedy. Quantum speed-up of Markov chain based algorithms. In Proceedings of FOCS, pages 32-41, 2004.
[18] Virginia Vassilevska Williams and Ryan Williams. Finding, minimizing, and counting weighted subgraphs. SIAM J. Comput. 42(3): 831-854, 2013.
[19] Virginia Vassilevska Williams and Ryan Williams. Subcubic equivalences between path, matrix and triangle problems. In Proceedings of FOCS, pages 645-654, 2010.
[20] Ryan Williams. A new algorithm for optimal 2-constraint satisfaction and its implications. Theor. Comput. Sci. 348(2-3): 357-365, 2005.
[21] Ryan Williams. Algorithms and resource requirements for fundamental problems. Ph.D. Thesis, Carnegie Mellon University, 2007.
[22] Yechao Zhu. Quantum query complexity of constant-sized subgraph containment. Int. J. Quant. Inf. 10(3): 1250019, 2012.


[^0]:    ${ }^{1}$ We stress that, while this quantum algorithm can also be used to find with the same complexity a 4-clique of maximal weight, this does not currently lead to a better algorithm for Max-3SAT since our algorithm is only query-efficient.

[^1]:    ${ }^{2}$ Note that a better estimation of the accuracy of the approximation can be obtained, but in this proof approximation within inverse polynomial will be enough for our purpose. In consequence, while stronger tail bounds can be proved, the statements of Lemmas 9 and 10 will be enough for our purpose.

