# First-Order Interpolation Derived from Propositional Interpolation 

Matthias Baaz ${ }^{1}$ and Anela Lolic ${ }^{2}$<br>${ }^{1}$ Institut für Diskrete Mathematik und Geometrie 104<br>Technische Universität Wien<br>Vienna, Austria<br>baaz@logic.at<br>${ }^{2}$ Institut für Diskrete Mathematik und Geometrie 104<br>Technische Universität Wien<br>Vienna, Austria<br>anela@logic.at


#### Abstract

This paper develops a general methodology to connect propositional and first-order interpolation. In fact, the existence of suitable skolemizations and of Herbrand expansions together with a propositional interpolant suffice to construct a first-order interpolant. This methodology is realized for lattice-based finitely-valued logics, the top element representing true. It is shown that interpolation is decidable for these logics.


Keywords: Proof theory, Interpolation, Lattice-based many-valued logics, Gödel logics

## 1 Introduction

Ever since Craig's seminal paper on interpolation [7, interpolation properties have been recognized as important properties of logical systems. Recall that a logic $L$ has interpolation if whenever $A \rightarrow B$ holds in $L$ there exists a formula $I$ in the common language of $A$ and $B$ such that $A \rightarrow I$ and $I \rightarrow B$ hold in $L$.

Propositional interpolation properties can be determined and classified with relative ease using the ground-breaking results of Maksimova cf. [12[11]10. This approach is based on an algebraic analysis of the logic in question. In contrast first-order interpolation properties are notoriously hard to determine, even for logics where propositional interpolation is more or less obvious. For example it is unknown whether $\mathrm{G}_{[0,1]}^{\mathrm{QF}}$ (first-order infinitely-valued Gödel logic) interpolates (cf [1) and even for $\mathrm{MC}^{\mathrm{QF}}$, the logic of constant domain Kripke frames of 3 worlds with 2 top worlds (an extension of MC), interpolation proofs are very hard cf. Ono [15]. This situation is due to the lack of an adequate algebraization of non-classical first-order logics.

In this paper we present a proof theoretic methodology to reduce first-order interpolation to propositional interpolation:

$$
\left.\begin{array}{c}
\text { existence of suitable skolemizations }+ \\
\text { existence of Herbrand expansions }+ \\
\text { propositional interpolance }
\end{array}\right\} \rightarrow \begin{gathered}
\text { first-order } \\
\text { interpolation. }
\end{gathered}
$$

The construction of the first-order interpolant from the propositional interpolant follows this procedure:

1. Develop a validity equivalent skolemization replacing all strong quantifiers (negative existential or positive universal quantifiers) in the valid formula $A \supset B$ to obtain the valid formula $A_{1} \supset B_{1}$.
2. Construct a valid Herbrand expansion $A_{2} \supset B_{2}$ for $A_{1} \supset B_{1}$. Occurrences of $\exists x B(x)$ and $\forall x A(x)$ are replaced by suitable finite disjunctions $\bigvee B\left(t_{i}\right)$ and conjunctions $\bigwedge B\left(t_{i}\right)$, respectively.
3. Interpolate the propositionally valid formula $A_{2} \supset B_{2}$ with the propositional interpolant $I^{*}$ :

$$
A_{2} \supset I^{*} \quad \text { and } \quad I^{*} \supset B_{2}
$$

are propositionally valid.
4. Reintroduce weak quantifiers to obtain valid formulas

$$
A_{1} \supset I^{*} \quad \text { and } \quad I^{*} \supset B_{1} .
$$

5. Eliminate all function symbols and constants not in the common language of $A_{1}$ and $B_{1}$ by introducing suitable quantifiers in $I^{*}$ (note that no Skolem functions are in the common language, therefore they are eliminated). Let $I$ be the result.
6. $I$ is an interpolant for $A_{1} \supset B_{1} . A_{1} \supset I$ and $I \supset B_{1}$ are skolemizations of $A \supset I$ and $I \supset B$. Therefore $I$ is an interpolant of $A \supset B$.

We apply this methodology to lattice based finitely-valued logics and the weak quantifier and subprenex fragments of infinitely-valued first-order Gödel logic.

Note that finitely-valued first-order logics admit variants of Maehara's Lemma and therefore interpolate if all truth values are quantifier free definable [14. For logics where not all truth-values are represented by quantifier-free formulas this argument does not hold, which explains the necessity of different interpolation arguments for e.g. $\mathrm{MC}^{\mathrm{QF}}$ (the result for $\mathrm{MC}^{\mathrm{QF}}$ is covered by our framework, cf. Example(4). We provide a decision algorithm for the interpolation property for lattice based finitely-valued logics.

Most results in interpolation are concerned with the question whether a given logic interpolates but not with the more general question, to check the minimal extensions with that property. Our framework allows for the calculation of the relevant first-order extensions, which is given by the calculation of the relevant propositional extensions. For classical logic we show in this way that the fragment with $\top, \wedge, \vee, \forall, \exists, \supset$ interpolates, see Example 10 .

## 2 Lattice Based Finitely-Valued Logics

Definition 1 (signature, cf [6]). A signature with polarities $\mathcal{L}^{\rightarrow}$ (or simply signature) consists of a finite set $C_{\mathcal{L}} \rightarrow$ of symbols (called connectives), where each connective $c$ has an assigned arity $n_{c} \in \mathbb{N}$ and polarity $p_{c}:\left\{1,2, \ldots, n_{c}\right\} \rightarrow$ $\{-,+\}$. It is called lattice-oriented if $C_{\mathcal{L} \rightarrow}$ contains three binary connectives $\vee$, $\wedge$ and $\rightarrow$ with $p_{\vee}(i)=p_{\wedge}(i)=+$ for $i \in\{1,2\}$ and $p_{\rightarrow}(1)=-$ and $p_{\rightarrow}(2)=+$.

Definition 2 ( $\mathcal{L}^{\rightarrow}$-lattice). Given any lattice-oriented signature $\mathcal{L} \rightarrow$, a finite $\mathcal{L}^{\rightarrow}$-lattice is an algebraic structure $\left\langle L,\left\{c^{L}\right\}_{c \in C_{\mathcal{L}} \rightarrow}\right\rangle$ satisfying

1. $\left\langle L, \vee^{L}, \wedge^{L}\right\rangle$ is a lattice with an order defined by $x \leq^{L} y \leftrightarrow x \wedge^{L} y=x$.
2. $c^{L}$ is an $n_{c}$-ary operator on $L$ for each $c \in C_{\mathcal{L} \rightarrow}$ such that for $1 \leq i \leq n_{c}$,
(a) if $p_{c}(i)=+$, then $c^{L}$ is monotone in the $i$-th argument
(b) if $p_{c}(i)=-$, then $c^{L}$ is antitone in the $i$-th argument.
3. $1 \leq^{L} A \rightarrow B$ iff $A \leq^{L} B$.

We abbreviate $A \rightarrow^{L} B \wedge^{L} B \rightarrow^{L} A$ with $A \leftrightarrow^{L} B$. We write $\models_{0} A$ for $1 \leq A$ for all elements ( $A$ is valid) and $A_{1}, \ldots, A_{n} \models_{0} A^{\prime}$ for $\models_{0} A_{1} \wedge^{L} \ldots \wedge^{L} A_{n} \rightarrow^{L} A^{\prime}$. The logic $\mathbf{L}^{\mathbf{0}}\left(\mathcal{L}^{\rightarrow}\right)$ is defined as the set of valid sentences $A$. The monotony and antitony of connectives is iterated as usual.

Proposition 1. For all logics the following hold

1. $\models A \rightarrow A$,
2. If $\models B$ then $\models A \rightarrow B$,
3. If $\models A \rightarrow B$ and $\models C \rightarrow D$ then $\models(B \rightarrow C) \supset(A \rightarrow D)$.

Let $L^{\prime} \subseteq L$ and $\mathcal{D}_{L^{\prime}}=\left\{c_{i} \mid c_{i}\right.$ constant with value $\left.i \in L^{\prime}\right\} . \mathcal{L}^{\rightarrow}\left(\mathcal{D}_{L^{\prime}}\right)$ is a lattice with extended signature by $\mathcal{D}_{L^{\prime}}$ where $c_{i}$ has value $i$.

We will omit $L$ from the connectives and $\leq$ when the semantical context is obvious.

Definition 3. Let $L$ be the domain of $\mathcal{L} \rightarrow$. A function $f: L^{n} \rightarrow L$ is representable if there is a word $w\left(x_{1}, \ldots, x_{n}\right)$ such that $w\left(d_{1}, \ldots, d_{n}\right)$ evaluates to $d$ if $f\left(d_{1}, \ldots, d_{n}\right)=d$.

Let $V\left(\mathcal{L}^{\rightarrow}\right)$ be the set of values of constant functions of $\mathcal{L}$.

## Proposition 2.

1. If $V\left(\mathcal{L}^{\rightarrow}\right)=\emptyset$ then $\mathcal{L} \rightarrow$ does not admit the interpolation property.
2. If $V(\mathcal{L} \rightarrow)=L$ then $\mathcal{L}^{\rightarrow}$ admits the interpolation property.

Proof.

1. $x \leq(y \rightarrow y)$ has as only possible interpolant a closed word denoting 1 .
2. Consider $a\left(x_{1}, \ldots, x_{n}\right) \leq b$ valid with left variables $x_{1}, \ldots, x_{n}$.

$$
I=\bigvee_{\left\langle v_{i_{1}}, \ldots v_{i_{n}}\right\rangle \in V(\mathcal{L} \rightarrow) \times F(\mathcal{L} \rightarrow)} a\left(v_{i_{1}}, \ldots, v_{i_{n}}\right)
$$

is an interpolant as $a\left(x_{1}, \ldots, x_{n}\right) \leq I$ and $I \leq b$.

Suppose that $\langle L, \vee, \wedge, \rightarrow, \&, 0,1\rangle$ is an $\mathcal{L} \rightarrow$ lattice. If $\langle L, \&, \overline{1}\rangle$ is a commutative monoid and $\rightarrow$ is the residuum of $\&(x \& y \leq z \leftrightarrow x \leq y \rightarrow z$ for all $x, y, z \in L)$, then $L$ is a commutative pointed residual lattice. Note that $p_{\&}(i)=+$ for $i \in$ $\{1,2\}$.
Proposition 3. The condition on implication given by Definition 2 implies the definition of implication by residuation.

Proof. The definition of implication by residuation implies Definition 2 3:

$$
\begin{aligned}
& 1 \leq A \rightarrow B \quad \Rightarrow \quad 1 \& A \leq B \quad \Rightarrow \quad A \leq B \\
& A \leq B \quad \Rightarrow \quad 1 \& A \leq B \quad \Rightarrow \quad 1 \leq A \rightarrow B
\end{aligned}
$$

Remark 1. Note that the condition on implication given by Definition 23 is more general, i.e. there are implications fulfilling (iii) which do not correspond to a residuation of any monoid.

Example 1. Consider $\left\langle\left\{0, u_{1}, u_{2}, 1\right\}, \vee, \wedge, \rightarrow\right\rangle$ given by

with

$$
u \rightarrow v= \begin{cases}1 & u \leq v \\ 0 & u \not \leq v\end{cases}
$$

Then $\rightarrow$ is not given by residuation.
Assume on the contrary there were a monoid with $\&$ and 1 on $\left\{u_{1}, u_{2}, 1\right\}$

$$
\begin{array}{lll}
u_{1} \& u_{2}=0 \Rightarrow u_{1} \leq u_{2} \rightarrow 0 & \Rightarrow u_{1} \leq 0 & \text { contradiction. } \\
u_{1} \& u_{2}=u_{1} \Rightarrow u_{1} \leq u_{2} \rightarrow u_{1} & \Rightarrow u_{1} \leq 0 & \text { contradiction. } \\
u_{1} \& u_{2}=u_{2} \Rightarrow u_{2} \leq u_{1} \rightarrow u_{2} & \Rightarrow u_{2} \leq 0 & \text { contradiction. } \\
u_{1} \& u_{2}=1 \Rightarrow 1 \leq u_{1} \& u_{2}, u_{2} \leq 1 \Rightarrow 1 \leq u_{1} \& 1 \Rightarrow 1 \leq u_{1} & \text { contradiction. }
\end{array}
$$

Proposition 4. $F\left(\mathcal{L}^{\rightarrow}\right)$ is decidable.
Proof. Use the following algorithm, which determines all functions with $\leq n$ variables for a finite set of finitely-valued matrices of $\leq m$ values:
Level 0: Start with the $n$ input columns of the lattice for an $n$-placed connective
and constant columns for the constants.
Level $n+1$ : Apply the connectives to all existing columns in all possible ways and add columns if a new column occurs.

This algorithm terminates in $\leq m^{n}$ rounds.
Example 2. Values $=\left\{0, \frac{1}{2}, 1\right\}$

| $\rightarrow$ | 0 | $\frac{1}{2}$ | 1 |
| :---: | :---: | :---: | :---: |
| 0 | 1 | 1 | 1 |
| $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | 1 |
| 1 | 0 | $\frac{1}{2}$ | 1 |


|  | $\overline{0}$ |
| :---: | :---: |
| 0 | 0 |
| $\frac{1}{2}$ | 0 |
| 1 | 0 |

0 variables
Level 0: $\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$
Level 1: Level $0+\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$
$\leq 1$ variables
Level 0: $\left(\begin{array}{c}0 \\ \frac{1}{2} \\ 1\end{array}\right)\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right) \quad$ Level 1: Level $0+\left(\begin{array}{c}1 \\ 1 \\ 1\end{array}\right)\left(\begin{array}{c}1 \\ \frac{1}{2} \\ 0\end{array}\right)$
Level 2: Level $1+\left(\begin{array}{l}1 \\ 1 \\ 0\end{array}\right)\left(\begin{array}{l}0 \\ 1 \\ 1\end{array}\right) \quad$ Level 3: Level $2+\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)\left(\begin{array}{l}1 \\ 0 \\ 0\end{array}\right)\left(\begin{array}{l}1 \\ \frac{1}{2} \\ 1\end{array}\right)$
Level 4: Level $3+\left(\begin{array}{c}0 \\ \frac{1}{2} \\ 0\end{array}\right)\left(\begin{array}{l}1 \\ 0 \\ 1\end{array}\right) \quad$ Level 5: Level $4+\left(\begin{array}{c}0 \\ 1 \\ 0\end{array}\right)$
All functions $f(x)$ are representable, where

$$
\left\langle f(0), f\left(\frac{1}{2}\right), f(1)\right\rangle \leq\left\langle\{0,1\},\left\{0, \frac{1}{2}, 1\right\},\{0,1\}\right\rangle
$$

## 3 Interpolation for Finitely-Valued Lattice Based Logics is Decidable

Definition 4. $\mathcal{L} \rightarrow$ has the interpolation property iff $a(\bar{x}, \bar{y}) \leq b(\bar{y}, \bar{z})$ valid implies $a(\bar{x}, \bar{y}) \leq i(\bar{y})$ and $i(\bar{y}) \leq b(\bar{y}, \bar{z})$ for an interpolant $i(\bar{y})$ (all variables are indicated). We call the variables occurring only in a left variables, the variables occurring only in $b$ the right variables and the variables occurring in $a$ and in $b$ the intersection variables.
$\mathcal{L}^{\rightarrow}$ admits the Craig interpolation property $1 \leq a(\bar{x}, \bar{y}) \rightarrow b(\bar{y}, \bar{z})$ valid implies $1 \leq a(\bar{x}, \bar{y}) \rightarrow i(\bar{y})$ and $1 \leq i(\bar{y}) \rightarrow b(\bar{y}, \bar{z})$ for an interpolant $i(\bar{y})$ (all variables are indicated).

Proposition 5. A logic based on $\mathcal{L} \rightarrow$ interpolates iff $\mathcal{L} \rightarrow$ interpolates.

Example 3． $\mathcal{L}^{\rightarrow}=\langle\{0,1, a\}, \vee, \wedge, \rightarrow, \overline{0}, \overline{1}\rangle, \overline{0}=0, \overline{1}=1,0<a$ and $a<1$

$$
u \rightarrow v= \begin{cases}1 & u \leq v \\ 0 & u=1 \text { and } v=0 \\ a & \text { else }\end{cases}
$$

$\mathbf{L}^{\mathbf{0}}\left(\mathcal{L}^{\rightarrow}\right)$ does not interpolate as

$$
\models^{0}(x \wedge(x \rightarrow \overline{0})) \rightarrow(y \vee(y \rightarrow \overline{0}))
$$

does not admit an interpolant，as the only possible interpolant is a constant with value $a$（there are no common variables in the antecedent and the succedent）．

Let $\mathcal{L}^{\rightarrow}=\langle\{0,1, a\}, \vee, \wedge, \rightarrow, \overline{0}, \bar{a}\rangle, \overline{0}=0, \bar{a}=a, 0<a$ and $a<1 \mathbf{L}^{0}\left(\mathcal{L}^{\rightarrow}\right)$ interpolates as all truth constants are representable， 1 by $\overline{0} \rightarrow \overline{0}$（c．f．Section 7 ）．

Example 4．Finite propositional and constant－domain Kripke frames can be un－ derstood as lattice－based finitely valued logics：Consider upwards closed subsets $\Gamma \subseteq W, W$ is the set of worlds，and order them by inclusion．A formula $A$ is assigned the truth value $\Gamma$ iff $A$ is true at exactly the worlds in $\Gamma$ ．

The constant－domain intuitionistic Kripke frame $\mathcal{K}$ in Fig．$⿴ 囗 十 ⺝$ is represented by the lattice $\mathcal{L} \rightarrow\left(\left\{\left(\begin{array}{cc}1 & 1\end{array}\right),\left(\begin{array}{cc}1 & 1 \\ l^{1}\end{array}\right),\left(\begin{array}{cc}0 & 1 \\ 0\end{array}\right),\left(\begin{array}{cc}1 & 0\end{array}\right),\left(\begin{array}{cc}0 & 0 \\ & 0\end{array}\right)\right\}, \vee, \wedge, \rightarrow, \overline{\left(\begin{array}{cc}0 & 0\end{array}\right)}\right)$ in Fig．2．


Fig．1．Constant－domain intuitionistic Kripke frame $\mathcal{K}$ ．
where

$$
u \rightarrow v= \begin{cases}1 & u \leq v \\ v & \text { else }\end{cases}
$$

$\mathrm{MC}=\mathbf{L}^{0}\left(\mathcal{L}^{\rightarrow}\right)$ is the set of valid propositional sentences．
Propositional interpolation is easily demonstrated for MC，one of the seven in－ termediate logics which admit propositional interpolation［11］．Previous proofs for the interpolation of $\mathrm{MC}^{\mathrm{QF}}$ ，the first－order variant of MC ，are quite involved， ［15］．In fact，in Section 7，Example 9 we will show that this interpolation result is a corollary of the main theorem of this paper．

## Proposition 6.

$$
\left(A\left(x_{1}, \ldots, x_{n}\right) \wedge \bigwedge_{i=1}^{n} x_{i} \leftrightarrow x_{i}^{\prime}\right) \leftrightarrow A\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right) \wedge \bigwedge_{i=1}^{n} x_{i} \leftrightarrow x_{i}^{\prime}
$$



Fig. 2. The lattice.

Proof. By induction on the complexity of $A$.
Theorem 1. It is decidable if a given finite $\mathcal{L} \rightarrow$ admits the interpolation property.

This theorem follows from the following three lemmas.
Definition 5. Let $\pi^{n}$ be a partition of $X$ with $\leq n$ equivalence classes $E_{i}$ and let $x^{E_{i}}$ be a representative of $E_{i} . \sigma_{\pi^{n}}: X \rightarrow X$ be defined by $\sigma(x)=x^{E_{i}}$ for $x \in E_{i}$. Let $\Sigma^{n}(x)$ be the set of all such substitutions.

Lemma 1. $\mathcal{L} \rightarrow$ with domain $A$, where $|A| \leq n$ admits the interpolation property iff it admits the interpolation property for all $a \leq b$ where the number of left variables is $\leq n$.

Proof. Let $a \leq b$ be valid and let $X$ be the set of left variables. Then $a \sigma \leq b \sigma$ is valid and consequently $a \sigma \leq b$ is valid for all $\sigma \in \Sigma^{n}(X)$. Then $\left(\bigvee \sigma \in \Sigma^{n}(X) a\right) \leq$ $b$ is valid. As $a \leq\left(\bigvee \sigma \in \Sigma^{n}(X) a\right)$ is valid because the number of classes of variables identified by any valuation is $\leq n$. Therefore the interpolant for $\left(\bigvee \sigma \in \Sigma^{n}(X) a\right) \leq b$ is an interpolant for $a \leq b$.

Lemma 2. $\mathcal{L} \rightarrow$ with domain $A$, where $|A| \leq n$ admits the interpolation property iff it admits the interpolation property for all $a \leq b$ where the number of right variables is $\leq n$.

Proof. Let $a \leq b$ be valid and let $X$ be the set of right variables. Then $a \sigma \leq$ $b \sigma$ is valid and consequently $a \leq b \sigma$ is valid for all $\sigma \in \Sigma^{n}(X)$. Then $a \leq$
$\left(\bigwedge_{\sigma \in \Sigma^{n}(X)} b\right)$ is valid. As $\left(\bigwedge_{\sigma \in \Sigma^{n}(X)} b\right) \leq b$ is valid because the number of classes of variables identified by any valuation is $\leq n$. Therefore the interpolant for $a \leq\left(\bigwedge_{\sigma \in \Sigma^{n}(X)} b\right)$ is an interpolant for $a \leq b$.

Lemma 3. $\mathcal{L}^{\rightarrow}$ with domain $A$, where $|A| \leq n$ admits the interpolation property iff it admits the interpolation property for all $a \leq b$ where the number of intersection variables is $\leq n$.

Proof. Let $a \leq b$ be valid and let $X$ be the set of intersection variables. For $\sigma \in \Sigma^{n}(X)$ let $C_{\sigma}$ be $\bigwedge_{x \in X}(x \sigma \rightarrow x) \wedge(x \rightarrow x \sigma)$. Then

$$
a \leq \bigvee_{\sigma \in \Sigma^{n}(X)}\left(a \wedge C_{\sigma}\right)
$$

as under any valuation at least one of $C_{\sigma}$ is evaluated to 1 . Therefore

$$
a \leq \bigvee_{\sigma \in \Sigma^{n}(X)}\left(a \sigma \wedge C_{\sigma}\right)
$$

by Proposition 6. Now $a \sigma \leq b \sigma$ for all $\sigma \in \Sigma^{n}(X),|X \sigma| \leq n$. In case there is always an interpolant $I_{\sigma}$ we obtain

$$
\bigvee_{\sigma \in \Sigma^{n}(X)}\left(a \sigma \wedge C_{\sigma}\right) \leq \bigvee_{\sigma \in \Sigma^{n}(X)}\left(I_{\sigma} \wedge C_{\sigma}\right)
$$

and

$$
\bigvee_{\sigma \in \Sigma^{n}(X)}\left(I_{\sigma} \wedge C_{\sigma}\right) \leq \bigvee_{\sigma \in \Sigma^{n}(X)}\left(b \sigma \wedge C_{\sigma}\right)
$$

as $a \sigma \leq I_{\sigma}$ and $I_{\sigma} \leq b \sigma$ for all $\sigma \in \Sigma^{n}(X)$.

$$
\bigvee_{\sigma \in \Sigma^{n}(X)}\left(b \sigma \wedge C_{\sigma}\right) \leq \bigvee_{\sigma \in \Sigma^{n}(X)}\left(b \wedge C_{\sigma}\right)
$$

again by Proposition 6
Finally, $\bigvee_{\sigma \in \Sigma^{n}(X)}\left(b \wedge C_{\sigma}\right) \leq b$. Therefore, $\bigvee_{\sigma \in \Sigma^{n}(X)} I_{\sigma} \wedge C_{\sigma}$ is a suitable interpolant for $a \leq b$.

Proof (Proof of Theorem 1). By Lemma 1, 2 and 3 the number of left variables, right variables and intersection variables is bound by $n$. Consider all pairs of words denoting the representable functions with the limitation of variable occurrences as above. $(a, b)$ : check whether $a \leq b$ is valid. In case it is valid check whether there is a representable function whose representation might serve as interpolant.

Corollary 1. It is decidable if a given finite $\mathcal{L} \rightarrow$ admits the Craig interpolation property.

Example 5. Consider $\mathcal{L} \rightarrow\left\langle\left\{0, \frac{1}{2}, 1\right\}, \vee, \wedge, \rightarrow, \overline{0}\right\rangle$ with $0<1 / 2$ and $1 / 2<1$.

$$
a \rightarrow b= \begin{cases}1 & a \leq b \\ b-a & \text { else }\end{cases}
$$

( $b-a$ in the usual sense). $x \wedge(x \rightarrow 0) \rightarrow y \vee(y \rightarrow 0)$ interpolates iff a constant for $\frac{1}{2}$ is added.

Remark 2. Note that both extension and reduction of the signature may influence interpolation.

Example 6. Consider $\mathcal{L}^{\rightarrow}=\left\langle\left\{0, \frac{1}{2}, 1\right\}, \vee, \wedge, \rightarrow, \overline{0}\right\rangle$, where $\overline{0}=0,0<\frac{1}{2}$ and $\frac{1}{2}<1$

$$
a \rightarrow b= \begin{cases}1 & a \leq b \\ b & \text { else }\end{cases}
$$

$\mathbf{L}^{\mathbf{0}}\left(\mathcal{L}^{\rightarrow}\right)$ interpolates $\left(\mathbf{L}^{\mathbf{0}}\left(\mathcal{L}^{\rightarrow}\right)\right.$ is a three-valued Gödel logic).
Extend $\mathcal{L}^{\rightarrow}$ to $\mathcal{L}^{\prime} \rightarrow=\left\langle\left\{0, \frac{1}{2}, 1\right\}, \vee, \wedge, \rightarrow, \square_{\text {up }}, \square_{\text {down }}, \overline{0}\right\rangle . \mathcal{L}^{\prime} \rightarrow$ is defined as before, with the addition of

$$
\begin{gathered}
\square_{\text {up }}(i)= \begin{cases}i & i \leq \frac{1}{2} \\
\frac{1}{2} & \text { else }\end{cases} \\
\square_{\text {down }}(i)= \begin{cases}i & \frac{1}{2} \leq i \\
\frac{1}{2} & \text { else }\end{cases}
\end{gathered}
$$

$\mathbf{L}^{\mathbf{0}}\left(\mathcal{L}^{\prime \rightarrow}\right)$ does not interpolate as $\square_{\text {down }}(x) \rightarrow \square_{\text {up }}(y)$ does not interpolate. Note that the addition of constants alone does not weaken the interpolation property.

Proposition 2 2. makes it possible to characterize all extensions of a lattice by constants which admit interpolation. SPECTRUM $\left(\mathcal{L}^{\rightarrow}\right)=$

$$
\left\{V \mid \mathcal{L}^{\rightarrow} \text { extended by constants representing the values in } V \text { interpolates }\right\}
$$

Corollary 2. SPECTRUM is a calculable function.

## 4 First-Order Logic

Definition 6 (predicate language). A (countable) predicate language $\mathcal{P}$ is a triple $\langle\mathbf{P}, \mathbf{F}, \mathbf{a r}\rangle$ consisting of disjoint countable sets $\mathbf{P}$ and $\mathbf{F}$ of predicate and function symbols, and a function $\mathbf{a r}: \mathbf{P} \cup \mathbf{F} \rightarrow \mathbb{N}$ assigning arities to these symbols. We call nullary function symbols object constants and nullary predicate symbols propositional atoms. For convenience, a predicate language containing only propositional atoms will be called propositional.

Let us fix a lattice-oriented signature $\mathcal{L} \rightarrow$ and a predicate language $\mathcal{P}=\langle\mathbf{P}, \mathbf{F}$, ar $\rangle$. We define $\mathcal{P}$-terms, atomic $\mathcal{P}$-formulas, and $\langle\mathcal{L}, \mathcal{P}\rangle$-formulas as in classical logic using a fixed countably infinite set $O V$ of object variables $x, y, \ldots$, the quantifiers $\forall$ and $\exists$ and the connectives in $\mathcal{L} \rightarrow .\langle\mathcal{L}, \mathcal{P}\rangle$-formulas are denoted with $\varphi, \psi, \ldots$ The notions of bound and free variables, closed terms, sentences, prenex formulas, and substitutability in formulas are defined in the standard way.

An $\langle\mathcal{L} \rightarrow, \mathcal{P}\rangle$-structure $\mathcal{S}$ is a pair $\langle\mathbf{A}, \mathbf{S}\rangle$ such that

1. $\mathbf{A}$ is a finite $\mathcal{L}^{\rightarrow}$-lattice,
2. $\mathbf{S}$ is a triple $\left\langle S,\left\{P^{\mathbf{S}}\right\}_{P \in \mathbf{P}},\left\{f^{\mathbf{S}}\right\}_{f \in \mathbf{F}}\right\rangle$ where
$-S$ is a non-empty set,
$-P^{\mathbf{S}}: S^{n} \rightarrow A$ is a function for each $n$-ary predicate symbol $P \in \mathbf{P}$,
$-f^{\mathbf{S}}: S^{n} \rightarrow S$ is a function for each $n$-ary function symbol $f \in \mathbf{F}$.
An $\mathcal{S}$-evaluation is a mapping $v: O V \rightarrow S$. For any $\mathcal{S}$-evaluation $v$ we denote by $v[x \rightarrow a]$ the $\mathcal{S}$-evaluation satisfying $v[x \rightarrow a](x)=a$ and $v[x \rightarrow a](y)=v(y)$ for each $y \neq x$. Terms and formulas are evaluated in $\mathcal{S}$ with respect to an $\mathcal{S}$ evaluation $v$ according to the following conditions, where $f \in \mathbf{F}, P \in P b$ and $c \in C_{\mathcal{L} \rightarrow}$ :
$-\|x\|_{v}^{\mathcal{S}}=v(x)$,
$-\left\|f\left(t_{1}, \ldots, t_{n}\right)\right\|_{v}^{\mathcal{S}}=f^{\mathcal{S}}\left(\left\|t_{1}\right\|_{v}^{\mathcal{S}}, \ldots,\left\|t_{n}\right\|_{v}^{\mathcal{S}}\right)$,
$-\left\|P\left(t_{1}, \ldots, t_{n}\right)\right\|_{v \mathcal{S}}^{\mathcal{S}}=P^{\mathcal{S}}\left(\left\|t_{1}\right\|_{v}^{\mathcal{S}}, \ldots,\left\|t_{n}\right\|_{v}^{\mathcal{S}}\right)$,
$-\left\|c\left(\varphi_{1}, \ldots, \varphi_{n}\right)\right\|_{v}^{\mathcal{S}}=c^{\mathbf{A}}\left(\left\|\varphi_{1}\right\|_{v}^{\mathcal{S}}, \ldots,\left\|\varphi_{n}\right\|_{v}^{\mathcal{S}}\right)$,
$-\|\forall x \varphi\|_{v}^{\mathcal{S}}=\bigwedge\left\{\|\varphi\|_{v[x \rightarrow a]}^{\mathcal{S}} \mid a \in S\right\}$,
$-\|\exists x \varphi\|_{v}^{\mathcal{S}}=\bigvee\left\{\|\varphi\|_{v[x \rightarrow a]}^{\mathcal{S}} \mid a \in S\right\}$.
We write $\models_{1} C\left(C\right.$ is valid in $\left.\mathcal{L}^{\rightarrow}\right)$ iff for every structure $\langle\mathbf{A}, \mathbf{S}\rangle$ and every $v$ $1 \leq\|C\|_{v}^{\mathbf{S}}$ and $C_{1}, \ldots, C_{n} \models_{1} A^{\prime}$ for $\models_{1} \forall \bar{x} \bigwedge_{i=1}^{n} C_{i} \rightarrow C^{\prime}$. The first-order logic $\mathbf{L}^{1}\left(\mathcal{L}^{\rightarrow}\right)$ is defined as the set of valid sentences $C$ in $\mathcal{L}^{\rightarrow}$.

Monotony and antitony of first-order contexts are iterated as usual, quantifiers do not change the polarity.
Lemma 4. For formulas $A, B$ and a corresponding context $C(\circ)$ it holds

$$
\text { if } \quad \models_{1} A \rightarrow B \quad \text { then } \quad \models_{1} C(A) \rightarrow C(B)
$$

if $\circ$ occurs positively and

$$
\text { if } \quad \models_{1} A \rightarrow B \quad \text { then } \quad \models_{1} C(B) \rightarrow C(A)
$$

if $\circ$ occurs negatively.
Proof. By iteration of the polarity of the connectives.
Definition 7 (weak interpolation property). $\mathbf{L}^{\mathbf{1}}\left(\mathcal{L}^{\rightarrow}\right)$ has the weak interpolation property if for every $\models_{1} A \rightarrow B$ there is a $I$ with predicate symbols occurring in both $A$ and $B$ such that $\models_{1} A \rightarrow I$ and $\models_{1} I \rightarrow B$ hold.

Definition 8 (strong interpolation property). $\mathbf{L}^{\mathbf{1}}\left(\mathcal{L}^{\rightarrow}\right)$ has the strong interpolation property if for every $\models_{1} A \rightarrow B$ there is a $I$ with predicate and function symbols occurring in both $A$ and $B$ such that $\models_{1} A \rightarrow I$ and $\models_{1} I \rightarrow B$ hold.

## 5 Skolemization

We use skolemization to replace strong quantifiers in valid formulas such that the original formulas can be recovered. Note that several Skolem functions for the replacement of a single quantifier are necessary to represent proper suprema and proper infima.

Definition 9. Consider a formula $B$ in a context $A(B)$. Then its skolemization $A(s k(B))$ is defined as follows:

Replace all strong quantifier occurrences (positive occurrence of $\forall$ and negative occurrence of $\exists$ ) (note that no quantifiers in $A$ bind variables in $B$ ) of the form $\exists x C(x)$ (or $\forall x C(x)$ ) in $B$ by $\bigvee_{i=1}^{|W|} C\left(f_{i}(\bar{x})\right.$ ) (or $\bigwedge_{i=1}^{|W|} C\left(f_{i}(\bar{x})\right.$ )), where $f_{i}$ are new function symbols and $\bar{x}$ are the weakly quantified variables of the scope.

Skolem axioms are closed sentences

$$
\forall \bar{x}\left(\exists y A(y, \bar{x}) \supset \bigvee_{i=1}^{|W|} A\left(f_{i}(\bar{x}), \bar{x}\right) \quad \text { and } \quad \forall \bar{x}\left(\bigwedge_{i=1}^{|W|} A\left(f_{i}(\bar{x}), \bar{x}\right) \supset \forall y A(y, \bar{x})\right)\right.
$$

where $f_{i}$ are new function symbols (Skolem functions).

## Lemma 5.

1. If $\models^{1} A(B)$ then $\models^{1} A(\operatorname{sk}(B))$.
2. If $S_{1} \ldots S_{k} \models^{1} A(s k(B))$ then $S_{1} \ldots S_{k} \models^{1} A(B)$, for suitable Skolem axioms $S_{1} \ldots S_{k}$.
3. If $S_{1} \ldots S_{k} \models^{1} A$, where $S_{1} \ldots S_{k}$ are $S k o l e m$ axioms and $A$ does not contain Skolem functions then $\models^{1} A$.

Proof.

1. Note that

$$
\text { if } \quad \models A(D) \quad \text { then } \quad \models A(D \vee D)
$$

and

$$
\text { if } \quad \models A(D) \quad \text { then } \quad \models A(D \wedge D) \text {. }
$$

Use Lemma 4 and

$$
\models^{1} D^{\prime}(t) \supset \exists x D^{\prime}(x), \quad \models^{1} \forall x D^{\prime}(x) \supset D^{\prime}(t)
$$

2. Use Lemma 4 and suitable Skolem axioms to reconstruct strong quantifiers.
3. Assume $\forall^{1} A$. As usual, we have to extend the valuation to the Skolem functions to verify the Skolem axioms. There is a valuation in $\left\langle D_{\Phi^{1}}, \Omega_{\Phi^{1}}\right\rangle$ s.t. $\Phi^{1}(A) \neq 1$. Using at most $|W|$ Skolem functions and $A C$ we can always pick witnesses as values for the Skolem functions such that the first-order suprema and infima are reconstructed on the propositional level. ( $A C$ is applied to sets of objects where the corresponding truth value is taken.)

$$
\sup \left\{\Phi^{1}\left(B\left(f_{i}(\bar{t}), \bar{t}\right)\right)|1 \leq i \leq|W|\}=\right.
$$

$$
\sup \left\{\Phi^{1}\left(B(d, \bar{t}) \mid d \in D_{\Phi^{1}}\right\}=\Phi^{1}(\exists y B(y, \bar{t}))\right.
$$

and

$$
\begin{gathered}
\inf \left\{\Phi^{1}\left(B\left(f_{i}(\bar{t}), \bar{t}\right)\right)|1 \leq i \leq|W|\}=\right. \\
\inf \left\{\Phi^{1}(B(d, \bar{t})) \mid d \in D_{\Phi^{1}}\right\}=\Phi^{1}(\forall y B(y, \bar{t}))
\end{gathered}
$$

Example 7. We continue with the logic MC and its first-order variant $\mathrm{MC}^{\mathrm{QF}}$ introduced in Example $4 \mathrm{MC}=\mathbf{L}^{0}\left(\mathcal{L}^{\rightarrow}\right)$ is the set of valid propositional sentences and $\mathrm{MC}^{\mathrm{QF}}=\mathbf{L}^{1}\left(\mathcal{L}^{\rightarrow}\right)$ the set of valid first-order sentences. For the given logic $\mathrm{MC}^{\mathrm{QF}}$

$$
\exists x B(x) \supset s k(\exists y \forall z C(y, z)) \equiv \exists x B(x) \supset \exists y \bigwedge_{i=1}^{5} C\left(y, f_{i}(y)\right)
$$

## 6 Expansions

Expansions, first introduced in [13, are natural structures representing the instantiated variables for quantified formulas. They record the substitutions for quantifiers in an effort to recover a sound proof of the original formulation of Herbrand's Theorem. As we work with skolemized formulas, in this paper we we consider only expansions for formulas with weak quantifiers. Consequently the arguments are simplified.

In the following we assume that a constant $c$ is present in the language and that $t_{1}, t_{2}, \ldots$ is a fixed ordering of all closed terms (terms not containing variables).

Definition 10. A term structure is a structure $\langle D, \Omega\rangle$ such that $D$ is the set of all closed terms.

Proposition 7. Let $\Phi^{1}(\exists x A(x))=v$ in a term structure. Then $\Phi^{1}(\exists x A(x)=$ $\Phi^{1}\left(\bigvee_{i=1}^{n} A\left(t_{i}\right)\right)$ for some $n$. Analogously for $\forall x A(x)$, i.e. let $\Phi^{1}(\forall x A(x))=v$ in a term structure, then $\Phi^{1}(\forall x A(x))=\Phi^{1}\left(\bigwedge_{i=1}^{n} A\left(t_{i}\right)\right)$ for some $n$.

Proof. Only finitely many truth values exists, therefore there is an $n$ such that the valuation becomes stable on $\bigvee_{i=1}^{n} A\left(t_{i}\right)\left(\bigwedge_{i=1}^{n} A\left(t_{i}\right)\right)$.

Definition 11. Let $E$ be a formula with weak quantifiers only. The n-th expansion $E_{n}$ of $E$ is obtained from $E$ by replacing inside out all subformulas $\exists x A(x)$ $(\forall x A(x))$ by $\bigvee_{i=1}^{n} A\left(t_{i}\right)\left(\bigwedge_{i=1}^{n} A\left(t_{i}\right)\right)$. $E_{n}$ is a Herbrand expansion iff $E_{n}$ is valid. In case there are only $m$ terms $E_{m+k}=E_{m}$.

Lemma 6. Let $\Phi^{1}(E)=v$ in a term structure. Then there is an $n$ such that for all $m \geq n \Phi^{1}\left(E_{m}\right)=v$.

Proof. We apply Proposition 7 outside in to replace subformulas $\exists x A(x)(\forall x$ $A(x))$ stepwise by $\bigvee_{i=1}^{n} A\left(t_{i}\right)\left(\bigwedge_{i=1}^{n} A\left(t_{i}\right)\right)$ without changing the truth value. The disjunctions and conjunctions can be extended to common maximal disjunctions and conjunctions.

Theorem 2. Let $E$ contain only weak quantifiers. Then $\models E$ iff there is a Herbrand expansion $E_{n}$ of $E$.
Proof. $\Rightarrow$ : Assume $\models E$ but $\not \vDash E_{n}$ for all $n$. Let $\Gamma_{i}=\left\{\Phi_{i, v}^{0} \mid \Phi_{i, v}^{0}\left(E_{i}\right) \neq 1\right\}$ and define $\Gamma=\bigcup \Gamma_{i}$. Note that the first index in $\Phi_{i, v}^{0}$ relates to the expansion level and the second index to all counter-valuations at this level. Assign a partial order $<$ to $\Gamma$ by $\Phi_{i, v}^{0}<\Phi_{j, w}^{0}$ for $\Phi_{i, v}^{0} \in \Gamma_{i}, \Phi_{j, w}^{0} \in \Gamma_{j}$ and $i<j$ iff $\Phi_{i, v}^{0}$ and $\Phi_{j, w}^{0}$ coincide on the atoms of $E_{i}$. By König's Lemma there is an infinite branch $\Phi_{1, i_{1}}^{0}<\Phi_{2, i_{2}}^{0}<\ldots$. Define a term structure induced by an evaluation on atoms $P$ :

$$
\Phi^{1}(P)= \begin{cases}v & P \text { occurs in some } E_{n} \text { and } \Phi_{n, i_{n}}(P)=v \\ 1 & \text { else }\end{cases}
$$

$\Phi^{1}(E) \neq 1$ by Lemma 6
$\Leftarrow$ : Use Lemma 4 and $\models A(t) \supset \exists x A(x)$ and $\models \forall x A(x) \supset A(t)$. Note that

$$
\text { if } \quad \models A(D \vee D) \quad \text { then } \quad \models A(D)
$$

and

$$
\text { if } \quad \models A(D \wedge D) \quad \text { then } \quad \models A(D) \text {. }
$$

Example 8. Consider the lattice in Example 4. Fig. 2 and the term ordering $c<d$. The expansion sequence of $P(c, d, d) \supset \exists x P(c, x, d)$ is

$$
E_{1}=P(c, d, d) \supset P(c, c, d), E_{2}=P(c, d, d) \supset P(c, c, d) \vee P(c, d, d), E_{2+k}=E_{2}
$$

The second formula is a Herbrand expansion.

## 7 The Interpolation Theorem

Theorem 3. Interpolation holds for $\mathbf{L}^{0}(\mathcal{L} \rightarrow)$ iff interpolation holds for $\mathbf{L}^{1}(\mathcal{L} \rightarrow)$.
Proof.
$\Leftarrow$ : trivial.
$\Rightarrow$ : Assume $A \supset B$ is in the language of $\mathcal{L} \rightarrow$ and $\models A \supset B$.

$$
\models s k(A) \supset s k(B) \quad \text { by Lemma } 51
$$

Construct a Herbrand expansion $A_{H} \supset B_{H}$ of $s k(A) \supset s k(B)$ by Theorem 2 Construct the propositional interpolant $I^{*}$ of $A_{H} \supset B_{H}$,

$$
\models A_{H} \supset I^{*} \quad \text { and } \quad \models I^{*} \supset B_{H} .
$$

Use Lemma 4 and

$$
\models A(t) \supset \exists x A(x), \quad \models \forall x A(x) \supset A(t)
$$

to obtain

$$
\models s k(A) \supset I^{*} \quad \text { and } \quad \models I^{*} \supset \operatorname{sk}(B)
$$

Order all terms $f(t)$ in $I^{*}$ by inclusion where $f$ is not in the common language. Let $f^{*}(\bar{t})$ be the maximal term.
i. $f^{*}$ is not in $s k(A)$. Replace $f^{*}(\bar{t})$ by a fresh variable $x$ to obtain

$$
\models \operatorname{sk}(A) \supset I^{*}\left\{x / f^{*}(\bar{t})\right\} .
$$

But then also

$$
\models \operatorname{sk}(A) \supset \forall x I^{*}\left\{x / f^{*}(\bar{t})\right\}
$$

and

$$
\models \forall x I^{*}\left\{x / f^{*}(\bar{t})\right\} \supset \operatorname{sk}(B)
$$

by

$$
\models \forall x I^{*}\left\{x / f^{*}(\bar{t})\right\} \supset I^{*} .
$$

ii. $f^{*}$ is not in $s k(B)$. Replace $f^{*}(\bar{t})$ by a fresh variable $x$ to obtain

$$
\models I^{*}\left\{x / f^{*}(\bar{t})\right\} \supset \operatorname{sk}(B) .
$$

But then also

$$
\models \exists x I^{*}\left\{x / f^{*}(\bar{t})\right\} \supset \operatorname{sk}(B)
$$

and

$$
\models \operatorname{sk}(A) \supset \exists x I^{*}\left\{x / f^{*}(\bar{t})\right\}
$$

by

$$
\models I^{*} \rightarrow \exists x I^{*}\left\{x \backslash f^{*}(\bar{t})\right\}
$$

Repeat this procedure till all functions and constants not in the common language (among them the Skolem functions) are eliminated from the middle formula. Let $I$ be the result. $I$ is an interpolant of $s k(A) \supset s k(B)$. By Lemma 52,3 $I$ is an interpolant of $A \rightarrow B$. For a similar construction for classical first-order logic see Chapter 8.2 of [3].

Corollary 3. If interpolation holds for $\mathbf{L}^{0}\left(\mathcal{L}^{\rightarrow}\right), \quad \models A \rightarrow B$ and $A \rightarrow B$ contains only weak quantifiers, then there is a quantifier-free weak interpolant with common predicates for $A \rightarrow B$.

Remark 3. Corollary 3 cannot be strengthened to provide a quantifier-free interpolant with common predicate symbols and common function symbols for $A \rightarrow B$. Consider

$$
Q_{\forall} A\left(x_{1}, f_{1}\left(x_{1}\right), x_{2}, f_{2}\left(x_{1}, x_{2}\right), \ldots\right) \rightarrow Q_{\exists} A\left(g_{1}, y_{1}, g_{2}\left(y_{1}\right), y_{2}, g_{3}\left(y_{1}, y_{2}\right), \ldots\right)
$$

where $Q_{\forall}=\forall x_{1} \forall x_{2} \ldots$ and $Q_{\exists}=\exists y_{1} \exists y_{2} \ldots$ This is the skolemization of

$$
\forall x_{1} \exists x_{1}^{\prime} \forall x_{2} \ldots A\left(x_{1}, x_{1}^{\prime}, x_{2}, \ldots\right) \rightarrow \forall x_{1} \exists x_{1}^{\prime} \forall x_{2} \ldots A\left(x_{1}, x_{1}^{\prime}, x_{2}, \ldots\right)
$$

where $\forall x_{1} \exists x_{1}^{\prime} \forall x_{2} \exists x_{2}^{\prime} \ldots A\left(x_{1}, x_{1}^{\prime}, x_{2}, x_{2}^{\prime}, \ldots\right)$ is the only possible interpolant modulo provable equivalence with common predicate and function symbols.

Example 9. Example 4 continued. For the given logic we calculate the interpolant for

$$
\exists x(B(x) \wedge \forall y C(y)) \rightarrow \exists x(A(x) \vee B(x))
$$

1. Skolemization

$$
\bigvee_{i=1}^{5}\left(B\left(c_{i}\right) \wedge \forall y C(y)\right) \rightarrow \exists x(A(x) \vee B(x))
$$

2. Herbrand expansion

$$
\bigvee_{i=1}^{5}\left(B\left(c_{i}\right) \wedge C\left(c_{1}\right)\right) \rightarrow \bigvee_{i=1}^{5}\left(A\left(c_{i}\right) \vee B\left(c_{i}\right)\right)
$$

3. Propositional interpolant

$$
\bigvee_{i=1}^{5}\left(B\left(c_{i}\right) \wedge C\left(c_{1}\right)\right) \rightarrow \bigvee_{i=1}^{5} B\left(c_{i}\right) \bigvee_{i=1}^{5} B\left(c_{i}\right) \rightarrow \bigvee_{i=1}^{5}\left(A\left(c_{i}\right) \vee B\left(c_{i}\right)\right)
$$

4. Back to the Skolem form

$$
\bigvee_{i=1}^{5}\left(B\left(c_{i}\right) \wedge \forall y C(y)\right) \rightarrow \bigvee_{i=1}^{5} B\left(c_{i}\right) \quad \bigvee_{i=1}^{5} B\left(c_{i}\right) \rightarrow \exists x(A(x) \vee B(x))
$$

5. Elimination of function symbols and constants not in the common language from $\bigvee_{i=1}^{5} B\left(c_{i}\right)$. Result:

$$
\exists z_{1} \ldots \exists z_{5} \bigvee B\left(z_{i}\right)
$$

6. Use the Skolem axiom

$$
\exists x(B(x) \wedge \forall y C(y)) \rightarrow \bigvee_{i=1}^{5} B\left(c_{i}\right) \wedge \forall y C(y)
$$

to reconstruct the original first-order form.
7. The Skolem axiom can be deleted.

Example 10. $\mathcal{L}^{\rightarrow}=\langle\{0,1\}, \leq, \vee, \wedge, \rightarrow\rangle$ be the lattice of classical logic.

$$
\operatorname{SPECTRUM}\left(\mathcal{L}^{\rightarrow}\right)=\{\{0\},\{1\},\{0,1\}\}
$$

This is the maximal possible spectrum by Proposition 21.
Let $\mathcal{L}^{\prime \rightarrow}$ and $\mathcal{L}^{\prime \prime} \rightarrow$ be the extension of $\mathcal{L}^{\rightarrow}$ by $\{\overline{0}\}$ and $\{\overline{0}, \overline{1}\}$, respectively. $\mathbf{L}^{\mathbf{1}}\left(\mathcal{L}^{\prime \rightarrow}\right)$ and $\mathbf{L}^{\mathbf{1}}\left(\mathcal{L}^{\prime \prime \rightarrow}\right)$ interpolate as all truth constants are representable by closed formulas. This is Craig's result, which does however not cover $\mathbf{L}^{\mathbf{1}}\left(\mathcal{L}^{\prime \prime \prime} \rightarrow\right)$, where $\mathcal{L}^{\prime \prime \prime} \rightarrow$ is the extension of $\mathcal{L}^{\rightarrow}$ with $\{\overline{1}\}$. We have only to show that $\mathbf{L}^{0}\left(\mathcal{L}^{\prime \prime \prime} \rightarrow\right)$ interpolates.

First note in general that

$$
\bigvee_{i} E_{i} \rightarrow \bigwedge_{j} F_{j}
$$

interpolates iff there are interpolants

$$
E_{i} \rightarrow I_{i j} \quad I_{i j} \rightarrow F_{j} .
$$

$\bigwedge_{j} \bigvee_{i} I_{i j}$ is a suitable interpolant. Now use the value preserving transformations

$$
\begin{aligned}
D(A \wedge B \rightarrow C) & \Leftrightarrow D(A \rightarrow C \vee B \rightarrow C) \\
D(A \vee B \rightarrow C) & \Leftrightarrow D(A \rightarrow C \wedge B \rightarrow C) \\
D((A \rightarrow B) \rightarrow C) & \Leftrightarrow D(C \vee(A \wedge(B \rightarrow C))) \\
D(x) & \Leftrightarrow D(\top \rightarrow x)
\end{aligned}
$$

for variables $x$ together with distributions and simplifications, to reduce the problem to

$$
\bigwedge_{i}\left(u_{i} \rightarrow v_{i}\right) \rightarrow \bigvee_{j}\left(s_{j} \rightarrow t_{j}\right)
$$

$v_{i}, t_{j}$ variables, $u_{i}, s_{j}$ variables or $\top$. We assume that the succedent is not valid (otherwise $T$ is the interpolant). So any variable occurs either in the $s_{j}$ group or in the $t_{j}$ group. Close the antecedents under transitivity of $\rightarrow$. There is a common implication $u \rightarrow v$, an interpolant (Otherwise there is a counter valuation by assigning 0 to all $t_{j}$ and extending this assignment in the antecedent such that if $v_{i}$ is assigned 0 also $u_{i}$ is assigned 0 . No $s_{j}$ is assigned 0 by this procedure. Assign 1 to all other variables and derive a contradiction to the assumption, that the initial implication is valid). Therefore, $\mathbf{L}^{\mathbf{1}}\left(\mathcal{L}^{\prime \prime \prime}\right)$ interpolates.

Example 11. $n$-valued Gödel logics.
A finitely-valued Gödel logic extended by constants interpolates if there are no consecutive two truth values different to 0,1 not expressible by closed terms (see Theorem 11 in (5]). Note that $\overline{0}$ is able to express $\overline{1}$ and vice versa but no other truth constant is expressible by any other truth constants.

Let $\mathcal{G}_{n}=\left\langle W_{n}, \vee, \wedge, \rightarrow, \overline{0}\right\rangle$, where $\overline{0}=0$ and $W_{n}=\left\{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\right\}$.

$$
u \rightarrow v= \begin{cases}1 & u \leq v \\ v & \text { else }\end{cases}
$$

Let $T_{n}$ be the set of non-empty subsets of $W_{n}$ not containing 0,1 such that besides 0,1 no two consecutive truth values lack.

$$
\operatorname{SPECTRUM}\left(\mathcal{G}_{n}\right)=\left\{\Gamma \mid \Delta \leq \Gamma \text { for some } \Delta \in T_{n}\right\}
$$

Example 12. Finitely-valued Łukasiewicz logic.
Let $\mathrm{E}_{n}=\left\langle W_{n}, \vee, \wedge, \rightarrow, \overline{0}\right\rangle$, where $W_{n}=\left\{a_{1}, \ldots, a_{n}\right\}, a_{i} \leq a_{j} \Leftrightarrow i \leq j, \overline{0}=a_{1}$. Note that $\mathrm{£}_{n}$ interpolates iff all truth values are representable (see Theorem 17 in [5). The first-order SPECTRUM is given by the propositional SPECTRUM. Let $T_{n}$ be the set of non-empty subsets of $W_{n}$ such that the greatest common divisor of the indices of elements of the set is 1 . The SPECTRUM of $\mathrm{L}_{n}$ is the set of all supersets of $T_{n}$.

## 8 Conclusion

Extending the notion of expansion to formulas containing strong quantifiers might be possible to cover logics which do not admit skolemization, e.g. logics based on non-constant domain Kripke frames (such notions of expansion are in the spirit of Herbrand's original proof of Herbrand's Theorem). 4] contains an application to the prenex fragment of first-order Gödel logic.

Another possibility is to develop unusual skolemizations e.g. based on existence assumptions [2] or on the addition of Skolem predicates instead of Skolem functions as in 9 .

The methodology of this paper can also be used to obtain negative results. First-order $S 5$ does not interpolate by a well-known result of Fine [8]. As propositional $S 5$ interpolates, first-order $S 5$ cannot admit skolemization together with expansions in general.

## References

1. Juan P Aguilera and Matthias Baaz. Ten problems in Gödel logic. Soft Computing, 21(1):149-152, 2017.
2. Matthias Baaz and Rosalie Iemhoff. The skolemization of existential quantifiers in intuitionistic logic. Annals of Pure and Applied Logic, 142(1-3):269-295, 2006.
3. Matthias Baaz and Alexander Leitsch. Methods of Cut-elimination, volume 34. Springer Science \& Business Media, 2011.
4. Matthias Baaz and Anela Lolic. First-order interpolation of non-classical logics derived from propositional interpolation. In FroCoS, volume 10483 of Lecture Notes in Computer Science, pages 265-280. Springer, 2017.
5. Matthias Baaz and Helmut Veith. Interpolation in fuzzy logic. Arch. Math. Log., 38(7):461-489, 1999.
6. Petr Cintula, Denisa Diaconescu, and George Metcalfe. Skolemization and herbrand theorems for lattice-valued logics. Theor. Comput. Sci., 768:54-75, 2019.
7. William Craig. Three uses of the Herbrand-Gentzen theorem in relating model theory and proof theory. The Journal of Symbolic Logic, 22(03):269-285, 1957.
8. Kit Fine. Failures of the interpolation lemma in quantified modal logic. The Journal of Symbolic Logic, 44(02):201-206, 1979.
9. Kurt Gödel. Die Vollständigkeit der Axiome des logischen Funktionenkalküls. Monatshefte für Mathematik, 37(1):349-360, 1930.
10. Larisa Maksimova. Intuitionistic logic and implicit definability. Annals of Pure and Applied Logic, 105(1-3):83-102, 2000.
11. Larisa L Maksimova. Craig's theorem in superintuitionistic logics and amalgamable varieties of pseudo-Boolean algebras. Algebra and Logic, 16(6):427-455, 1977.
12. Larisa L Maksimova. Interpolation properties of superintuitionistic logics. Studia Logica, 38(4):419-428, 1979.
13. Dale A Miller. A compact representation of proofs. Studia Logica, 46(4):347-370, 1987.
14. T Miyama. The interpolation theorem and Beth's theorem in many-valued logics. Mathematica Japonica, 19:341-355, 1974.
15. Hiroakira Ono. Model extension theorem and Craig's interpolation theorem for intermediate predicate logics. Reports on Mathematical Logic, 15:41-58, 1983.
