# Lyndon words versus inverse Lyndon words: queries on suffixes and bordered words 

Paola Bonizzoni ${ }^{1}$, Clelia De Felice ${ }^{2}$, Rocco Zaccagnino ${ }^{2}$, Rosalba Zizza ${ }^{2}$<br>${ }^{1}$ Università degli Studi di Milano Bicocca<br>${ }^{2}$ Università degli Studi di Salerno

November 6, 2019


#### Abstract

Lyndon words have been largely investigated and showned to be a useful tool to prove interesting combinatorial properties of words. In this paper we state new properties of both Lyndon and inverse Lyndon factorizations of a word $w$, with the aim of exploring their use in some classical queries on $w$.

The main property we prove is related to a classical query on words. We prove that there are relations between the length of the longest common extension (or longest common prefix) $\operatorname{lcp}(x, y)$ of two different suffixes $x, y$ of a word $w$ and the maximum length $\mathcal{M}$ of two consecutive factors of the inverse Lyndon factorization of $w$. More precisely, $\mathcal{M}$ is an upper bound on the length of $\operatorname{lcp}(x, y)$. This result is in some sense stronger than the compatibility property, proved by Mantaci, Restivo, Rosone and Sciortino for the Lyndon factorization and here for the inverse Lyndon factorization. Roughly, the compatibility property allows us to extend the mutual order between local suffixes of (inverse) Lyndon factors to the suffixes of the whole word.

A main tool used in the proof of the above results is a property that we state for factors $m_{i}$ with nonempty borders in an inverse Lyndon factorization: a nonempty border of $m_{i}$ cannot be a prefix of the next factor $m_{i+1}$. The last property we prove shows that if two words share a common overlap, then their Lyndon factorizations can be used to capture the common overlap of the two words.


The above results open to the study of new applications of Lyndon words and inverse Lyndon words in the field of string comparison.

Keywords: Lyndon words, Lyndon factorization, Combinatorial algorithms on words.
2010 Mathematics Subject Classification: 68R15, 68 W 32 .

## 1 Introduction

The Lyndon factorization of a word $w$ is a unique factorization of $w$ into a sequence of Lyndon words in nonincreasing lexicographic ordering. This factorization is one of the most known factorizations and it has been extensively studied in different contexts, from formal languages to algorithmic stringology and string compression. In particular the notion of a Lyndon word has been shown to be useful in theoretical applications, such as the well known proof of the Runs Theorem [3] as well in string compression analysis. A connection between the Lyndon factorization and the Lempel-Ziv (LZ) factorization has been given in [29], where it is shown that in general the size of the LZ factorization is larger than the size of the Lyndon factorization, and in any case the size of the Lyndon factorization cannot be larger than a factor of 2 with respect to the size of LZ. This result has been further extended in [46] to overlapping LZ factorizations.

The Lyndon factorization has recently revealed to be a useful tool also in investigating queries related to sorting suffixes of a word, with strong potentialities for string comparison that have not been completely explored and understood 40, 41]. Relations between Lyndon words and the Burrows-Wheeler Transform (BWT) have been discovered first in [13, 38] and more recently in [32. A main property of the Lyndon factorization is that it can be efficiently computed. Linear-time algorithms for computing the Lyndon factorization can be found in [20, 21] whereas an $\mathcal{O}(\lg n)$-time parallel algorithm has been proposed in [1, 16].

More recently Lyndon words found a renewed theoretical interest and several variants of them have been introduced and investigated with different motivations [9, 18, 19]. A related field studies the combinatorial and algorithmic properties of necklaces, that are powers of Lyndon words, and their prefixes or prenecklaces [8]. In [6], the notion of an inverse Lyndon word (a word which is strictly greater than each of its proper suffixes) has been introduced to define a new factorization, called the inverse Lyndon factorization. A word which is not an inverse Lyndon word may have different factorizations with inverse Lyndon words as factors but each word $w$ admits a unique canonical inverse Lyndon factorization, denoted $\operatorname{ICFL}(w)$. This factorization has the property that a word is factorized in a sequence of inverse Lyndon words, in an increasing and prefix-order-free lexicographic ordering, where prefix-order-free means that a factor cannot be a prefix of the consecutive one. Moreover $\operatorname{ICFL}(w)$ can be still computed in linear time and it is uniquely determined by $w$.

Differently from Lyndon words, inverse Lyndon words may be bordered. As a main result in this paper, we show that if a factor $m_{i}$ in $\operatorname{ICFL}(w)$ has a nonempty border, then such a border cannot be inherited by the consecutive factor, since it cannot be the prefix of the consecutive factor $m_{i+1}$. In other words, the longest common prefix between $m_{i}$ and $m_{i+1}$ is shorter than the border of $m_{i}$. This result is proved by a further investigation on the connection between the Lyndon factorization and the canonical inverse Lyndon factorization of a word, given in [6] through the grouping property. Indeed, given a word $w$ which is not an inverse Lyndon word, the factors in $\operatorname{ICFL}(w)$ are obtained by grouping together consecutive factors of the anti-Lyndon factorization of $w$ that form a chain for the prefix order.

Thanks to the properties of $\operatorname{ICFL}(w)$, the longest common extensions (or longest common prefix) of two distinct factors in $\operatorname{ICFL}(w)$ appear to have different properties than in the Lyndon factorization. In this framework, a natural question is whether and how the longest common extensions of two factors of $w$ are related to the size of the factors in $\operatorname{ICFL}(w)$. We prove that there are relations between the length of the longest common extension (or longest common prefix) $\operatorname{lcp}(x, y)$ of two different factors $x, y$ of a word $w$ and the maximum length $\mathcal{M}$ of two consecutive factors of the inverse Lyndon factorization of $w$. More precisely, $\mathcal{M}$ is an upper bound on the length of $\operatorname{lcp}(x, y)$. This result is in some sense stronger than the compatibility property, proved in [39, 40] for the Lyndon factorization and here for the inverse Lyndon factorization. Roughly, the compatibility property allows us to extend the mutual order between local suffixes of (inverse) Lyndon factors to the suffixes of the whole word. Another natural question is the following.

Given two words having a common overlap, can we use their Lyndon factorizations to capture the similarity of these words?

A partial positive answer to this question is provided here: given a word $w$ and a factor $x$ of $w$, we prove that their Lyndon factorizations share factors, except for the first and last term of the Lyndon factorization of $x$.

The paper is organized as follows. In Sections 2, (4, 5, 6, we gathered the basic definitions and known results we need. Relations between the Lyndon factorizations of two words that
share a common overlap are proved in Section 3. Borders of inverse Lyndon words are discussed in Section 7. The compatibility property for $\operatorname{ICFL}(w)$ is proved in Section 8. Finally the upper bound on the length of the longest common prefix of two factors of $w$ in terms of factors in $\operatorname{ICFL}(w)$ is stated in Section 9 .

## 2 Preliminaries

For the material in this section see [5, 11, 35, 36, 43].

### 2.1 Words

Let $\Sigma^{*}$ be the free monoid generated by a finite alphabet $\Sigma$ and let $\Sigma^{+}=\Sigma^{*} \backslash 1$, where 1 is the empty word. For a set $X, \operatorname{Card}(X)$ denotes the cardinality of $X$. For a word $w \in \Sigma^{*}$, we denote by $|w|$ its length. A word $x \in \Sigma^{*}$ is a factor of $w \in \Sigma^{*}$ if there are $u_{1}, u_{2} \in \Sigma^{*}$ such that $w=u_{1} x u_{2}$. If $u_{1}=1$ (resp. $u_{2}=1$ ), then $x$ is a prefix (resp. suffix) of $w$. A factor (resp. prefix, suffix) $x$ of $w$ is proper if $x \neq w$. Given a language $L \subseteq A^{*}$, we denote by $\operatorname{Pref}(L)$ (resp. $\operatorname{Suff}(L), \operatorname{Fact}(L))$ the set of all prefixes (resp. suffixes, factors) of its elements. Two words $x, y$ are incomparable for the prefix order, denoted as $x \bowtie y$, if neither $x$ is a prefix of $y$ nor $y$ is a prefix of $x$. Otherwise, $x, y$ are comparable for the prefix order. We write $x \leq_{p} y$ if $x$ is a prefix of $y$ and $x \geq_{p} y$ if $y$ is a prefix of $x$. The notion of a pair of words comparable (or incomparable) for the suffix order is defined symmetrically.

We recall that two words $x, y$ are called conjugate if there exist words $u, v$ such that $x=$ $u v, y=v u$. The conjugacy relation is an equivalence relation. A conjugacy class is a class of this equivalence relation. The following is Proposition 1.3.4 in [34].

Proposition 2.1 Two words $x, y \in \Sigma^{+}$are conjugate if and only if there exists $r \in \Sigma^{*}$ such that

$$
\begin{equation*}
x r=r y \tag{2.1}
\end{equation*}
$$

More precisely, equality (2.1) holds if and only if there exist $u, v \in \Sigma^{*}$ such that

$$
\begin{equation*}
x=u v, \quad y=v u, \quad r \in u(v u)^{*} \tag{2.2}
\end{equation*}
$$

A sesquipower of a word $x$ is a word $w=x^{n} p$ where $p$ is a proper prefix of $x$ and $n \geq 1$. A nonempty word $w$ is unbordered if no proper nonempty prefix of $w$ is a suffix of $w$. Otherwise, $w$ is bordered. A nonempty word $w$ is primitive if $w=x^{k}$ implies $k=1$. An unbordered word is primitive.

The following is a part of Proposition 1.3.2 in [34.
Proposition 2.2 Two words $u, v \in \Sigma^{+}$commute if and only if they are powers of the same word.

### 2.2 Lexicographic order and Lyndon words

Definition 2.1 Let $(\Sigma,<)$ be a totally ordered alphabet. The lexicographic (or alphabetic order) $\prec$ on $\left(\Sigma^{*},<\right)$ is defined by setting $x \prec y$ if

- $x$ is a proper prefix of $y$, or
- $x=r a s, y=r b t, a<b$, for $a, b \in \Sigma$ and $r, s, t \in \Sigma^{*}$.

In the next part of the paper we will implicitly refer to totally ordered alphabets. For two nonempty words $x, y$, we write $x \ll y$ if $x \prec y$ and $x$ is not a proper prefix of $y$ [2]. We also write $y \succ x$ if $x \prec y$. Basic properties of the lexicographic order are recalled below.

Lemma 2.1 For $x, y \in \Sigma^{*}$, the following properties hold.
(1) $x \prec y$ if and only if $z x \prec z y$, for every word $z$.
(2) If $x \ll y$, then $x u \ll y v$ for all words $u, v$.
(3) If $x \prec y \prec x z$ for a word $z$, then $y=x y^{\prime}$ for some word $y^{\prime}$ such that $y^{\prime} \prec z$.

Lemma 2.2 Let $x, y \in \Sigma^{*}$. If $x \ll y$, then $y \nprec x$.
Proof :
Suppose, contrary to our claim, that there would be $x, y \in \Sigma^{*}$ such that $y \prec x \ll y$. By definition there are $a, b \in \Sigma$ and $r, s, t \in \Sigma^{*}$ such that $x=r a s, y=r b t$. Thus $y$ cannot be a prefix of $x$, hence there are $a^{\prime}, b^{\prime} \in \Sigma$ and $r^{\prime}, s^{\prime}, t^{\prime} \in \Sigma^{*}$ such that $y=r^{\prime} a^{\prime} s^{\prime}, x=r^{\prime} b^{\prime} t^{\prime}, a^{\prime}<b^{\prime}$. By $x=$ ras $=r^{\prime} b^{\prime} t^{\prime}$ we have that the words $r a, r^{\prime} b^{\prime}$ are comparable for the prefix order. If $r^{\prime} b^{\prime}$ would be a prefix of $r$, then $r^{\prime} b^{\prime}$ were a prefix of $y=r^{\prime} a^{\prime} s^{\prime}$, which is impossible. Analogously, if $r a$ would be a prefix of $r^{\prime}$, then $r a$ were a prefix of $y=r b t$, once again a contradiction. Hence $r a=r^{\prime} b^{\prime}$, which implies $r=r^{\prime}, a=b^{\prime}$, therefore $a^{\prime}=b>a=b^{\prime}>a^{\prime}$, a contradiction.

Definition 2.2 A Lyndon word $w \in \Sigma^{+}$is a word which is primitive and the smallest one in its conjugacy class for the lexicographic order.

Example 2.1 Let $\Sigma=\{a, b\}$ with $a<b$. The words $a, b$, $a a a b, a b b b$, $a a b a b$ and $a a b a b a a b b$ are Lyndon words. On the contrary, $a b a b, a b a$ and $a b a a b$ are not Lyndon words. Indeed, $a b a b$ is a non-primitive word, $a a b \prec a b a$ and $a a b a b \prec a b a a b$.

Lyndon words are also called prime words and their prefixes are also called preprime words in 30. Some properties of Lyndon words are recalled below.

Proposition 2.3 Each Lyndon word $w$ is unbordered.
Proposition 2.4 $A$ word $w \in \Sigma^{+}$is a Lyndon word if and only if $w \prec s$, for each nonempty proper suffix $s$ of $w$.

The following is Proposition 5.1.3 in [34] and gives a second characterization of Lyndon words.

Proposition 2.5 A word $w \in \Sigma^{+}$is a Lyndon word if and only if $w \in \Sigma$ or $w=\ell m$ with $\ell, m$ Lyndon words, $\ell \prec m$.

Finally, in 18 the authors credited to folklore the following third characterization of Lyndon words: $w \in \Sigma^{+}$is a Lyndon word if and only if for each nontrivial factorization $w=u v$ one has $u \prec v$.

A class of conjugacy is also called a necklace and often identified with the minimal word for the lexicographic order in it. We will adopt this terminology. Then a word is a necklace if and only if it is a power of a Lyndon word. A prenecklace is a prefix of a necklace. Then clearly any nonempty prenecklace $w$ has the form $w=(u v)^{k} u$, where $u v$ is a Lyndon word, $u \in \Sigma^{*}, v \in \Sigma^{+}$, $k \geq 1$, that is, $w$ is a sesquipower of a Lyndon word $u v$. The following result has been proved in 20 .

Proposition 2.6 $A$ word is a nonempty preprime word if and only if it is a sesquipower of a Lyndon word distinct of the maximal letter.

The proof of Proposition [2.6 uses the following result which characterizes, for a given nonempty prenecklace $w$ and a letter $b$, whether $w b$ is still a prenecklace or not and, in the first case, whether $w b$ is a Lyndon word or not [20, Lemma 1.6].

Theorem 2.1 Let $w=\left(u a v^{\prime}\right)^{k} u$ be a nonempty prenecklace, where uav is a Lyndon word, $u, v^{\prime} \in \Sigma^{*}, a \in \Sigma, k \geq 1$. For any $b \in \Sigma$, the word $w b$ is a prenecklace if and only if $b \geq a$. Moreover $w b \in L$ if and only if $b>a$.

A direct consequence of Theorem [2.1 is reported below (see [8, Theorem 2.1] which states both Theorem 2.1 and Corollary (2.1).

Corollary 2.1 Let $w=\left(u a v^{\prime}\right)^{k} u$ be a nonempty prenecklace, where uav' is a Lyndon word, $u, v^{\prime} \in \Sigma^{*}, a \in \Sigma, k \geq 1$. Let $b \in \Sigma$ with $b \geq a$ and let $y$ be the longest prefix of $w b$ which is $a$ Lyndon word. Then

$$
y= \begin{cases}u a v^{\prime} & \text { if } b=a \\ w b & \text { if } b>a\end{cases}
$$

### 2.3 The Lyndon factorization

A family $\left(X_{i}\right)_{i \in I}$ of subsets of $\Sigma^{+}$, indexed by a totally ordered set $I$, is a factorization of the free monoid $\Sigma^{*}$ if each word $w \in \Sigma^{*}$ has a unique factorization $w=x_{1} \cdots x_{n}$, with $n \geq 0, x_{i} \in X_{j_{i}}$ and $j_{1} \geq j_{2} \geq \ldots \geq j_{n}$ 5. A factorization $\left(X_{i}\right)_{i \in I}$ is called complete if each $X_{i}$ is reduced to a singleton $x_{i}$ [5]. In the following $L=L_{\left(\Sigma^{*},<\right)}$ will be the set of Lyndon words, totally ordered by the relation $\prec$ on $\left(\Sigma^{*},<\right)$. The following theorem shows that the family $(\ell)_{\ell \in L}$ is a complete factorization of $\Sigma^{*}$.

Theorem 2.2 Any word $w \in \Sigma^{+}$can be written in a unique way as a nonincreasing product $w=\ell_{1} \ell_{2} \cdots \ell_{h}$ of Lyndon words, i.e., in the form

$$
\begin{equation*}
w=\ell_{1} \ell_{2} \cdots \ell_{h} \text {, with } \ell_{j} \in L \text { and } \ell_{1} \succeq \ell_{2} \succeq \ldots \succeq \ell_{h} \tag{2.3}
\end{equation*}
$$

The sequence $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{h}\right)$ in Eq. (2.3) is called the Lyndon decomposition (or Lyndon factorization) of $w$. It is denoted by $\operatorname{CFL}(w)$ because Theorem 2.2 is usually credited to Chen, Fox and Lyndon [10]. Uniqueness of the above factorization is a consequence of the following result, proved in [20].

Lemma 2.3 Let $w \in \Sigma^{+}$and let $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{h}\right)$. Then the following properties hold:
(i) $\ell_{h}$ is the nonempty suffix of $w$ which is the smallest with respect to the lexicographic order.
(ii) $\ell_{h}$ is the longest suffix of $w$ which is a Lyndon word.
(iii) $\ell_{1}$ is the longest prefix of $w$ which is a Lyndon word.

A direct consequence is stated below and it is necessary for our aims.
Corollary 2.2 Let $w \in \Sigma^{+}$, let $\ell_{1}$ be its longest prefix which is a Lyndon word and let $w^{\prime}$ be such that $w=\ell_{1} w^{\prime}$. If $w^{\prime} \neq 1$, then $\operatorname{CFL}(w)=\left(\ell_{1}, \operatorname{CFL}\left(w^{\prime}\right)\right)$.

As a consequence of Theorem 2.2, for any word $w$ there is a factorization

$$
w=\ell_{1}^{n_{1}} \cdots \ell_{r}^{n_{r}}
$$

where $r>0, n_{1}, \ldots, n_{r} \geq 1$, and $\ell_{1} \succ \ldots \succ \ell_{r}$ are Lyndon words, also named Lyndon factors of $w$. In the next, $\operatorname{CFL}(w)=\left(\ell_{1}^{n_{1}}, \ldots, \ell_{h}^{n_{r}}\right)$ will be an alternative notation for the Lyndon factorization of $w$. There is a linear time algorithm to compute the pair $\left(\ell_{1}, n_{1}\right)$ and thus, by iteration, the Lyndon factorization of $w$. It is due to Fredricksen and Maiorana [21] and it is also reported in [36]. It can also be used to compute the Lyndon word in the conjugacy class of a primitive word in linear time [36]. Linear time algorithms may also be found in [20] and in the more recent paper 25$]$.

## 3 Lyndon factorizations of factors of a word

Let $w \in \Sigma^{+}$be a word and let $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ be its Lyndon factorization, $k \geq 1$. Let $x$ be a proper factor (resp. prefix, suffix) of $w$. We say that $x$ is a simple factor of $w$ if, for each occurrence of $x$ as a factor of $w$, there is $j$, with $1 \leq j \leq k$, such that $x$ is a factor of $\ell_{j}$. We say that $x$ is a simple prefix (resp. suffix) of $w$ if $x$ is a proper prefix (resp. suffix) of $\ell_{1}$ (resp $\ell_{k}$ ). In this section we compare the Lyndon factorization of $w$ and that of its non-simple factors.

The following result is a direct consequence of Theorem 2.2.

Lemma 3.1 Let $w \in \Sigma^{+}$be a word and let $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ be its Lyndon factorization. For any $i, j$, with $1 \leq i<j \leq k$, one has

$$
\operatorname{CFL}\left(\ell_{i} \cdots \ell_{j}\right)=\left(\ell_{i}, \ldots, \ell_{j}\right)
$$

If $x$ is a non-simple factor of $w$ and $x$ does not satisfy the hypotheses of Lemma 3.1, then there are $i, j$ with $1 \leq i<j \leq k$, a suffix $\ell_{i}^{\prime \prime}$ of $\ell_{i}$ and a prefix $\ell_{j}^{\prime}$ of $\ell_{j}$, with $\ell_{i}^{\prime \prime} \ell_{j}^{\prime} \neq 1$, such that

$$
x=\ell_{i}^{\prime \prime} \ell_{i+1} \cdots \ell_{j-1} \ell_{j}^{\prime}
$$

where it is understood that if $j=i+1$, then $\ell_{i+1}, \ldots, \ell_{j-1}=1$ and $\ell_{i}^{\prime \prime} \neq 1, \ell_{j}^{\prime} \neq 1, \ell_{i}^{\prime \prime} \ell_{j}^{\prime} \neq \ell_{i} \ell_{j}$. We say that the sequence $\ell_{i}^{\prime \prime}, \ell_{i+1}, \ldots, \ell_{j-1}, \ell_{j}^{\prime}$ is associated with $x$. The following result gives relations between $\operatorname{CFL}(x)$ and $\operatorname{CFL}(w)$.

Lemma 3.2 Let $w \in \Sigma^{+}$be a word and let $\mathrm{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ be its Lyndon factorization. Let $x$ be a non-simple factor of $w$ such that $x$ does not satisfy the hypotheses of Lemma 3.1 and let $\ell_{i}^{\prime \prime}, \ell_{i+1}, \ldots, \ell_{j-1}, \ell_{j}^{\prime}$ be the sequence associated with $x$. We have

$$
\operatorname{CFL}(x)=\left(\operatorname{CFL}\left(\ell_{i}^{\prime \prime}\right), \ell_{i+1}, \ldots, \ell_{j-1}, \operatorname{CFL}\left(\ell_{j}^{\prime}\right)\right)
$$

where it is understood that if $\ell_{i}^{\prime \prime}=1$ (resp. $\ell_{j}^{\prime}=1$ ), then the first term $\mathrm{CFL}\left(\ell_{i}^{\prime \prime}\right)$ (resp. last term $\left.\mathrm{CFL}\left(\ell_{j}^{\prime}\right)\right)$ vanishes.

Proof :
Let $w, x, \ell_{1}, \ldots, \ell_{k}, \ell_{i}^{\prime \prime}, \ell_{j}^{\prime}$ be as in the statement. Set $\operatorname{CFL}\left(\ell_{i}^{\prime \prime}\right)=\left(m_{1}, \ldots, m_{h}\right)$ if $\ell_{i}^{\prime \prime} \neq 1$ and set $\operatorname{CFL}\left(\ell_{j}^{\prime}\right)=\left(v_{1}, \ldots, v_{t}\right)$ if $\ell_{j}^{\prime} \neq 1$. By Theorem [2.2, we shall have established the lemma if we prove the following claims
(1) if $\ell_{j}^{\prime} \neq 1$ and $j>i+1$, then $\ell_{j-1} \succeq v_{1}$;
(2) if $\ell_{i}^{\prime \prime} \neq 1$ and $j>i+1$, then $m_{h} \succeq \ell_{i+1}$;
(3) if $j=i+1$, then $m_{h} \succeq v_{1}$.

We preliminary observe that $\operatorname{CFL}\left(\ell_{j-1} \ell_{j} \cdots \ell_{k}\right)=\left(\ell_{j-1}, \ldots, \ell_{k}\right)$ (Lemma 3.1), hence $\ell_{j-1}$ is the longest prefix of $\ell_{j-1} \ell_{j} \cdots \ell_{k}$ which is a Lyndon word (Lemma 2.3).
(1) If $\ell_{j-1} \prec v_{1}$, then $\ell_{j-1} v_{1}$ would be a Lyndon word, by Proposition 2.5, and a prefix of $\ell_{j-1} \ell_{j} \cdots \ell_{k}$, longer than $\ell_{j-1}$, a contradiction.
(2) If $\ell_{i}^{\prime \prime}=\ell_{i}$, then $\left(m_{1}, \ldots, m_{h}\right)=\left(\ell_{i}\right)$ and we are done. Otherwise, $m_{h}$ is a suffix of $\ell_{i}^{\prime \prime}$ which is a proper nonempty suffix of $\ell_{i}$. By Proposition [2.4, we know that $\ell_{i} \prec m_{h}$. If $m_{h} \prec \ell_{i+1}$, then $\ell_{i} \prec \ell_{i+1}$, in contradiction with Eq.(2.3).
(3) Recall that in this case $x=\ell_{i}^{\prime \prime} \ell_{j}^{\prime}, \ell_{i}^{\prime \prime} \neq 1, \ell_{j}^{\prime} \neq 1, \ell_{i}^{\prime \prime} \ell_{j}^{\prime} \neq \ell_{i} \ell_{j}$. We claim that if $m_{h} \prec v_{1}$, then $\ell_{i}=\ell_{j-1} \prec v_{1}$. This is obvious if $\ell_{i}^{\prime \prime}=\ell_{i}$ because $\left(m_{1}, \ldots, m_{h}\right)=\left(\ell_{i}\right)$. Otherwise $m_{h}$ is a suffix of $\ell_{i}^{\prime \prime}$ which is a proper nonempty suffix of $\ell_{i}$. By Proposition [2.4, we know that $\ell_{i} \prec m_{h}$, thus if $m_{h} \prec v_{1}$, then $\ell_{i}=\ell_{j-1} \prec v_{1}$. Hence, $\ell_{j-1} v_{1}$ would be a Lyndon word, by Proposition [2.5, and a prefix of $\ell_{j-1} \ell_{j} \cdots \ell_{k}$, longer than $\ell_{j-1}$, a contradiction.

Let $x, y, z, w, w^{\prime} \in \Sigma^{+}$. The following result, which is a consequence of Lemma 3.2, gives relations between the Lyndon factorizations of two overlapping words $w, w^{\prime}$, i.e., such that $w=x y, w^{\prime}=y z$, and the Lyndon factorization of the overlap $y$, when $y$ is non-simple (as a suffix of $w$ and as a prefix of $w^{\prime}$ ).

Lemma 3.3 Let $w, w^{\prime} \in \Sigma^{+}$, let $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ and $\operatorname{CFL}\left(w^{\prime}\right)=\left(f_{1}, f_{2}, \ldots, f_{h}\right)$. If $y$ is a non-simple suffix of $w$ and a non-simple prefix of $w^{\prime}$, then there are $i, j$, with $1 \leq i<k$, $1<j \leq h$, such that one of the following cases holds.
(1) $\operatorname{CFL}(y)=\left(f_{1}, \ldots, f_{j-1}, \ell_{i+1}, \ldots, \ell_{k}\right)$
(2) There exists $j^{\prime}, 1<j^{\prime}<j$ such that $\operatorname{CFL}(y)=\left(f_{1}, \ldots, f_{j^{\prime}-1}, \ell_{i+1}, \ldots, \ell_{k}\right)$ and $f_{j^{\prime}+r}=$ $\ell_{i+1+r}$, for any $r, 0 \leq r \leq j-j^{\prime}-1$
(3) There is $i^{\prime}, i<i^{\prime}<k$ such that $\operatorname{CFL}(y)=\left(f_{1}, \ldots, f_{j-1}, \ell_{i^{\prime}+1}, \ldots, \ell_{k}\right)$ and $\ell_{i^{\prime}-r}=f_{j-r-1}$, for any $r, 0 \leq r \leq i^{\prime}-i-1$

Proof :
Let $w, w^{\prime} \in \Sigma^{+}$, let $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{k}\right)$ and $\operatorname{CFL}\left(w^{\prime}\right)=\left(f_{1}, f_{2}, \ldots, f_{h}\right)$. If $y$ is a non-simple suffix of $w$ and a non-simple prefix of $w^{\prime}$, then there are $i, j$, with $1 \leq i<k, 1<j \leq h$, such that

$$
y=\ell_{i}^{\prime \prime} \ell_{i+1} \cdots \ell_{k}=f_{1} \cdots f_{j-1} f_{j}^{\prime}
$$

where $\ell_{i}^{\prime \prime}$ is a suffix of $\ell_{i}$ and $f_{j}^{\prime}$ is a prefix of $f_{j}$. By Lemma 3.2 we have

$$
\operatorname{CFL}(y)=\left(\operatorname{CFL}\left(\ell_{i}^{\prime \prime}\right), \ell_{i+1}, \ldots, \ell_{k}\right)=\left(f_{1}, \ldots, f_{j-1} \operatorname{CFL}\left(f_{j}^{\prime}\right)\right)
$$

Thus the conclusion follows by Theorem 2.2.
Since Lyndon factorizations can be computed in linear time, the above result leads to efficient measures of similarities between words. These measures can be used to capture words that may be overlapping.

## 4 Anti-Lyndon words, inverse Lyndon words and anti-prenecklaces

For the material in this section see [6].

### 4.1 Inverse lexicographic order and anti-Lyndon words

Inverse Lyndon words are related to the inverse alphabetic order. Its definition is recalled below.
Definition 4.1 Let $(\Sigma,<)$ be a totally ordered alphabet. The inverse $<_{\text {in }}$ of $<$ is defined by

$$
\forall a, b \in \Sigma \quad b<_{\text {in }} a \Leftrightarrow a<b
$$

The inverse lexicographic or inverse alphabetic order on $\left(\Sigma^{*},<\right)$, denoted $\prec_{i n}$, is the lexicographic order on $\left(\Sigma^{*},<_{i n}\right)$.

Example 4.1 Let $\Sigma=\{a, b, c, d\}$ with $a<b<c<d$. Then $d a b \prec d a b d$ and $d a b d a \prec d a c$. We have $d<_{i n} c<_{i n} b<_{i n} a$. Therefore $d a b \prec_{\text {in }} d a b d$ and $d a c \prec_{\text {in }} d a b d a$.

The following proposition justifies the adopted terminology.
Proposition 4.1 Let $(\Sigma,<)$ be a totally ordered alphabet. For all $x, y \in \Sigma^{*}$ such that $x \bowtie y$,

$$
y \prec_{\text {in }} x \Leftrightarrow x \prec y .
$$

Moreover, in this case $x \ll y$.
From now on, $L_{i n}=L_{\left(\Sigma^{*},<_{i n}\right)}$ denotes the set of the Lyndon words on $\Sigma^{*}$ with respect to the inverse lexicographic order. A word $w \in L_{i n}$ will be named an anti-Lyndon word. Correspondingly, an anti-prenecklace will be a prefix of an anti-necklace, which in turn will be a necklace with respect to the inverse lexicographic order. The following proposition characterizes $L_{i n}=L_{\left(\Sigma^{*},<_{i n}\right)}$.

Proposition 4.2 $A$ word $w \in \Sigma^{+}$is in $L_{\text {in }}$ if and only if $w$ is primitive and $w \succ v u$, for each $u, v \in \Sigma^{+}$such that $w=u v$.

We state below a slightly modified dual version of Proposition 2.4.
Proposition 4.3 $A$ word $w \in \Sigma^{+}$is in $L_{\text {in }}$ if and only if $w$ is unbordered and $w \succ v$, for each proper nonempty suffix $v$.

The following result give more precise relations between words in $L_{i n}$ and their proper nonempty suffixes.

Proposition 4.4 If $v$ is a proper nonempty suffix of $w \in L_{i n}$, then $v \ll w$.
In the following, we denote by $\mathrm{CFL}_{i n}(w)$ the Lyndon factorization of $w$ with respect to the inverse order $<_{i n}$.

### 4.2 Inverse Lyndon words and anti-prenecklaces

Definition 4.2 $A$ word $w \in \Sigma^{+}$is an inverse Lyndon word if $s \prec w$, for each nonempty proper suffix $s$ of $w$.

Example 4.2 The words $a, b$, $a a a a a, b b b a, b a a a b, b b a b a$ and bbababbaa are inverse Lyndon words on $\{a, b\}$, with $a<b$. On the contrary, $a a b a$ is not an inverse Lyndon word since $a a b a \prec b a$. Analogously, $a a b b a \prec b a$ and thus $a a b b a$ is not an inverse Lyndon word.

The following result is a direct consequence of Proposition 4.3,

Proposition 4.5 $A$ word $w \in \Sigma^{+}$is an anti-Lyndon word if and only if it is an unbordered inverse Lyndon word.

In turn, by Proposition 4.5 it is clear that the set of anti-Lyndon words is a proper subset of the set of inverse Lyndon words since there are inverse Lyndon words which are not anti-Lyndon words. For instance consider $\Sigma=\{a, b\}$, with $a<b$. The word $b a b$ is an inverse Lyndon word but it is bordered, thus it is not an anti-Lyndon word.

Inverse Lyndon words and anti-prenecklaces are strongly related, as the following result shows.

Proposition 4.6 $A$ word $w \in \Sigma^{+}$is an inverse Lyndon word if and only if $w$ is a nonempty anti-prenecklace.

The following result is a direct consequence of Proposition 4.6.
Lemma 4.1 Any nonempty prefix of an inverse Lyndon word is an inverse Lyndon word.

## 5 A canonical inverse Lyndon factorization: $\operatorname{ICFL}(w)$

An inverse Lyndon factorization of a word $w \in \Sigma^{+}$is a sequence ( $m_{1}, \ldots, m_{k}$ ) of inverse Lyndon words such that $m_{1} \cdots m_{k}=w$ and $m_{i} \ll m_{i+1}, 1 \leq i \leq k-1$. The canonical inverse Lyndon factorization, denoted $\operatorname{ICFL}(w)$, is a special inverse Lyndon factorization that maintains the main properties of the Lyndon factorization. Its definition and properties are based on other notions and results recalled below.

Definition 5.1 Let $w \in \Sigma^{+}$, let $p$ be an inverse Lyndon word which is a nonempty proper prefix of $w=p v$. The bounded right extension $\bar{p}_{w}$ of $p$ (relatively to $w$ ), denoted by $\bar{p}$ when it is understood, is a nonempty prefix of $v$ such that:
(1) $\bar{p}$ is an inverse Lyndon word,
(2) $p z^{\prime}$ is an inverse Lyndon word, for each proper nonempty prefix $z^{\prime}$ of $\bar{p}$,
(3) $p \bar{p}$ is not an inverse Lyndon word,
(4) $p \ll \bar{p}$.

Moreover, we set

$$
\begin{aligned}
\operatorname{Pref}_{\text {bre }}(w)= & \{(p, \bar{p}) \mid p \text { is an inverse Lyndon word } \\
& \text { which is a nonempty proper prefix of } w\} .
\end{aligned}
$$

It has been proved that either $\operatorname{Pref}_{\text {bre }}(w)=\emptyset$ or $\operatorname{Card}\left(\operatorname{Pref}_{\text {bre }}(w)\right)=1$. $\operatorname{Moreover}^{\operatorname{Pref}}{ }_{\text {bre }}(w)$ is empty if and only if $w$ is an inverse Lyndon word. Another useful property of $\operatorname{Pref}_{b r e}(w)$ is recalled below.

Proposition 5.1 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $z$ be the shortest nonempty prefix of $w$ which is not an inverse Lyndon word. Then,
(1) $z=p \bar{p}$, with $(p, \bar{p}) \in \operatorname{Pref}_{b r e}(w)$.
(2) $p=$ ras and $\bar{p}=r b$, where $r, s \in \Sigma^{*}, a, b \in \Sigma$ and $r$ is the shortest prefix of $p \bar{p}$ such that $p \bar{p}=$ rasrb, with $a<b$.

We now give the recursive definition of $\operatorname{ICFL}(w)$.
Definition 5.2 Let $w \in \Sigma^{+}$.
(Basis Step) If $w$ is an inverse Lyndon word, then $\operatorname{ICFL}(w)=(w)$.
(Recursive Step) If $w$ is not an inverse Lyndon word, let $(p, \bar{p}) \in \operatorname{Pref}_{\text {bre }}(w)$ and let $v \in \Sigma^{*}$ such that $w=p v$. Let $\operatorname{ICFL}(v)=\left(m_{1}^{\prime}, \ldots, m_{k}^{\prime}\right)$ and let $r, s \in \Sigma^{*}, a, b \in \Sigma$ such that $p=r a s, \bar{p}=r b$ with $a<b$.

$$
\operatorname{ICFL}(w)= \begin{cases}(p, \operatorname{ICFL}(v)) & \text { if } \bar{p}=r b \leq_{p} m_{1}^{\prime} \\ \left(p m_{1}^{\prime}, m_{2}^{\prime}, \ldots, m_{k}^{\prime}\right) & \text { if } m_{1}^{\prime} \leq_{p} r\end{cases}
$$

## 6 Groupings

Let $w \in \Sigma^{+}$. There are relations between $\operatorname{ICFL}(w)$, the Lyndon factorization $\operatorname{CFL}_{i n}(w)$ of $w$ with respect to the inverse order $<_{i n}$ and some special inverse Lyndon factorizations of $w$, called groupings of $\mathrm{CFL}_{i n}(w)$. We first give some needed definitions and results.

Definition 6.1 Let $w \in \Sigma^{+}$, let $\operatorname{CFL}_{i n}(w)=\left(\ell_{1}, \ldots, \ell_{h}\right)$ and let $1 \leq r<s \leq h$. We say that $\ell_{r}, \ell_{r+1} \ldots, \ell_{s}$ is a non-increasing maximal chain for the prefix order in $\mathrm{CFL}_{i n}(w)$, abbreviated $\mathcal{P M C}$, if $\ell_{r} \geq_{p} \ell_{r+1} \geq_{p} \ldots \geq_{p} \ell_{s}$. Moreover, if $r>1$, then $\ell_{r-1} \not ¥_{p} \ell_{r}$, if $s<h$, then $\ell_{s} \not ¥_{p} \ell_{s+1}$. Two $\mathcal{P M C} \mathcal{C}_{1}=\ell_{r}, \ell_{r+1} \ldots, \ell_{s}, \mathcal{C}_{2}=\ell_{r^{\prime}}, \ell_{r^{\prime}+1} \ldots, \ell_{s^{\prime}}$ are consecutive if $r^{\prime}=s+1\left(\right.$ or $\left.r=s^{\prime}+1\right)$.

The definition of a grouping of $\mathrm{CFL}_{i n}(w)$ is given below in two steps. We first define the grouping of a $\mathcal{P M C}$. Then a grouping of $\mathrm{CFL}_{i n}(w)$ is obtained by changing each $\mathcal{P M C}$ with one of its groupings.

Definition 6.2 Let $\ell_{1}, \ldots, \ell_{h}$ be words in $L_{i n}$ such that $\ell_{i}$ is a prefix of $\ell_{i-1}, 1<i \leq h$. We say that $\left(m_{1}, \ldots, m_{k}\right)$ is a grouping of $\left(\ell_{1}, \ldots, \ell_{h}\right)$ if the following conditions are satisfied.
(1) $m_{j}$ is an inverse Lyndon word,
(2) $\ell_{1} \cdots \ell_{h}=m_{1} \cdots m_{k}$. More precisely, there are $i_{0}, i_{1}, \ldots, i_{k}, i_{0}=0,1 \leq i_{j} \leq h, i_{k}=h$, such that $m_{j}=\ell_{i_{j-1}+1} \cdots \ell_{i_{j}}, 1 \leq j \leq k$,
(3) $m_{1} \ll \ldots \ll m_{k}$.

We now extend Definition 6.2 to $\mathrm{CFL}_{i n}(w)$.
Definition 6.3 Let $w \in \Sigma^{+}$and let $\operatorname{CFL}_{i n}(w)=\left(\ell_{1}, \ldots, \ell_{h}\right)$. We say that $\left(m_{1}, \ldots, m_{k}\right)$ is a grouping of $\mathrm{CFL}_{i n}(w)$ if it can be obtained by replacing any $\mathcal{P M C} \mathcal{C}$ in $\mathrm{CFL}_{\text {in }}(w)$ by a grouping of $\mathcal{C}$.

Groupings of $\mathrm{CFL}_{i n}(w)$ are inverse Lyndon factorizations of $w$ but there are inverse Lyndon factorizations which are not groupings. As stated below, $\operatorname{ICFL}(w)$ is a grouping of $\mathrm{CFL}_{i n}(w)$. We first consider the special case of an inverse Lyndon word.

Proposition 6.1 Let $(\Sigma,<)$ be a totally ordered alphabet. Let $w \in \Sigma^{+}$and let $\mathrm{CFL}_{\text {in }}(w)=$ $\left(\ell_{1}, \ldots, \ell_{h}\right)$. If $w$ is an inverse Lyndon word, then either $w$ is unbordered or $\ell_{1}, \ldots, \ell_{h}$ is a $\mathcal{P M C}$ in $\mathrm{CFL}_{i n}(w)$. In both cases $\operatorname{ICFL}(w)=(w)$ is the unique grouping of $\mathrm{CFL}_{i n}(w)$.

Proposition 6.2 Let $(\Sigma,<)$ be a totally ordered alphabet. For any $w \in \Sigma^{+}, \operatorname{ICFL}(w)$ is a grouping of $\mathrm{CFL}_{i n}(w)$.

## 7 Borders

We recall that, given a nonempty word $w$, a border of $w$ is a word which is both a proper prefix and a suffix of $w$ [14]. The longest proper prefix of $w$ which is a suffix of $w$ is also called the border of $w$ [14, 36. Thus a word $w \in \Sigma^{+}$is unbordered if and only if it has a nonempty border. Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word, let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. The aim of this section is to show that any nonempty border of $m_{i}$ is not a prefix of $m_{i+1}$, $1 \leq i \leq k-1$. Some preliminary results are needed.

Proposition 7.1 Let $w \in \Sigma^{+}$, let $\operatorname{CFL}_{i n}(w)=\left(\ell_{1}, \ldots, \ell_{h}\right)$ and let $\ell_{r}, \ldots, \ell_{s}, 1 \leq r<s \leq h$, be a non-increasing chain for the prefix order in $\mathrm{CFL}_{i n}(w)$. For any nonempty border $x$ of $y=\ell_{r} \cdots \ell_{s}$ there is $t, r \leq t<s$, such that $x=\ell_{t+1} \cdots \ell_{s}$. Consequently, $\ell_{s}$ is a prefix of any nonempty border of $\ell_{r} \cdots \ell_{s}$.

Proof :
Let $w, \ell_{1}, \ldots, \ell_{h}, r, s$ be as in the statement. By hypothesis, for each $t$, with $r \leq t \leq s$, the word $\ell_{t}$ is a prefix of $\ell_{r}$. Let $x$ be a nonempty border of $y=\ell_{r} \cdots \ell_{s}$. If there were a nonempty proper suffix $x^{\prime}$ of $\ell_{t}, r \leq t \leq s$, such that $x=x^{\prime} \ell_{t+1} \cdots \ell_{s}$, then $x^{\prime}$ would be both a prefix and a nonempty proper suffix of $\ell_{t}$, thus a nonempty border of $\ell_{t}$, in contradiction with $\ell_{t}$ being an anti-Lyndon word.

Lemma 7.1 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word, let $\operatorname{CFL}_{i n}(w)=$ $\left(\ell_{1}^{n_{1}}, \ldots, \ell_{h}^{n_{h}}\right)$, with $h>0, n_{1}, \ldots, n_{h} \geq 1$. For all $z \in \Sigma^{+}$and $b \in \Sigma$ such that $z$ is an antiprenecklace, $z b$ is not an anti-prenecklace and $z b$ is a prefix of $w$, there is an integer $g$ such that

$$
z b=\left(u_{1} v_{1}\right)^{n_{1}} \cdots\left(u_{g} v_{g}\right)^{n_{g}} u_{g} b,
$$

where $u_{j} v_{j}=u_{j} a_{j} v_{j}^{\prime}=\ell_{j}, 1 \leq j \leq g, a_{j}<b$ and $u_{g} b$ is an anti-prenecklace.
Proof :
We prove the statement by induction on $|w|$. If $|w|=1$, then $w$ is an inverse Lyndon word and we are done. Hence assume $|w|>1$. If $w$ is an inverse Lyndon word, then again the proof is ended. Therefore, assume that $w$ is not an inverse Lyndon word. Let $\mathrm{CFL}_{i n}(w)=\left(\ell_{1}^{n_{1}}, \ldots, \ell_{h}^{n_{h}}\right)$, with $h>0, n_{1}, \ldots, n_{h} \geq 1$.

Let $z \in \Sigma^{+}, b \in \Sigma$ be such that $z$ is an anti-prenecklace, $z b$ is not an anti-prenecklace and $z b$ is a prefix of $w$. By Theorem 2.1 and Corollary 2.1, there are words $u, v, v^{\prime} \in \Sigma^{*}, a \in \Sigma$, with $a<b$, and an integer $k \geq 1$, such that $z b=(u v)^{k} u b, v=a v^{\prime}$ and where $u v$ is the longest anti-Lyndon prefix of $z b$.

We claim that $u v$ is also the longest anti-Lyndon prefix of $w$. Indeed, if $y$ is a prefix of $w$ such that $|y|>|z b|$, then $y=z b z^{\prime}=\left(u a v^{\prime}\right)^{k} u b z^{\prime}$, with $z^{\prime} \in \Sigma^{*}$. Thus, $y \ll u b z^{\prime}$ and $y$ is not an anti-Lyndon word. Consequently, by Lemma [2.3, we have $u v=\ell_{1}$. Moreover, $k=n_{1}$ because $u b$ is not a prefix of $\ell_{1}$.

If $u b$ is an anti-prenecklace the proof is ended. Otherwise, let $w^{\prime} \in \Sigma^{*}$ be such that $w=$ $\ell_{1}^{n_{1}} w^{\prime}$. We have $0<\left|w^{\prime}\right|<|w|$ since $u b$ is a prefix of $w^{\prime}$ and $\ell_{1} \neq 1$. By Theorem 2.2, we have $\mathrm{CFL}_{i n}\left(w^{\prime}\right)=\left(\ell_{2}^{n_{2}}, \ldots, \ell_{h}^{n_{h}}\right)$. The word $u$ an anti-prenecklace whereas $u b$ is not an antiprenecklace. By induction hypothesis there is an integer $g$ such that

$$
u b=\left(u_{2} v_{2}\right)^{n_{2}} \cdots\left(u_{g} v_{g}\right)^{n_{g}} u_{g} b,
$$

where $u_{j} v_{j}=u_{j} a_{j} v_{j}^{\prime}=\ell_{j}, 2 \leq j \leq g, a_{j}<b$ and $u_{g} b$ is an anti-prenecklace. Thus, there is an integer $g$ such that

$$
z b=\left(u_{1} v_{1}\right)^{n_{1}} \cdots\left(u_{g} v_{g}\right)^{n_{g}} u_{g} b,
$$

where $u_{1}=u, v_{1}=v, v_{1}^{\prime}=v^{\prime}, a_{1}=a, u_{j} v_{j}=u_{j} a_{j} v_{j}^{\prime}=\ell_{j}, 1 \leq j \leq g, a_{j}<b$ and $u_{g} b$ is an anti-prenecklace.

Proposition 7.2 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word, let $(p, \bar{p}) \in$ $\operatorname{Pref}_{\text {bre }}(w)$ and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $\operatorname{CFL}_{i n}(w)=\left(\ell_{1}^{n_{1}}, \ldots, \ell_{h}^{n_{h}}\right)$, with $h>0$, $n_{1}, \ldots, n_{h} \geq 1$ and let $\ell_{1}^{n_{1}}, \ldots, \ell_{q}^{n_{q}}$ be a $\mathcal{P M C}$ in $\mathrm{CFL}_{\text {in }}(w), 1 \leq q \leq h$. Then the following properties hold.

$$
\text { (1) } p=\ell_{1}^{n_{1}} \cdots \ell_{g}^{n_{g}} \text {, for some } g, 1 \leq g \leq q \text {. }
$$

(2) $\ell_{g}=u_{g} v_{g}=u_{g} a_{g} v_{g}^{\prime}, \bar{p}=u_{g} b, a_{g}<b$.

Proof:
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word, let $(p, \bar{p}) \in \operatorname{Pref}_{b r e}(w)$. Let $\mathrm{CFL}_{i n}(w)=\left(\ell_{1}^{n_{1}}, \ldots, \ell_{h}^{n_{h}}\right)$, with $h>0, n_{1}, \ldots, n_{h} \geq 1$ and let $\ell_{1}^{n_{1}}, \ldots, \ell_{q}^{n_{q}}$ be a $\mathcal{P M C}$ in $\mathrm{CFL}_{i n}(w), 1 \leq q \leq h$.

By Proposition 4.6, the word $p \bar{p}$ is not an anti-prenecklace but its longest proper prefix is an anti-prenecklace. Thus, by Lemma 7.1 there is an integer $g$ such that

$$
p \bar{p}=\left(u_{1} v_{1}\right)^{n_{1}} \cdots\left(u_{g} v_{g}\right)^{n_{g}} u_{g} b,
$$

where $u_{j} v_{j}=u_{j} a_{j} v_{j}^{\prime}=\ell_{j}, 1 \leq j \leq g, a_{j}<b$ and $u_{g} b$ is an anti-prenecklace. Let

$$
\beta=\left(u_{1} v_{1}\right)^{n_{1}}\left(u_{2} v_{2}\right)^{n_{2}} \cdots\left(u_{g} v_{g}\right)^{n_{g}}, \quad \beta^{\prime}=\beta u_{g} .
$$

By Definition [5.1, the words $\beta^{\prime}$ and $\beta$ are inverse Lyndon words, therefore $g \leq q$ (otherwise $\ell_{q}$ would be a prefix of $\beta$ and there would be a word $z^{\prime}$ such that $\ell_{q+1} z^{\prime}$ is a suffix of $\beta$, a contradiction since $\beta$ is an inverse Lyndon word and $\ell_{q} \ll \ell_{q+1}$ ). Moreover, $\beta \ll u_{g} b$.

Let $r, s \in \Sigma^{*}, a^{\prime}, b \in \Sigma$ be such that $p=r a^{\prime} s, \bar{p}=r b, a^{\prime}<b$. Then $p \bar{p}=\beta u_{g} b=r a^{\prime} s r b$. By Proposition [5.1, $r$ is a suffix of $u_{g}$. Consequently, $\ell_{g}=u_{g} v_{g}$ and $u_{g}$ are prefixes of $p$. Moreover, we know that $u_{g}$ and $u_{g} b$ are both anti-prenecklaces. Thus, by Proposition [2.6, Theorem 2.1 and Corollary 2.1, there are $x, y \in \Sigma^{*}$, an integer $t \geq 1, c \in \Sigma$ such that $x y$ is an anti-Lyndon word, $u_{g}=(x y)^{t} x, y=c y^{\prime}$ with $c \geq b$.

The words $\ell_{g+1}$ and $u_{g} b=(x y)^{t} x b$ are both prefixes of the same word $\gamma$, hence they are comparable for the prefix order. Since $\ell_{g+1}$ is the longest anti-Lyndon prefix of $\gamma$, we have $\left|\ell_{g+1}\right| \geq|x y|$ and since $\ell_{g+1}$ is unbordered, either $\ell_{g+1}=x y$ is a prefix of $\ell_{g}$ and $g+1 \leq q$, or the word $u_{g} b=(x y)^{t} x b$ is a prefix of $\ell_{g+1}$. By Proposition 6.2, the first case holds, otherwise $m_{1}$ would not be a product of anti-Lyndon words because $m_{1}$ is a prefix of $\beta u_{g}$.

If $r=u_{g}$, then $p=\beta$ and the proof is ended. By contradiction, assume that $r$ is a proper suffix of $u_{g}$. Therefore $r$ is a border of $u_{g}$ because $r$ is a prefix of $p$ and $u_{g}$ is nonempty. Of course $r \neq x$ because $u_{g}$ starts with $r a^{\prime}$ and also with $x c$, with $c \geq b>a^{\prime}$. If $r$ would be shorter than $x$, then $r$ would be a border of $x$. This is impossible because $r c y^{\prime}(x y)^{t-1} x$ would be a suffix of the inverse Lyndon word $u_{g}$ and $u_{g}$ starts with $r a^{\prime}$, with $c \geq b>a^{\prime}$. Thus $|r|>|x| \geq 0$. Since $r$ is a nonempty border of $u_{g}=(x y)^{t} x$ and $|r|>|x| \geq 0$, one of the following three cases holds:

$$
\begin{align*}
& r=(x y)^{t^{\prime}} x, \quad 0<t^{\prime}<t  \tag{7.1}\\
& r=y_{1}(x y)^{t^{\prime}} x, \quad y_{1} \text { nonempty suffix of } y, \quad 0 \leq t^{\prime}<t  \tag{7.2}\\
& r=x_{1}(y x)^{t^{\prime}}, \quad x_{1} \text { nonempty suffix of } x, \quad 0<t^{\prime} \leq t \tag{7.3}
\end{align*}
$$

Assume that Eq. (7.1) holds. Then $p$ starts with $r a^{\prime}=(x y)^{t^{\prime}} x a^{\prime}, a^{\prime}<b$, and $p$ also starts with $u_{g}=(x y)^{t} x$. Since $t^{\prime}<t$, the letter $a^{\prime}$ should be the first letter of $y=c y^{\prime}, c \geq b>a^{\prime}$. Therefore, Eq. (7.1) cannot hold.

Assume that Eq. (7.2) holds. Therefore $y_{1}=y$, otherwise $y_{1} x$ would be a proper prefix of $x y$, hence a nonempty border of $x y$, which is impossible since $x y$ is an anti-Lyndon word. Moreover $x=1$, otherwise $y x=x y$ and $x y$ would not be primitive (Proposition 2.2), which is impossible since $x y$ is an anti-Lyndon word. As above, $p$ starts with $r a^{\prime}=y^{t^{\prime}} a^{\prime}, a^{\prime}<b$, and $p$ also starts with $u_{g}=y^{t}$. Since $t^{\prime}<t$, the letter $a^{\prime}$ should be the first letter of $y=c y^{\prime}, c \geq b>a^{\prime}$. Therefore, Eq. (7.2) cannot hold.

Finally, assume that Eq. (7.3) holds. If $x_{1} \neq x$, then $x_{1} y$ would be both a proper nonempty suffix and a prefix of $x y$, hence a nonempty border of $x y$, which is impossible since $x y$ is an anti-Lyndon word. Therefore $x_{1}=x$. If $t^{\prime}<t$, then $r$ satisfies Eq. (7.1) and we proved that this is impossible. Thus $t^{\prime}=t$, which implies $r=u_{g}$, a contradiction.

Proposition 7.3 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $(p, \bar{p}) \in$ $\operatorname{Pref}_{\text {bre }}(w)$. For each nonempty border $z$ of $p$, one has that $z$ and $\bar{p}$ are incomparable for the prefix order.

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $(p, \bar{p}) \in \operatorname{Pref}_{b r e}(w)$. By Proposition 5.1, there are $r, s \in \Sigma^{*}, a, b \in \Sigma$, with $a<b$, such that $p=r a s$ and $\bar{p}=r b$. Let $\mathrm{CFL}_{i n}(w)=\left(\ell_{1}^{n_{1}}, \ldots, \ell_{h}^{n_{h}}\right)$, with $h>0, n_{1}, \ldots, n_{h} \geq 1$ and let $\ell_{1}^{n_{1}}, \ldots, \ell_{q}^{n_{q}}$ be a $\mathcal{P M C}$ in $\mathrm{CFL}_{i n}(w), 1 \leq q \leq h$.

Let $z$ be a nonempty border of $p$. Of course $\bar{p}$ cannot be a prefix of $z$ because $\bar{p}$ is not a prefix of $p$. By contradiction, suppose that $z$ is a prefix of $\bar{p}$. By Proposition 7.2, there is $g$, $1 \leq g \leq q$ such that $p=\ell_{1}^{n_{1}} \cdots \ell_{g}^{n_{g}}$ and $\ell_{g}=u_{g} v_{g}=u_{g} a_{g} v_{g}^{\prime}, \bar{p}=u_{g} b, a_{g}<b$.

By Proposition 7.1, $\ell_{g}$ is a prefix of any nonempty border of $p$, hence $\ell_{g}$ is a prefix of $z$. Moreover $z$ is a prefix of $\bar{p}$, thus $\ell_{g}=u_{g} a_{g} v_{g}^{\prime}$ would be a prefix of $\bar{p}=u_{g} b$. This is impossible because $a_{g}<b$.

Proposition 7.4 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. If $z$ is a nonempty border of $m_{1}$, then $z$ is not a prefix of $m_{2}$.

## Proof :

Let $w \in \Sigma^{+}$and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. We prove the statement by induction on $|w|$.
If $|w|=1$, then $w$ is an inverse Lyndon word and we are done. Hence assume $|w|>1$. If $w$ is an inverse Lyndon word, then again the proof is ended. Therefore, assume that $w$ is not an inverse Lyndon word. Let $(p, \bar{p}) \in \operatorname{Pref}_{b r e}(w)$ and let $r, s \in \Sigma^{*}, a, b \in \Sigma$ be such that $p=r a s$, $\bar{p}=r b, a<b$. Let $v \in \Sigma^{*}$ be such that $w=m_{1} v$. Of course $0<|v|<|w|$ because $w$ is not an inverse Lyndon word. Let $\operatorname{ICFL}(v)=\left(m_{1}^{\prime}, \ldots, m_{k^{\prime}}^{\prime}\right)$. By Definition 5.2, one of the following two cases holds
(1) $m_{1}=p$ if $\bar{p}$ is a prefix of $m_{1}^{\prime}=m_{2}$
(2) $m_{1}=p m_{1}^{\prime}, m_{2}=m_{2}^{\prime}, \ldots, m_{k}=m_{k}^{\prime}, k^{\prime}=k$, if $m_{1}^{\prime}$ is a prefix of $r$.

Let $z$ be a nonempty border of $m_{1}$. In case (1), if $z$ would be a prefix of $m_{2}$, then $z$ and $\bar{p}$ would be comparable for the prefix order, in contradiction with Proposition 7.3.

In case (2), $m_{1}^{\prime}$ is a prefix of $m_{1}$. By contradiction, suppose that $z$ is a prefix of $m_{2}$. We have either $|z| \geq\left|m_{1}^{\prime}\right|$ or $|z|<\left|m_{1}^{\prime}\right|$. If $|z| \geq\left|m_{1}^{\prime}\right|$, then $m_{1}^{\prime}$ would be a prefix of $z$ and thus
of $m_{2}=m_{2}^{\prime}$, in contradiction with $m_{1}^{\prime} \ll m_{2}^{\prime}$. If $|z|<\left|m_{1}^{\prime}\right|$, then $z$ would be a suffix of $m_{1}^{\prime}$, hence $z$ would be a nonempty border of $m_{1}^{\prime}$. Thus a nonempty border of $m_{1}^{\prime}$ would be a prefix of $m_{2}=m_{2}^{\prime}$, in contradiction with the induction hypothesis.

Proposition 7.5 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. If $z$ is a nonempty border of $m_{i}$, then $z$ is not a prefix of $m_{i+1}, 1 \leq i \leq k-1$.

## Proof :

Let $w \in \Sigma^{+}$and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. We prove the statement by induction on $|w|$.
If $|w|=1$, then $w$ is an inverse Lyndon word and we are done. Hence assume $|w|>1$. If $w$ is an inverse Lyndon word, then again the proof is ended. Therefore, assume that $w$ is not an inverse Lyndon word. Let $(p, \bar{p}) \in \operatorname{Pref}_{b r e}(w)$ and let $r, s \in \Sigma^{*}, a, b \in \Sigma$ be such that $p=$ ras, $\bar{p}=r b, a<b$. Let $v \in \Sigma^{*}$ be such that $w=m_{1} v$. Of course $0<|v|<|w|$ because $w$ is not an inverse Lyndon word. Let $\operatorname{ICFL}(v)=\left(m_{1}^{\prime}, \ldots, m_{k^{\prime}}^{\prime}\right)$. By Definition 5.2, one of the following two cases holds
(1) $m_{1}=p, m_{i}=m_{i-1}^{\prime}, 1<i \leq k=k^{\prime}+1$, if $\bar{p} \leq_{p} m_{1}^{\prime}$
(2) $m_{1}=p m_{1}^{\prime}, m_{i}=m_{i}^{\prime}, 1 \leq i \leq k=k^{\prime}$, if $m_{1}^{\prime} \leq_{p} r$.

If $z$ is a nonempty border of $m_{1}$, then $z$ is not a prefix of $m_{2}$, by Proposition 7.4. Thus assume that $z$ is a nonempty border of $m_{i}, 1<i \leq k-1$. In case ( 1 ), $z$ is a nonempty border of $m_{i-1}^{\prime}$, hence, by induction hypothesis, $z$ is not a prefix of $m_{i}^{\prime}=m_{i+1}$. Analogously, in case (2), $z$ is a nonempty border of $m_{i}^{\prime}$, therefore, by induction hypothesis, $z$ is not a prefix of $m_{i+1}^{\prime}=m_{i+1}$.

## 8 Suffixes compatibility

In this section we use the same notation and terminology as in [39, 40], where the authors found interesting relations between the sorting of the suffixes of a word $w$ and that of its factors. Here we prove a similar property when $\operatorname{ICFL}(w)$ is considered.

Let $w, x, u, y \in \Sigma^{*}$, and let $u$ be a nonempty factor of $w=x u y$. Let first(u) and last( $u$ ) denote the position of the first and the last symbol of $u$ in $w$, respectively. If $w=a_{1} \cdots a_{n}$, $a_{i} \in \Sigma, 1 \leq i \leq j \leq n$, then we also set $w[i, j]=a_{i} \cdots a_{j}$. A local suffix of $w$ is a suffix of a factor of $w$, specifically $s u f_{u}(i)=w[i, \operatorname{last}(u)]$ denotes the local suffix of $w$ at the position $i$ with respect to $u, i \geq f i r s t(u)$. The corresponding global suffix suff $(i) y$ of $w$ at the position $i$ is denoted by $s u f_{w}(i)=w[i, \operatorname{last}(w)]$ (or simply $\operatorname{suf}(i)$ when it is understood). We say that $s u f_{u}(i) y$ is associated with $s u f_{u}(i)$.

Definition 8.1 [39, 40] Let $w \in \Sigma^{+}$and let $u$ be a nonempty factor of $w$. We say that the sorting of the nonempty local suffixes of $w$ with respect to $u$ is compatible with the sorting of the corresponding nonempty global suffixes of $w$ if for all $i, j$ with $\operatorname{first}(u) \leq i<j \leq \operatorname{last}(u)$,

$$
\operatorname{suf}_{u}(i) \prec s u f_{u}(j) \Longleftrightarrow \operatorname{suf}(i) \prec \operatorname{suf}(j) .
$$

The following result has been proved in [39, 40].
Theorem 8.1 Let $w \in \Sigma^{+}$and let $\operatorname{CFL}(w)=\left(\ell_{1}, \ldots, \ell_{h}\right)$ be its Lyndon factorization. Then, for any $i, g, 1 \leq i \leq g \leq h$, the sorting of the nonempty local suffixes of $w$ with respect to $u=\ell_{i} \cdots \ell_{g}$ is compatible with the sorting of the corresponding nonempty global suffixes of $w$.

In [6] the same compatibility property as in Theorem 8.1 has been proved for the sorting of the nonempty suffixes of a word $w$ with respect to $\prec_{i n}$, when we replace CFL $(w)$ with $\operatorname{ICFL}(w)$.

Proposition 8.1 Let $w$ be a word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Then, for any $i, h, 1 \leq$ $i \leq h \leq k$, the sorting with respect to $\prec_{\text {in }}$ of the nonempty local suffixes of $w$ with respect to $u=m_{i} \cdots m_{h}$ is compatible with the sorting with respect to $\prec_{\text {in }}$ of the corresponding nonempty global suffixes of $w$.

The following result proves another compatibility property for the sorting of the nonempty suffixes of a word $w$ with respect to $\prec$, when we replace $\operatorname{CFL}(w)$ with $\operatorname{ICFL}(w)$.

Proposition 8.2 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $u=m_{i} m_{i+1} \cdots m_{h}$ with $1 \leq i<h \leq k$. Assume that suf $f_{u}\left(j_{1}\right) \prec s u f_{u}\left(j_{2}\right)$, where $\operatorname{first}(u) \leq j_{1} \leq \operatorname{last}(u)$, $\operatorname{first}(u) \leq j_{2} \leq \operatorname{last}(u), j_{1} \neq j_{2}$.

If $\operatorname{suf}_{u}\left(j_{1}\right)$ is a proper prefix of $\operatorname{suf} f_{u}\left(j_{2}\right)$ and $h<k$ then $\operatorname{suf}\left(j_{2}\right) \prec \operatorname{suf}\left(j_{1}\right)$, otherwise $\operatorname{suf}\left(j_{1}\right) \prec \operatorname{suf}\left(j_{2}\right)$.

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $u=m_{i} m_{i+1} \cdots m_{h}$ with $1 \leq i<h \leq k$. Assume that $s u f_{u}\left(j_{1}\right) \prec s u f_{u}\left(j_{2}\right)$, where $\operatorname{first}(u) \leq$ $j_{1} \leq \operatorname{last}(u), \operatorname{first}(u) \leq j_{2} \leq \operatorname{last}(u)$.

If $h=k$, then $\operatorname{suf}\left(j_{1}\right)=s u f_{u}\left(j_{1}\right) \prec s u f_{u}\left(j_{2}\right)=\operatorname{suf}\left(j_{2}\right)$ and we are done. Thus assume $h<k$. If $s u f_{u}\left(j_{1}\right)$ is not a proper prefix of $s u f_{u}\left(j_{2}\right)$, then $s u f_{u}\left(j_{1}\right) \ll s u f_{u}\left(j_{2}\right)$. Hence, by item (2) in Lemma 2.1, we have $\operatorname{suf}\left(j_{1}\right) \ll \operatorname{suf}\left(j_{2}\right)$ and we are done again.

Therefore, assume that $\operatorname{su} f_{u}\left(j_{1}\right)$ is a proper prefix of $s u f_{u}\left(j_{2}\right)$. Thus $j_{2}<j_{1}$ because $\left|s u f_{u}\left(j_{1}\right)\right|<\left|s u f_{u}\left(j_{1}\right)\right|$. Set $x=\operatorname{suf} f_{u}\left(j_{1}\right)=w\left[j_{1}, \operatorname{last}\left(m_{h}\right)\right]$ and $y=w\left[j_{2}, j_{2}+|x|-1\right]$. We have $x=y$ because $x, y$ are prefixes of $s u f_{u}\left(j_{2}\right)$ with the same length. Let $g$ be the minimum integer such that $j_{2}+|x| \leq \operatorname{last}\left(m_{g}\right), g \leq h<k$, and let $z=w\left[j_{2}+|x|\right.$, $\left.\operatorname{last}\left(m_{g}\right)\right]$. Therefore,

$$
\operatorname{suf}\left(j_{2}\right)=x z m_{g+1} \cdots m_{k}, \quad \operatorname{suf}\left(j_{1}\right)=x m_{h+1} \cdots m_{k}
$$

and we distinguish two cases:
(1) $z=1$
(2) $z \neq 1$
(Case (1)) If $z=1$, then $g<h$ because $j_{2} \neq j_{1}$ and thus $\operatorname{suf}\left(j_{2}\right) \neq \operatorname{suf}\left(j_{1}\right)$. Therefore

$$
\operatorname{suf}\left(j_{2}\right)=x m_{g+1} \cdots m_{k} \ll x m_{h+1} \cdots m_{k}=\operatorname{suf}\left(j_{1}\right)
$$

(Case (2)) Assume $z \neq 1$. If $z=m_{g}$, we apply the above argument again and we obtain

$$
\operatorname{suf}\left(j_{2}\right)=x m_{g} m_{g+1} \cdots m_{k} \ll x m_{h+1} \cdots m_{k}=\operatorname{suf}\left(j_{1}\right)
$$

Thus assume that $z$ is a nonempty proper suffix of $m_{g}$. Hence $z \prec m_{g}$ and we have one of the following two cases.
(2a) $z \ll m_{g}$
(2b) $z<_{p} m_{g}$
(Case (2a)) If $z \ll m_{g}$, then we have

$$
\operatorname{suf}\left(j_{2}\right)=x z m_{g+1} \cdots m_{k} \ll x m_{g} m_{g+1} \cdots m_{k} \ll x m_{h+1} \cdots m_{k}=\operatorname{suf}\left(j_{1}\right)
$$

(Case (2b)) Let $r, s \in \Sigma^{*}, a, b \in \Sigma$ be such that $m_{g}=r a s, m_{g+1}=r b t, a<b$. Assume that $z<_{p} m_{g}$. Since $z$ is also a nonempty proper suffix of $m_{g}$, we have that $z$ is a border of $m_{g}$. Then, by Proposition 7.5, $z$ cannot be a prefix of $m_{g+1}$, hence there is a prefix $s^{\prime}$ of $s$ such that $z=$ ras $^{\prime}$. Therefore we have

$$
\operatorname{suf}\left(j_{2}\right)=x z m_{g+1} \cdots m_{k} \ll x m_{g+1} \cdots m_{k} \preceq x m_{h+1} \cdots m_{k}=\operatorname{suf}\left(j_{1}\right)
$$

and the proof is complete.
Example 8.1 Let $w=a^{12} b b a b \in\{a, b\}^{+}$with $a<b$. We have $\operatorname{ICFL}(w)=\left(m_{1}, m_{2}\right)=$ $\left(a^{12}, b b a b\right)$. Let $u=m_{1}=a^{12}$. Consider $s u f_{u}(4)=a^{9}$ and $s u f_{u}(12)=a$. We have $s u f_{u}(12)=$ $a \prec a^{9}=s u f_{u}(4)$. We are in the first case of Lemma 8.2 and then $\operatorname{suf}(4)=a^{9} b b a b \prec a b b a b=$ suf(12).

Example 8.2 Let $w=$ dabadabdabdadac $\in\{a, b, c, d\}^{+}$with $a<b<c<d$. We have $\operatorname{ICFL}(w)=\left(m_{1}, m_{2}, m_{3}\right)=(d a b a, d a b d a b, d a d a c)$. Let $u=m_{2}$. Consider $s u f_{m_{2}}(8)=d a b$ and $s u f_{m_{2}}(5)=d a b d a b$. We have $s u f_{m_{2}}(8)=d a b \prec s u f_{m_{2}}(5)=d a b d a b=(d a b)^{2}$. We are in the first case of Lemma 8.2 and then $s u f(5)=d a b d a b d a d a c \prec s u f(8)=d a b d a d c$.

Consider now $s u f_{m_{2}}(9)=a b \prec s u f_{m_{2}}(8)=d a b$. Since $s u f_{m_{2}}(9)$ is not a proper prefix of $\left.s u f_{m_{2}}(8)\right)$, we are in the second case of Lemma 8.2 and we have $\operatorname{suf}(9)=a b d a d a c \prec s u f(8)=$ dabdadac.

## 9 Sorting Suffixes via ICFL

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. The aim of this section is to define an integer related to $\operatorname{ICFL}(w)$ and then to prove that it is an upper bound to the lengths $\mathrm{LCP}(x, y)$ of the longest common prefix $\operatorname{lcp}(x, y)$ of two factors $x, y$ of $w$. Some preliminary results are needed and proved below.

### 9.1 Technical Results

Lemma 9.1 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Then $m_{i} \notin \operatorname{Fact}\left(m_{1} \cdots m_{i-1}\right)$, for each $1<i \leq k$.

Proof:
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Suppose the lemma were false. Then there would be $i, 1<i \leq k$, such that $m_{i} \in \operatorname{Fact}\left(m_{1} \cdots m_{i-1}\right)$. Thus one of the following three cases holds.
(1) There are an integer $j, 1 \leq j<i$, and $x, y \in \Sigma^{*}$ such that $m_{j}=x m_{i} y$
(2) There is an integer $j, 1 \leq j<i$, such that $m_{j}$ is a prefix of $m_{i}$.
(3) There are integers $j, h, 1 \leq j<i, h \geq 0$, a proper nonempty suffix $x$ of $m_{j}$, and a proper prefix $y$ of $m_{j+h+1}$ such that $m_{i}=x m_{j+1} \cdots m_{j+h} y$, where it is understood that $m_{j+1} \cdots m_{j+h}=1$ for $h=0$.

Assume that case (1) holds. If $x=1$, then $m_{i} \preceq m_{j} \ll m_{i}$ which contradicts Lemma 2.2, Otherwise, $m_{i} y$ is a proper suffix of $m_{j}$, hence $m_{i} y \preceq m_{j} \ll m_{i}$. Therefore $m_{i} y \preceq m_{j} \ll m_{i} y$ (Lemma 2.1) which is impossible, once again by Lemma 2.2. Case (2) leads also to a contradiction since in this case we would have $m_{j}<_{p} m_{i}$ whereas $m_{j} \ll m_{i}$.

Assume that case (3) holds. We know that $x \preceq m_{j}$. If $x \ll m_{j}$, then $m_{i} \ll m_{j}$ (Lemma [2.1) which is impossible since $m_{j} \ll m_{i}$ and then $m_{j} \ll m_{i} \ll m_{j}$, in contradiction with Lemma 2.2, Thus $x$ is a proper prefix, thus a border of $m_{j}$. By Proposition 7.5, $x$ is not a prefix of $m_{j+1}$. Thus $j+1<i$ and there are $r, s, t \in \Sigma^{*}, a, b \in \Sigma$ be such that $m_{j+1}=r a s, m_{i}=r b t, a<b$. The words $x, r$ are comparable for the prefix order and $x$ is not a prefix of $m_{j+1}$. Therefore there is $t^{\prime} \in \Sigma^{*}$ such that $x=r b t^{\prime}$. Consequently, $m_{j+1} \ll x$, hence $m_{j+1} \ll m_{j}$ (Lemma 2.1). Since $m_{j} \ll m_{j+1}$, we would have $m_{j+1} \ll m_{j} \ll m_{j+1}$, once again in contradiction with Lemma 2.2

Lemma 9.2 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i, h, j$ be integers such that $1 \leq i<h<j \leq k$. Let $r_{i}, s_{i}, t_{i}, r_{h}, s_{h}, t_{h} \in \Sigma^{*}$, $a_{i}, b_{i}, a_{h}, b_{h} \in \Sigma$ be such that $m_{i}=r_{i} a_{i} s_{i}, m_{h}=r_{h} a_{h} s_{h}, m_{j}=r_{i} b_{i} t_{i}=r_{h} b_{h} t_{h}, a_{i}<b_{i}, a_{h}<b_{h}$. Then, the word $r_{i}$ is a prefix of $r_{h}$.

## Proof:

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i, h, j$ be integers such that $1 \leq i<h<j \leq k$. Let $r_{i}, s_{i}, t_{i}, r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{i}, b_{i}, a_{h}, b_{h} \in \Sigma$ be such that $m_{i}=r_{i} a_{i} s_{i}, m_{h}=r_{h} a_{h} s_{h}, m_{j}=r_{i} b_{i} t_{i}=r_{h} b_{h} t_{h}, a_{i}<b_{i}, a_{h}<b_{h}$. The words $r_{i}$ and $r_{h}$ are comparable for the prefix order. If $r_{h}$ would be a proper prefix of $r_{i}$, then $r_{h} b_{h}$ were a prefix of $r_{i}$. Thus there would be $u \in \Sigma^{*}$ such that $r_{i}=r_{h} b_{h} u$, and consequently $m_{h}=r_{h} a_{h} s_{h} \ll r_{h} b_{h} u a_{i} s_{i}=m_{i} \ll m_{h}$, which is impossible (Lemma [2.2). Thus $r_{i}$ is a prefix of $r_{h}$.

Corollary 9.1 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i, h, j$ be integers such that $1 \leq i<h<j \leq k$. Let $r, s, t \in \Sigma^{*}$, be such that $m_{i}=r s$ and $m_{j}=r$. Then, the word $r$ is a prefix of $m_{h}$.

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i, h, j$ be integers such that $1 \leq i<h<j \leq k$. Let $r, s, t \in \Sigma^{*}$, be such that $m_{i}=$ $r s$ and $m_{j}=r t$. Let $r_{i}, s_{i}, t_{i}, r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{i}, b_{i}, a_{h}, b_{h} \in \Sigma$ be such that $m_{i}=r_{i} a_{i} s_{i}$, $m_{h}=r_{h} a_{h} s_{h}, m_{j}=r_{i} b_{i} t_{i}=r_{h} b_{h} t_{h}, a_{i}<b_{i}, a_{h}<b_{h}$. Of course $r$ is a prefix of $r_{i}$, because $r \in \operatorname{Pref}\left(m_{i}\right) \cap \operatorname{Pref}\left(m_{j}\right)$. Thus $r$ is a prefix of $r_{h}$, by Lemma 9.2, hence $r \in \operatorname{Pref}\left(m_{h}\right)$.

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be such that $m_{h}=r_{h} a_{h} s_{h}$, $m_{i}=r_{h} b_{h} t_{h}, a_{h}<b_{h}, 1<h \leq i-1$. The following strengthening of Lemma 0.1 is proved below: $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{1} \cdots m_{i-1}\right)$ (Lemma 9.5). We have divided the proof of this result into a sequence of lemmas. We first prove that $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{h}\right), 1 \leq h \leq i-1$ (Lemma 9.3). Then, we prove that $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{h} m_{h+1}\right), 1 \leq h<i-1$ (Lemma 9.4). Finally, we prove Lemma 9.5

Lemma 9.3 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be such that $m_{h}=r_{h} a_{h} s_{h}, m_{i}=r_{h} b_{h} t_{h}, a_{h}<b_{h}, 1 \leq h \leq i-1$. Then, for each $h$, with $1 \leq h \leq i-1$, we have $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{h}\right)$.

## Proof :

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be as in the statement.

Suppose, contrary to our claim, that there exists $h$, with $1 \leq h \leq i-1$, such that $r_{i-1} b_{i-1} \in$ Fact $\left(m_{h}\right)$. Therefore, there are $u, v \in \Sigma^{*}$ such that $m_{h}=u r_{i-1} b_{i-1} v$. If $r_{i-1} b_{i-1}$ were a prefix of $m_{h}$, then necessarily $h<i-1$, because $m_{i-1}$ starts with $r_{i-1} a_{i-1}$. Thus, by $r_{i-1} a_{i-1} \ll r_{i-1} b_{i-1}$ we would have $m_{i-1} \ll m_{h}$, with $h<i-1$, which is impossible. Hence, $r_{i-1} b_{i-1} v$ is a proper nonempty suffix of $m_{h}$. Since $r_{i-1} b_{i-1} v$ is a proper nonempty suffix of $m_{h}$ and $r_{i-1} b_{i-1} v \notin$ $\operatorname{Pref}\left(m_{h}\right)$, we have $r_{i-1} b_{i-1} v \ll m_{h}$. By definition, there are $r, s, t \in \Sigma^{*}, a, b \in \Sigma$, such that $r_{i-1} b_{i-1} v=r a s, m_{h}=r b t, a<b$. The words $r_{i-1}$ and $r$ are comparable for the prefix order. Moreover, $r_{i-1}$ cannot be a proper prefix of $r$ because $r_{i-1} b_{i-1} \notin \operatorname{Pref}\left(m_{h}\right)$. Hence, $r$ is a prefix of $r_{i-1}$, thus $r a$ is a prefix of $r_{i-1} b_{i-1}$. As a consequence we have $r_{i-1} b_{i-1} \ll m_{h}$ which yields $m_{i} \ll m_{h}$, with $h<i$, once again a contradiction.

Lemma 9.4 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be such that $m_{h}=r_{h} a_{h} s_{h}, m_{i}=r_{h} b_{h} t_{h}, a_{h}<b_{h}, 1 \leq h \leq i-1$. Then, for each $h$, with $1 \leq h<i-1$, we have $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{h} m_{h+1}\right)$.

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be such that $m_{h}=r_{h} a_{h} s_{h}$, $m_{i}=r_{h} b_{h} t_{h}, a_{h}<b_{h}, 1 \leq h \leq i-1$.

Suppose the lemma were false. Then we could find $h$, with $1 \leq h<i-1$, such that $r_{i-1} b_{i-1} \in \operatorname{Fact}\left(m_{h} m_{h+1}\right)$. Therefore, there are $u, v \in \Sigma^{*}$ such that $u r_{i-1} b_{i-1} v=m_{h} m_{h+1}$. The words $u$ and $m_{h}$ (resp. $v$ and $m_{h+1}$ ) are comparable for the prefix (resp. suffix) order. Moreover, by Lemma 9.3, $m_{h}$ (resp. $m_{h+1}$ ) is not a prefix (resp. suffix) of $u$ (resp. $v$ ). Consequently there are $r, s \in \Sigma^{+}$such that

$$
\begin{equation*}
m_{h}=u r, \quad r_{i-1} b_{i-1}=r s, \quad m_{h+1}=s v \tag{9.1}
\end{equation*}
$$

In addition, $u \neq 1$, otherwise $m_{h} \in \operatorname{Pref}\left(m_{i}\right)$, in contradiction with $m_{h} \ll m_{i}$. Therefore, $r$ is a proper nonempty suffix of $m_{h}$. Moreover, notice that $r$ is a prefix of $m_{i}$. We claim that $r \notin \operatorname{Pref}\left(m_{h}\right)$. Indeed, if $r$ were a prefix of $m_{h}$, it would be a nonempty border of $m_{h}$. Thus, on one hand $r \notin \operatorname{Pref}\left(m_{h+1}\right)$ by Proposition 7.5. On the other hand, $r$ would be a prefix both of $m_{h}$ and $m_{i}$, hence $r \in \operatorname{Pref}\left(m_{h+1}\right)$ by Corollary 9.1, a contradiction.

Since $r$ is a proper nonempty suffix of $m_{h}$ and $r \notin \operatorname{Pref}\left(m_{h}\right)$, we have $r \ll m_{h}$ which yields $m_{i} \ll m_{h}$, because $r \in \operatorname{Pref}\left(m_{i}\right)$. This is impossible since $m_{h} \ll m_{i}$.

Lemma 9.5 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be such that $m_{h}=r_{h} a_{h} s_{h}, m_{i}=r_{h} b_{h} t_{h}, a_{h}<b_{h}, 1 \leq h \leq i-1$. Then, we have $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{1} \cdots m_{i-1}\right)$.

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i$ be an integer such that $1<i \leq k$. Let $r_{h}, s_{h}, t_{h} \in \Sigma^{*}, a_{h}, b_{h} \in \Sigma$ be such that $m_{h}=r_{h} a_{h} s_{h}$, $m_{i}=r_{h} b_{h} t_{h}, a_{h}<b_{h}, 1 \leq h \leq i-1$.

By contradiction, suppose that $r_{i-1} b_{i-1} \in \operatorname{Fact}\left(m_{1} \cdots m_{i-1}\right)$. Thus there are $z, z^{\prime} \in \Sigma^{*}$ such that $z r_{i-1} b_{i-1} z^{\prime}=m_{1} \cdots m_{i-1}$. By Lemma 9.4, for each $h$, with $1 \leq h<i-1$, we have $r_{i-1} b_{i-1} \notin \operatorname{Fact}\left(m_{h} m_{h+1}\right)$. Therefore, there is $h, 1 \leq h \leq i-1$, such that $m_{h} \in \operatorname{Fact}\left(r_{i-1} b_{i-1}\right)$.

Take $h$ minimal with respect to this condition. Then, there would be $u, v \in \Sigma^{*}$ such that $u m_{h} v=r_{i-1} b_{i-1}$ which implies $z u m_{h} v z^{\prime}=m_{1} \cdots m_{i-1}$. We have $u \neq 1$, otherwise $m_{h} \in$ $\operatorname{Pref}\left(m_{i}\right)$, in contradiction with $m_{h} \ll m_{i}$. Thus $h>1$. The words $m_{h-1}$ and $u$ are comparable for the suffix order. In addition, $m_{h-1}$ is not a suffix of $u$ by the minimality of $h$. Hence $u$ would be a nonempty proper suffix of $m_{h-1}$. Moreover, $h<i-1$, since $m_{i-1}$ starts with $r_{i-1} a_{i-1}$. Notice that $u$ is a prefix of $m_{i}$.

We now use the same argument as in Lemma 9.4. We claim that $u \notin \operatorname{Pref}\left(m_{h-1}\right)$. Indeed, if $u$ were a prefix of $m_{h-1}$, then $u$ would be a nonempty border of $m_{h-1}$. Thus, on one hand $u \notin \operatorname{Pref}\left(m_{h}\right)$ by Proposition 7.5. On the other hand, $u$ would be a prefix both of $m_{h-1}$ and $m_{i}$, hence $u \in \operatorname{Pref}\left(m_{h}\right)$ by Corollary 9.1, a contradiction.

Since $u$ is a proper nonempty suffix of $m_{h-1}$ and $u \notin \operatorname{Pref}\left(m_{h-1}\right)$, we have $u \ll m_{h-1}$ which yields $m_{i} \ll m_{h-1}$, because $u \in \operatorname{Pref}\left(m_{i}\right)$. This is impossible since $m_{h-1} \ll m_{i}$.

### 9.2 The Main Result

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. For any suffix $x$ of $m_{i}, 1 \leq i \leq k$, we set $x_{w}=x m_{i+1} \cdots m_{k}$. In this section we compare a pair of suffixes $x, y$ of factors in $\operatorname{ICFL}(w)$ and the corresponding pair of suffixes $x_{w}, y_{w}$ of $w$, with respect to lcp. First we handle suffixes of the same factor $m_{i}$ (Lemma 9.61), then we focus on suffixes of two different factors $m_{i}, m_{j}$ (Lemma 9.8).

Lemma 9.6 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $r, s, t \in \Sigma^{*}, a, b \in \Sigma$ be such that $m_{i-1}=r a s, m_{i}=r b t, a<b, 1<i \leq k$. If $x, y$ are different nonempty suffixes of $m_{i-1}$, then $\operatorname{lcp}\left(x_{w}, y_{w}\right)=\operatorname{lcp}(x r, y r)$.

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $r, s, t \in \Sigma^{*}, a, b \in \Sigma$ be such that $m_{i-1}=r a s, m_{i}=r b t, a<b, 1<i \leq k$. Let $x, y$ be different nonempty suffixes of $m_{i-1}$. Set $z=\operatorname{lcp}\left(x_{w}, y_{w}\right)$. If $|z| \leq \min \{|x r|,|y r|\}$, then clearly $\operatorname{lcp}\left(x_{w}, y_{w}\right)=\operatorname{lcp}(x r, y r)$.

Assume $|z|>\min \{|x r|,|y r|\}$. Thus, the words $x r, y r$ are comparable for the prefix order. Let $u \in \Sigma^{+}$be such that $y r=x r u$ (a symmetric argument applies if $y r$ is a proper prefix of $x r$ ). Thus $|x|<|y|$. Since $z$ is a prefix of $\operatorname{xrbtm}_{i+1} \cdots m_{k}$ and $|z|>\min \{|x r|,|y r|\}=|x r|$, there is $v \in \Sigma^{*}$ such that $z=x r b v$. Therefore the words $x r b$ and $y$ are comparable for the prefix order, because they are both prefixes of the same word $y_{w}$. Hence there is $v_{1} \in \Sigma^{*}$ such that one of the following two cases holds.

$$
\begin{align*}
y & =x r b v_{1}  \tag{9.2}\\
x r b & =y v_{1}, \quad v_{1} \neq 1 \tag{9.3}
\end{align*}
$$

Both cases lead to a contradiction. If Eq. (9.2) holds, then $r b v_{1}$ is a suffix of $m_{i-1}=r a s$ and $m_{i-1} \ll r b v_{1}$, which is impossible. If Eq. (9.3) holds, since $|x|<|y|$ and $v_{1} \neq 1$, we have $y=x r^{\prime}$, where $r^{\prime}$ is a nonempty prefix of $r$. Thus $r^{\prime}$ is a nonempty border of $m_{i-1}$ and $r^{\prime}$ is a prefix of $m_{i}$, in contradiction with Proposition 7.5.

Lemma 9.7 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i, j$ be integers such that $1 \leq i<j \leq k$. If $x$ is a nonempty suffix of $m_{i-1}$ and $y$ is a nonempty suffix of $m_{j-1}$ such that $x$ is a proper prefix of $y$, then $\operatorname{lcp}\left(x_{w}, y_{w}\right)$ is a prefix of $y m_{j}$.

## Proof :

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i, j$ be integers such that $1 \leq i<j \leq k$. Let $x$ be a nonempty suffix of $m_{i-1}$ and let $y$ be a nonempty suffix of $m_{j-1}$ such that $x$ is a proper prefix of $y$. Let $r_{j-1}, s_{j-1}, t_{j-1} \in \Sigma^{*}$, $a_{j-1}, b_{j-1} \in \Sigma$ be such that $m_{j-1}=r_{j-1} a_{j-1} s_{j-1}, m_{j}=r_{j-1} b_{j-1} t_{j-1}, a_{j-1}<b_{j-1}$.

Set $z=\operatorname{lcp}\left(x_{w}, y_{w}\right)$. Since $z$ and $y m_{j}$ are prefixes of the same word $y_{w}$, they are comparable for the prefix order. By contradiction, assume that $z$ is not a prefix of $y m_{j}$. Thus $y m_{j}$ is a proper prefix of $z$, hence of $x_{w}$. Since $y m_{j}$ and $x m_{i} \cdots m_{j-1} m_{j}$ are both prefixes of the same word $x_{w}$, they are comparable for the prefix order. Moreover $\left|x m_{i} \cdots m_{j-1} m_{j}\right|>\left|y m_{j}\right|$ because $y$ is a suffix of $m_{j-1}$ and $x$ is nonempty. Hence there exists $v_{j} \in \Sigma^{+}$such that

$$
\begin{equation*}
y m_{j} v_{j}=y r_{j-1} b_{j-1} t_{j-1} v_{j}=x m_{i} \cdots m_{j-1} m_{j} \tag{9.4}
\end{equation*}
$$

Since $x$ is a proper prefix of $y$, there is $x^{\prime} \in \Sigma^{+}$such that $y=x x^{\prime}$. Therefore, by Eq. (9.4) we have

$$
\begin{equation*}
x^{\prime} r_{j-1} b_{j-1} t_{j-1} v_{j}=m_{i} \cdots m_{j-1} m_{j} \tag{9.5}
\end{equation*}
$$

If $\left|m_{j}\right| \leq\left|t_{j-1} v_{j}\right|$, then by Eq. (9.5) we have $r_{j-1} b_{j-1} \in \operatorname{Fact}\left(m_{i} \cdots m_{j-1}\right)$, in contradiction with Lemma 9.5. Hence $\left|b_{j-1} t_{j-1} v_{j}\right| \leq\left|m_{j}\right|<\left|r_{j-1} b_{j-1} t_{j-1} v_{j}\right|$. Thus, by Eq. (9.4), there are $r_{j-1}^{\prime} \in \Sigma^{+}, r_{j-1}^{\prime \prime} \in \Sigma^{*}$ such that $r_{j-1}=r_{j-1}^{\prime} r_{j-1}^{\prime \prime}$ and

$$
\begin{equation*}
y r_{j-1}^{\prime}=x m_{i} \cdots m_{j-1} \tag{9.6}
\end{equation*}
$$

The word $r_{j-1}^{\prime}$ is a proper prefix of $m_{j-1}$, thus, by Eq. (9.6), $r_{j-1}^{\prime}$ is a nonempty border of $m_{j-1}$. Since $r_{j-1}^{\prime}$ is a prefix of $m_{j}$, this is in contradiction with Proposition 7.5.

Lemma 9.8 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i, j$ be integers such that $1<i<j \leq k$. If $x$ is a nonempty suffix of $m_{i-1}$ and $y$ is a nonempty suffix of $m_{j-1}$, then $\operatorname{lcp}\left(x_{w}, y_{w}\right)$ is a prefix of $y m_{j}$.

## Proof:

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word. Let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $r_{i}, s_{i}, t_{i} \in \Sigma^{*}, a_{i}, b_{i} \in \Sigma$ be such that $m_{i}=r_{i} a_{i} s_{i}, m_{j}=r_{i} b_{i} t_{i}, a_{i}<b_{i}, 1<i<j \leq k$. Let $r_{i-1}, s_{i-1}, t_{i-1} \in \Sigma^{*}, a_{i-1}, b_{i-1} \in \Sigma$ be such that $m_{i-1}=r_{i-1} a_{i-1} s_{i-1}, m_{j}=r_{i-1} b_{i-1} t_{i-1}$, $a_{i-1}<b_{i-1}$. By Lemma 9.2, $r_{i-1}$ is a prefix of $r_{i}$.

Let $x$ be a nonempty suffix of $m_{i-1}$ and let $y$ be a nonempty suffix of $m_{j-1}$. If $x$ is a proper prefix of $y$, then by Lemma 9.7 we are done. Thus assume that $x$ is not a prefix of $y$. Set $z=\operatorname{lcp}\left(x_{w}, y_{w}\right)$. If $|z| \leq\left|y r_{i-1}\right|$, then $z$ is a prefix of $y m_{j} \cdots m_{k}$ shorter than $y r_{i-1} b_{i-1} t_{i-1}=y m_{j}$, thus $z$ is a prefix of $y m_{j}$. Assume $|z|>\left|y r_{i-1}\right|$.

Since $z$ is a prefix of $y m_{j} \cdots m_{k}=y r_{i-1} b_{i-1} t_{i-1} m_{j+1} \cdots m_{k}$ and $|z|>\left|y r_{i-1}\right|$, there is $v \in \Sigma^{*}$ such that $z=y r_{i-1} b_{i-1} v$. Therefore the words $y r_{i-1} b_{i-1}$ and $x$ are comparable for the prefix order, because they are both prefixes of the same word $x_{w}$. Hence there is $v_{1} \in \Sigma^{*}$ such that one of the following two cases holds.

$$
\begin{align*}
x & =y r_{i-1} b_{i-1} v_{1}  \tag{9.7}\\
y r_{i-1} b_{i-1} & =x v_{1}, \quad v_{1} \neq 1 \tag{9.8}
\end{align*}
$$

Both cases lead to a contradiction. If Eq. (9.7) holds, then $r_{i-1} b_{i-1} v_{1}$ is a suffix of $m_{i-1}=$ $r_{i-1} a_{i-1} s_{i-1}$ and $m_{i-1} \ll r_{i-1} b_{i-1} v_{1}$, which is impossible. Assume that Eq. (9.8) holds. The
words $x$ and $y$ are comparable for the prefix order and $x$ is not a prefix of $y$. Therefore we have $x=y r_{i-1}^{\prime}$, where $r_{i-1}^{\prime}$ is a nonempty prefix of $r_{i-1}$. Thus $r_{i-1}^{\prime}$ is a nonempty proper prefix of both $m_{i-1}$ and $m_{i}$. Since $x=y r_{i-1}^{\prime}$, the word $r_{i-1}^{\prime}$ is a nonempty border of $m_{i-1}$ and $r_{i-1}^{\prime}$ is a prefix of $m_{i}$, in contradiction with Proposition 7.5,

Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. We set

$$
\mathcal{M}=\max \left\{\left|m_{i} m_{i+1}\right| \mid 1 \leq i<k\right\}
$$

Proposition 9.1 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let ICFL $(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. Let $i, j$ be integers such that $1<i<j \leq k$. If $x$ is a nonempty suffix of $m_{i-1}$ and $y$ is a nonempty suffix of $m_{j-1}$, then

$$
\operatorname{LCP}\left(x_{w}, y_{w}\right)=\left|\operatorname{lcp}\left(x_{w}, y_{w}\right)\right| \leq \mathcal{M}
$$

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $i, j$ be integers such that $1 \leq i<j \leq k$. Let $x$ be a nonempty suffix of $m_{i-1}$ and let $y$ be a nonempty suffix of $m_{j-1}$. By Lemma 9.8, $\operatorname{lcp}\left(x_{w}, y_{w}\right)$ is a prefix of $y m_{j}$, hence

$$
\operatorname{LCP}\left(x_{w}, y_{w}\right)=\left|\operatorname{lcp}\left(x_{w}, y_{w}\right)\right| \leq\left|y m_{j}\right| \leq\left|m_{j-1} m_{j}\right| \leq \mathcal{M}
$$

Proposition 9.2 Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=$ $\left(m_{1}, \ldots, m_{k}\right)$. For each nonempty proper factors $u, v$ of $w$, we have

$$
\operatorname{LCP}(u, v)=|\operatorname{lcp}(u, v)| \leq \mathcal{M}
$$

Proof :
Let $w \in \Sigma^{+}$be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w)=\left(m_{1}, \ldots, m_{k}\right)$. Let $u, v$ be nonempty proper factors of $w$. Let $u_{1}, u_{2}, v_{1}, v_{2} \in \Sigma^{*}$ be such that $w=u_{1} u u_{2}=v_{1} v v_{2}$. Let $x$ be a nonempty suffix of $m_{i-1}$ and let $y$ be a nonempty suffix of $m_{j-1}$ such that $u u_{2}=x_{w}$, $v v_{2}=y_{w}$, with $1<i \leq k, 1<j \leq k$. If $i=j$, then by Lemma 9.6 we have

$$
\operatorname{LCP}(u, v)=|\operatorname{lcp}(u, v)| \leq\left|\operatorname{lcp}\left(x_{w}, y_{w}\right)\right| \leq\left|m_{i-1} m_{i}\right| \leq \mathcal{M}
$$

If $i \neq j$, then by Proposition 9.1, we have

$$
\operatorname{LCP}(u, v)=|\operatorname{lcp}(u, v)| \leq\left|\operatorname{lcp}\left(x_{w}, y_{w}\right)\right| \leq \mathcal{M}
$$

## References

[1] A. Apostolico, M. Crochemore, Fast parallel Lyndon factorization with applications, Math. Systems Theory 28 (1995) 89-108.
[2] H. Bannai, I Tomohiro, S. Inenaga, Y. Nakashima, M. Takeda, K. Tsuruta, A new characterization of maximal repetitions by Lyndon trees, in: P. Indyk (Ed.), 26th ACM-SIAM Symposium on Discrete Algorithms (SODA), San Diego, CA, USA, 2015, pp. 562-571.
[3] H. Bannai, I Tomohiro, S. Inenaga, Y. Nakashima, M. Takeda, K. Tsuruta, The "Runs" Theorem, SIAM J. Comput. 46 (2017) 1501-1514.
[4] J. Berstel, D. Perrin, The origins of combinatorics on words, European J. Combin. 28 (2007) 996-1022.
[5] J. Berstel, D. Perrin, C. Reutenauer, Codes and Automata, Encyclopedia Math. Appl., vol. 129, Cambridge Univ. Press, Cambridge, 2009.
[6] P. Bonizzoni, C. De Felice, R. Zaccagnino, R. Zizza, Inverse Lyndon words and inverse Lyndon factorizations of words, Advances in Applied Mathematics, 101 (2018) 281319.
[7] S. Bonomo, S. Mantaci, A. Restivo, G. Rosone, M. Sciortino, Suffixes, conjugates and Lyndon words, in: M.-P. Béal, O. Carton (Eds.), DLT 2013, Lecture Notes in Comp. Sci., vol. 7907, Springer, Berlin, 2013, pp. 131-142.
[8] K. Cattell, F. Ruskey, J. Sawada, M. Serra, C. R. Miers, Fast algorithms to generate necklaces, unlabeled necklaces, and irreducible polynomials over GF(2), J. Algorithms 37 (2000) 267-282.
[9] E. Charlier, M. Philibert, M. Stipulanti, Nyldon words, https://arxiv.org/abs/1804.09735, 2018.
[10] K.-T. Chen, R. H. Fox, R. C. Lyndon, Free differential calculus. IV: The quotient groups of the lower central series, Ann. of Math. 68 (1958) 81-95.
[11] C. Choffrut, J. Karhumäki, Combinatorics on words, in: G. Rozenberg, A. Salomaa (Eds.), Handbook of Formal Languages, vol. 1, Springer Verlag, Berlin, 1997, pp. 329-438.
[12] T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein, Introduction to Algorithms, Third Edition, The MIT Press, Cambridge, Massachusetts, USA, 2009.
[13] M. Crochemore, J. Désarménien, D. Perrin, A note on the Burrows-Wheeler transformation, Theoret. Comput. Sci. 332 (2005) 567-572.
[14] M. Crochemore, C. Hancart, T. Lecroq, Algorithms on Strings, Cambridge Univ. Press, Cambridge, 2007.
[15] M. Crochemore, W. Rytter, Text Algorithms, Oxford University Press, Oxford, 1994.
[16] J. W. Daykin, C. S. Iliopoulos, W. F. Smyth, Parallel RAM algorithms for factorizing words, Theoret. Comput. Sci. 127 (1994) 53-67.
[17] J. W. Daykin, W. F. Smyth, A bijective variant of the Burrows-Wheeler Transform using V-order, Theoret. Comput. Sci. 531 (2014) 77-89.
[18] F. Dolce, A. Restivo, C. Reutenauer, On generalized Lyndon words, Theoret. Comput. Sci. 777 (2019) 232-242.
[19] F. Dolce, A. Restivo, C. Reutenauer, Some Variations on Lyndon words, 30th Annual Symposium on Combinatorial Pattern Matching (CPM 2019), June 18-20, 2019, Pisa, Italy.
[20] J.-P. Duval, Factorizing words over an ordered alphabet, J. Algorithms 4 (1983) 363381.
[21] H. Fredricksen, J. Maiorana, Necklaces of beads in $k$ colors and $k$-ary de Brujin sequences, Discrete Math. 23 (1978) 207-210.
[22] I. M. Gessel, A. Restivo, C. Reutenauer, A bijection between words and multisets of necklaces, European J. Combin. 33 (2012) 1537-1546.
[23] I. M. Gessel, C. Reutenauer, Counting permutations with given cycle structure and descent set, J. Comb. Theory Ser. A 64 (1993) 189-215.
[24] D. A. Gewurza, F. Merola, Numeration and enumeration, European J. Combin. 33 (2012) 1547-1556.
[25] S. S. Ghuman, E. Giaquinta, J. Tarhio, Alternative algorithms for Lyndon factorization, in: J. Holub, J. Zdárek (Eds.), Prague Stringology Conference (PSC) 2014, Czech Technical University in Prague, Czech Republic, 2014, pp. 169-178.
[26] J. Yossi Gil, D. A. Scott, A bijective string sorting transform, CoRR abs/1201.3077, 2012, https://arxiv.org/abs/1201.3077.
[27] D. Grinberg, "Nyldon words": understanding a class of words factorizing the free monoid increasingly, https://mathoverflow.net/questions/187451/, 2014.
[28] C. Hohlweg, C. Reutenauer, Lyndon words, permutations and trees, Theoret. Comput. Sci. 307 (2003) 173-178.
[29] J. Kärkkäinen, D. Kempa, Y. Nakashima, S. J. Puglisi, A. M. Shur, On the size of Lempel-Ziv and Lyndon factorizations, in: H. Vollmer, B. Vallée (Eds.), STACS 2017, Leibniz International Proceedings in Informatics (LIPIcs), vol. 66, Dagstuhl Publishing, Dagsthul, 2017, pp. 1-13.
[30] D. E. Knuth, The Art of Computer Programming, vol. 4A, Combinatorial Algorithms: Part I, Addison Wesley, 2012.
[31] D. E. Knuth, J. H. Morris Jr., V. R. Pratt, Fast pattern matching in strings, SIAM J. Computing 6 (1977) 323-350.
[32] M. Kufleitner, On bijective variants of the Burrows-Wheeler Transform, in: J. Holub, J. Zdárek (Eds.), Prague Stringology Conference (PSC) 2009, Czech Technical University in Prague, Czech Republic, 2009, pp. 65-79.
[33] H. Li, N. Homer, A survey of sequence alignment algorithms for next-generation sequencing, Briefings in Bioinformatics 11 (2010) 473-483.
[34] M. Lothaire, Combinatorics on Words, Cambridge Math. Lib., Cambridge Univ. Press, Cambridge, 1997.
[35] M. Lothaire, Algebraic Combinatorics on Words, Encyclopedia Math. Appl., vol. 90, Cambridge Univ. Press, Cambridge, 2002.
[36] M. Lothaire, Applied Combinatorics on Words, Encyclopedia Math. Appl., vol. 105, Cambridge Univ. Press, Cambridge, 2005.
[37] R. C. Lyndon, On Burnside problem I, Trans. Amer. Math. Soc. 77 (1954) 202-215.
[38] S. Mantaci, A. Restivo, G. Rosone, M. Sciortino, An extension of the Burrows-Wheeler Transform, Theoret. Comput. Sci. 387 (2007) 298-312.
[39] S. Mantaci, A. Restivo, G. Rosone, M. Sciortino, Sorting suffixes of a text via its Lyndon factorization, in: J. Holub, J. Zdárek (Eds.), Prague Stringology Conference (PSC) 2013, Czech Technical University in Prague, Czech Republic, 2013, pp. 119127.
[40] S. Mantaci, A. Restivo, G. Rosone, M. Sciortino, Suffix array and Lyndon factorization of a text, J. Discr. Algorithms 28 (2014) 2-8.
[41] M. Mucha, Lyndon words and short superstrings, in: S. Khanna (Ed.), 24th ACMSIAM Symposium on Discrete Algorithms (SODA), San Diego, CA, USA, 2013, pp. 958-972.
$[42]$ D. Perrin, A. Restivo, Words, in: M. Bóna (Ed.), Handbook of Enumerative Combinatorics, Discrete Mathematics and Its Applications Series, Chapman and Hall/CRC, 2015, pp. 485-539.
[43] C. Reutenauer, Free Lie Algebras, London Mathematical Society Monographs, vol. 7, Oxford Science Publications, Oxford, 1993.
[44] C. Reutenauer, Mots de Lyndon qénéralisés, Sém. Lothar. Combin. 54 (2006) B54h.
[45] M.-P. Schützenberger, On a factorization of free monoids, Proc. Amer. Math. Soc. 16 (1965) 21-24.
[46] Y. Urabe, D. Kempa, Y. Nakashima, S. Inenaga, H. Bannai, M. Takeda, On the Size of Overlapping Lempel-Ziv and Lyndon Factorizations, 30th Annual Symposium on Combinatorial Pattern Matching (CPM 2019), June 18-20, 2019, Pisa, Italy.

