Lyndon words versus inverse Lyndon words: queries on suffixes and bordered words

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Abstract

Lyndon words have been largely investigated and showned to be a useful tool to prove interesting combinatorial properties of words. In this paper we state new properties of both Lyndon and inverse Lyndon factorizations of a word w, with the aim of exploring their use in some classical queries on w.

The main property we prove is related to a classical query on words. We prove that there are relations between the length of the longest common extension (or longest common prefix) lcp(x, y) of two different suffixes x, y of a word w and the maximum length \mathcal{M} of two consecutive factors of the inverse Lyndon factorization of w. More precisely, \mathcal{M} is an upper bound on the length of lcp(x, y). This result is in some sense stronger than the compatibility property, proved by Mantaci, Restivo, Rosone and Sciortino for the Lyndon factorization and here for the inverse Lyndon factorization. Roughly, the compatibility property allows us to extend the mutual order between local suffixes of (inverse) Lyndon factors to the suffixes of the whole word.

A main tool used in the proof of the above results is a property that we state for factors m_i with nonempty borders in an inverse Lyndon factorization: a nonempty border of m_i cannot be a prefix of the next factor m_{i+1} . The last property we prove shows that if two words share a common overlap, then their Lyndon factorizations can be used to capture the common overlap of the two words.

The above results open to the study of new applications of Lyndon words and inverse Lyndon words in the field of string comparison.

Keywords: Lyndon words, Lyndon factorization, Combinatorial algorithms on words.

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1 Introduction

The Lyndon factorization of a word w is a unique factorization of w into a sequence of Lyndon words in nonincreasing lexicographic ordering. This factorization is one of the most known factorizations and it has been extensively studied in different contexts, from formal languages to algorithmic stringology and string compression. In particular the notion of a Lyndon word has been shown to be useful in theoretical applications, such as the well known proof of the *Runs Theorem* [3] as well in string compression analysis. A connection between the Lyndon factorization and the Lempel-Ziv (LZ) factorization has been given in [29], where it is shown that in general the size of the LZ factorization is larger than the size of the Lyndon factorization, and in any case the size of the Lyndon factorization cannot be larger than a factor of 2 with respect to the size of LZ. This result has been further extended in [46] to overlapping LZ factorizations. The Lyndon factorization has recently revealed to be a useful tool also in investigating queries related to sorting suffixes of a word, with strong potentialities for string comparison that have not been completely explored and understood [40, 41]. Relations between Lyndon words and the Burrows-Wheeler Transform (BWT) have been discovered first in [13, 38] and more recently in [32]. A main property of the Lyndon factorization is that it can be efficiently computed. Linear-time algorithms for computing the Lyndon factorization can be found in [20, 21] whereas an $\mathcal{O}(\lg n)$ -time parallel algorithm has been proposed in [1, 16].

More recently Lyndon words found a renewed theoretical interest and several variants of them have been introduced and investigated with different motivations [9, 18, 19]. A related field studies the combinatorial and algorithmic properties of *necklaces*, that are powers of Lyndon words, and their prefixes or *prenecklaces* [8]. In [6], the notion of an inverse Lyndon word (a word which is strictly greater than each of its proper suffixes) has been introduced to define a new factorization, called the *inverse Lyndon factorization*. A word which is not an inverse Lyndon word may have different factorizations with inverse Lyndon words as factors but each word wadmits a unique canonical inverse Lyndon factorization, denoted ICFL(w). This factorization has the property that a word is factorized in a sequence of inverse Lyndon words, in an increasing and *prefix-order-free* lexicographic ordering, where prefix-order-free means that a factor cannot be a prefix of the consecutive one. Moreover ICFL(w) can be still computed in linear time and it is uniquely determined by w.

Differently from Lyndon words, inverse Lyndon words may be bordered. As a main result in this paper, we show that if a factor m_i in ICFL(w) has a nonempty border, then such a border cannot be inherited by the consecutive factor, since it cannot be the prefix of the consecutive factor m_{i+1} . In other words, the longest common prefix between m_i and m_{i+1} is shorter than the border of m_i . This result is proved by a further investigation on the connection between the Lyndon factorization and the canonical inverse Lyndon factorization of a word, given in [6] through the grouping property. Indeed, given a word w which is not an inverse Lyndon word, the factors in ICFL(w) are obtained by grouping together consecutive factors of the anti-Lyndon factorization of w that form a chain for the prefix order.

Thanks to the properties of ICFL(w), the longest common extensions (or longest common prefix) of two distinct factors in ICFL(w) appear to have different properties than in the Lyndon factorization. In this framework, a natural question is whether and how the longest common extensions of two factors of w are related to the size of the factors in ICFL(w). We prove that there are relations between the length of the longest common extension (or longest common prefix) lcp(x, y) of two different factors x, y of a word w and the maximum length \mathcal{M} of two consecutive factors of the inverse Lyndon factorization of w. More precisely, \mathcal{M} is an upper bound on the length of lcp(x, y). This result is in some sense stronger than the compatibility property, proved in [39, 40] for the Lyndon factorization and here for the inverse Lyndon factorization. Roughly, the compatibility property allows us to extend the mutual order between local suffixes of (inverse) Lyndon factors to the suffixes of the whole word. Another natural question is the following.

Given two words having a common overlap, can we use their Lyndon factorizations to capture the similarity of these words?

A partial positive answer to this question is provided here: given a word w and a factor x of w, we prove that their Lyndon factorizations share factors, except for the first and last term of the Lyndon factorization of x.

The paper is organized as follows. In Sections 2, 4, 5, 6, we gathered the basic definitions and known results we need. Relations between the Lyndon factorizations of two words that share a common overlap are proved in Section 3. Borders of inverse Lyndon words are discussed in Section 7. The compatibility property for ICFL(w) is proved in Section 8. Finally the upper bound on the length of the longest common prefix of two factors of w in terms of factors in ICFL(w) is stated in Section 9.

2 Preliminaries

For the material in this section see [5, 11, 35, 36, 43].

2.1 Words

Let Σ^* be the *free monoid* generated by a finite alphabet Σ and let $\Sigma^+ = \Sigma^* \setminus 1$, where 1 is the empty word. For a set X, $\operatorname{Card}(X)$ denotes the cardinality of X. For a word $w \in \Sigma^*$, we denote by |w| its *length*. A word $x \in \Sigma^*$ is a *factor* of $w \in \Sigma^*$ if there are $u_1, u_2 \in \Sigma^*$ such that $w = u_1 x u_2$. If $u_1 = 1$ (resp. $u_2 = 1$), then x is a *prefix* (resp. *suffix*) of w. A factor (resp. prefix, suffix) x of w is *proper* if $x \neq w$. Given a language $L \subseteq A^*$, we denote by $\operatorname{Pref}(L)$ (resp. $\operatorname{Suff}(L)$, $\operatorname{Fact}(L)$) the set of all prefixes (resp. suffixes, factors) of its elements. Two words x, yare *incomparable* for the prefix order, denoted as $x \bowtie y$, if neither x is a prefix of y nor y is a prefix of x. Otherwise, x, y are *comparable* for the prefix order. We write $x \leq_p y$ if x is a prefix of y and $x \geq_p y$ if y is a prefix of x. The notion of a pair of words comparable (or incomparable) for the suffix order is defined symmetrically.

We recall that two words x, y are called *conjugate* if there exist words u, v such that x = uv, y = vu. The conjugacy relation is an equivalence relation. A conjugacy class is a class of this equivalence relation. The following is Proposition 1.3.4 in [34].

Proposition 2.1 Two words $x, y \in \Sigma^+$ are conjugate if and only if there exists $r \in \Sigma^*$ such that

$$xr = ry \tag{2.1}$$

More precisely, equality (2.1) holds if and only if there exist $u, v \in \Sigma^*$ such that

$$x = uv, \quad y = vu, \quad r \in u(vu)^*.$$

$$(2.2)$$

A sesquipower of a word x is a word $w = x^n p$ where p is a proper prefix of x and $n \ge 1$. A nonempty word w is unbordered if no proper nonempty prefix of w is a suffix of w. Otherwise, w is bordered. A nonempty word w is primitive if $w = x^k$ implies k = 1. An unbordered word is primitive.

The following is a part of Proposition 1.3.2 in [34].

Proposition 2.2 Two words $u, v \in \Sigma^+$ commute if and only if they are powers of the same word.

2.2 Lexicographic order and Lyndon words

Definition 2.1 Let $(\Sigma, <)$ be a totally ordered alphabet. The lexicographic (or alphabetic order) \prec on $(\Sigma^*, <)$ is defined by setting $x \prec y$ if

- x is a proper prefix of y, or
- $x = ras, y = rbt, a < b, for a, b \in \Sigma and r, s, t \in \Sigma^*$.

In the next part of the paper we will implicitly refer to totally ordered alphabets. For two nonempty words x, y, we write $x \ll y$ if $x \prec y$ and x is not a proper prefix of y [2]. We also write $y \succ x$ if $x \prec y$. Basic properties of the lexicographic order are recalled below.

Lemma 2.1 For $x, y \in \Sigma^*$, the following properties hold.

(1) $x \prec y$ if and only if $zx \prec zy$, for every word z.

(2) If $x \ll y$, then $xu \ll yv$ for all words u, v.

(3) If $x \prec y \prec xz$ for a word z, then y = xy' for some word y' such that $y' \prec z$.

Lemma 2.2 Let $x, y \in \Sigma^*$. If $x \ll y$, then $y \not\prec x$.

Proof:

Suppose, contrary to our claim, that there would be $x, y \in \Sigma^*$ such that $y \prec x \ll y$. By definition there are $a, b \in \Sigma$ and $r, s, t \in \Sigma^*$ such that x = ras, y = rbt. Thus y cannot be a prefix of x, hence there are $a', b' \in \Sigma$ and $r', s', t' \in \Sigma^*$ such that y = r'a's', x = r'b't', a' < b'. By x = ras = r'b't' we have that the words ra, r'b' are comparable for the prefix order. If r'b' would be a prefix of r, then r'b' were a prefix of y = r'a's', which is impossible. Analogously, if ra would be a prefix of r', then ra were a prefix of y = rbt, once again a contradiction. Hence ra = r'b', which implies r = r', a = b', therefore a' = b > a = b' > a', a contradiction.

Definition 2.2 A Lyndon word $w \in \Sigma^+$ is a word which is primitive and the smallest one in its conjugacy class for the lexicographic order.

Example 2.1 Let $\Sigma = \{a, b\}$ with a < b. The words a, b, aaab, abbb, aabab and aababaabb are Lyndon words. On the contrary, abab, aba and abaab are not Lyndon words. Indeed, abab is a non-primitive word, $aab \prec aba$ and $aabab \prec abaab$.

Lyndon words are also called *prime words* and their prefixes are also called *preprime words* in [30]. Some properties of Lyndon words are recalled below.

Proposition 2.3 Each Lyndon word w is unbordered.

Proposition 2.4 A word $w \in \Sigma^+$ is a Lyndon word if and only if $w \prec s$, for each nonempty proper suffix s of w.

The following is Proposition 5.1.3 in [34] and gives a second characterization of Lyndon words.

Proposition 2.5 A word $w \in \Sigma^+$ is a Lyndon word if and only if $w \in \Sigma$ or $w = \ell m$ with ℓ, m Lyndon words, $\ell \prec m$.

Finally, in [18] the authors credited to folklore the following third characterization of Lyndon words: $w \in \Sigma^+$ is a Lyndon word if and only if for each nontrivial factorization w = uv one has $u \prec v$.

A class of conjugacy is also called a *necklace* and often identified with the minimal word for the lexicographic order in it. We will adopt this terminology. Then a word is a necklace if and only if it is a power of a Lyndon word. A *prenecklace* is a prefix of a necklace. Then clearly any nonempty prenecklace w has the form $w = (uv)^k u$, where uv is a Lyndon word, $u \in \Sigma^*$, $v \in \Sigma^+$, $k \ge 1$, that is, w is a sesquipower of a Lyndon word uv. The following result has been proved in [20]. **Proposition 2.6** A word is a nonempty preprime word if and only if it is a sesquipower of a Lyndon word distinct of the maximal letter.

The proof of Proposition 2.6 uses the following result which characterizes, for a given nonempty prenecklace w and a letter b, whether wb is still a prenecklace or not and, in the first case, whether wb is a Lyndon word or not [20, Lemma 1.6].

Theorem 2.1 Let $w = (uav')^k u$ be a nonempty prenecklace, where uav' is a Lyndon word, $u, v' \in \Sigma^*$, $a \in \Sigma$, $k \ge 1$. For any $b \in \Sigma$, the word wb is a prenecklace if and only if $b \ge a$. Moreover $wb \in L$ if and only if b > a.

A direct consequence of Theorem 2.1 is reported below (see [8, Theorem 2.1] which states both Theorem 2.1 and Corollary 2.1).

Corollary 2.1 Let $w = (uav')^k u$ be a nonempty prenecklace, where uav' is a Lyndon word, $u, v' \in \Sigma^*$, $a \in \Sigma$, $k \ge 1$. Let $b \in \Sigma$ with $b \ge a$ and let y be the longest prefix of wb which is a Lyndon word. Then

$$y = \begin{cases} uav' & \text{if } b = a \\ wb & \text{if } b > a \end{cases}$$

2.3 The Lyndon factorization

A family $(X_i)_{i \in I}$ of subsets of Σ^+ , indexed by a totally ordered set I, is a factorization of the free monoid Σ^* if each word $w \in \Sigma^*$ has a unique factorization $w = x_1 \cdots x_n$, with $n \ge 0$, $x_i \in X_{j_i}$ and $j_1 \ge j_2 \ge \ldots \ge j_n$ [5]. A factorization $(X_i)_{i \in I}$ is called *complete* if each X_i is reduced to a singleton x_i [5]. In the following $L = L_{(\Sigma^*, <)}$ will be the set of Lyndon words, totally ordered by the relation \prec on $(\Sigma^*, <)$. The following theorem shows that the family $(\ell)_{\ell \in L}$ is a complete factorization of Σ^* .

Theorem 2.2 Any word $w \in \Sigma^+$ can be written in a unique way as a nonincreasing product $w = \ell_1 \ell_2 \cdots \ell_h$ of Lyndon words, i.e., in the form

$$w = \ell_1 \ell_2 \cdots \ell_h, \text{ with } \ell_i \in L \text{ and } \ell_1 \succeq \ell_2 \succeq \ldots \succeq \ell_h$$

$$(2.3)$$

The sequence $CFL(w) = (\ell_1, \ldots, \ell_h)$ in Eq. (2.3) is called the Lyndon decomposition (or Lyndon factorization) of w. It is denoted by CFL(w) because Theorem 2.2 is usually credited to Chen, Fox and Lyndon [10]. Uniqueness of the above factorization is a consequence of the following result, proved in [20].

Lemma 2.3 Let $w \in \Sigma^+$ and let $CFL(w) = (\ell_1, \ldots, \ell_h)$. Then the following properties hold:

(i) ℓ_h is the nonempty suffix of w which is the smallest with respect to the lexicographic order.

- (ii) ℓ_h is the longest suffix of w which is a Lyndon word.
- (iii) ℓ_1 is the longest prefix of w which is a Lyndon word.

A direct consequence is stated below and it is necessary for our aims.

Corollary 2.2 Let $w \in \Sigma^+$, let ℓ_1 be its longest prefix which is a Lyndon word and let w' be such that $w = \ell_1 w'$. If $w' \neq 1$, then $CFL(w) = (\ell_1, CFL(w'))$.

As a consequence of Theorem 2.2, for any word w there is a factorization

$$w = \ell_1^{n_1} \cdots \ell_r^{n_r}$$

where $r > 0, n_1, \ldots, n_r \ge 1$, and $\ell_1 \succ \ldots \succ \ell_r$ are Lyndon words, also named Lyndon factors of w. In the next, $\operatorname{CFL}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_r})$ will be an alternative notation for the Lyndon factorization of w. There is a linear time algorithm to compute the pair (ℓ_1, n_1) and thus, by iteration, the Lyndon factorization of w. It is due to Fredricksen and Maiorana [21] and it is also reported in [36]. It can also be used to compute the Lyndon word in the conjugacy class of a primitive word in linear time [36]. Linear time algorithms may also be found in [20] and in the more recent paper [25].

3 Lyndon factorizations of factors of a word

Let $w \in \Sigma^+$ be a word and let $CFL(w) = (\ell_1, \ldots, \ell_k)$ be its Lyndon factorization, $k \ge 1$. Let x be a proper factor (resp. prefix, suffix) of w. We say that x is a *simple* factor of w if, for each occurrence of x as a factor of w, there is j, with $1 \le j \le k$, such that x is a factor of ℓ_j . We say that x is a *simple* prefix (resp. suffix) of w if x is a proper prefix (resp. suffix) of ℓ_1 (resp ℓ_k). In this section we compare the Lyndon factorization of w and that of its non-simple factors.

The following result is a direct consequence of Theorem 2.2.

Lemma 3.1 Let $w \in \Sigma^+$ be a word and let $CFL(w) = (\ell_1, \ldots, \ell_k)$ be its Lyndon factorization. For any i, j, with $1 \le i < j \le k$, one has

$$\operatorname{CFL}(\ell_i \cdots \ell_j) = (\ell_i, \dots, \ell_j).$$

If x is a non-simple factor of w and x does not satisfy the hypotheses of Lemma 3.1, then there are i, j with $1 \le i < j \le k$, a suffix ℓ''_i of ℓ_i and a prefix ℓ'_j of ℓ_j , with $\ell''_i \ell'_j \ne 1$, such that

$$x = \ell_i'' \ell_{i+1} \cdots \ell_{j-1} \ell_j',$$

where it is understood that if j = i + 1, then $\ell_{i+1}, \ldots, \ell_{j-1} = 1$ and $\ell''_i \neq 1$, $\ell'_j \neq 1$, $\ell''_i \ell'_j \neq \ell_i \ell_j$. We say that the sequence $\ell''_i, \ell_{i+1}, \ldots, \ell_{j-1}, \ell'_j$ is associated with x. The following result gives relations between CFL(x) and CFL(w).

Lemma 3.2 Let $w \in \Sigma^+$ be a word and let $CFL(w) = (\ell_1, \ldots, \ell_k)$ be its Lyndon factorization. Let x be a non-simple factor of w such that x does not satisfy the hypotheses of Lemma 3.1 and let $\ell''_i, \ell_{i+1}, \ldots, \ell_{j-1}, \ell'_j$ be the sequence associated with x. We have

$$CFL(x) = (CFL(\ell_i''), \ell_{i+1}, \dots, \ell_{j-1}, CFL(\ell_j'))$$

where it is understood that if $\ell''_i = 1$ (resp. $\ell'_j = 1$), then the first term $CFL(\ell''_i)$ (resp. last term $CFL(\ell'_i)$) vanishes.

Proof :

Let $w, x, \ell_1, \ldots, \ell_k, \ell''_i, \ell'_j$ be as in the statement. Set $\text{CFL}(\ell''_i) = (m_1, \ldots, m_h)$ if $\ell''_i \neq 1$ and set $\text{CFL}(\ell'_j) = (v_1, \ldots, v_l)$ if $\ell'_j \neq 1$. By Theorem 2.2, we shall have established the lemma if we prove the following claims

- (1) if $\ell'_i \neq 1$ and j > i+1, then $\ell_{j-1} \succeq v_1$;
- (2) if $\ell_i'' \neq 1$ and j > i+1, then $m_h \succeq \ell_{i+1}$;

(3) if j = i + 1, then $m_h \succeq v_1$.

We preliminary observe that $CFL(\ell_{j-1}\ell_j\cdots\ell_k) = (\ell_{j-1},\ldots,\ell_k)$ (Lemma 3.1), hence ℓ_{j-1} is the longest prefix of $\ell_{j-1}\ell_j\cdots\ell_k$ which is a Lyndon word (Lemma 2.3).

(1) If $\ell_{j-1} \prec v_1$, then $\ell_{j-1}v_1$ would be a Lyndon word, by Proposition 2.5, and a prefix of $\ell_{j-1}\ell_j \cdots \ell_k$, longer than ℓ_{j-1} , a contradiction.

(2) If $\ell_i'' = \ell_i$, then $(m_1, \ldots, m_h) = (\ell_i)$ and we are done. Otherwise, m_h is a suffix of ℓ_i'' which is a proper nonempty suffix of ℓ_i . By Proposition 2.4, we know that $\ell_i \prec m_h$. If $m_h \prec \ell_{i+1}$, then $\ell_i \prec \ell_{i+1}$, in contradiction with Eq.(2.3).

(3) Recall that in this case $x = \ell''_i \ell'_j$, $\ell''_i \neq 1$, $\ell'_j \neq 1$, $\ell''_i \ell'_j \neq \ell_i \ell_j$. We claim that if $m_h \prec v_1$, then $\ell_i = \ell_{j-1} \prec v_1$. This is obvious if $\ell''_i = \ell_i$ because $(m_1, \ldots, m_h) = (\ell_i)$. Otherwise m_h is a suffix of ℓ''_i which is a proper nonempty suffix of ℓ_i . By Proposition 2.4, we know that $\ell_i \prec m_h$, thus if $m_h \prec v_1$, then $\ell_i = \ell_{j-1} \prec v_1$. Hence, $\ell_{j-1}v_1$ would be a Lyndon word, by Proposition 2.5, and a prefix of $\ell_{j-1}\ell_j \cdots \ell_k$, longer than ℓ_{j-1} , a contradiction.

Let $x, y, z, w, w' \in \Sigma^+$. The following result, which is a consequence of Lemma 3.2, gives relations between the Lyndon factorizations of two overlapping words w, w', i.e., such that w = xy, w' = yz, and the Lyndon factorization of the overlap y, when y is non-simple (as a suffix of w and as a prefix of w').

Lemma 3.3 Let $w, w' \in \Sigma^+$, let $CFL(w) = (\ell_1, \ldots, \ell_k)$ and $CFL(w') = (f_1, f_2, \ldots, f_h)$. If y is a non-simple suffix of w and a non-simple prefix of w', then there are i, j, with $1 \le i < k$, $1 < j \le h$, such that one of the following cases holds.

- (1) CFL(y) = $(f_1, \ldots, f_{j-1}, \ell_{i+1}, \ldots, \ell_k)$
- (2) There exists j', 1 < j' < j such that $CFL(y) = (f_1, \ldots, f_{j'-1}, \ell_{i+1}, \ldots, \ell_k)$ and $f_{j'+r} = \ell_{i+1+r}$, for any $r, 0 \le r \le j j' 1$
- (3) There is i', i < i' < k such that $CFL(y) = (f_1, \dots, f_{j-1}, \ell_{i'+1}, \dots, \ell_k)$ and $\ell_{i'-r} = f_{j-r-1}$, for any $r, 0 \le r \le i' i 1$

Proof :

Let $w, w' \in \Sigma^+$, let $CFL(w) = (\ell_1, \ldots, \ell_k)$ and $CFL(w') = (f_1, f_2, \ldots, f_h)$. If y is a non-simple suffix of w and a non-simple prefix of w', then there are i, j, with $1 \le i < k, 1 < j \le h$, such that

$$y = \ell_i'' \ell_{i+1} \cdots \ell_k = f_1 \cdots f_{j-1} f_j',$$

where ℓ''_i is a suffix of ℓ_i and f'_j is a prefix of f_j . By Lemma 3.2 we have

$$\operatorname{CFL}(y) = (\operatorname{CFL}(\ell_i''), \ell_{i+1}, \dots, \ell_k) = (f_1, \dots, f_{j-1} \operatorname{CFL}(f_j')).$$

Thus the conclusion follows by Theorem 2.2.

Since Lyndon factorizations can be computed in linear time, the above result leads to efficient measures of similarities between words. These measures can be used to capture words that may be overlapping.

4 Anti-Lyndon words, inverse Lyndon words and anti-prenecklaces

For the material in this section see [6].

4.1 Inverse lexicographic order and anti-Lyndon words

Inverse Lyndon words are related to the inverse alphabetic order. Its definition is recalled below.

Definition 4.1 Let $(\Sigma, <)$ be a totally ordered alphabet. The inverse $<_{in}$ of < is defined by

$$\forall a, b \in \Sigma \quad b <_{in} a \Leftrightarrow a < b$$

The inverse lexicographic or inverse alphabetic order on $(\Sigma^*, <)$, denoted \prec_{in} , is the lexicographic order on $(\Sigma^*, <_{in})$.

Example 4.1 Let $\Sigma = \{a, b, c, d\}$ with a < b < c < d. Then $dab \prec dabd$ and $dabda \prec dac$. We have $d <_{in} c <_{in} b <_{in} a$. Therefore $dab \prec_{in} dabd$ and $dac \prec_{in} dabda$.

The following proposition justifies the adopted terminology.

Proposition 4.1 Let $(\Sigma, <)$ be a totally ordered alphabet. For all $x, y \in \Sigma^*$ such that $x \bowtie y$,

$$y \prec_{in} x \Leftrightarrow x \prec y.$$

Moreover, in this case $x \ll y$.

From now on, $L_{in} = L_{(\Sigma^*, \leq_{in})}$ denotes the set of the Lyndon words on Σ^* with respect to the inverse lexicographic order. A word $w \in L_{in}$ will be named an *anti-Lyndon word*. Correspondingly, an *anti-prenecklace* will be a prefix of an *anti-necklace*, which in turn will be a necklace with respect to the inverse lexicographic order. The following proposition characterizes $L_{in} = L_{(\Sigma^*, \leq_{in})}$.

Proposition 4.2 A word $w \in \Sigma^+$ is in L_{in} if and only if w is primitive and $w \succ vu$, for each $u, v \in \Sigma^+$ such that w = uv.

We state below a slightly modified dual version of Proposition 2.4.

Proposition 4.3 A word $w \in \Sigma^+$ is in L_{in} if and only if w is unbordered and $w \succ v$, for each proper nonempty suffix v.

The following result give more precise relations between words in L_{in} and their proper nonempty suffixes.

Proposition 4.4 If v is a proper nonempty suffix of $w \in L_{in}$, then $v \ll w$.

In the following, we denote by $CFL_{in}(w)$ the Lyndon factorization of w with respect to the inverse order \leq_{in} .

4.2 Inverse Lyndon words and anti-prenecklaces

Definition 4.2 A word $w \in \Sigma^+$ is an inverse Lyndon word if $s \prec w$, for each nonempty proper suffix s of w.

Example 4.2 The words $a, b, aaaaa, bbba, baaab, bbaba and bbababbaa are inverse Lyndon words on <math>\{a, b\}$, with a < b. On the contrary, *aaba* is not an inverse Lyndon word since $aaba \prec ba$. Analogously, $aabba \prec ba$ and thus aabba is not an inverse Lyndon word.

The following result is a direct consequence of Proposition 4.3.

Proposition 4.5 A word $w \in \Sigma^+$ is an anti-Lyndon word if and only if it is an unbordered inverse Lyndon word.

In turn, by Proposition 4.5 it is clear that the set of anti-Lyndon words is a proper subset of the set of inverse Lyndon words since there are inverse Lyndon words which are not anti-Lyndon words. For instance consider $\Sigma = \{a, b\}$, with a < b. The word *bab* is an inverse Lyndon word but it is bordered, thus it is not an anti-Lyndon word.

Inverse Lyndon words and anti-prenecklaces are strongly related, as the following result shows.

Proposition 4.6 A word $w \in \Sigma^+$ is an inverse Lyndon word if and only if w is a nonempty anti-prenecklace.

The following result is a direct consequence of Proposition 4.6.

Lemma 4.1 Any nonempty prefix of an inverse Lyndon word is an inverse Lyndon word.

5 A canonical inverse Lyndon factorization: ICFL(w)

An inverse Lyndon factorization of a word $w \in \Sigma^+$ is a sequence (m_1, \ldots, m_k) of inverse Lyndon words such that $m_1 \cdots m_k = w$ and $m_i \ll m_{i+1}$, $1 \le i \le k-1$. The canonical inverse Lyndon factorization, denoted ICFL(w), is a special inverse Lyndon factorization that maintains the main properties of the Lyndon factorization. Its definition and properties are based on other notions and results recalled below.

Definition 5.1 Let $w \in \Sigma^+$, let p be an inverse Lyndon word which is a nonempty proper prefix of w = pv. The bounded right extension \overline{p}_w of p (relatively to w), denoted by \overline{p} when it is understood, is a nonempty prefix of v such that:

- (1) \overline{p} is an inverse Lyndon word,
- (2) pz' is an inverse Lyndon word, for each proper nonempty prefix z' of \overline{p} ,
- (3) $p\overline{p}$ is not an inverse Lyndon word,
- (4) $p \ll \overline{p}$.

Moreover, we set

 $\operatorname{Pref}_{bre}(w) = \{(p,\overline{p}) \mid p \text{ is an inverse Lyndon word} \\ which is a nonempty proper prefix of w\}.$

It has been proved that either $\operatorname{Pref}_{bre}(w) = \emptyset$ or $\operatorname{Card}(\operatorname{Pref}_{bre}(w)) = 1$. Moreover $\operatorname{Pref}_{bre}(w)$ is empty if and only if w is an inverse Lyndon word. Another useful property of $\operatorname{Pref}_{bre}(w)$ is recalled below.

Proposition 5.1 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let z be the shortest nonempty prefix of w which is not an inverse Lyndon word. Then,

- (1) $z = p\overline{p}$, with $(p,\overline{p}) \in \operatorname{Pref}_{bre}(w)$.
- (2) p = ras and $\overline{p} = rb$, where $r, s \in \Sigma^*$, $a, b \in \Sigma$ and r is the shortest prefix of $p\overline{p}$ such that $p\overline{p} = rasrb$, with a < b.

We now give the recursive definition of ICFL(w).

Definition 5.2 Let $w \in \Sigma^+$.

(Basis Step) If w is an inverse Lyndon word, then ICFL(w) = (w).

(Recursive Step) If w is not an inverse Lyndon word, let $(p,\overline{p}) \in \operatorname{Pref}_{bre}(w)$ and let $v \in \Sigma^*$ such that w = pv. Let $\operatorname{ICFL}(v) = (m'_1, \ldots, m'_k)$ and let $r, s \in \Sigma^*$, $a, b \in \Sigma$ such that p = ras, $\overline{p} = rb$ with a < b.

$$\operatorname{ICFL}(w) = \begin{cases} (p, \operatorname{ICFL}(v)) & \text{if } \overline{p} = rb \leq_p m'_1 \\ (pm'_1, m'_2, \dots, m'_k) & \text{if } m'_1 \leq_p r \end{cases}$$

6 Groupings

Let $w \in \Sigma^+$. There are relations between ICFL(w), the Lyndon factorization CFL_{in}(w) of w with respect to the inverse order \langle_{in} and some special inverse Lyndon factorizations of w, called groupings of CFL_{in}(w). We first give some needed definitions and results.

Definition 6.1 Let $w \in \Sigma^+$, let $\operatorname{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$ and let $1 \leq r < s \leq h$. We say that $\ell_r, \ell_{r+1} \ldots, \ell_s$ is a non-increasing maximal chain for the prefix order in $\operatorname{CFL}_{in}(w)$, abbreviated \mathcal{PMC} , if $\ell_r \geq_p \ell_{r+1} \geq_p \ldots \geq_p \ell_s$. Moreover, if r > 1, then $\ell_{r-1} \not\geq_p \ell_r$, if s < h, then $\ell_s \not\geq_p \ell_{s+1}$. Two $\mathcal{PMC} \ \mathcal{C}_1 = \ell_r, \ell_{r+1} \ldots, \ell_s, \ \mathcal{C}_2 = \ell_{r'}, \ell_{r'+1} \ldots, \ell_{s'}$ are consecutive if r' = s+1 (or r = s'+1).

The definition of a grouping of $CFL_{in}(w)$ is given below in two steps. We first define the grouping of a \mathcal{PMC} . Then a grouping of $CFL_{in}(w)$ is obtained by changing each \mathcal{PMC} with one of its groupings.

Definition 6.2 Let ℓ_1, \ldots, ℓ_h be words in L_{in} such that ℓ_i is a prefix of $\ell_{i-1}, 1 < i \leq h$. We say that (m_1, \ldots, m_k) is a grouping of (ℓ_1, \ldots, ℓ_h) if the following conditions are satisfied.

- (1) m_j is an inverse Lyndon word,
- (2) $\ell_1 \cdots \ell_h = m_1 \cdots m_k$. More precisely, there are $i_0, i_1, \dots, i_k, i_0 = 0, 1 \le i_j \le h, i_k = h$, such that $m_j = \ell_{i_{j-1}+1} \cdots \ell_{i_j}, 1 \le j \le k$,
- (3) $m_1 \ll \ldots \ll m_k$.

We now extend Definition 6.2 to $CFL_{in}(w)$.

Definition 6.3 Let $w \in \Sigma^+$ and let $CFL_{in}(w) = (\ell_1, \ldots, \ell_h)$. We say that (m_1, \ldots, m_k) is a grouping of $CFL_{in}(w)$ if it can be obtained by replacing any $\mathcal{PMC} \ \mathcal{C}$ in $CFL_{in}(w)$ by a grouping of \mathcal{C} .

Groupings of $CFL_{in}(w)$ are inverse Lyndon factorizations of w but there are inverse Lyndon factorizations which are not groupings. As stated below, ICFL(w) is a grouping of $CFL_{in}(w)$. We first consider the special case of an inverse Lyndon word.

Proposition 6.1 Let $(\Sigma, <)$ be a totally ordered alphabet. Let $w \in \Sigma^+$ and let $CFL_{in}(w) = (\ell_1, \ldots, \ell_h)$. If w is an inverse Lyndon word, then either w is unbordered or ℓ_1, \ldots, ℓ_h is a \mathcal{PMC} in $CFL_{in}(w)$. In both cases ICFL(w) = (w) is the unique grouping of $CFL_{in}(w)$.

Proposition 6.2 Let $(\Sigma, <)$ be a totally ordered alphabet. For any $w \in \Sigma^+$, ICFL(w) is a grouping of $\text{CFL}_{in}(w)$.

7 Borders

We recall that, given a nonempty word w, a *border* of w is a word which is both a proper prefix and a suffix of w [14]. The longest proper prefix of w which is a suffix of w is also called *the border* of w [14, 36]. Thus a word $w \in \Sigma^+$ is unbordered if and only if it has a nonempty border. Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. The aim of this section is to show that any nonempty border of m_i is not a prefix of m_{i+1} , $1 \leq i \leq k - 1$. Some preliminary results are needed.

Proposition 7.1 Let $w \in \Sigma^+$, let $\operatorname{CFL}_{in}(w) = (\ell_1, \ldots, \ell_h)$ and let $\ell_r, \ldots, \ell_s, 1 \leq r < s \leq h$, be a non-increasing chain for the prefix order in $\operatorname{CFL}_{in}(w)$. For any nonempty border x of $y = \ell_r \cdots \ell_s$ there is $t, r \leq t < s$, such that $x = \ell_{t+1} \cdots \ell_s$. Consequently, ℓ_s is a prefix of any nonempty border of $\ell_r \cdots \ell_s$.

Proof:

Let $w, \ell_1, \ldots, \ell_h, r, s$ be as in the statement. By hypothesis, for each t, with $r \leq t \leq s$, the word ℓ_t is a prefix of ℓ_r . Let x be a nonempty border of $y = \ell_r \cdots \ell_s$. If there were a nonempty proper suffix x' of ℓ_t , $r \leq t \leq s$, such that $x = x'\ell_{t+1}\cdots\ell_s$, then x' would be both a prefix and a nonempty proper suffix of ℓ_t , thus a nonempty border of ℓ_t , in contradiction with ℓ_t being an anti-Lyndon word.

Lemma 7.1 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $CFL_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with h > 0, $n_1, \ldots, n_h \ge 1$. For all $z \in \Sigma^+$ and $b \in \Sigma$ such that z is an antiprenecklace, zb is not an anti-prenecklace and zb is a prefix of w, there is an integer g such that

$$zb = (u_1v_1)^{n_1}\cdots(u_gv_g)^{n_g}u_gb,$$

where $u_j v_j = u_j a_j v'_j = \ell_j, 1 \le j \le g, a_j < b$ and $u_g b$ is an anti-prenecklace.

Proof :

We prove the statement by induction on |w|. If |w| = 1, then w is an inverse Lyndon word and we are done. Hence assume |w| > 1. If w is an inverse Lyndon word, then again the proof is ended. Therefore, assume that w is not an inverse Lyndon word. Let $\text{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with $h > 0, n_1, \ldots, n_h \ge 1$.

Let $z \in \Sigma^+$, $b \in \Sigma$ be such that z is an anti-prenecklace, zb is not an anti-prenecklace and zb is a prefix of w. By Theorem 2.1 and Corollary 2.1, there are words $u, v, v' \in \Sigma^*$, $a \in \Sigma$, with a < b, and an integer $k \ge 1$, such that $zb = (uv)^k ub$, v = av' and where uv is the longest anti-Lyndon prefix of zb.

We claim that uv is also the longest anti-Lyndon prefix of w. Indeed, if y is a prefix of w such that |y| > |zb|, then $y = zbz' = (uav')^k ubz'$, with $z' \in \Sigma^*$. Thus, $y \ll ubz'$ and y is not an anti-Lyndon word. Consequently, by Lemma 2.3, we have $uv = \ell_1$. Moreover, $k = n_1$ because ub is not a prefix of ℓ_1 .

If ub is an anti-prenecklace the proof is ended. Otherwise, let $w' \in \Sigma^*$ be such that $w = \ell_1^{n_1}w'$. We have 0 < |w'| < |w| since ub is a prefix of w' and $\ell_1 \neq 1$. By Theorem 2.2, we have $\operatorname{CFL}_{in}(w') = (\ell_2^{n_2}, \ldots, \ell_h^{n_h})$. The word u an anti-prenecklace whereas ub is not an anti-prenecklace. By induction hypothesis there is an integer g such that

$$ub = (u_2v_2)^{n_2}\cdots(u_gv_g)^{n_g}u_gb,$$

where $u_j v_j = u_j a_j v'_j = \ell_j, 2 \leq j \leq g, a_j < b$ and $u_g b$ is an anti-prenecklace. Thus, there is an integer g such that

$$zb = (u_1v_1)^{n_1}\cdots(u_qv_q)^{n_g}u_qb,$$

where $u_1 = u$, $v_1 = v$, $v'_1 = v'$, $a_1 = a$, $u_j v_j = u_j a_j v'_j = \ell_j$, $1 \le j \le g$, $a_j < b$ and $u_g b$ is an anti-prenecklace.

Proposition 7.2 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $(p, \bar{p}) \in \operatorname{Pref}_{bre}(w)$ and let $\operatorname{ICFL}(w) = (m_1, \ldots, m_k)$. Let $\operatorname{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with h > 0, $n_1, \ldots, n_h \ge 1$ and let $\ell_1^{n_1}, \ldots, \ell_q^{n_q}$ be a \mathcal{PMC} in $\operatorname{CFL}_{in}(w)$, $1 \le q \le h$. Then the following properties hold.

(1) $p = \ell_1^{n_1} \cdots \ell_g^{n_g}$, for some $g, 1 \le g \le q$.

(2)
$$\ell_g = u_g v_g = u_g a_g v'_g, \ \bar{p} = u_g b, \ a_g < b$$

Proof :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word, let $(p,\bar{p}) \in \operatorname{Pref}_{bre}(w)$. Let $\operatorname{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with $h > 0, n_1, \ldots, n_h \ge 1$ and let $\ell_1^{n_1}, \ldots, \ell_q^{n_q}$ be a \mathcal{PMC} in $\operatorname{CFL}_{in}(w), 1 \le q \le h$.

By Proposition 4.6, the word $p\bar{p}$ is not an anti-prenecklace but its longest proper prefix is an anti-prenecklace. Thus, by Lemma 7.1 there is an integer g such that

$$p\bar{p} = (u_1v_1)^{n_1}\cdots(u_gv_g)^{n_g}u_gb_g$$

where $u_j v_j = u_j a_j v'_j = \ell_j, 1 \le j \le g, a_j < b$ and $u_g b$ is an anti-prenecklace. Let

$$\beta = (u_1 v_1)^{n_1} (u_2 v_2)^{n_2} \cdots (u_g v_g)^{n_g}, \quad \beta' = \beta u_g.$$

By Definition 5.1, the words β' and β are inverse Lyndon words, therefore $g \leq q$ (otherwise ℓ_q would be a prefix of β and there would be a word z' such that $\ell_{q+1}z'$ is a suffix of β , a contradiction since β is an inverse Lyndon word and $\ell_q \ll \ell_{q+1}$). Moreover, $\beta \ll u_g b$.

Let $r, s \in \Sigma^*$, $a', b \in \Sigma$ be such that p = ra's, $\bar{p} = rb$, a' < b. Then $p\bar{p} = \beta u_g b = ra'srb$. By Proposition 5.1, r is a suffix of u_g . Consequently, $\ell_g = u_g v_g$ and u_g are prefixes of p. Moreover, we know that u_g and $u_g b$ are both anti-prenecklaces. Thus, by Proposition 2.6, Theorem 2.1 and Corollary 2.1, there are $x, y \in \Sigma^*$, an integer $t \ge 1$, $c \in \Sigma$ such that xy is an anti-Lyndon word, $u_g = (xy)^t x, y = cy'$ with $c \ge b$.

The words ℓ_{g+1} and $u_g b = (xy)^t x b$ are both prefixes of the same word γ , hence they are comparable for the prefix order. Since ℓ_{g+1} is the longest anti-Lyndon prefix of γ , we have $|\ell_{g+1}| \ge |xy|$ and since ℓ_{g+1} is unbordered, either $\ell_{g+1} = xy$ is a prefix of ℓ_g and $g+1 \le q$, or the word $u_g b = (xy)^t x b$ is a prefix of ℓ_{g+1} . By Proposition 6.2, the first case holds, otherwise m_1 would not be a product of anti-Lyndon words because m_1 is a prefix of βu_g .

If $r = u_g$, then $p = \beta$ and the proof is ended. By contradiction, assume that r is a proper suffix of u_g . Therefore r is a border of u_g because r is a prefix of p and u_g is nonempty. Of course $r \neq x$ because u_g starts with ra' and also with xc, with $c \geq b > a'$. If r would be shorter than x, then r would be a border of x. This is impossible because $rcy'(xy)^{t-1}x$ would be a suffix of the inverse Lyndon word u_g and u_g starts with ra', with $c \geq b > a'$. Thus $|r| > |x| \geq 0$. Since r is a nonempty border of $u_g = (xy)^t x$ and $|r| > |x| \geq 0$, one of the following three cases holds:

$$r = (xy)^{t'}x, \quad 0 < t' < t \tag{7.1}$$

$$r = y_1(xy)^{t'}x, \quad y_1 \text{ nonempty suffix of } y, \quad 0 \le t' < t$$
(7.2)

$$r = x_1(yx)^{t'}, \quad x_1 \text{ nonempty suffix of } x, \quad 0 < t' \le t$$

$$(7.3)$$

Assume that Eq. (7.1) holds. Then p starts with $ra' = (xy)^{t'}xa'$, a' < b, and p also starts with $u_g = (xy)^t x$. Since t' < t, the letter a' should be the first letter of y = cy', $c \ge b > a'$. Therefore, Eq. (7.1) cannot hold.

Assume that Eq. (7.2) holds. Therefore $y_1 = y$, otherwise y_1x would be a proper prefix of xy, hence a nonempty border of xy, which is impossible since xy is an anti-Lyndon word. Moreover x = 1, otherwise yx = xy and xy would not be primitive (Proposition 2.2), which is impossible since xy is an anti-Lyndon word. As above, p starts with $ra' = y^{t'}a'$, a' < b, and palso starts with $u_g = y^t$. Since t' < t, the letter a' should be the first letter of y = cy', $c \ge b > a'$. Therefore, Eq. (7.2) cannot hold.

Finally, assume that Eq. (7.3) holds. If $x_1 \neq x$, then x_1y would be both a proper nonempty suffix and a prefix of xy, hence a nonempty border of xy, which is impossible since xy is an anti-Lyndon word. Therefore $x_1 = x$. If t' < t, then r satisfies Eq. (7.1) and we proved that this is impossible. Thus t' = t, which implies $r = u_q$, a contradiction.

Proposition 7.3 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $(p, \bar{p}) \in \operatorname{Pref}_{bre}(w)$. For each nonempty border z of p, one has that z and \bar{p} are incomparable for the prefix order.

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $(p, \bar{p}) \in \operatorname{Pref}_{bre}(w)$. By Proposition 5.1, there are $r, s \in \Sigma^*$, $a, b \in \Sigma$, with a < b, such that p = ras and $\overline{p} = rb$. Let $\operatorname{CFL}_{in}(w) = (\ell_1^{n_1}, \ldots, \ell_h^{n_h})$, with $h > 0, n_1, \ldots, n_h \ge 1$ and let $\ell_1^{n_1}, \ldots, \ell_q^{n_q}$ be a \mathcal{PMC} in $\operatorname{CFL}_{in}(w), 1 \le q \le h$.

Let z be a nonempty border of p. Of course \overline{p} cannot be a prefix of z because \overline{p} is not a prefix of p. By contradiction, suppose that z is a prefix of \overline{p} . By Proposition 7.2, there is g, $1 \leq g \leq q$ such that $p = \ell_1^{n_1} \cdots \ell_g^{n_g}$ and $\ell_g = u_g v_g = u_g a_g v'_g$, $\overline{p} = u_g b$, $a_g < b$. By Proposition 7.1, ℓ_g is a prefix of any nonempty border of p, hence ℓ_g is a prefix of z.

By Proposition 7.1, ℓ_g is a prefix of any nonempty border of p, hence ℓ_g is a prefix of z. Moreover z is a prefix of \overline{p} , thus $\ell_g = u_g a_g v'_g$ would be a prefix of $\overline{p} = u_g b$. This is impossible because $a_g < b$.

Proposition 7.4 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. If z is a nonempty border of m_1 , then z is not a prefix of m_2 .

PROOF :

Let $w \in \Sigma^+$ and let $ICFL(w) = (m_1, \ldots, m_k)$. We prove the statement by induction on |w|.

If |w| = 1, then w is an inverse Lyndon word and we are done. Hence assume |w| > 1. If w is an inverse Lyndon word, then again the proof is ended. Therefore, assume that w is not an inverse Lyndon word. Let $(p,\bar{p}) \in \operatorname{Pref}_{bre}(w)$ and let $r, s \in \Sigma^*$, $a, b \in \Sigma$ be such that p = ras, $\overline{p} = rb$, a < b. Let $v \in \Sigma^*$ be such that $w = m_1 v$. Of course 0 < |v| < |w| because w is not an inverse Lyndon word. Let $\operatorname{ICFL}(v) = (m'_1, \ldots, m'_{k'})$. By Definition 5.2, one of the following two cases holds

- (1) $m_1 = p$ if \overline{p} is a prefix of $m'_1 = m_2$
- (2) $m_1 = pm'_1, m_2 = m'_2, \dots, m_k = m'_k, k' = k$, if m'_1 is a prefix of r.

Let z be a nonempty border of m_1 . In case (1), if z would be a prefix of m_2 , then z and \overline{p} would be comparable for the prefix order, in contradiction with Proposition 7.3.

In case (2), m'_1 is a prefix of m_1 . By contradiction, suppose that z is a prefix of m_2 . We have either $|z| \ge |m'_1|$ or $|z| < |m'_1|$. If $|z| \ge |m'_1|$, then m'_1 would be a prefix of z and thus

of $m_2 = m'_2$, in contradiction with $m'_1 \ll m'_2$. If $|z| < |m'_1|$, then z would be a suffix of m'_1 , hence z would be a nonempty border of m'_1 . Thus a nonempty border of m'_1 would be a prefix of $m_2 = m'_2$, in contradiction with the induction hypothesis.

Proposition 7.5 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. If z is a nonempty border of m_i , then z is not a prefix of m_{i+1} , $1 \le i \le k-1$.

Proof :

Let $w \in \Sigma^+$ and let $ICFL(w) = (m_1, \ldots, m_k)$. We prove the statement by induction on |w|.

If |w| = 1, then w is an inverse Lyndon word and we are done. Hence assume |w| > 1. If w is an inverse Lyndon word, then again the proof is ended. Therefore, assume that w is not an inverse Lyndon word. Let $(p, \bar{p}) \in \operatorname{Pref}_{bre}(w)$ and let $r, s \in \Sigma^*$, $a, b \in \Sigma$ be such that p = ras, $\bar{p} = rb$, a < b. Let $v \in \Sigma^*$ be such that $w = m_1 v$. Of course 0 < |v| < |w| because w is not an inverse Lyndon word. Let $\operatorname{ICFL}(v) = (m'_1, \ldots, m'_{k'})$. By Definition 5.2, one of the following two cases holds

- (1) $m_1 = p, m_i = m'_{i-1}, 1 < i \le k = k' + 1$, if $\overline{p} \le_p m'_1$
- (2) $m_1 = pm'_1, m_i = m'_i, 1 \le i \le k = k'$, if $m'_1 \le_p r$.

If z is a nonempty border of m_1 , then z is not a prefix of m_2 , by Proposition 7.4. Thus assume that z is a nonempty border of m_i , $1 < i \le k-1$. In case (1), z is a nonempty border of m'_{i-1} , hence, by induction hypothesis, z is not a prefix of $m'_i = m_{i+1}$. Analogously, in case (2), z is a nonempty border of m'_i , therefore, by induction hypothesis, z is not a prefix of $m'_{i+1} = m_{i+1}$.

8 Suffixes compatibility

In this section we use the same notation and terminology as in [39, 40], where the authors found interesting relations between the sorting of the suffixes of a word w and that of its factors. Here we prove a similar property when ICFL(w) is considered.

Let $w, x, u, y \in \Sigma^*$, and let u be a nonempty factor of w = xuy. Let first(u) and last(u)denote the position of the first and the last symbol of u in w, respectively. If $w = a_1 \cdots a_n$, $a_i \in \Sigma$, $1 \le i \le j \le n$, then we also set $w[i, j] = a_i \cdots a_j$. A local suffix of w is a suffix of a factor of w, specifically $suf_u(i) = w[i, last(u)]$ denotes the local suffix of w at the position iwith respect to $u, i \ge first(u)$. The corresponding global suffix $suf_u(i)y$ of w at the position i is denoted by $suf_w(i) = w[i, last(w)]$ (or simply suf(i) when it is understood). We say that $suf_u(i)y$ is associated with $suf_u(i)$.

Definition 8.1 [39, 40] Let $w \in \Sigma^+$ and let u be a nonempty factor of w. We say that the sorting of the nonempty local suffixes of w with respect to u is compatible with the sorting of the corresponding nonempty global suffixes of w if for all i, j with $first(u) \leq i < j \leq last(u)$,

$$suf_u(i) \prec suf_u(j) \iff suf(i) \prec suf(j).$$

The following result has been proved in [39, 40].

Theorem 8.1 Let $w \in \Sigma^+$ and let $CFL(w) = (\ell_1, \ldots, \ell_h)$ be its Lyndon factorization. Then, for any $i, g, 1 \leq i \leq g \leq h$, the sorting of the nonempty local suffixes of w with respect to $u = \ell_i \cdots \ell_g$ is compatible with the sorting of the corresponding nonempty global suffixes of w. In [6] the same compatibility property as in Theorem 8.1 has been proved for the sorting of the nonempty suffixes of a word w with respect to \prec_{in} , when we replace CFL(w) with ICFL(w).

Proposition 8.1 Let w be a word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Then, for any $i, h, 1 \leq i \leq h \leq k$, the sorting with respect to \prec_{in} of the nonempty local suffixes of w with respect to $u = m_i \cdots m_h$ is compatible with the sorting with respect to \prec_{in} of the corresponding nonempty global suffixes of w.

The following result proves another compatibility property for the sorting of the nonempty suffixes of a word w with respect to \prec , when we replace CFL(w) with ICFL(w).

Proposition 8.2 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let ICFL $(w) = (m_1, \ldots, m_k)$. Let $u = m_i m_{i+1} \cdots m_h$ with $1 \le i < h \le k$. Assume that $suf_u(j_1) \prec suf_u(j_2)$, where $first(u) \le j_1 \le last(u)$, $first(u) \le j_2 \le last(u)$, $j_1 \ne j_2$.

If $suf_u(j_1)$ is a proper prefix of $suf_u(j_2)$ and h < k then $suf(j_2) \prec suf(j_1)$, otherwise $suf(j_1) \prec suf(j_2)$.

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let $u = m_i m_{i+1} \cdots m_h$ with $1 \le i < h \le k$. Assume that $suf_u(j_1) \prec suf_u(j_2)$, where $first(u) \le j_1 \le last(u)$, $first(u) \le j_2 \le last(u)$.

If h = k, then $suf(j_1) = suf_u(j_1) \prec suf_u(j_2) = suf(j_2)$ and we are done. Thus assume h < k. If $suf_u(j_1)$ is not a proper prefix of $suf_u(j_2)$, then $suf_u(j_1) \ll suf_u(j_2)$. Hence, by item (2) in Lemma 2.1, we have $suf(j_1) \ll suf(j_2)$ and we are done again.

Therefore, assume that $suf_u(j_1)$ is a proper prefix of $suf_u(j_2)$. Thus $j_2 < j_1$ because $|suf_u(j_1)| < |suf_u(j_1)|$. Set $x = suf_u(j_1) = w[j_1, last(m_h)]$ and $y = w[j_2, j_2 + |x| - 1]$. We have x = y because x, y are prefixes of $suf_u(j_2)$ with the same length. Let g be the minimum integer such that $j_2 + |x| \le last(m_g)$, $g \le h < k$, and let $z = w[j_2 + |x|, last(m_g)]$. Therefore,

$$suf(j_2) = xzm_{g+1}\cdots m_k, \quad suf(j_1) = xm_{h+1}\cdots m_k$$

and we distinguish two cases:

(1) z = 1

(2)
$$z \neq 1$$

(Case (1)) If z = 1, then g < h because $j_2 \neq j_1$ and thus $suf(j_2) \neq suf(j_1)$. Therefore

$$suf(j_2) = xm_{q+1}\cdots m_k \ll xm_{h+1}\cdots m_k = suf(j_1)$$

(Case (2)) Assume $z \neq 1$. If $z = m_g$, we apply the above argument again and we obtain

$$suf(j_2) = xm_g m_{g+1} \cdots m_k \ll xm_{h+1} \cdots m_k = suf(j_1)$$

Thus assume that z is a nonempty proper suffix of m_g . Hence $z \prec m_g$ and we have one of the following two cases.

(2a) $z \ll m_g$ (2b) $z <_p m_g$ (Case (2a)) If $z \ll m_q$, then we have

$$suf(j_2) = xzm_{g+1}\cdots m_k \ll xm_gm_{g+1}\cdots m_k \ll xm_{h+1}\cdots m_k = suf(j_1)$$

(Case (2b)) Let $r, s \in \Sigma^*$, $a, b \in \Sigma$ be such that $m_g = ras$, $m_{g+1} = rbt$, a < b. Assume that $z <_p m_g$. Since z is also a nonempty proper suffix of m_g , we have that z is a border of m_g . Then, by Proposition 7.5, z cannot be a prefix of m_{g+1} , hence there is a prefix s' of s such that z = ras'. Therefore we have

$$suf(j_2) = xzm_{q+1}\cdots m_k \ll xm_{q+1}\cdots m_k \preceq xm_{h+1}\cdots m_k = suf(j_1)$$

and the proof is complete.

Example 8.1 Let $w = a^{12}bbab \in \{a, b\}^+$ with a < b. We have $\text{ICFL}(w) = (m_1, m_2) = (a^{12}, bbab)$. Let $u = m_1 = a^{12}$. Consider $suf_u(4) = a^9$ and $suf_u(12) = a$. We have $suf_u(12) = a \prec a^9 = suf_u(4)$. We are in the first case of Lemma 8.2 and then $suf(4) = a^9bbab \prec abbab = suf(12)$.

Example 8.2 Let $w = dabadabdabdadac \in \{a, b, c, d\}^+$ with a < b < c < d. We have ICFL $(w) = (m_1, m_2, m_3) = (daba, dabdab, dadac)$. Let $u = m_2$. Consider $suf_{m_2}(8) = dab$ and $suf_{m_2}(5) = dabdab$. We have $suf_{m_2}(8) = dab \prec suf_{m_2}(5) = dabdab = (dab)^2$. We are in the first case of Lemma 8.2 and then $suf(5) = dabdabdadac \prec suf(8) = dabdadc$.

Consider now $suf_{m_2}(9) = ab \prec suf_{m_2}(8) = dab$. Since $suf_{m_2}(9)$ is not a proper prefix of $suf_{m_2}(8)$), we are in the second case of Lemma 8.2 and we have $suf(9) = abdadac \prec suf(8) = dabdadac$.

9 Sorting Suffixes via ICFL

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. The aim of this section is to define an integer related to ICFL(w) and then to prove that it is an upper bound to the lengths LCP(x, y) of the *longest common prefix* lcp(x, y) of two factors x, y of w. Some preliminary results are needed and proved below.

9.1 Technical Results

Lemma 9.1 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let $ICFL(w) = (m_1, \ldots, m_k)$. Then $m_i \notin Fact(m_1 \cdots m_{i-1})$, for each $1 < i \leq k$.

Proof :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let $ICFL(w) = (m_1, \ldots, m_k)$. Suppose the lemma were false. Then there would be $i, 1 < i \leq k$, such that $m_i \in Fact(m_1 \cdots m_{i-1})$. Thus one of the following three cases holds.

- (1) There are an integer $j, 1 \leq j < i$, and $x, y \in \Sigma^*$ such that $m_j = xm_i y$
- (2) There is an integer $j, 1 \leq j < i$, such that m_j is a prefix of m_i .
- (3) There are integers $j, h, 1 \leq j < i, h \geq 0$, a proper nonempty suffix x of m_j , and a proper prefix y of m_{j+h+1} such that $m_i = xm_{j+1}\cdots m_{j+h}y$, where it is understood that $m_{j+1}\cdots m_{j+h} = 1$ for h = 0.

Assume that case (1) holds. If x = 1, then $m_i \leq m_j \ll m_i$ which contradicts Lemma 2.2. Otherwise, $m_i y$ is a proper suffix of m_j , hence $m_i y \leq m_j \ll m_i$. Therefore $m_i y \leq m_j \ll m_i y$ (Lemma 2.1) which is impossible, once again by Lemma 2.2. Case (2) leads also to a contradiction since in this case we would have $m_j <_p m_i$ whereas $m_j \ll m_i$.

Assume that case (3) holds. We know that $x \leq m_j$. If $x \ll m_j$, then $m_i \ll m_j$ (Lemma 2.1) which is impossible since $m_j \ll m_i$ and then $m_j \ll m_i \ll m_j$, in contradiction with Lemma 2.2. Thus x is a proper prefix, thus a border of m_j . By Proposition 7.5, x is not a prefix of m_{j+1} . Thus j+1 < i and there are $r, s, t \in \Sigma^*$, $a, b \in \Sigma$ be such that $m_{j+1} = ras, m_i = rbt, a < b$. The words x, r are comparable for the prefix order and x is not a prefix of m_{j+1} . Therefore there is $t' \in \Sigma^*$ such that x = rbt'. Consequently, $m_{j+1} \ll x$, hence $m_{j+1} \ll m_j$ (Lemma 2.1). Since $m_j \ll m_{j+1}$, we would have $m_{j+1} \ll m_j \ll m_{j+1}$, once again in contradiction with Lemma 2.2.

Lemma 9.2 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let i, h, j be integers such that $1 \le i < h < j \le k$. Let $r_i, s_i, t_i, r_h, s_h, t_h \in \Sigma^*$, $a_i, b_i, a_h, b_h \in \Sigma$ be such that $m_i = r_i a_i s_i$, $m_h = r_h a_h s_h$, $m_j = r_i b_i t_i = r_h b_h t_h$, $a_i < b_i$, $a_h < b_h$. Then, the word r_i is a prefix of r_h .

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let ICFL $(w) = (m_1, \ldots, m_k)$. Let i, h, j be integers such that $1 \leq i < h < j \leq k$. Let $r_i, s_i, t_i, r_h, s_h, t_h \in \Sigma^*$, $a_i, b_i, a_h, b_h \in \Sigma$ be such that $m_i = r_i a_i s_i$, $m_h = r_h a_h s_h$, $m_j = r_i b_i t_i = r_h b_h t_h$, $a_i < b_i$, $a_h < b_h$. The words r_i and r_h are comparable for the prefix order. If r_h would be a proper prefix of r_i , then $r_h b_h$ were a prefix of r_i . Thus there would be $u \in \Sigma^*$ such that $r_i = r_h b_h u$, and consequently $m_h = r_h a_h s_h \ll r_h b_h u a_i s_i = m_i \ll m_h$, which is impossible (Lemma 2.2). Thus r_i is a prefix of r_h .

Corollary 9.1 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let i, h, j be integers such that $1 \le i < h < j \le k$. Let $r, s, t \in \Sigma^*$, be such that $m_i = rs$ and $m_j = rt$. Then, the word r is a prefix of m_h .

Proof :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let i, h, j be integers such that $1 \leq i < h < j \leq k$. Let $r, s, t \in \Sigma^*$, be such that $m_i = rs$ and $m_j = rt$. Let $r_i, s_i, t_i, r_h, s_h, t_h \in \Sigma^*$, $a_i, b_i, a_h, b_h \in \Sigma$ be such that $m_i = r_i a_i s_i$, $m_h = r_h a_h s_h, m_j = r_i b_i t_i = r_h b_h t_h, a_i < b_i, a_h < b_h$. Of course r is a prefix of r_i , because $r \in \text{Pref}(m_i) \cap \text{Pref}(m_j)$. Thus r is a prefix of r_h , by Lemma 9.2, hence $r \in \text{Pref}(m_h)$.

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let ICFL $(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \leq k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be such that $m_h = r_h a_h s_h$, $m_i = r_h b_h t_h$, $a_h < b_h$, $1 < h \leq i-1$. The following strengthening of Lemma 9.1 is proved below: $r_{i-1}b_{i-1} \notin \text{Fact}(m_1 \cdots m_{i-1})$ (Lemma 9.5). We have divided the proof of this result into a sequence of lemmas. We first prove that $r_{i-1}b_{i-1} \notin \text{Fact}(m_h)$, $1 \leq h \leq i-1$ (Lemma 9.3). Then, we prove that $r_{i-1}b_{i-1} \notin \text{Fact}(m_h m_{h+1})$, $1 \leq h < i-1$ (Lemma 9.4). Finally, we prove Lemma 9.5.

Lemma 9.3 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \leq k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be such that $m_h = r_h a_h s_h$, $m_i = r_h b_h t_h$, $a_h < b_h$, $1 \leq h \leq i-1$. Then, for each *h*, with $1 \leq h \leq i-1$, we have $r_{i-1}b_{i-1} \notin \text{Fact}(m_h)$.

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \leq k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be as in the statement.

Suppose, contrary to our claim, that there exists h, with $1 \leq h \leq i-1$, such that $r_{i-1}b_{i-1} \in Fact(m_h)$. Therefore, there are $u, v \in \Sigma^*$ such that $m_h = ur_{i-1}b_{i-1}v$. If $r_{i-1}b_{i-1}$ were a prefix of m_h , then necessarily h < i-1, because m_{i-1} starts with $r_{i-1}a_{i-1}$. Thus, by $r_{i-1}a_{i-1} \ll r_{i-1}b_{i-1}$ we would have $m_{i-1} \ll m_h$, with h < i-1, which is impossible. Hence, $r_{i-1}b_{i-1}v$ is a proper nonempty suffix of m_h and $r_{i-1}b_{i-1}v \notin Pref(m_h)$, we have $r_{i-1}b_{i-1}v \ll m_h$. By definition, there are $r, s, t \in \Sigma^*$, $a, b \in \Sigma$, such that $r_{i-1}b_{i-1}v = ras$, $m_h = rbt$, a < b. The words r_{i-1} and r are comparable for the prefix order. Moreover, r_{i-1} cannot be a proper prefix of r because $r_{i-1}b_{i-1} \notin Pref(m_h)$. Hence, r is a prefix of $r_{i-1}b_{i-1}$. As a consequence we have $r_{i-1}b_{i-1} \ll m_h$ which yields $m_i \ll m_h$, with h < i, once again a contradiction.

Lemma 9.4 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \leq k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be such that $m_h = r_h a_h s_h$, $m_i = r_h b_h t_h$, $a_h < b_h$, $1 \leq h \leq i-1$. Then, for each *h*, with $1 \leq h < i-1$, we have $r_{i-1}b_{i-1} \notin Fact(m_h m_{h+1})$.

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \leq k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be such that $m_h = r_h a_h s_h$, $m_i = r_h b_h t_h, a_h < b_h, 1 \leq h \leq i - 1$.

Suppose the lemma were false. Then we could find h, with $1 \leq h < i - 1$, such that $r_{i-1}b_{i-1} \in \operatorname{Fact}(m_h m_{h+1})$. Therefore, there are $u, v \in \Sigma^*$ such that $ur_{i-1}b_{i-1}v = m_h m_{h+1}$. The words u and m_h (resp. v and m_{h+1}) are comparable for the prefix (resp. suffix) order. Moreover, by Lemma 9.3, m_h (resp. m_{h+1}) is not a prefix (resp. suffix) of u (resp. v). Consequently there are $r, s \in \Sigma^+$ such that

$$m_h = ur, \quad r_{i-1}b_{i-1} = rs, \quad m_{h+1} = sv$$
(9.1)

In addition, $u \neq 1$, otherwise $m_h \in \operatorname{Pref}(m_i)$, in contradiction with $m_h \ll m_i$. Therefore, r is a proper nonempty suffix of m_h . Moreover, notice that r is a prefix of m_i . We claim that $r \notin \operatorname{Pref}(m_h)$. Indeed, if r were a prefix of m_h , it would be a nonempty border of m_h . Thus, on one hand $r \notin \operatorname{Pref}(m_{h+1})$ by Proposition 7.5. On the other hand, r would be a prefix both of m_h and m_i , hence $r \in \operatorname{Pref}(m_{h+1})$ by Corollary 9.1, a contradiction.

Since r is a proper nonempty suffix of m_h and $r \notin \operatorname{Pref}(m_h)$, we have $r \ll m_h$ which yields $m_i \ll m_h$, because $r \in \operatorname{Pref}(m_i)$. This is impossible since $m_h \ll m_i$.

Lemma 9.5 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \le k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be such that $m_h = r_h a_h s_h$, $m_i = r_h b_h t_h$, $a_h < b_h$, $1 \le h \le i-1$. Then, we have $r_{i-1}b_{i-1} \notin Fact(m_1 \cdots m_{i-1})$.

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let *i* be an integer such that $1 < i \leq k$. Let $r_h, s_h, t_h \in \Sigma^*$, $a_h, b_h \in \Sigma$ be such that $m_h = r_h a_h s_h$, $m_i = r_h b_h t_h, a_h < b_h, 1 \leq h \leq i - 1$.

By contradiction, suppose that $r_{i-1}b_{i-1} \in \operatorname{Fact}(m_1 \cdots m_{i-1})$. Thus there are $z, z' \in \Sigma^*$ such that $zr_{i-1}b_{i-1}z' = m_1 \cdots m_{i-1}$. By Lemma 9.4, for each h, with $1 \leq h < i-1$, we have $r_{i-1}b_{i-1} \notin \operatorname{Fact}(m_h m_{h+1})$. Therefore, there is $h, 1 \leq h \leq i-1$, such that $m_h \in \operatorname{Fact}(r_{i-1}b_{i-1})$. Take h minimal with respect to this condition. Then, there would be $u, v \in \Sigma^*$ such that $um_h v = r_{i-1}b_{i-1}$ which implies $zum_h vz' = m_1 \cdots m_{i-1}$. We have $u \neq 1$, otherwise $m_h \in Pref(m_i)$, in contradiction with $m_h \ll m_i$. Thus h > 1. The words m_{h-1} and u are comparable for the suffix order. In addition, m_{h-1} is not a suffix of u by the minimality of h. Hence u would be a nonempty proper suffix of m_{h-1} . Moreover, h < i-1, since m_{i-1} starts with $r_{i-1}a_{i-1}$. Notice that u is a prefix of m_i .

We now use the same argument as in Lemma 9.4. We claim that $u \notin \operatorname{Pref}(m_{h-1})$. Indeed, if u were a prefix of m_{h-1} , then u would be a nonempty border of m_{h-1} . Thus, on one hand $u \notin \operatorname{Pref}(m_h)$ by Proposition 7.5. On the other hand, u would be a prefix both of m_{h-1} and m_i , hence $u \in \operatorname{Pref}(m_h)$ by Corollary 9.1, a contradiction.

Since u is a proper nonempty suffix of m_{h-1} and $u \notin \operatorname{Pref}(m_{h-1})$, we have $u \ll m_{h-1}$ which yields $m_i \ll m_{h-1}$, because $u \in \operatorname{Pref}(m_i)$. This is impossible since $m_{h-1} \ll m_i$.

9.2 The Main Result

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. For any suffix x of m_i , $1 \le i \le k$, we set $x_w = xm_{i+1} \cdots m_k$. In this section we compare a pair of suffixes x, y of factors in ICFL(w) and the corresponding pair of suffixes x_w, y_w of w, with respect to lcp. First we handle suffixes of the same factor m_i (Lemma 9.6), then we focus on suffixes of two different factors m_i, m_j (Lemma 9.8).

Lemma 9.6 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let $r, s, t \in \Sigma^*$, $a, b \in \Sigma$ be such that $m_{i-1} = ras$, $m_i = rbt$, a < b, $1 < i \le k$. If x, y are different nonempty suffixes of m_{i-1} , then $\text{lcp}(x_w, y_w) = \text{lcp}(xr, yr)$.

PROOF :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let $r, s, t \in \Sigma^*$, $a, b \in \Sigma$ be such that $m_{i-1} = ras$, $m_i = rbt$, a < b, $1 < i \leq k$. Let x, y be different nonempty suffixes of m_{i-1} . Set $z = \text{lcp}(x_w, y_w)$. If $|z| \leq \min\{|xr|, |yr|\}$, then clearly $\text{lcp}(x_w, y_w) = \text{lcp}(xr, yr)$.

Assume $|z| > \min\{|xr|, |yr|\}$. Thus, the words xr, yr are comparable for the prefix order. Let $u \in \Sigma^+$ be such that yr = xru (a symmetric argument applies if yr is a proper prefix of xr). Thus |x| < |y|. Since z is a prefix of $xrbtm_{i+1} \cdots m_k$ and $|z| > \min\{|xr|, |yr|\} = |xr|$, there is $v \in \Sigma^*$ such that z = xrbv. Therefore the words xrb and y are comparable for the prefix order, because they are both prefixes of the same word y_w . Hence there is $v_1 \in \Sigma^*$ such that one of the following two cases holds.

$$y = xrbv_1 \tag{9.2}$$

$$xrb = yv_1, \quad v_1 \neq 1 \tag{9.3}$$

Both cases lead to a contradiction. If Eq. (9.2) holds, then rbv_1 is a suffix of $m_{i-1} = ras$ and $m_{i-1} \ll rbv_1$, which is impossible. If Eq. (9.3) holds, since |x| < |y| and $v_1 \neq 1$, we have y = xr', where r' is a nonempty prefix of r. Thus r' is a nonempty border of m_{i-1} and r' is a prefix of m_i , in contradiction with Proposition 7.5.

Lemma 9.7 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\operatorname{ICFL}(w) = (m_1, \ldots, m_k)$. Let i, j be integers such that $1 \leq i < j \leq k$. If x is a nonempty suffix of m_{i-1} and y is a nonempty suffix of m_{j-1} such that x is a proper prefix of y, then $\operatorname{lcp}(x_w, y_w)$ is a prefix of ym_j .

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let i, j be integers such that $1 \leq i < j \leq k$. Let x be a nonempty suffix of m_{i-1} and let y be a nonempty suffix of m_{j-1} such that x is a proper prefix of y. Let $r_{j-1}, s_{j-1}, t_{j-1} \in \Sigma^*$, $a_{j-1}, b_{j-1} \in \Sigma$ be such that $m_{j-1} = r_{j-1}a_{j-1}s_{j-1}, m_j = r_{j-1}b_{j-1}t_{j-1}, a_{j-1} < b_{j-1}$.

Set $z = lcp(x_w, y_w)$. Since z and ym_j are prefixes of the same word y_w , they are comparable for the prefix order. By contradiction, assume that z is not a prefix of ym_j . Thus ym_j is a proper prefix of z, hence of x_w . Since ym_j and $xm_i \cdots m_{j-1}m_j$ are both prefixes of the same word x_w , they are comparable for the prefix order. Moreover $|xm_i \cdots m_{j-1}m_j| > |ym_j|$ because y is a suffix of m_{j-1} and x is nonempty. Hence there exists $v_j \in \Sigma^+$ such that

$$ym_jv_j = yr_{j-1}b_{j-1}t_{j-1}v_j = xm_i\cdots m_{j-1}m_j$$
(9.4)

Since x is a proper prefix of y, there is $x' \in \Sigma^+$ such that y = xx'. Therefore, by Eq. (9.4) we have

$$x'r_{j-1}b_{j-1}t_{j-1}v_j = m_i \cdots m_{j-1}m_j \tag{9.5}$$

If $|m_j| \leq |t_{j-1}v_j|$, then by Eq. (9.5) we have $r_{j-1}b_{j-1} \in \text{Fact}(m_i \cdots m_{j-1})$, in contradiction with Lemma 9.5. Hence $|b_{j-1}t_{j-1}v_j| \leq |m_j| < |r_{j-1}b_{j-1}t_{j-1}v_j|$. Thus, by Eq. (9.4), there are $r'_{j-1} \in \Sigma^+$, $r''_{j-1} \in \Sigma^+$ such that $r_{j-1} = r'_{j-1}r''_{j-1}$ and

$$yr'_{j-1} = xm_i \cdots m_{j-1}$$
 (9.6)

The word r'_{j-1} is a proper prefix of m_{j-1} , thus, by Eq. (9.6), r'_{j-1} is a nonempty border of m_{j-1} . Since r'_{j-1} is a prefix of m_j , this is in contradiction with Proposition 7.5.

Lemma 9.8 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let i, j be integers such that $1 < i < j \leq k$. If x is a nonempty suffix of m_{i-1} and y is a nonempty suffix of m_{j-1} , then $lcp(x_w, y_w)$ is a prefix of ym_j .

PROOF :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word. Let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let $r_i, s_i, t_i \in \Sigma^*$, $a_i, b_i \in \Sigma$ be such that $m_i = r_i a_i s_i$, $m_j = r_i b_i t_i$, $a_i < b_i$, $1 < i < j \leq k$. Let $r_{i-1}, s_{i-1}, t_{i-1} \in \Sigma^*$, $a_{i-1}, b_{i-1} \in \Sigma$ be such that $m_{i-1} = r_{i-1} a_{i-1} s_{i-1}$, $m_j = r_{i-1} b_{i-1} t_{i-1}$, $a_{i-1} < b_{i-1}$. By Lemma 9.2, r_{i-1} is a prefix of r_i .

Let x be a nonempty suffix of m_{i-1} and let y be a nonempty suffix of m_{j-1} . If x is a proper prefix of y, then by Lemma 9.7 we are done. Thus assume that x is not a prefix of y. Set $z = lcp(x_w, y_w)$. If $|z| \leq |yr_{i-1}|$, then z is a prefix of $ym_j \cdots m_k$ shorter than $yr_{i-1}b_{i-1}t_{i-1} = ym_j$, thus z is a prefix of ym_j . Assume $|z| > |yr_{i-1}|$.

Since z is a prefix of $ym_j \cdots m_k = yr_{i-1}b_{i-1}t_{i-1}m_{j+1}\cdots m_k$ and $|z| > |yr_{i-1}|$, there is $v \in \Sigma^*$ such that $z = yr_{i-1}b_{i-1}v$. Therefore the words $yr_{i-1}b_{i-1}$ and x are comparable for the prefix order, because they are both prefixes of the same word x_w . Hence there is $v_1 \in \Sigma^*$ such that one of the following two cases holds.

$$x = yr_{i-1}b_{i-1}v_1 (9.7)$$

$$yr_{i-1}b_{i-1} = xv_1, \quad v_1 \neq 1$$
(9.8)

Both cases lead to a contradiction. If Eq. (9.7) holds, then $r_{i-1}b_{i-1}v_1$ is a suffix of $m_{i-1} = r_{i-1}a_{i-1}s_{i-1}$ and $m_{i-1} \ll r_{i-1}b_{i-1}v_1$, which is impossible. Assume that Eq. (9.8) holds. The

words x and y are comparable for the prefix order and x is not a prefix of y. Therefore we have $x = yr'_{i-1}$, where r'_{i-1} is a nonempty prefix of r_{i-1} . Thus r'_{i-1} is a nonempty proper prefix of both m_{i-1} and m_i . Since $x = yr'_{i-1}$, the word r'_{i-1} is a nonempty border of m_{i-1} and r'_{i-1} is a prefix of m_i , in contradiction with Proposition 7.5.

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. We set

$$\mathcal{M} = \max\{|m_i m_{i+1}| \mid 1 \le i < k\}$$

Proposition 9.1 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. Let i, j be integers such that $1 < i < j \leq k$. If x is a nonempty suffix of m_{i-1} and y is a nonempty suffix of m_{j-1} , then

$$LCP(x_w, y_w) = |lcp(x_w, y_w)| \le \mathcal{M}$$

Proof:

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let i, j be integers such that $1 \leq i < j \leq k$. Let x be a nonempty suffix of m_{i-1} and let y be a nonempty suffix of m_{j-1} . By Lemma 9.8, $\text{lcp}(x_w, y_w)$ is a prefix of ym_j , hence

$$\operatorname{LCP}(x_w, y_w) = |\operatorname{lcp}(x_w, y_w)| \le |ym_j| \le |m_{j-1}m_j| \le \mathcal{M}$$

Proposition 9.2 Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $ICFL(w) = (m_1, \ldots, m_k)$. For each nonempty proper factors u, v of w, we have

$$LCP(u, v) = |lcp(u, v)| \le \mathcal{M}$$

Proof :

Let $w \in \Sigma^+$ be a word which is not an inverse Lyndon word and let $\text{ICFL}(w) = (m_1, \ldots, m_k)$. Let u, v be nonempty proper factors of w. Let $u_1, u_2, v_1, v_2 \in \Sigma^*$ be such that $w = u_1 u u_2 = v_1 v v_2$. Let x be a nonempty suffix of m_{i-1} and let y be a nonempty suffix of m_{j-1} such that $uu_2 = x_w$, $vv_2 = y_w$, with $1 < i \le k$, $1 < j \le k$. If i = j, then by Lemma 9.6 we have

$$\operatorname{LCP}(u, v) = |\operatorname{lcp}(u, v)| \le |\operatorname{lcp}(x_w, y_w)| \le |m_{i-1}m_i| \le \mathcal{M}$$

If $i \neq j$, then by Proposition 9.1, we have

$$\operatorname{LCP}(u, v) = |\operatorname{lcp}(u, v)| \le |\operatorname{lcp}(x_w, y_w)| \le \mathcal{M}$$

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