# Extremal Set Theory and LWE Based Access Structure Hiding Verifiable Secret Sharing with Malicious-Majority and Free Verification 

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#### Abstract

Secret sharing allows a dealer to distribute a secret among a set of parties such that only authorized subsets, specified by an access structure, can reconstruct the secret. Sehrawat and Desmedt (COCOON 2020) introduced hidden access structures, that remain secret until some authorized subset of parties collaborate. However, their scheme assumes semi-honest parties and supports only restricted access structures. We address these shortcomings by constructing a novel access structure hiding verifiable secret sharing scheme that supports all monotone access structures. Our scheme is the first secret sharing solution to support malicious behavior identification and share verifiability in malicious-majority settings. Furthermore, the verification procedure of our scheme incurs no communication overhead, and is therefore "free". As the building blocks of our scheme, we introduce and construct the following:


- a set-system with greater than $\exp \left(c \frac{2(\log h)^{2}}{(\log \log h)}\right)+2 \exp \left(c \frac{(\log h)^{2}}{(\log \log h)}\right)$ subsets of a set of $h$ elements. Our set-system, $\mathcal{H}$, is defined over $\mathbb{Z}_{m}$, where $m$ is a non-prime-power. The size of each set in $\mathcal{H}$ is divisible by $m$ while the sizes of the pairwise intersections of different sets are not divisible by $m$ unless one set is a (proper) subset of the other,
- a new variant of the learning with errors (LWE) problem, called PRIM-LWE, wherein the secret matrix is sampled such that its determinant is a generator of $\mathbb{Z}_{q}^{*}$, where $q$ is the LWE modulus.
Our scheme arranges parties as nodes of a directed acyclic graph and employs modulus switching during share generation and secret reconstruction. For a setting with $\ell$ parties, our (non-linear) scheme supports all $2^{2^{\ell-O(\log \ell)}}$ monotone access structures, and its security relies on the hardness of the LWE problem. Our scheme's maximum share size, for any access structure, is:

$$
(1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+\sqrt{q}+\Theta(h)\right)
$$

where $\varrho \leq 1$ is a constant. We provide directions for future work to reduce the maximum share size to:

$$
\frac{1}{l+1}\left((1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+2 \sqrt{q}\right)\right)
$$

where $l \geq 2$. We also discuss three applications of our secret sharing scheme.

[^0]Keywords: Learning with Errors • Hidden Access Structures • General Access Structures • Verifiable • PRIM-LWE • Extremal Set Theory.

## 1 Introduction

A secret sharing scheme is a method by which a dealer distributes shares of a secret to a set of parties such that only authorized subsets of parties, specified by an access structure, can combine their shares to reconstruct the secret. As noted by Shamir [268], the mechanical approach to secret sharing, involving multiple locks to a mechanical safe, was already known to researchers in combinatorics (see [187], Example 1-11). Digital secret sharing schemes were introduced in the late 1970s by Shamir [268] and Blakley [38] for the $t$-out-of- $\ell$ threshold access structure, wherein all subsets of cardinality at least $t(t \in[\ell])$ are authorized. Ito et al. [156] showed the existence of a secret sharing scheme for every monotone access structure. A number of strengthenings of secret sharing, such as verifiable secret sharing [70], identifiable secret sharing [201], robust secret sharing [247], rational secret sharing [139], ramp secret sharing [37], evolving secret sharing [172], proactive secret sharing [144], dynamic secret sharing [180], secret sharing with veto capability [35], anonymous secret sharing [279], evolving ramp secret-sharing [30], locally repairable secret sharing [2] and leakage-resilient secret sharing [34, 130], have been proposed under varying settings and assumptions. Quantum versions have also been developed for some secret sharing variants (e.g., see [145, 185, 71, 243, 75, 193, 192, 157, 281]). Secret sharing is the foundation of multiple cryptographic constructs and applications, including threshold cryptography [92, 94, 259, 246], (secure) multiparty computation [33, 65, 82, 83, 146, 124, 67], secure distributed storage [272], attribute-based encryption [131, 293, 97], generalized oblivious transfer [283, 269], perfectly secure message transmission [95, 73, 200, 297], access control [216, 151, 173], anonymous communications [267], leakage-resilient circuit compilers [107, 155, 255], e-voting [263, 164, 152], e-auctions [141, 42], secure cloud computing [226, 282], witness pseudorandom functions [173], cloud data security [19, 296], distributed storage blockchain [203, 250, 249, 168, 86], copyright protection [150, 295], indistinguishability obfuscation [173], multimedia applications [137], and private (linear and logistic) regression [120, 270, 66], tree-based models [106] and general machine learning algorithms [91, 212, 213].

### 1.1 Motivation

Hidden Access Structures. Traditional secret sharing models require the access structure to be known to the parties. Since secret reconstruction requires shares of any authorized subset from the access structure, having a public access structure reveals the high-value targets, which can lead to compromised security in the presence of malicious parties. Having a public access structure also implies that some parties must publicly consent to the fact that they themselves are not trusted. As a motivating example, consider a scenario where Alice dictates her will/testament and instructs her lawyer that each of her 10 family members should receive a valid share of the will. In addition, the shares should be indistinguishable from each other in terms of size and entropy. She also insists that to reconstruct her will, \{Bob, Tom, Catherine \} or \{Bob, Cristine, Brad, Roger\} or \{Rob, Eve\} must be part of the collaborating set. However, Alice does not want to be in the bad books of her other, less trusted family members. Therefore, she demands that the shares of her will and the procedure to reconstruct it back from the shares must not reveal her "trust structures", until after the will is successfully reconstructed. This problem can be generalized to secret sharing with hidden access structures, that remain secret until some authorized subset of parties assembles. However, the (only) known access structure hiding secret sharing scheme does not support all $2^{2^{\ell-O(\log \ell)}}$ monotone access
structures, where $\ell$ denotes the number of parties [266, 264], but only those access structures where the smallest authorized subset contains at least half of the total number of parties.

Superpolynomial Size Set-Systems and Efficient Cryptography. In this work, we consider the application of set-systems with specific intersections towards enhancing existing cryptographic protocols for distributed security. To minimize the overall computational and space overhead of such protocols, it is desirable that the parameters such as exponents, moduli and dimensions do not grow too large. For a set-system whose size is superpolynomial in the number of elements over which it is defined, achieving a sufficiently large size requires a smaller modulus and fewer elements, which translates into smaller dimensions, exponents and moduli for its cryptographic applications. Hence, quickly growing set-systems are well-suited for the purpose of constructing (relatively) efficient cryptographic protocols.

Lattice Based Secret Sharing for General Access Structures. Lattice-based cryptosystems are among the leading "post-quantum" cryptographic candidates that are plausibly secure from large-scale quantum computers. For a thorough review of the various implementations of lattice-based cryptosystems, we refer the interested reader to the survey by Nejatollahi et al. [218]. With NIST's latest announcements [10], the transition towards widespread deployment of lattice-based cryptography is expected to pick up even more steam. However, existing lattice-based secret sharing schemes support only threshold access structures [278, 239]. Hence, there is a need to develop lattice-based secret sharing schemes for general (i.e., all monotone) access structures.
(Im)possibility of Verifiable Secret Sharing for Malicious-Majority. In its original form, secret sharing assumes a fault-free system, wherein the dealer and parties are honest. Verifiable secret sharing (VSS) relaxes this assumption, guaranteeing that there is some unique secret that a malicious dealer must "commit" to. The objective of VSS is to resist malicious parties, which are classified as follows:

- a dealer sending incorrect shares,
- malicious parties submitting incorrect shares for secret reconstruction.

VSS is a fundamental building block for many secure distributed computing protocols, such as (secure) multiparty computation and byzantine agreement [1, 62, 108, 161, 230]. Tompa and Woll [285], and McEliece and Sarwate [202] gave the first (partial) solutions to realize VSS, but the notion was defined and fully realized first by Chor et al. [70]. Since then, multiple solutions, under various assumptions, have been proposed $[70,60,65,33,109,126,231,247,121,25,160,64,20,275]$. VSS typically assumes that the parties are connected pairwise by authenticated private channels and they all have a broadcast channel, which allows one party to send a consistent message to all other parties, guaranteeing consistency even if the broadcaster itself is malicious. However, even probabilistically, broadcast cannot be simulated on a point-to-point network when more than a third of the parties are malicious. Therefore, it is infeasible to construct VSS protocols when more than a third of the parties are malicious [182]. Hence, relaxed definitions of verifiability must be explored to design efficient schemes that:

- do not fail when more than a third of the parties are malicious,
- unlike VSS and related concepts, do not require additional communication or cryptographic protocols.


### 1.2 Related Work

A limited number of attempts have been made to introduce privacy-preserving features to secret sharing. The first solution that focused on bolstering privacy for secret sharing was called anonymous secret sharing, wherein the secret can be reconstructed without the knowledge of which parties hold which shares [279]. In such schemes, secret reconstruction can be performed by giving the shares to a black box that does not know the identities of the parties holding those shares. As pointed out by Guillermo et al. [136], anonymous secret sharing does not provide cryptographic anonymity. Existing anonymous secret sharing schemes either operate in restricted settings (e.g., $\ell$-out-of- $\ell, 2$-out-of- $\ell$ threshold) or use difficult to generate underlying primitives [279, 238, 40, 169, 242, 303]. For instance, the constructions from [279, 40] use resolvable Steiner systems [277], which are non-trivial to achieve and have only a few known results in restricted settings [59, 284, 76, 90, 179, 214, 240, 251, 110, 174, 300, 301]. There are also known impossibility results concerning the existence of certain desirable Steiner systems [229]. For an introduction to Steiner systems, we refer the interested reader to [78, 77]. Kishimoto et al. [169] employed combinatorics to realize anonymous secret sharing, thereby avoiding the difficult to generate primitives. However, their scheme also works for only certain specific thresholds.

Daza and Domingo-Ferrer [89] aimed at achieving a weaker form of anonymous secret sharing wherein the notion of privacy is analogous to that for ring signatures [253], i.e., instead of a party's identity, only its subset membership is leaked. Recently, Sehrawat and Desmedt [266] introduced access structure hiding secret sharing for restricted access structures, wherein no non-negligible information about the access structure gets revealed until some authorized subset of parties assembles. They constructed novel set-systems and vector families to "encode" the access structures such that deterministic and private assessments can be conducted to test whether a given subset of parties is authorized for secret reconstruction.

### 1.3 Our Contributions

The access structure hiding secret sharing scheme from [266] has the following limitations:

1. It assumes semi-honest polynomial-time adversaries, which try to gain additional information while correctly following the protocol. Hence, the scheme fails in the presence of malicious adversaries, which are not guaranteed to follow the protocol correctly.
2. It requires that the smallest authorized subset contain at least half of the total number of parties.

We address these limitations by introducing access structure hiding verifiable secret sharing, which supports all monotone access structures and remains "verifiable" even when a majority of the parties are malicious. Our detailed contributions follow:

Novel Superpolynomial Sized Set-Systems and Vector Families. In order to build our access structure hiding verifiable secret sharing scheme, we construct a set-system that is described by Theorem 1 in the following text.

Definition 1. We say that a family of sets $\left\{G_{1}, G_{2}, \ldots, G_{t}\right\}$ is non-degenerate if there does not exist $1 \leq i \leq t$ such that $G_{i} \subseteq G_{j}$ for all $1 \leq j \leq t$.

Definition 2. Let $m \geq 2, t \geq 2$ be integers and $\mathcal{H}$ be a set-system. We shall say that $\mathcal{H}$ has $t$-wise restricted intersections modulo $m$ if the following two conditions hold:

1. $\forall H \in \mathcal{H},|H|=0 \bmod m$,
2. $\forall t^{\prime}$ satisfying $2 \leq t^{\prime} \leq t$, and $\forall H_{1}, H_{2}, \ldots, H_{t^{\prime}} \in \mathcal{H}$ with $\left\{H_{1}, H_{2}, \ldots, H_{t^{\prime}}\right\}$ non-degenerate, it holds that:

$$
\left|\bigcap_{\tau=1}^{t^{\prime}} H_{\tau}\right| \neq 0 \bmod m
$$

Theorem 1. Let $\left\{\alpha_{i}\right\}_{i=1}^{r}$ be $r>1$ positive integers and $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r$ different odd prime divisors: $p_{1}, \ldots, p_{r}$, and $l \geq 2$ be an integer such that $l<\min \left(p_{1}, \ldots, p_{r}\right)$. Then, there exists $c>0$ such that for all integers $t \geq 2$ and $h \geq l m$, there exists an explicitly constructible non-uniform ${ }^{\boldsymbol{\sigma}}$ set-system $\mathcal{H}$, defined over a universe of $h$ elements, such that

$$
\text { 1. }|\mathcal{H}|>\exp \left(c \frac{l(\log h)^{r}}{(\log \log h)^{r-1}}\right)+l \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right) \text {, }
$$

2. $\forall H_{1}, H_{2} \in \mathcal{H}$, either $\left|H_{1}\right|=\left|H_{2}\right|,\left|H_{1}\right|=l\left|H_{2}\right|$ or $l\left|H_{1}\right|=\left|H_{2}\right|$,
3. $\mathcal{H}$ has $t$-wise restricted intersections modulo $m$.

Recall that $a \bmod m$ denotes the smallest non-negative $b=a \bmod m$. Since the access structure $\Gamma$ is monotone, it holds that if $\mathcal{B} \supseteq \mathcal{A}$, for some $\mathcal{A} \in \Gamma$, then $\mathcal{B} \in \Gamma$. We derive a family of vectors $\mathcal{V} \in\left(\mathbb{Z}_{m}\right)^{h}$ from our set-system $\mathcal{H}$, that captures the superset-subset relations in $\mathcal{H}$ as (vector) inner products in $\mathcal{V}$. This capability allows us to capture special information about each authorized subset $\mathcal{A} \in \Gamma$ in the form of an inner product, enabling us to devise an efficient test for whether a given subset of parties $\mathcal{B}$ is a superset of any $\mathcal{A} \in \Gamma$.

PRIM-LWE. Informally, (the multi-secret version of) the learning with errors (LWE) problem [252] asks for the solution of a system of noisy linear modular equations: given positive integers $n, w=\operatorname{poly}(n)$ and $q \geq 2$, an LWE sample consists of $(\mathbf{A}, \mathbf{B}=\mathbf{A S}+\mathbf{E} \bmod q)$ for a fixed secret $\mathbf{S} \in \mathbb{Z}_{q}^{n \times n}$ with small entries, and $\mathbf{A} \stackrel{\$}{\leftrightarrows} \mathbb{Z}_{q}^{w \times n}$. The error term $\mathbf{E} \in \mathbb{Z}^{w \times n}$ is sampled from some distribution supported on small numbers, typically a (discrete or rounded) Gaussian distribution with standard deviation $\alpha q$ for $\alpha=o(1)$. We introduce a new variant of the LWE problem, called PRIM-LWE, wherein the matrix $\mathbf{S}$ can be sampled from the set of matrices whose determinants are generators of $\mathbb{Z}_{q}^{*}$. We prove that, up to a constant factor, PRIM-LWE is as hard as the plain LWE problem.

Access Structure Hiding Verifiable Secret Sharing Scheme. We use our novel set-system and vector family to generate PRIM-LWE instances, and thereby construct the first access structure hiding verifiable (computational) secret sharing scheme that guarantees secrecy, correctness and verifiability (with high probability) even when a majority of the parties are malicious. To detect malicious behavior, we postpone the verification procedure until after secret reconstruction. The idea of delaying verification till secret reconstruction is also used in identifiable secret sharing [201] wherein parties only interact with a trusted external stateless server and the goal is to inform each honest player of the correct set of cheaters. However, unlike the identifiable secret sharing solutions [ $72,178,227,201,228,154,142,69]$, our scheme supports share verification and does not require any digital signature or message authentication subroutines. Furthermore, our scheme does not require any dedicated round to verify whether the reconstructed secret is consistent with all participating shares. Our scheme is graph-based with the parties represented by nodes in a directed acyclic graph (DAG). For a setting with $\ell$ parties, our (non-linear) scheme supports all monotone access

[^1]structures, and its security relies on the hardness of the LWE problem. The maximum share size of our scheme is $(1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+\sqrt{q}+\Theta(h)\right)$, where $q$ is the LWE modulus and $\varrho \leq 1$ is a constant. We also describe improvements that will lead to an access structure hiding verifiable secret sharing scheme with maximum share size equal to:
$$
\frac{1}{l+1}\left((1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+2 \sqrt{q}\right)\right),
$$
where $l \geq 2$ (as defined by Theorem 1 ).

## 1.4 "Free" Verification at the Expense of Larger Shares

In the first secret sharing scheme for general (monotone) access structures [156], the share size is proportional to the depth 2 complexity of the access structure when viewed as a Boolean function; hence, shares are exponential for most access structures. While for specific access structures, the share size of the later schemes $[57,158,271]$ is less than the share size for the scheme from [156], the share size of all schemes for general access structures remained $2^{\ell-o(\ell)}$ ( $\ell$ denotes the number of parties) until 2018, when Liu and Vaikuntanathan [188] (using results from [190]) constructed a secret sharing scheme for general access structures with a share size of $2^{0.944 \ell}$. Applebaum et al. [16] (using the results of [15, 190]) constructed a secret sharing scheme for general access structures with a share size of $2^{0.637 \ell+o(\ell)}$. Whether the share size for general access structures can be improved to $2^{o(\ell)}$ (or even smaller) remains an important open problem. On the other hand, multiple works [41, 63, 84, 85, 288] have proved various lower bounds on the share size of secret sharing for general access structures, with the best being $\Omega\left(\ell^{2} / \log \ell\right)$ from Csirmaz [84].

The maximum share size of our access structure hiding verifiable secret sharing scheme is:

$$
(1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+\sqrt{q}+\Theta(h)\right)
$$

where $q$ is the LWE modulus and $\varrho \leq 1$ is a constant. Therefore, the maximum share size of our scheme is larger than the best known upper bound of $2^{0.637 \ell+o(\ell)}$ on the share size for secret sharing over general access structures. However, at the expense of the larger share size, our scheme achieves "free" verification because unlike the existing VSS protocols, whose verification procedures incur at least $O\left(\ell^{2}\right)$ communication overhead $[60,20]$, the verification procedure of our scheme does not incur any communication overhead.

### 1.5 Applications

In this section, we discuss three example applications of our access structure hiding verifiable secret sharing scheme.

Frameproof Secret Sharing. In secret sharing, any authorized subset of parties can compute the shares of another authorized subset of parties and use the latter's shares to perform non licet activities. For example, they can reconstruct a key and sign a message on behalf of their organization, and later, during audit, they can put the blame on the other authorized subset. By doing this they may escape the accountability of using the key inappropriately. Recently, Desmedt et al. [93] captured this threat by defining framing in secret sharing schemes as the ability of some subset $\mathcal{A} \subset \mathcal{P}$ to compute the share of any participant $P_{i} \in \mathcal{P} \backslash \mathcal{A}$.

In our access structure hiding verifiable secret sharing scheme, the share of each party $P_{i}$ is sealed as a PRIM-LWE instance such that the lattice basis, $\mathbf{A}_{i}$, used to generate it is known only to $P_{i}$. Since $\mathbf{A}_{i}$ is required to generate $P_{i}$ 's share, it is infeasible for any coalition of polynomial-time parties $\mathcal{A} \subset \mathcal{P}$ to compute the share of $P_{i} \in \mathcal{P} \backslash \mathcal{A}$ without solving the LWE problem. Hence, our access structure hiding verifiable secret sharing scheme is not vulnerable to framing, and is therefore frameproof.

Eternity Service. Eternity service aims to use redundancy and scattering techniques to replicate data across a large set of machines (such as the Internet), and add anonymity mechanisms to increase the cost of selective service denial attacks [14]. We know that secret sharing is a provably secure scattering technique. Moreover, hidden access structures guarantee that neither an insider nor outsider (polynomial) adversary can know the access structure without collecting all shares of some authorized subset, making it impossible for the adversary to identify targets for selective service denial attacks. Hence, access structure hiding secret sharing fits the requirements for realizing eternity service.

Undetectable Honeypots. Honeypots are information systems resources conceived to attract, detect, and gather attack information. Honeypots serve several purposes, including the following:

- distracting adversaries from more valuable machines on a network,
- providing early warning about new attack and exploitation trends,
- allowing in-depth examination of adversaries during and after exploitation of the honeypot.

The value of a honeypot is determined by the information that we can obtain from it. Monitoring the data that enters and leaves a honeypot lets us gather information that is not available to network intrusion detection systems. For example, we can log the key strokes of an interactive session even if encryption is used to protect the network traffic. Although the concept is not new [280], interest in protection and countermeasure mechanisms using honeypots has become popular only during the past two decades [26, 177, 287, 18]. For an introduction to the topic, we refer the interested reader to [274]. Unfortunately, honeypots are easy to detect and avoid $[286,176,217,291,304,143,257,256,148,96,289]$.

In scenarios wherein secret sharing is used to distribute a secret (e.g., encryption keys) among multiple servers, hidden access structures would allow the dealer to provide all servers with legitimate shares while enforcing zero or negligible information leakage without some authorized subset's shares. Moreover, each share corresponds to the same secret and the entropy of all shares is equal. Since access structures are hidden, the dealer can keep servers out of the minimal authorized subsets without revealing this information. Shares from such servers are "useless" since their participation is only optional for successful secret reconstruction. Because their shares do not hold any value without the participation of an authorized subset, these servers can be exposed to attackers and turned into honeypots. Furthermore, since the protocol allows all servers to participate in secret reconstruction, identifying honeypots is impossible until successful secret reconstruction.

### 1.6 Organization

The rest of the paper is organized as follows: Section 2 recalls necessary definitions and constructs that are required for our constructions and solutions. Section 3 formally defines access structure hiding verifiable secret sharing scheme. In Section 4, we construct the first building block for our secret sharing scheme, i.e., our superpolynomial size set-systems and vector families. Section 5 establishes that our set-systems can be operated upon via the vector families. Section 6 extends the idea from Section 5 by introducing
access structure tokens and giving an example procedure to generate access structure tokens to "encode" any monotone access structure. In Section 7, we introduce a new variant of LWE, called PRIM-LWE. We present our access structure hiding verifiable secret sharing scheme in Section 8. We conclude with a conclusion in Section 9.

## 2 Preliminaries

For a positive integer $n$, let $[n]$ denote the set of the first $n$ positive integers, i.e., $[n]=\{1, \ldots, n\}$.
Theorem 2 (Dirichlet's Theorem). For all coprime integers $c$ and $q$, there are infinitely many primes, $p$, of the form $p=c \bmod q$.

Theorem 3 (Fermat's Little Theorem). If $p$ is a prime and $c$ is any number coprime to $p$, then $c^{p-1}=1 \bmod p$.

Theorem 4 (Euler's Theorem). Let $y$ be a positive integer and $\mathbb{Z}_{y}^{*}$ denote the multiplicative group modulo $y$. Then for every integer $c$ that is coprime to $y$, it holds that: $c^{\varphi(y)}=1 \bmod y$, where $\varphi(y)=\left|\mathbb{Z}_{y}^{*}\right|$ denotes Euler's totient function.

For a detailed background on Theorems 2 to 4, we refer the interested reader to [140].
Definition 3 (Hadamard/Schur product). Hadamard/Schur product of two vectors $\mathbf{u}, \mathbf{v} \in \mathcal{R}^{n}$, denoted by $\mathbf{u} \circ \mathbf{v}$, returns a vector in the same linear space whose $i$-th element is defined as: $(\mathbf{u} \circ \mathbf{v})[i]=\mathbf{u}[i] \cdot \mathbf{v}[i]$, for all $i \in[n]$.

Definition 4 (Negligible Function). For security parameter $\omega$, a function $\epsilon(\omega)$ is called negligible if for all $c>0$, there exists a $\omega_{0}$ such that $\epsilon(\omega)<1 / \omega^{c}$ for all $\omega>\omega_{0}$.

Definition 5 (Computational Indistinguishability [128]). Let $X=\left\{X_{\lambda}\right\}_{\lambda \in \mathbb{N}}$ and $Y=\left\{Y_{\lambda}\right\}_{\lambda \in \mathbb{N}}$ be ensembles, where $X_{\lambda}$ 's and $Y_{\lambda}$ 's are probability distributions over $\{0,1\}^{\kappa(\lambda)}$ for some polynomial $\kappa(\lambda)$. We say that $\left\{X_{\lambda}\right\}_{\lambda \in \mathbb{N}}$ and $\left\{Y_{\lambda}\right\}_{\lambda \in \mathbb{N}}$ are polynomially/computationally indistinguishable if the following holds for every (probabilistic) polynomial-time algorithm $\mathcal{D}$ and all $\lambda \in \mathbb{N}$ :

$$
\left|\operatorname{Pr}\left[t \leftarrow X_{\lambda}: \mathcal{D}(t)=1\right]-\operatorname{Pr}\left[t \leftarrow Y_{\lambda}: \mathcal{D}(t)=1\right]\right| \leq \epsilon(\lambda),
$$

where $\epsilon$ is a negligible function.
Definition 6 (Access Structure). Let $\mathcal{P}=\left\{P_{1}, \ldots, P_{\ell}\right\}$ be a set of parties. A collection $\Gamma \subseteq 2^{\mathcal{P}}$ is monotone if $\mathcal{A} \in \Gamma$ and $\mathcal{A} \subseteq \mathcal{B}$ imply that $\mathcal{B} \in \Gamma$. An access structure $\Gamma \subseteq 2^{\mathcal{P}}$ is a monotone collection of non-empty subsets of $\mathcal{P}$. Sets in $\Gamma$ are called authorized, and sets not in $\Gamma$ are called unauthorized.

If $\Gamma$ consists of all subsets of $\mathcal{P}$ with size greater than or equal to a fixed threshold $t(1 \leq t \leq \ell)$, then $\Gamma$ is called a $t$-threshold access structure. In its most general form, an access structure can be any monotone NP language. This was first observed by Steven Rudich in private communications with Moni Naor [27, 215].

Definition 7 (Closure). Let $\mathcal{P}$ be a set of participants and $\mathcal{A} \in 2^{\mathcal{P}}$. The closure of $\mathcal{A}$, denoted by $\operatorname{cl}(\mathcal{A})$, is the set

$$
\operatorname{cl}(\mathcal{A})=\left\{\mathcal{C}: \mathcal{C}^{*} \subseteq \mathcal{C} \subseteq \mathcal{P} \text { for some } \mathcal{C}^{*} \in \mathcal{A}\right\}
$$

Definition 8 (Minimal Authorized Subset). For an access structure $\Gamma$, a family of minimal authorized subsets $\Gamma_{0} \in \Gamma$ is defined as:

$$
\Gamma_{0}=\{\mathcal{A} \in \Gamma: \mathcal{B} \not \subset \mathcal{A} \text { for all } \mathcal{B} \in \Gamma \backslash\{\mathcal{A}\}\}
$$

Hence, the family of minimal access subsets $\Gamma_{0}$ uniquely determines the access structure $\Gamma$, and it holds that: $\Gamma=\operatorname{cl}\left(\Gamma_{0}\right)$, where cl denotes closure.

Definition 9 (Computational Secret Sharing [175]). A computational secret sharing scheme with respect to an access structure $\Gamma$, security parameter $\omega$, a set of $\ell$ polynomial-time parties $\mathcal{P}=\left\{P_{1}, \ldots, P_{\ell}\right\}$, and a set of secrets $\mathcal{K}$, consists of a pair of polynomial-time algorithms, (Share, Recon), where:

- Share is a randomized algorithm that gets a secret $k \in \mathcal{K}$ and access structure $\Gamma$ as inputs, and outputs $\ell$ shares, $\left\{\Pi_{1}^{(k)}, \ldots, \Pi_{\ell}^{(k)}\right\}$, of $k$,
- Recon is a deterministic algorithm that gets as input the shares of a subset $\mathcal{A} \subseteq \mathcal{P}$, denoted by $\left\{\Pi_{i}^{(k)}\right\}_{i \in \mathcal{A}}$, and outputs a string in $\mathcal{K}$,
such that, the following two requirements are satisfied:

1. Perfect Correctness: for all secrets $k \in \mathcal{K}$ and every authorized subset $\mathcal{A} \in \Gamma$, it holds that: $\operatorname{Pr}\left[\operatorname{Recon}\left(\left\{\Pi_{i}^{(k)}\right\}_{i \in \mathcal{A}}, \mathcal{A}\right)=k\right]=1$,
2. Computational Secrecy: for every unauthorized subset $\mathcal{B} \notin \Gamma$ and all different secrets $k_{1}, k_{2} \in \mathcal{K}$, it holds that the distributions $\left\{\Pi_{i}^{\left(k_{1}\right)}\right\}_{i \in \mathcal{B}}$ and $\left\{\Pi_{i}^{\left(k_{2}\right)}\right\}_{i \in \mathcal{B}}$ are computationally indistinguishable (w.r.t. $\omega$ ).
Remark 1 (Perfect Secrecy). If $\forall k_{1}, k_{2} \in \mathcal{K}$ with $k_{1} \neq k_{2}$, the distributions $\left\{\Pi_{i}^{\left(k_{1}\right)}\right\}_{i \in \mathcal{B}}$ and $\left\{\Pi_{i}^{\left(k_{2}\right)}\right\}_{i \in \mathcal{B}}$ are identical, then the scheme is called a perfect secret sharing scheme.

Steven Rudich proved that if NP $\neq$ coNP, then efficient (i.e., polynomial-time) perfect secret sharing is impossible for Hamiltonian and monotone NP access structures, and efficient computational secret sharing is the best that we can do [171].

Definition 10 (Disjoint Union). Let $n \geq 2$ be an integer and $\mathcal{H}=\left\{H_{i}: i \in[n]\right\}$ be a family of sets. Then, disjoint union of $\mathcal{H}$ is given as:

$$
\bigsqcup_{i \in[n]} H_{i}=\bigcup_{i \in[n]}\left\{(h, i): h \in H_{i}\right\} .
$$

The Hybrid Argument. The hybrid argument [129], which is essentially the triangle inequality, is one of the most fundamental tools used in security proofs [211]. In cryptography, the canonical application of the hybrid argument is towards constructing the (inductive) arguments underlying various pseudorandom generators [39, 298, 125, 149, 223, 225, 224, 153]. Here, we give an informal introduction to the hybrid argument. For a formal account, we refer the interested reader to [112].

The hybrid argument is a technique to bound the closeness of two distributions, $D_{0}$ and $D_{n}$, via a polynomially long sequence of "hybrids", $D_{0}, D_{1}, \ldots, D_{n}$, which are constructed such that any two consecutive hybrids differ in exactly one feature. The central idea behind the hybrid argument is that if a (bounded or unbounded) distinguisher can distinguish the "extreme hybrids" $D_{0}$ and $D_{n}$, then it can also distinguish any adjacent hybrids $D_{i}$ and $D_{i+1}$, which it cannot do by the design of the hybrids. Therefore, the triangle inequality (for statistical or computational distance) can be used to obtain a bound on the distance between $D_{0}$ and $D_{n}$ by bounding the distance between neighboring distributions $D_{i}$ and $D_{i+1}$ for all $i \in\{0\} \cup[n-1]$.

Set Systems with Restricted Intersections. Extremal set theory is a field within combinatorics which deals with determining or estimating the size of set-systems, satisfying certain restrictions. The first result in extremal set theory was from Sperner [273] in 1928, establishing the maximum size of an antichain, i.e., a set-system where no member is a superset of another. But, it was Erdős et al.'s pioneering work in 1961 [103] that started systematic research on extremal set theory problems. Our work in this paper concerns a subfield of extremal set theory, called intersection theorems, wherein set-systems under certain intersection restrictions are constructed, and bounds on their sizes are derived. We shall not give a full account of the known intersection theorems and mention only the results that are relevant to our set-system and its construction. For a broader account of intersection theorems over finite sets, we refer the interested reader to the comprehensive survey by Frankl and Tokushige [114]. For an introduction to intersecting and cross-intersecting families related to hypergraph coloring, please see [248].

Lemma 1 ( [134]). Let $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r>1$ different prime divisors. Then there exists an explicitly constructible polynomial $Q$ with n variables and degree $O\left(n^{1 / r}\right)$, which is equal to 0 on $z=(1,1, \ldots, 1) \in\{0,1\}^{n}$ but is nonzero mod $m$ on all other $z \in\{0,1\}^{n}$. Furthermore, $\forall z \in\{0,1\}^{n}$ and $\forall i \in\{1, \ldots, r\}$, it holds that: $Q(z) \in\{0,1\} \bmod p_{i}^{\alpha_{i}}$.

Theorem 5 ( [134]). Let $m$ be a positive integer, and suppose that $m$ has $r>1$ different prime divisors: $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$. Then there exists $c=c(m)>0$, such that for every integer $h>0$, there exists an explicitly constructible uniform set-system $\mathcal{H}$ over a universe of $h$ elements such that:

1. $|\mathcal{H}| \geq \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right)$,
2. $\forall H \in \mathcal{H}:|H|=0 \bmod m$,
3. $\forall G, H \in \mathcal{H}, G \neq H:|G \cap H| \neq 0 \bmod m$.

Matching Vectors. A matching vector family is a combinatorial object that is defined as:
Definition 11 ( $[\mathbf{1 0 0}])$. Let $S \subseteq \mathbb{Z}_{m} \backslash\{0\}$, and $\langle\cdot, \cdot\rangle$ denote the inner product. We say that subsets $\mathcal{U}=\left\{\mathbf{u}_{i}\right\}_{i=1}^{N}$ and $\mathcal{V}=\left\{\mathbf{v}_{i}\right\}_{i=1}^{N}$ of vectors in $\left(\mathbb{Z}_{m}\right)^{h}$ form an $S$-matching family if the following two conditions are satisfied:

- $\forall i \in[N]$, it holds that: $\left\langle\mathbf{u}_{i}, \mathbf{v}_{i}\right\rangle=0 \bmod m$,
$-\forall i, j \in[N]$ such that $i \neq j$, it holds that: $\left\langle\mathbf{u}_{i}, \mathbf{v}_{j}\right\rangle \bmod m \in S$.
The question of bounding the size of matching vector families is closely related to the well-known extremal set theory problem of constructing set systems with restricted modular intersections. Matching vectors have found applications in the context of private information retrieval [28, 29, 101, 100, 102, 299, 189], conditional disclosure of secrets [189], secret sharing [190] and coding theory [100]. The first super-polynomial size matching vector family follows directly from the set-system constructed by Grolmusz [134]. If each set $H$ in the set-system $\mathcal{H}$ defined by Theorem 5 is represented by a vector $\mathbf{u} \in\left(\mathbb{Z}_{m}\right)^{h}$, then it leads to the following family of $S$-matching vectors:

Corollary 1 (to Theorem 5). For $h>0$, suppose that a positive integer $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ has $r>1$ different prime divisors: $p_{1}, \ldots, p_{r}$. Then, there exists a set $S$ of size $2^{r}-1$ and a family of $S$-matching vectors $\left\{\mathbf{u}_{i}\right\}_{i=1}^{N}, \mathbf{u}_{i} \in\left(\mathbb{Z}_{m}\right)^{h}$, such that, $N \geq \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right)$.

Lattices. A lattice $\Lambda$ of $\mathbb{R}^{w}$ is defined as a discrete subgroup of $\mathbb{R}^{w}$. In cryptography, we are interested in integer lattices, i.e., $\Lambda \subseteq \mathbb{Z}^{w}$. Given $w$-linearly independent vectors $\mathbf{b}_{1}, \ldots, \mathbf{b}_{w} \in \mathbb{R}^{w}$, a basis of the lattice generated by them can be represented as the matrix $\mathbf{B}=\left(\mathbf{b}_{1}, \ldots, \mathbf{b}_{w}\right) \in \mathbb{R}^{w \times w}$. The lattice generated by $\mathbf{B}$ is the following set of vectors:

$$
\Lambda=\mathcal{L}(\mathbf{B})=\left\{\sum_{i=1}^{w} c_{i} \mathbf{b}_{i}: c_{i} \in \mathbb{Z}\right\} .
$$

The lattices that are of particular interest in lattice-based cryptography are called $q$-ary lattices, and they satisfy the following condition:

$$
q \mathbb{Z}^{w} \subseteq \Lambda \subseteq \mathbb{Z}^{w}
$$

for some (possibly prime) integer $q$. In other words, the membership of a vector $\mathbf{x}$ in $\Lambda$ is determined by $\mathbf{x} \bmod q$. Given a matrix $\mathbf{A} \in \mathbb{Z}_{q}^{w \times n}$ for some integers $q, w, n$, we can define the following two $n$-dimensional q-ary lattices,

$$
\begin{aligned}
& \Lambda_{q}(\mathbf{A})=\left\{\mathbf{y} \in \mathbb{Z}^{n}: \mathbf{y}=\mathbf{A}^{T} \mathbf{s} \bmod q \text { for some } \mathbf{s} \in \mathbb{Z}^{w}\right\}, \\
& \Lambda_{q}^{\perp}(\mathbf{A})=\left\{\mathbf{y} \in \mathbb{Z}^{n}: \mathbf{A y}=\mathbf{0} \bmod q\right\} .
\end{aligned}
$$

The first $q$-ary lattice is generated by the rows of $\mathbf{A}$; the second contains all vectors that are orthogonal (modulo $q$ ) to the rows of $\mathbf{A}$. Hence, the first $q$-ary lattice, $\Lambda_{q}(\mathbf{A})$, corresponds to the code generated by the rows of $\mathbf{A}$ whereas the second, $\Lambda_{q}^{\perp}(\mathbf{A})$, corresponds to the code whose parity check matrix is $\mathbf{A}$. For a complete introduction to lattices, we refer the interested reader to the monographs by Grätzer [132, 133].

Lattices and Cryptography. Problems in lattices have been of interest to cryptographers for decades with the earliest work dating back to 1997 when Ajtai and Dwork [6] proposed a lattice-based public key cryptosystem following Ajtai's [4] seminal worst-case to average-case reductions for lattice problems, wherein he showed that if there is no efficient algorithm that approximates the decision version of the Shortest Vector Problem (SVP) with a polynomial approximation factor, then it is hard to solve the associated search problem exactly over a random choice of the underlying lattice [206]. This reduction gave us the first cryptographically meaningful lattice-based hardness assumption, which became an essential component in proving the security of numerous lattice-based cryptographic constructions. For a detailed introduction to lattice-based cryptography, we refer the interested reader to [162, 302, 205, 204, 220].

Learning with Errors. The learning with errors (LWE) problem [252] has emerged as the most popular hard problem for constructing lattice-based cryptographic solutions. The majority of practical LWE-based cryptosystems are derived from its variants such as ring LWE [197], module LWE [183], cyclic LWE [135], continuous LWE [58], middle-product LWE [254], group LWE [116], entropic LWE [56] and polynomial-ring LWE [276]. Many cryptosystems have been constructed whose security can be proved under the hardness of the LWE problem, including (identity-based, attribute-based, leakage-resilient, fully homomorphic, functional, public-key/key-encapsulation) encryption [12, 165, 292, 252, 123, 9, 197, 3, 54, 127, 87, 36, 45, 46, 47, $51,105,48,191,194]$, oblivious transfer [236, 50, 244], (blind) signatures [123, 195, 258, 196, 11, 98, 113], pseudorandom functions with special algebraic properties [23, 44, 22, 21, 55, 265, 43, 53, 61, 166, $167,245]$, hash functions [163, 234], secure matrix multiplication [99, 290], classically verifiable quantum computation [199], noninteractive zero-knowledge proof system for (any) NP language [235], obfuscation [186, $122,138,56,13,81]$, multilinear maps [119, 122, 74], lossy-trapdoor functions [32, 237, 294], and many more [233, 218].

Definition 12 (Decision-LWE [252]). For positive integers $n$ and $q \geq 2$, and an error (probability) distribution $\chi=\chi(n)$ over $\mathbb{Z}_{q}$, the decision-LWE ${ }_{n, q, \chi}$ problem is to distinguish between the following pairs of distributions:

$$
(\mathbf{A}, \mathbf{A s}+\mathbf{e}) \quad \text { and } \quad(\mathbf{A}, \mathbf{u}),
$$

where $\mathbf{A} \stackrel{\$}{\leftarrow} \mathbb{Z}_{q}^{w \times n}, w=\operatorname{poly}(n), \mathbf{s} \in \mathbb{Z}_{q}^{n}, \mathbf{e} \stackrel{\$}{\leftarrow} \chi^{w}$ and $\mathbf{u} \stackrel{\$}{\leftarrow} \mathbb{Z}_{q}^{w}$.
Definition 13 (Search-LWE [252]). For positive integers $n$ and $q \geq 2$, and an error (probability) distribution $\chi=\chi(n)$ over $\mathbb{Z}_{q}$, the search- $\operatorname{LWE}_{n, q, \chi}$ problem is to recover $\mathbf{s} \in \mathbb{Z}_{q}^{n}$, given $(\mathbf{A}, \mathbf{A s}+\mathbf{e})$, where $\mathbf{A} \stackrel{\$}{\leftarrow} \mathbb{Z}_{q}^{w \times n}, \mathbf{s} \in \mathbb{Z}_{q}^{n}, \mathbf{e} \stackrel{\$}{\leftarrow} \chi^{w}$ and $w=\operatorname{poly}(n)$.

Regev [252] showed that for certain noise distributions and a sufficiently large $q$, the LWE problem is as hard as the worst-case SIVP and GapSVP under a quantum reduction (see [232,52] for classical hardness arguments). Regev's results were extended to establish that the fixed vector $\mathbf{s}$ can be sampled from a low norm distribution (in particular, from the noise distribution $\chi$ ) and the resulting problem is as hard as the original LWE problem [17]. Later, it was discovered that $\chi$ can also be a simple low-norm distribution [208]. Therefore, a standard hybrid argument can be used to get to multi-secret LWE, which asks to distinguish $(\mathbf{A}, \mathbf{B}=\mathbf{A S}+\mathbf{E})$ from $(\mathbf{A}, \mathbf{U})$ for $\mathbf{A} \stackrel{\$}{\leftarrow} \mathbb{Z}_{q}^{w \times n}, \mathbf{S} \in \mathbb{Z}_{q}^{n \times n}$ or $\mathbf{S} \in \chi^{n \times n}, \mathbf{E} \stackrel{\$}{\leftarrow} \chi^{w \times n}$, and a uniformly sampled $\mathbf{U} \in \mathbb{Z}_{q}^{w \times n}$. It is easy to verify that up to a $w$ factor loss in the distinguishing advantage, multi-secret LWE is equivalent to plain (single-secret) decision-LWE. Lattice reduction algorithms, which are the most powerful tools against LWE, remain (practically) inefficient in solving LWE [8, 111, 115, 117, 118, 184, 221, 222, 210, 209, 241, 260, 261, 262, 219].

Trapdoors for Lattices. Trapdoors for lattices have been studied in [5, 207, 123, 68, 49, 147, 237, 198]. We recall the definition from [207] as that is the algorithm used in our scheme.
Definition 14. Let $n \geq w d$ be an integer and $\bar{n}=n-w d$. For $\mathbf{A} \in \mathbb{Z}_{q}^{w \times n}$, we say that $\mathbf{R} \in \mathbb{Z}_{q}^{\bar{n} \times w d}$ is a trapdoor for $\mathbf{A}$ with $\operatorname{tag} \mathbf{H} \in \mathbb{Z}_{q}^{w \times w}$ if $\mathbf{A}\left[\begin{array}{c}\mathbf{R} \\ \mathbf{I}\end{array}\right]=\mathbf{H} \cdot \mathbf{G}$, where $\mathbf{G} \in \mathbb{Z}_{q}^{w \times w d}$ is a primitive matrix.

Given a trapdoor $\mathbf{R}$ for $\mathbf{A}$, and an LWE instance $\mathbf{B}=\mathbf{A S}+\mathbf{E} \bmod q$ for some "short" error matrix $\mathbf{E}$, the LWE inversion algorithm from [207] successfully recovers $\mathbf{S}$ (and $\mathbf{E}$ ) with overwhelming probability.

STCON. STCON (s-t connectivity) in a directed graph can be defined as the following function: the input is a directed graph $G$. The graph contains two designated nodes, $s$ and $t$. The function outputs 1 if and only if $G$ has a directed path from $s$ to $t$. Karchmer and Wigderson [158] showed that there exists an efficient linear secret sharing scheme for the analogous function where the graph is undirected. In a linear secret sharing scheme [159], share generation and secret reconstruction are performed by evaluating linear maps and solving linear systems of equations. Later, Beimel and Paskin [31] extended those results to linear secret sharing schemes for STCON in directed graphs. It is known that (linear) secret sharing schemes based on undirected STCON have strictly smaller share size than those based on directed STCON [7, 158, 31].

## 3 Access Structure Hiding Verifiable Secret Sharing

In this section, we give a formal definition of an access structure hiding verifiable (computational) secret sharing scheme.

Definition 15. An access structure hiding verifiable (computational) secret sharing scheme with respect to an access structure $\Gamma$, a set of $\ell$ polynomial-time parties $\mathcal{P}=\left\{P_{1}, \ldots, P_{\ell}\right\}$, a set of secrets $\mathcal{K}$ and a security parameter $\omega$, consists of two sets of polynomial-time algorithms, (HsGen, HsVer) and (VerShr, Recon, Ver), which are defined as:

1. VerShr is a randomized algorithm that gets a secret $k \in \mathcal{K}$ and access structure $\Gamma$ as inputs, and outputs $\ell$ shares, $\left\{\Psi_{1}^{(k)}, \ldots, \Psi_{\ell}^{(k)}\right\}$, of $k$,
2. Recon is a deterministic algorithm that gets as input the shares of a subset $\mathcal{A} \subseteq \mathcal{P}$, denoted by $\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}$, and outputs a string in $\mathcal{K}$,
3. Ver is a deterministic Boolean algorithm that gets $\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}$ and a secret $k^{\prime} \in \mathcal{K}$ as inputs and outputs $b \in\{0,1\}$,
such that the following three requirements are satisfied:
(a) Perfect Correctness: for all secrets $k \in \mathcal{K}$ and every authorized subset $\mathcal{A} \in \Gamma$, it holds that: $\operatorname{Pr}\left[\operatorname{Recon}\left(\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}, \mathcal{A}\right)=k\right]=1$,
(b) Computational Secrecy: for every unauthorized subset $\mathcal{B} \notin \Gamma$ and all different secrets $k_{1}, k_{2} \in \mathcal{K}$, it holds that the distributions $\left\{\Psi_{i}^{\left(k_{1}\right)}\right\}_{i \in \mathcal{B}}$ and $\left\{\Psi_{i}^{\left(k_{2}\right)}\right\}_{i \in \mathcal{B}}$ are computationally indistinguishable (w.r.t. $\omega$ ),
(c) Computational Verifiability: every authorized subset $\mathcal{A} \in \Gamma$ can use Ver to verify whether its set of shares $\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}$ is consistent with a given secret $k \in \mathcal{K}$. Formally, for a negligible function $\epsilon$, it holds that:
$-\operatorname{Pr}\left[\operatorname{Ver}\left(k,\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}\right)=1\right]=1-\epsilon(\omega)$ if all shares $\Psi_{i}^{(k)} \in\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}$ are consistent with the secret $k$,

- else, if any share $\Psi_{i}^{(k)} \in\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}$ is inconsistent with the secret $k$, then it holds that:

$$
\operatorname{Pr}\left[\operatorname{Ver}\left(k,\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{A}}\right)=0\right]=1-\epsilon(\omega),
$$

4. HsGen is a randomized algorithm that gets $\mathcal{P}$ and $\Gamma$ as inputs, and outputs $\ell$ access structure tokens $\left\{\mho_{1}^{(\Gamma)}, \ldots, \mho_{\ell}^{(\Gamma)}\right\}$,
5. HsVer is a deterministic algorithm that gets as input the access structure tokens of a subset $\mathcal{A} \subseteq \mathcal{P}$, denoted by $\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{A}}$, and outputs $b \in\{0,1\}$,
such that, the following three requirements are satisfied:
(a) Perfect Completeness: every authorized subset of parties $\mathcal{A} \in \Gamma$ can identify itself as a member of the access structure $\Gamma$, i.e., it holds that: $\operatorname{Pr}\left[\operatorname{HsVer}\left(\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{A}}\right)=1\right]=1$,
(b) Perfect Soundness: every unauthorized subset of parties $\mathcal{B} \notin \Gamma$ can identify itself to be outside of the access structure $\Gamma$, i.e., it holds that: $\operatorname{Pr}\left[\operatorname{HsVer}\left(\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}\right)=0\right]=1$,
(c) Statistical Hiding: for all access structures $\Gamma, \Gamma^{\prime} \subseteq 2^{\mathcal{P}}$, where $\Gamma \neq \Gamma^{\prime}$, and each subset of parties $\mathcal{B} \notin \Gamma, \Gamma^{\prime}$ that is unauthorized in both $\Gamma$ and $\Gamma^{\prime}$, it holds that:

$$
\left|\operatorname{Pr}\left[\Gamma \mid\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}},\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{B}}\right]-\operatorname{Pr}\left[\Gamma^{\prime} \mid\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}},\left\{\Psi_{i}^{(k)}\right\}_{i \in \mathcal{B}}\right]\right|=2^{-\omega} .
$$

## 4 Novel Set-Systems and Vector Families

In this section, we prove Theorem 1 by constructing a novel set-system.

Proposition 1. Let $l \geq 2$ be an integer, and $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r>1$ different prime divisors such that $\forall i \in\{1, \ldots, r\}: p_{i}>l$. Suppose there exists an integer $t \geq 2$ and a uniform set-system $\mathcal{G}$ satisfying the conditions:

1. $\forall G \in \mathcal{G}:|G|=0 \bmod m$,
2. $\forall t^{\prime}$ such that $2 \leq t^{\prime} \leq t$, and for all distinct $G_{1}, G_{2}, \ldots, G_{t^{\prime}} \in \mathcal{G}$, it holds that:

$$
\left|\bigcap_{\tau=1}^{t^{\prime}} G_{\tau}\right|=\mu \bmod m,
$$

where $\mu \neq 0 \bmod m$ and $\forall i \in\{1, \ldots, r\}: \mu \in\{0,1\} \bmod p_{i}$,
3. $\left|\bigcap_{G \in \mathcal{G}} G\right| \neq 0 \bmod m$.

Then, there exists a set-system $\mathcal{H}$ that is explicitly constructible from the set-system $\mathcal{G}$ such that:
(i) $\forall H_{1}, H_{2} \in \mathcal{H}$, either $\left|H_{1}\right|=\left|H_{2}\right|,\left|H_{1}\right|=l\left|H_{2}\right|$ or $l\left|H_{1}\right|=\left|H_{2}\right|$,
(ii) $\mathcal{H}$ has $t$-wise restricted intersections modulo $m$ (see Definition 2).

Proof. We start with $l$ uniform ${ }^{\|}$set systems $\mathcal{H}_{1}, \mathcal{H}_{2}, \ldots, \mathcal{H}_{l}$ satisfying the following properties:

1. $\forall H^{(i)} \in \mathcal{H}_{i}:\left|H^{(i)}\right|=0 \bmod m$,
2. $\forall t^{\prime}$ such that $2 \leq t^{\prime} \leq t$, and for all distinct $H_{1}^{(i)}, H_{2}^{(i)}, \ldots, H_{t^{\prime}}^{(i)} \in \mathcal{H}_{i}$, it holds that:

$$
\left|\bigcap_{\tau=1}^{t^{\prime}} H_{\tau}^{(i)}\right|=\mu \bmod m
$$

where $\mu \neq 0 \bmod m$ and $\forall z \in\{1, \ldots, r\}: \mu \in\{0,1\} \bmod p_{z}$,
3. $\forall i \in\{1, \ldots, l\}:\left|\bigcap_{H^{(i)} \in \mathcal{H}_{i}} H^{(i)}\right| \neq 0 \bmod m$,
4. $\left|H^{(i)}\right|=\left|H^{(j)}\right|$ for all $H^{(i)} \in \mathcal{H}_{i}, H^{(j)} \in \mathcal{H}_{j}$,
5. $\forall i, j \in\{1, \ldots, l\}:\left|\bigcap_{H^{(i)} \in \mathcal{H}_{i}} H^{(i)}\right|=\left|\bigcap_{H^{(j)} \in \mathcal{H}_{j}} H^{(j)}\right|$.

We begin by fixing bijections:

$$
f_{i, j}: \bigcap_{H^{(i)} \in \mathcal{H}_{i}} H^{(i)} \rightarrow \bigcap_{H^{(j)} \in \mathcal{H}_{j}} H^{(j)}
$$

such that $f_{i, i}$ is the identity and $f_{i, j} \circ f_{j, k}=f_{i, k}$ for all $1 \leq i, j, k \leq l$. Using these bijections, we can identify the sets $\bigcap_{H^{(i)} \in \mathcal{H}_{i}} H^{(i)}$ and $\bigcap_{H^{(j)} \in \mathcal{H}_{i}} H^{(j)}$ with each other. Let:

$$
A=\bigcap_{H^{(1)} \in \mathcal{H}_{1}} H^{(1)}=\bigcap_{H^{(2)} \in \mathcal{H}_{2}} H^{(2)}=\cdots=\bigcap_{H^{(l)} \in \mathcal{H}_{l}} H^{(l)}
$$

We shall treat the elements of the sets in $\mathcal{H}_{i}$ as being distinct from the elements of the sets in $\mathcal{H}_{j}$, except for the above identification of elements in $\bigcap_{H^{(i)} \in \mathcal{H}_{i}} H^{(i)}$ with elements in $\bigcap_{H^{(j)} \in \mathcal{H}_{j}} H^{(j)}$. Let $a=|A|$, and let $\beta_{1}, \beta_{2}, \ldots, \beta_{(l-1) a}$ be elements that are distinct from all the elements in the sets in $\mathcal{H}_{1}, \mathcal{H}_{2}, \ldots \mathcal{H}_{l}$. Define the set:

$$
B=\left\{\beta_{1}, \beta_{2}, \ldots, \beta_{(l-1) a}\right\},
$$

and consider a set system $\mathcal{H}$ which contains the following sets:

[^2]$-H^{(i)}$, where $H^{(i)} \in \mathcal{H}_{i}$ for some $i \in[l]$,
$-\bigcup_{i=1}^{l} H^{(i)} \cup B$, where $H^{(i)} \in \mathcal{H}_{i}$ for all $i \in[l]$.
Write the common size of the sets in the uniform set systems $\mathcal{H}_{i}(1 \leq i \leq l)$ as $k m$ for some $k>0$. Then, the following holds for all $H^{(i)} \in \mathcal{H}_{i}$,
\[

$$
\begin{aligned}
\left|\bigcup_{i=1}^{l} H^{(i)} \cup B\right| & =\left|\bigcup_{i=1}^{l} H^{(i)}\right|+|B|=\sum_{i=1}^{l}\left|H^{(i)}\right|-(l-1)|A|+|B| \\
& =l(k m)-(l-1) a+(l-1) a=l k m
\end{aligned}
$$
\]

where the second equality comes from the fact that $H^{(i)} \cap H^{(j)}=A$ for all $i \neq j$. This proves that Condition (i) holds. Moving on to the Condition (ii): let $t_{1}, t_{2}, \ldots, t_{l+1} \geq 0$ be such that $2 \leq t^{\prime}\left(=t_{1}+t_{2}+\cdots+t_{l+1}\right) \leq t$. We shall consider the intersection of the sets:
$-H_{\tau}^{(i)}$ where $1 \leq i \leq l, 1 \leq \tau \leq t_{i}$ and $H_{\tau}^{(i)} \in \mathcal{H}_{i}$,
$-\bigcup_{i=1}^{l} H_{\tau}^{\prime(i)} \cup B$ where $1 \leq \tau \leq t_{l+1}$ and $H_{\tau}^{\prime(i)} \in \mathcal{H}_{i}$.
Assume that these sets form a non-degenerate family. Let:

$$
\begin{aligned}
\sigma & =\left|\bigcap_{i=1}^{l} \bigcap_{\tau=1}^{t_{i}} H_{\tau}^{(i)} \cap \bigcap_{\tau=1}^{t_{l+1}}\left(H_{\tau}^{\prime(1)} \cup H_{\tau}^{\prime(2)} \cup \cdots \cup H_{\tau}^{\prime(l)} \cup B\right)\right| \\
& =\left|\bigcap_{i=1}^{l} \bigcap_{\tau=1}^{t_{i}} H_{\tau}^{(i)} \cap \bigcap_{\tau=1}^{t_{l+1}}\left(H_{\tau}^{\prime(1)} \cup H_{\tau}^{\prime(2)} \cup \cdots \cup H_{\tau}^{\prime(l)}\right)\right|+\epsilon|B|
\end{aligned}
$$

where $\epsilon=1$ if $t_{1}=t_{2}=\cdots=t_{l}=0$, and $\epsilon=0$ otherwise. If two or more of $t_{1}, t_{2}, \ldots, t_{l}$ are non-zero, then: $\sigma=|A|=a \neq 0 \bmod m$. On the other hand, if exactly one of $t_{1}, t_{2}, \ldots, t_{l}$ is non-zero, then:

$$
\sigma=\left|\bigcap_{\tau=1}^{t_{i}} H_{\tau}^{(i)} \cap \bigcap_{\tau=1}^{t_{l+1}} H_{\tau}^{\prime(i)}\right| \neq 0 \bmod m
$$

since $H_{\tau}^{(i)}$ (for $\left.1 \leq \tau \leq t_{i}\right)$ and $H_{\tau}^{\prime(i)}$ (for $1 \leq \tau \leq t_{l+1}$ ) are not all the same by the assumption of non-degeneracy. If $t_{1}=t_{2}=\cdots=t_{l}=0$, then we get:

$$
\begin{aligned}
\sigma & =\left|\bigcap_{\tau=1}^{t_{l+1}}\left(H_{\tau}^{\prime(1)} \cup H_{\tau}^{\prime(2)} \cup \cdots \cup H_{\tau}^{\prime(l)}\right)\right|+|B| \\
& =\sum_{i=1}^{l}\left|\bigcap_{\tau=1}^{t_{l+1}} H_{\tau}^{\prime(i)}\right|-(l-1)|A|+|B|=\sum_{i=1}^{l^{\prime}} \mu_{i} \bmod m
\end{aligned}
$$

for some integer $l^{\prime}$ such that $1 \leq l^{\prime} \leq l$, and some set $\left\{\mu_{i}\right\}_{i=1}^{l^{\prime}}$ such that for each $\mu_{i}$ and all primes $p$ such that $p \mid m$, it holds that: $\mu_{i} \in\{0,1\} \bmod p$. Since $\mu_{i} \neq 0 \bmod m$ for all $1 \leq i \leq l^{\prime}$, there must be some prime factor $p$ of $m$ for which at least one of the $\mu_{i}$ 's satisfy $\mu_{i}=1 \bmod p$. Since $p$ is a prime factor of $m$, it satisfies: $p>l \geq l^{\prime}$. Hence, for $p$, we get:

$$
\sigma=\sum_{i=1}^{l^{\prime}} \mu_{i} \neq 0 \bmod p
$$

This proves Condition (ii), and hence completes the proof.

Remark 2. Suppose that $|\mathcal{G}|=s$ and that the number of elements in the universe of $\mathcal{G}$ is $g$. Then, there are $l s$ sets of size $k m$ and $s^{l}$ sets of size $l k m$ in $\mathcal{H}$. Therefore, we get: $|\mathcal{H}|=s^{l}+l s$. The universe of $\mathcal{H}$ has $l g$ elements, and for each $H \in \mathcal{H}$, exactly one of the following is true:

- $H$ is a proper subset of exactly $s^{l-1}$ sets and not a proper superset of any sets in $\mathcal{H}$,
- $H$ is a proper superset of exactly $l$ sets and not a proper subset of any sets in $\mathcal{H}$.

In order to explicitly construct set systems which, in addition to having the properties in Proposition 1, have sizes superpolynomial in the number of elements, we first recall a result of Barrington et al. [24], which Grolmusz [134] used to construct a superpolynomial uniform set-system.

Theorem 6 ( [24], Theorem 2.1). Let $\left\{\alpha_{i}\right\}_{i=1}^{r}$ be $r>1$ positive integers and $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be $a$ positive integer with $r$ different prime divisors: $p_{1}, \ldots, p_{r}$. For every integer $n \geq 1$, there exists an explicitly constructible polynomial $P$ in $n$ variables such that

1. $P(0,0, \ldots, 0)=0 \bmod m$,
2. $P(x) \neq 0 \bmod m$ for all $x \in\{0,1\}^{n}$ such that $x \neq(0,0, \ldots, 0)$,
3. $\forall i \in[r]$ and $\forall x \in\{0,1\}^{n}$ such that $x \neq(0,0, \ldots, 0)$, it holds that: $P(x) \in\{0,1\} \bmod p_{i}$.

The polynomial $P$ has degree $d=\max \left(p_{1}^{e_{1}}, \ldots, p_{r}^{e_{r}}\right)-1$ where $e_{i}(\forall i \in[r])$ is the smallest integer that satisfies $p_{i}^{e_{i}}>\left\lceil n^{1 / r}\right\rceil$.

Define $Q\left(x_{1}, x_{2}, \ldots, x_{n}\right)=P\left(1-x_{1}, 1-x_{2}, \ldots, 1-x_{n}\right)$. Then:

1. $Q(1,1, \ldots, 1)=0 \bmod m$,
2. $Q(x) \neq 0 \bmod m$ for all $x \in\{0,1\}^{n}$ such that $x \neq(1,1, \ldots, 1)$.
3. $\forall i \in[r]$ and $\forall x \in\{0,1\}^{n}$ such that $x \neq(1,1, \ldots, 1)$, it holds that: $Q(x) \in\{0,1\} \bmod p_{i}$.

Theorem 7 ( [134], Theorem 1.4, Lemma 3.1). Let $\left\{\alpha_{i}\right\}_{i=1}^{r}$ be $r>1$ positive integers and $m=$ $\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r$ different prime divisors: $p_{1}, \ldots, p_{r}$. For every integer $n \geq 1$, there exists a uniform set system $\mathcal{G}$ over a universe of $g$ elements which is explicitly constructible from the polynomial $Q$ of degree $d$ such that

1. $g<\frac{2(m-1) n^{2 d}}{d!}$ if $n \geq 2 d$,
2. $|\mathcal{G}|=n^{n}$,
3. $\forall G \in \mathcal{G},|G|=0 \bmod m$,
4. $\forall G, H \in \mathcal{G}$ such that $G \neq H$, it holds that: $|G \cap H|=\mu \bmod m$, where $\mu \neq 0 \bmod m$ and $\mu \in\{0,1\} \bmod p_{i}$ for all $i \in[r]$,
5. $\left|\bigcap_{G \in \mathcal{G}} G\right| \neq 0 \bmod m$.

Note that Condition 5 follows from the fact that the following holds in Grolmusz's construction of superpolynomial set-systems:

$$
\left|\bigcap_{G \in \mathcal{G}} G\right|=Q(0,0, \ldots, 0) \neq 0 \bmod m .
$$

In fact, a straightforward generalization of the arguments in [134] proves the following theorem:
Theorem 8. Let $\left\{\alpha_{i}\right\}_{i=1}^{r}$ be $r>1$ positive integers and $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r$ different prime divisors: $p_{1}, \ldots, p_{r}$. For all integers $t \geq 2$ and $n \geq 1$, there exists a uniform set system $\mathcal{G}$ over $a$ universe of $g$ elements which is explicitly constructible from the polynomial $Q$ of degree $d$ such that

1. $g<\frac{2(m-1) n^{2 d}}{d!}$ if $n \geq 2 d$,
2. $|\mathcal{G}|=n^{n}$,
3. $\forall G \in \mathcal{G},|G|=0 \bmod m$,
4. $\forall t^{\prime}$ such that $2 \leq t^{\prime} \leq t$, and for all distinct $G_{1}, G_{2}, \ldots, G_{t^{\prime}} \in \mathcal{G}$, it holds that:

$$
\left|\bigcap_{\tau=1}^{t^{\prime}} G_{\tau}\right|=\mu \bmod m
$$

where $\mu \neq 0 \bmod m$ and $\mu \in\{0,1\} \bmod p_{i}$ for all $i \in[r]$,
5. $\left|\bigcap_{G \in \mathcal{G}} G\right| \neq 0 \bmod m$.

Proof. We will follow the proof of Theorem 1.4 in [134], but with a few minor changes. Write the polynomial $Q$ as

$$
Q\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i_{1}<i_{2}<\cdots<i_{l}} a_{i_{1}, i_{2}, \ldots, i_{l}} x_{i_{1}} x_{i_{2}} \cdots x_{i_{l}}
$$

Define

$$
\tilde{Q}\left(x_{1}, x_{2}, \ldots, x_{n}\right)=\sum_{i_{1}<i_{2}<\cdots<i_{l}} \tilde{a}_{i_{1}, i_{2}, \ldots, i_{l}} x_{i_{1}} x_{i_{2}} \cdots x_{i_{l}}
$$

where $\tilde{a}_{i_{1}, i_{2}, \ldots, i_{l}}$ is the remainder when $a_{i_{1}, i_{2}, \ldots, i_{l}}$ is divided by $m$.
Let $[0, n-1]=\{0,1, \ldots, n-1\}$. Define the function $\delta:[0, n-1]^{t} \rightarrow\{0,1\}$ as

$$
\delta\left(u_{1}, u_{2}, \ldots, u_{t}\right)= \begin{cases}1 & \text { if } u_{1}=u_{2}=\cdots=u_{t} \\ 0 & \text { otherwise }\end{cases}
$$

For $y_{1}, y_{2}, \ldots, y_{t} \in[0, n-1]^{n}$, let

$$
a^{y_{1}, y_{2}, \ldots, y_{t}}=\tilde{Q}\left(\delta\left(y_{1,1}, y_{2,1}, \ldots, y_{t, 1}\right), \ldots, \delta\left(y_{1, n}, y_{2, n}, \ldots, y_{t, n}\right)\right) \bmod m
$$

Then

$$
a^{y_{1}, y_{2}, \ldots, y_{t}}=\sum b_{i_{1}, i_{2}, \ldots, i_{l}}^{y_{1}, y_{2}, \ldots, y_{t}}
$$

where

$$
b_{i_{1}, i_{2}, \ldots, i_{l}}^{y_{1}, y_{2}, \ldots, y_{t}}=\prod_{j=1}^{l} \delta\left(y_{1, i_{j}}, y_{2, i_{j}}, \ldots, y_{t, i_{j}}\right)
$$

Each summand $b_{i_{1}, i_{2}, \ldots, i_{l}}^{y_{1}, y_{2}, \ldots, y_{t}}$ corresponds to a monomial of $\tilde{Q}$ and occurs with multiplicity $\tilde{a}_{i_{1}, i_{2}, \ldots, i_{l}}$ in the above sum.

It is easy to check that there exists partitions $\mathcal{P}_{i_{1}, i_{2}, \ldots, i_{l}}$ of $[0, n-1]^{n}$ such that for all $y_{1}, y_{2}, \ldots, y_{t} \in$ $[0, n-1]^{n}$,

$$
b_{i_{1}, i_{2}, \ldots, i_{l}}^{y_{1}, y_{2}, \ldots, y_{t}}= \begin{cases}1 & \text { if } y_{1}, y_{2}, \ldots, y_{t} \text { belong to the same block of } \mathcal{P}_{i_{1}, i_{2}}, \ldots, i_{l} \\ 0 & \text { otherwise }\end{cases}
$$

and that the equivalence classes defined by the partition $\mathcal{P}_{i_{1}, i_{2}, \ldots, i_{l}}$ each has size $n^{n-l}$. We say that a block in the partition $\mathcal{P}_{i_{1}, i_{2}, \ldots, i_{l}}$ covers $y \in[0, n-1]^{n}$ if $y$ is an element of the block.

We define a set system $\mathcal{G}$ as follows: the sets in $\mathcal{G}$ correspond to $y$ for $y \in[0, n-1]^{n}$, and the set corresponding to $y$ has elements given by the blocks that cover $y$.

The set $y$ in the set system $\mathcal{G}$ has size equal to the number of blocks that cover $y$, which is equal to

$$
a^{y, y, \ldots, y}=\tilde{Q}(1,1, \ldots, 1)=0 \bmod m .
$$

For any $2 \leq t^{\prime} \leq t$, and $y_{1}, y_{2}, \ldots, y_{t^{\prime}} \in[0, n-1]^{n}$ distinct, some block of $\mathcal{P}_{i_{1}, i_{2}, \ldots, i_{l}}$ covers all of $y_{1}, y_{2}, \ldots, y_{t^{\prime}}$ if and only if $b_{i_{1}, i_{2}, \ldots, i_{t}}^{\overline{y_{1}}, \ldots, y_{t^{\prime}}}=1$ (note that $y_{t^{\prime}}$ occurs in the superscript $t-t^{\prime}+1$ times). Hence, the number of such blocks is equal to:

$$
a^{y_{1}, y_{2}, \ldots, y_{t^{\prime}}, \ldots, y_{t^{\prime}}} \neq 0 \bmod m .
$$

Finally, we would like to have a bound on $g$, the number of elements in the universe of $\mathcal{G}$. By our construction, this is equal to the number of blocks. Since the partition $\mathcal{P}_{i_{1}, i_{2}, \ldots, i_{l}}$ defines $n^{l}$ equivalence classes, the number of blocks is given by

$$
\begin{aligned}
g=\sum_{i_{1}<i_{2}<\cdots<i_{l}} \tilde{a}_{i_{1}, i_{2}, \ldots, i_{l}} n^{l} & \leq \sum_{l=0}^{d}\binom{n}{l}(m-1) n^{l}<(m-1) \sum_{l=0}^{d} \frac{n^{2 l}}{l!} \\
& <\frac{2(m-1) n^{2 d}}{d!},
\end{aligned}
$$

provided that $n \geq 2 d$.
Theorem 9. Let $\left\{\alpha_{i}\right\}_{i=1}^{r}$ be $r>1$ positive integers and $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r$ different odd prime divisors: $p_{1}, \ldots, p_{r}$, and $l \geq 2$ be an integer such that $l<\min \left(p_{1}, \ldots, p_{r}\right)$. Then, for all integers $t \geq 2$ and $n \geq 1$, there exists an explicitly constructible non-uniform set-system $\mathcal{H}$, defined over a universe of $h$ elements, such that

$$
\text { 1. } h<2 l(m-1) n^{4 m n^{\frac{1}{r}}} \text { if } n \geq(4 m)^{1+\frac{1}{r-1}} \text {, }
$$

2. $|\mathcal{H}|=n^{l n}+l n^{n}$,
3. $\forall H_{1}, H_{2} \in \mathcal{H}$, either $\left|H_{1}\right|=\left|H_{2}\right|,\left|H_{1}\right|=l\left|H_{2}\right|$ or $l\left|H_{1}\right|=\left|H_{2}\right|$,
4. $\mathcal{H}$ has $t$-wise restricted intersections modulo $m$.

Proof. By Theorem 8, there exists a uniform set-system $\mathcal{G}$ that satisfies conditions 1-3 of Proposition 1, and is defined over a universe of $g$ elements, such that $|\mathcal{G}|=n^{n}$. Furthermore, we know that $g<\frac{2(m-1) n^{2 d}}{d!}$ provided the condition $n \geq 2 d$ is satisfied. From Theorem $6, d=\max \left(p_{1}^{e_{1}}, \ldots, p_{r}^{e_{r}}\right)-1$ where $e_{i}$ is the smallest integer that satisfies $p_{i}^{e_{i}}>\left\lceil n^{1 / r}\right\rceil$, from which we obtain the following inequality:

$$
d<\max \left(p_{1}, \ldots, p_{r}\right)\left\lceil n^{1 / r}\right\rceil<2 m n^{1 / r} .
$$

Hence if $n \geq(4 m)^{1+\frac{1}{r-1}}$, then $n^{\frac{r-1}{r}} \geq 4 m \Longrightarrow n \geq 4 m n^{1 / r}>2 d$, and thus we have:

$$
g<\frac{2(m-1) n^{2 d}}{d!}<2(m-1) n^{2 d}<2(m-1) n^{4 m n^{\frac{1}{r}}}
$$

Applying Proposition 1 with the set-system $\mathcal{G}$, we obtain a set-system $\mathcal{H}$ satisfying Conditions 3 and 4 . It follows from Remark 2, that the size of $\mathcal{H}$ is:

$$
|\mathcal{H}|=\left(n^{n}\right)^{l}+l\left(n^{n}\right)=n^{l n}+l n^{n},
$$

and the number of elements in the universe of $\mathcal{H}$ is $h=l g<2 l(m-1) n^{4 m n^{\frac{1}{r}}}$ for $n \geq(4 m)^{1+\frac{1}{r-1}}$.

Corollary 2 (Same as Theorem 1). Let $\left\{\alpha_{i}\right\}_{i=1}^{r}$ be $r>1$ positive integers and $m=\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ be a positive integer with $r$ different odd prime divisors: $p_{1}, \ldots, p_{r}$, and $l \geq 2$ be an integer such that $l<\min \left(p_{1}, \ldots, p_{r}\right)$. Then, there exists $c>0$ such that for all integers $t \geq 2$ and $h \geq l m$, there exists an explicitly constructible non-uniform ${ }^{* *}$ set-system $\mathcal{H}$, defined over a universe of $h$ elements, such that

1. $|\mathcal{H}|>\exp \left(c \frac{l(\log h)^{r}}{(\log \log h)^{r-1}}\right)+l \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right)$,
2. $\forall H_{1}, H_{2} \in \mathcal{H}$, either $\left|H_{1}\right|=\left|H_{2}\right|,\left|H_{1}\right|=l\left|H_{2}\right|$ or $l\left|H_{1}\right|=\left|H_{2}\right|$,
3. $\mathcal{H}$ has $t$-wise restricted intersections modulo $m$.

Proof. For small values of $h$, we can simply take $\mathcal{H}$ to be the set system

$$
\{[m-1] \cup\{m\},[m-1] \cup\{m+1\}, \ldots,[m-1] \cup\{m+l\},[l m]\},
$$

so it is enough to prove the statement for sufficiently large $h$. Choose $n$ as large as possible subject to the restriction $2 l(m-1) n^{4 m n^{\frac{1}{r}}} \leq h$. We may assume that $h$ is sufficiently large so that the condition $n \geq(4 m)^{1+\frac{1}{r-1}}$ is satisfied. For $N=n+1$, it holds that:

$$
h<2 l(m-1) N^{4 m N^{\frac{1}{r}}} \Longrightarrow N>e^{r W_{0}\left(\frac{1}{4 r m} \log \frac{h}{2 l(m-1)}\right)},
$$

where $W_{0}$ is the principal branch of the Lambert $W$ function [181]. Fix any $c_{1}$ such that $0<c_{1}<\frac{1}{4 r m}$. Then, for $h$ sufficiently large, $n>e^{r W_{0}\left(c_{1} \log h\right)}$. Corless et al. [80] proved the following:

$$
W_{0}(x)=\log x-\log \log x+o(1)
$$

hence, it follows that there exists some $c_{2}$ such that for all sufficiently large $h$, it holds that:

$$
\begin{aligned}
n & >\exp \left(r \log \log h-r \log \log \log h+c_{2}\right) \\
& =\frac{e^{c_{2}}(\log h)^{r}}{(\log \log h)^{r}} .
\end{aligned}
$$

This shows that there exists $c_{3}>0$ such that for sufficiently large $h$, we get:

$$
\begin{equation*}
n^{n}>\exp \left(\frac{c_{3}(\log h)^{r}}{(\log \log h)^{r-1}}\right) \tag{4.1}
\end{equation*}
$$

Since the size of $\mathcal{H}$ is $|\mathcal{H}|=n^{l n}+n^{n}$, it follows from Equation (4.1) that:

$$
|\mathcal{H}|>\exp \left(c \frac{l(\log h)^{r}}{(\log \log h)^{r-1}}\right)+l \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right) .
$$

Definition 16 (Covering Vectors [266]). Let $m, h>0$ be positive integers, $S \subseteq \mathbb{Z}_{m} \backslash\{0\}$, and w( $\cdot$ ) and $\langle\cdot, \cdot\rangle$ denote Hamming weight and inner product, respectively. We say that a subset $\mathcal{V}=\left\{\mathbf{v}_{i}\right\}_{i=1}^{N}$ of vectors in $\left(\mathbb{Z}_{m}\right)^{h}$ forms an $S$-covering family of vectors if the following two conditions are satisfied:
$-\forall i \in[N]$, it holds that: $\left\langle\mathbf{v}_{i}, \mathbf{v}_{i}\right\rangle=0 \bmod m$,

[^3]$-\forall i, j \in[N]$, where $i \neq j$, it holds that:
\[

\left\langle\mathbf{v}_{i}, \mathbf{v}_{j}\right\rangle \bmod m= $$
\begin{cases}0 & \text { if } \mathrm{w}\left(\mathbf{v}_{i} \circ \mathbf{v}_{j} \bmod m\right)=0 \bmod m, \\ \in S & \text { otherwise },\end{cases}
$$
\]

where $\circ$ denotes Hadamard/Schur product (see Definition 3).
Recall from Theorem 1 that $h, m, l$ are positive integers such that $2 \leq l<\min \left(p_{1}, \ldots, p_{r}\right)$ and $m=$ $\prod_{i=1}^{r} p_{i}^{\alpha_{i}}$ has $r>1$ different prime divisors: $p_{1}, \ldots, p_{r}$. Further, it follows trivially that the sizes of the pairwise intersections of the sets in $\mathcal{H}$ occupy at most $m-1$ residue classes modulo $m$. If each set $H_{i} \in \mathcal{H}$ is represented by a representative vector $\mathbf{v}_{i} \in\left(\mathbb{Z}_{m}\right)^{h}$, then for the resulting subset $\mathcal{V}$ of vectors in $\left(\mathbb{Z}_{m}\right)^{h}$, the following result follows from Theorem 1.

Corollary 3 (to Theorem 1). For the set-system $\mathcal{H}$ defined in Theorem 1, if each set $H_{i} \in \mathcal{H}$ is represented by a unique vector $\mathbf{v}_{i} \in\left(\mathbb{Z}_{m}\right)^{h}$, then for a set $S$ of size $m-1$, the set of vectors $\mathcal{V}=\left\{\mathbf{v}_{i}\right\}_{i=1}^{N}$, formed by the representative vectors of all sets in $\mathcal{H}$, forms an $S$-covering family such that

$$
N>\exp \left(c \frac{l(\log h)^{r}}{(\log \log h)^{r-1}}\right)+l \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right)
$$

and $\forall i, j \in[N]$ it holds that $\left\langle\mathbf{v}_{i}, \mathbf{v}_{j}\right\rangle=\left|H_{i} \cap H_{j}\right| \bmod m$.

## 5 Working Over Set-Systems via Vector Families

In this section, we explain how vector families and special inner products can be used to work with sets from different set-systems. We begin by recalling the following two properties (from Remark 2) that hold for all sets in any set-system $\mathcal{H}$ that is defined by Theorem 1 .

- $H$ is a proper subset of exactly $s^{l-1}$ sets and not a proper superset of any sets in $\mathcal{H}$,
- $H$ is a proper superset of exactly $l$ sets and not a proper subset of any sets in $\mathcal{H}$,
where $s \geq \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right)$.
Let $\mathcal{V} \subseteq\left(\mathbb{Z}_{m}\right)^{h}$ be a family of covering vectors, consisting of representative vectors for the sets in a set-system $\mathcal{H}$. For all $i \in|\mathcal{H}|(=|\mathcal{V}|)$, let $\mathbf{v}_{i} \in \mathcal{V}$ denote the representative vector for the set $H_{i} \in \mathcal{H}$. Recall from Corollary 3 that the following holds:

$$
\left\langle\mathbf{v}_{i}, \mathbf{v}_{j}\right\rangle=\left|H_{i} \cap H_{j}\right| \bmod m
$$

We define a $k$-multilinear form on $\mathcal{V}^{k}$ as:

$$
\begin{aligned}
\left\langle\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{k}\right\rangle_{k} & =\sum_{i=1}^{h} \mathbf{v}_{1}[i] \mathbf{v}_{2}[i] \cdots \mathbf{v}_{k}[i] \\
& =\left|\bigcap_{i=1}^{k} H_{i}\right|
\end{aligned}
$$

We fix a representative vector $\mathbf{v} \in \mathcal{V}$ for a fixed set $H \in \mathcal{H}$. For the rest of the sets $H_{i} \in \mathcal{H}$, we denote their respective representative vectors by $\mathbf{v}_{i} \in \mathcal{V}$. Let $\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2} \in \mathcal{V}$, and $\mathbf{v}_{i \cup j} \in \mathcal{V}$ denote the representative vector for the set $H_{i \cup j}=H_{i} \cup H_{j}$. Then, the following holds:

$$
\begin{align*}
\left\langle\mathbf{v}, \mathbf{v}_{1 \cup 2}\right\rangle & =\left|H \cap\left(H_{1} \cup H_{2}\right)\right|=\left|\left(H \cap H_{1}\right) \cup\left(H \cap H_{2}\right)\right| \\
& =\left|H \cap H_{1}\right|+\left|H \cap H_{2}\right|-\left|H \cap H_{1} \cap H_{2}\right| \\
& =\left\langle\mathbf{v}, \mathbf{v}_{1}\right\rangle+\left\langle\mathbf{v}, \mathbf{v}_{2}\right\rangle-\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}\right\rangle_{3} . \tag{5.1}
\end{align*}
$$

Define $F$ as:

$$
F(x, y, z)=x+y-z
$$

i.e., the following holds:

$$
F\left(\left\langle\mathbf{v}, \mathbf{v}_{1}\right\rangle,\left\langle\mathbf{v}, \mathbf{v}_{2}\right\rangle,\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}\right\rangle_{3}\right)=\left\langle\mathbf{v}, \mathbf{v}_{1 \cup 2}\right\rangle .
$$

Note that the following also holds:

$$
\begin{aligned}
\left|H \cap\left(H_{1} \cap H_{2}\right)\right| & =\left\langle\mathbf{v}, \mathbf{v}_{1}\right\rangle+\left\langle\mathbf{v}, \mathbf{v}_{2}\right\rangle-\left\langle\mathbf{v}, \mathbf{v}_{1 \cup 2}\right\rangle \\
& =\left|H \cap H_{1}\right|+\left|H \cap H_{2}\right|-\left|H \cap\left(H_{1} \cup H_{2}\right)\right| .
\end{aligned}
$$

Consider the following simple extension of Equation (5.1):

$$
\begin{aligned}
\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2 \cup 3}\right\rangle_{3} & =\left|H \cap H_{1} \cap\left(H_{2} \cup H_{3}\right)\right|=\left|\left(H \cap H_{1} \cap H_{2}\right) \cup\left(H \cap H_{1} \cap H_{3}\right)\right| \\
& =\left|H \cap H_{1} \cap H_{2}\right|+\left|H \cap H_{1} \cap H_{3}\right|-\left|H \cap H_{1} \cap H_{2} \cap H_{3}\right| \\
& =\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}\right\rangle_{3}+\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{3}\right\rangle_{3}-\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right\rangle_{4} .
\end{aligned}
$$

Therefore, we get:

$$
F\left(\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}\right\rangle_{3},\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{3}\right\rangle_{3},\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}, \mathbf{v}_{3}\right\rangle_{4}\right)=\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2 \cup 3}\right\rangle_{3} .
$$

Note that the following also holds:

$$
\begin{aligned}
\left|H \cap\left(H_{1} \cap H_{2} \cap H_{3}\right)\right| & =\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{2}\right\rangle_{3}+\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{3}\right\rangle_{3}-\left\langle\mathbf{v}, \mathbf{v}_{1}, \mathbf{v}_{1 \cup 2}\right\rangle_{4} \\
& =\left|H \cap H_{1} \cap H_{2}\right|+\left|H \cap H_{1} \cap H_{3}\right|-\left|H \cap H_{1} \cap\left(H_{2} \cup H_{3}\right)\right| .
\end{aligned}
$$

It follows by extension that $\left\langle\mathbf{v}, \mathbf{v}_{1 \cup 2 \cup} \cdots \cup \cup w\right\rangle_{w}$, can be computed from the $k$-multilinear forms $\left\langle\mathbf{v}_{1}, \mathbf{v}_{2}, \ldots, \mathbf{v}_{k}\right\rangle_{k}$, for all $k \in[w+1]$ and all $\mathbf{v}_{i} \in \mathcal{V}$. Hence, $\left\langle\mathbf{v}_{i}, \mathbf{v}_{j}\right\rangle=\left|H_{i} \cap H_{j}\right| \bmod m$ allows us to compute intersection of any sets $H_{i}, H_{j} \in \mathcal{H}$, and being able to compute the aforementioned function $F(x, y, z)$ allows us to perform unions and intersections of any arbitrary number of sets from $\mathcal{H}$.

Let $m=\prod_{i=1}^{r} p_{i}$ and $m^{\prime}=\prod_{i=1}^{r^{\prime}} p_{i}$ be positive integers, having $r$ and $r^{\prime}>r$ different odd prime divisors, respectively. Recall from Theorem 1 , that the universe of elements over which the set-system $\mathcal{H}$ is constructed is given by: $h \geq l m$, where $2 \leq l<\min \left(p_{1}, \ldots, p_{r}\right)$. We construct two set-systems $\mathcal{H}$ and $\mathcal{H}^{\prime}$ over $\mathbb{Z}_{m}$ and $\mathbb{Z}_{m^{\prime}}$. Let the sets of parameters $\{h, l, m\}$ and $\left\{h^{\prime}, l^{\prime}, m^{\prime}\right\}$ correspond to set-systems $\mathcal{H}$ and $\mathcal{H}^{\prime}$, respectively. In order to ensure that the number of elements is same for both $\mathcal{H}$ and $\mathcal{H}^{\prime}$, we set $h=h^{\prime}=\max \left(l m, l^{\prime} m^{\prime}\right)$. Since $m$ is a factor of $m^{\prime}$, the following holds for all $H \in \mathcal{H}^{\prime}$ :

$$
|H|=0 \bmod m^{\prime}=0 \bmod m .
$$

Note that for appropriate choice of the underlying set-system $\mathcal{G}$ (see Proposition 1), it holds that $\left|\mathcal{H} \cap \mathcal{H}^{\prime}\right|>0$. It follows from Remark 2 that the following two conditions hold for $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$ :


Fig. 1: Supersets and subsets of a set $H \in \mathcal{H}, \mathcal{H}^{\prime}$ within the two set-systems. $H_{i}---\rightarrow H_{j}$ denotes $H_{i} \subseteq H_{j}$, and $H_{j} \longrightarrow H_{k}$ denotes $H_{j} \supseteq H_{k}$. Since $\mathcal{H}$ and $\mathcal{H}^{\prime}$ are defined over the identical universe of $h$ elements, superset-subset relations can hold even between the sets that exclusively belong to different set-systems.


Fig. 2: Hopping between supersets and subsets of a set $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$.

- $H$ is a proper subset of exactly $s^{l-1}$ sets and not a proper superset of any sets in $\mathcal{H}^{\prime}$,
- $H$ is a proper superset of exactly $l$ sets and not a proper subset of any sets in $\mathcal{H}$.

Therefore, it holds for the representative $\mathbf{v}$ of $H$ that: $\mathbf{v} \in \mathcal{V} \cap \mathcal{V}^{\prime}$. Figure 1 gives graphical depiction of the various subset and superset relationships of $H$ in $\mathcal{H}$ and $\mathcal{H}^{\prime}$. Specifically, it shows the $s^{l-1}$ proper supersets $\left\{H_{1}^{\prime}, \ldots, H_{s^{l-1}}^{\prime}\right\} \in \mathcal{H}^{\prime}$ of $H$, along with its $l$ proper subsets $\left\{H_{1}, \ldots, H_{l}\right\} \in \mathcal{H}$.

Let $\mathcal{V} \in \mathbb{Z}_{m}^{h}$ and $\mathcal{V}^{\prime} \in \mathbb{Z}_{m^{\prime}}^{h}$ be the covering vectors families (see Definition 16) that correspond to the set systems $\mathcal{H}$ and $\mathcal{H}^{\prime}$, respectively. It is easy to see that for all $\mathbf{v} \in \mathcal{V}$ and $\mathbf{v}^{\prime} \in \mathcal{V}^{\prime}$, there exists some vector $\mathbf{v}_{\delta} \in \mathbb{Z}^{h}$ such that $\mathbf{v}+\mathbf{v}_{\delta}=\mathbf{v}^{\prime}$. Since vector inner products are additive in the second argument, we can
compute: $\langle\mathbf{u}, \mathbf{v}\rangle+\left\langle\mathbf{u}, \mathbf{v}_{\delta}\right\rangle=\left\langle\mathbf{u}, \mathbf{v}^{\prime}\right\rangle$, and hence "hop" between the set-systems $\mathcal{H}$ and $\mathcal{H}^{\prime}$. Having this ability along with our $k$-multilinear forms, allows us to "hop" within and between the set-systems $\mathcal{H}$ and $\mathcal{H}^{\prime}$ via inner products of the corresponding vectors from covering vectors families $\mathcal{V}$ and $\mathcal{V}^{\prime}$.

For instance, given a set $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$, we can "hop" between the subsets and supersets of $H$ within the two set-systems. Figure 2 shows all such hops between all supersets and subsets of $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$.

## 6 Access Structure Encoding

In this section, we give an example procedure to encode any access structure. Let $\mathcal{P}=\left\{P_{1}, \ldots, P_{\ell}\right\}$ be a set of $\ell$ polynomial-time parties and $\Omega \in \Gamma_{0}$ be any minimal authorized subset (see Definition 8). Hence, each party $P_{i} \in \mathcal{P}$ can be identified as $P_{i} \in \Omega$ or $P_{i} \in \mathcal{P} \backslash \Omega$. We begin by giving an overview of the central idea of our scheme.

### 6.1 Central Idea

If all parties in the minimal authorized subset $\Omega \subseteq \mathcal{P}$ combine their access structure tokens $\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \Omega}$, they should arrive at a fixed set $H$, which represents $\Omega$. From thereon, access structure token of each party $P_{j} \in \mathcal{P} \backslash \Omega$ is generated such that no combination of their access structure tokens can reach $H$. Finally, the result of combining the access structure tokens, $\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{A}}$, of any authorized subset $\mathcal{A} \in \Gamma$, where $\Gamma=$ $\operatorname{cl}(\Omega)$, takes us to some set $H_{\phi} \supseteq H$. As described in Section 5 , we can operate on the sets in our set-systems via their respective representative vectors, their inner products and $k$-multilinear forms.

### 6.2 Example Procedure

Generate an integer $m=\prod_{i=1}^{r} p_{r}$ with $r>1$ prime divisors: $p_{1}, \ldots, p_{r}$ such that $|\Omega| \ll \max \left(p_{1}, \ldots, p_{r}\right)$. Define a set-system $\mathcal{H}$ modulo $m$ (as described in Theorem 1) such that $l+|\Omega| \ll \max \left(p_{1}, \ldots, p_{r}\right)$. Pick a set $H \stackrel{\$}{\leftrightarrows} \mathcal{H}$ such that $H$ is a proper subset of exactly $s^{l-1}(>\ell)$ sets and not a proper superset of any sets in $\mathcal{H}$. Let $\Im \subset \mathcal{H}$ denote the collection of sets in $\mathcal{H}$ that are supersets of $H$. Randomly generate a positive integer $\kappa$ such that $l+|\Omega|+\kappa<\max \left(p_{1}, \ldots, p_{r}\right)$. Then, the following procedure is used to assign unique sets from $\mathcal{H}$ to the parties in $\mathcal{P}$. Without loss of generality, we assume that $\Omega=\left\{P_{1}, P_{2}, \ldots, P_{|\Omega|}\right\}$.

1. The set for party $P_{1}$ is generated as:

$$
S_{1}=H \sqcup([|\Omega|+\kappa] \backslash\{1\})
$$

2. For each party $P_{i}(2 \leq i \leq|\Omega|-1)$, generate its set as:

$$
S_{i}=H_{i} \sqcup([|\Omega|+\kappa] \backslash\{i\}),
$$

where $H_{i} \stackrel{\$}{\leftarrow} \Im$ is a superset of $H$.
3. The set for party $P_{|\Omega|}$ is generated as:

$$
S_{|\Omega|}=H_{|\Omega|} \sqcup([|\Omega|+\kappa] \backslash\{|\Omega|, \ldots,|\Omega|+\kappa\})=H_{|\Omega|} \sqcup[|\Omega|-1]
$$

where $H_{|\Omega|} \stackrel{\$}{\leftarrow} \Im$ is a superset of $H$.
4. For each party $P_{j} \in \mathcal{P} \backslash \Omega$, generate its set as:

$$
S_{j}=H_{j} \sqcup[|\Omega|+\kappa],
$$

where $H_{j} \stackrel{\$}{\leftrightarrows}$ is a superset of $H$.
5. Generate a "special" set $H_{0}=H_{\partial} \sqcup[|\Omega|+\kappa]$, where $H_{\partial} \stackrel{\$}{\leftarrow} \Im$ and $H_{\partial} \neq H_{i}$ for all $i \in[|\Omega|]$.
6. For each $i \in[\ell]$, compute $H_{0} \cap S_{i}$, and let $\mathbf{s}_{i}$ denote the elements in $H_{0} \cap S_{i}$. Let $\gamma$ a random permutation, then the access structure token for party $P_{i} \in \mathcal{P}$ is $\gamma\left(\mathbf{s}_{i}\right)$.

Generating access structure tokens in this manner allows any subset of parties $\mathcal{A} \in \mathcal{P}$ to compute intersections of their respective sets $\left\{S_{i}\right\}_{i \in \mathcal{A}}$ by simply computing the inner products of $\{\mathbf{s}\}_{i \in \mathcal{A}}$ modulo $m$. The way in which the sets $S_{i}$ are generated ensures that:

$$
\bigcap_{i \in \mathcal{A}} S_{i}=H
$$

if and only if $\Omega \subseteq \mathcal{A}$, i.e., $\mathcal{A} \in \Gamma$. By choosing a large enough maximum prime factor $\max \left(p_{i}\right)$, we can guarantee that the size of the intersection is never a multiple of $m$ unless $\bigcap_{i \in \mathcal{A}} S_{i}=H$.

Note 1 (From example procedure to a general procedure). The space overhead of sending one unique vector to each party is $\Theta(h)+\max \left(p_{i}\right)=\Theta(h)$, where $h$ is the number of elements in the universe over which $\mathcal{H}$ is defined. We know from Section 5 that instead of vectors, the parties can be provided with inner products along with sizes of various unions and intersections. This allows the parties to compute the sizes of the intersections of their respective sets without revealing any information about the sets themselves. However, in order to perform unions (and the respective intersections) of $\ell$ sets, the parties need sizes of the intersections and unions of various combinations of sets $S_{i}(1 \leq i \leq \ell)$, which increases the space overhead to $\approx 2^{\ell}$.

From hereon, we use $\mho_{i}^{(\Gamma)}$ to denote the access structure token of party $P_{i}(i \in[\ell])$.
Lemma 2. For every authorized subset of parties $\mathcal{A} \in \Gamma$, it holds that $\left|\bigcap_{i \in \mathcal{A}} S_{i}\right|=0 \bmod m$.
Proof. It follows from the generation of $\left\{S_{i}\right\}_{i=1}^{\ell}$ that $\bigcap_{i \in \mathcal{A}} S_{i}=H$. Hence, it follows that $\left|\bigcap_{i \in \mathcal{A}} S_{i}\right|=0 \bmod m$.
Lemma 3. For every unauthorized subset of parties $\mathcal{B} \notin \Gamma$, it holds that $\left|\bigcap_{i \in \mathcal{B}} S_{i}\right| \neq 0 \bmod m$.
Proof. Since $\mathcal{B}$ is unauthorized, there exists $1 \leq j \leq|\Omega|$ such that $P_{j} \notin \mathcal{B}$. Then $\bigcap_{i \in \mathcal{B}} S_{i}=K \sqcup K^{\prime}$, where $K$ is an intersection of certain supersets of $H$ (which might include $H$ itself), and $K^{\prime}$ is a non-empty subset of $[|\Omega|+\kappa]$. It follows that:

$$
1 \leq\left|\bigcap_{i \in \mathcal{B}} S_{i}\right| \bmod p \leq l+|\Omega|+\kappa
$$

for $p=\max \left(p_{1}, \ldots, p_{r}\right)$, from which we obtain $\left|\bigcap_{i \in \mathcal{B}} S_{i}\right| \neq 0 \bmod m$.

## 7 PRIM-LWE

In this section, we present a new variant of LWE, called PRIM-LWE. We begin by describing and deriving some relevant results.

Recall that $M_{n}\left(\mathbb{Z}_{p}\right)$ denotes the space of $n \times n$ matrices over $\mathbb{Z}_{p}$.
Proposition 2. Let $p$ be prime. Then, there are

$$
p^{n^{2}}-\prod_{k=0}^{n-1}\left(p^{n}-p^{k}\right)
$$

matrices in $M_{n}\left(\mathbb{Z}_{p}\right)$ with determinant 0 . And, for any non-zero $\alpha \in \mathbb{Z}_{p}$, there are

$$
p^{n-1} \prod_{k=0}^{n-2}\left(p^{n}-p^{k}\right)
$$

matrices with determinant $\alpha$.
Proof. Clearly, there are $p^{n^{2}}$ matrices in $M_{n}\left(\mathbb{Z}_{p}\right)$. A matrix $\mathbf{M} \in M_{n}\left(\mathbb{Z}_{p}\right)$ has non-zero determinant if and only if it has linearly independent columns. Hence there are

$$
\gamma(p):=\left(p^{n}-1\right)\left(p^{n}-p\right)\left(p^{n}-p^{2}\right) \cdots\left(p^{n}-p^{n-1}\right)=\prod_{k=0}^{n-1}\left(p^{n}-p^{k}\right)
$$

such matrices, which shows that there are

$$
p^{n^{2}}-\gamma(p)=p^{n^{2}}-\prod_{k=0}^{n-1}\left(p^{n}-p^{k}\right)
$$

matrices with determinant 0 .
To prove the second statement, consider the determinant map

$$
\operatorname{det}: G L_{n}\left(\mathbb{Z}_{p}\right) \rightarrow \mathbb{Z}_{p}^{*},
$$

where $G L_{n}(\cdot)$ denotes the general linear group (see [79], Chapter 2.1, p. x) and det is a group homomorphism. Hence, it follows that for any $\alpha \in \mathbb{Z}_{p}^{*}$, there are exactly

$$
\frac{\gamma(p)}{p-1}=p^{n-1} \prod_{k=0}^{n-2}\left(p^{n}-p^{k}\right)
$$

matrices with determinant $\alpha$.
Corollary 4. The fraction of matrices in $M_{n}\left(\mathbb{Z}_{p}\right)$ whose determinant generates $\mathbb{Z}_{p}^{*}$ is:

$$
\frac{\varphi(p-1) \prod_{k=2}^{n}\left(p^{k}-1\right)}{p^{\frac{1}{2} n(n+1)}},
$$

where $\varphi$ is Euler's totient function (see Theorem 4).

Proof. Note that $\mathbb{Z}_{p}^{*}$ is cyclic of order $p-1$, so it has exactly $\varphi(p-1)$ different generators. By Proposition 2 , the required fraction is

$$
\varphi(p-1) \times \frac{p^{n-1} \prod_{k=0}^{n-2}\left(p^{n}-p^{k}\right)}{p^{n^{2}}}=\varphi(p-1) \times \frac{\prod_{k=2}^{n}\left(p^{k}-1\right)}{p^{\frac{1}{2} n(n+1)}} .
$$

Let us recall two standard results, given as Propositions 3 and 4, from the theory of infinite products (see [170] for more details).

Proposition 3. The infinite product $\prod_{k=1}^{\infty} a_{k}$ converges to a non-zero limit if and only if $\sum_{k=1}^{\infty} \log a_{n}$ converges.

Proposition 4. $\sum_{k=1}^{\infty} \log a_{n}$ converges absolutely if and only if $\sum_{k=1}^{\infty}\left(1-a_{n}\right)$ converges absolutely. Hence, if $\sum_{k=1}^{\infty}\left(1-a_{n}\right)$ converges absolutely, then $\prod_{k=1}^{\infty} a_{k}$ converges to a non-zero limit.

Proposition 5. There exists a constant $c=c(p)>0$, independent of $n$, such that the fraction of matrices in $M_{n}\left(\mathbb{Z}_{p}\right)$ whose determinant generates $\mathbb{Z}_{p}^{*}$ is bounded below by $c$ for all $n$.

Proof. Let

$$
f_{p}(n)=\frac{\prod_{k=2}^{n}\left(p^{k}-1\right)}{p^{\frac{1}{2} n(n+1)}}=\frac{1}{p-1} \prod_{k=1}^{n} \frac{p^{k}-1}{p^{k}} .
$$

Since the infinite series

$$
\sum_{k=1}^{\infty}\left(1-\frac{p^{k}-1}{p^{k}}\right)=\sum_{k=1}^{\infty} \frac{1}{p^{k}}=\frac{1}{p-1}
$$

converges absolutely, by Proposition $4, \lim _{n \rightarrow \infty} f_{p}(n)$ exists and is non-zero. Let $c^{\prime}=\lim _{n \rightarrow \infty} f_{p}(n)>0$.
Note, furthermore, that $f_{p}(n+1)<f_{p}(n)$, so that $f_{p}(n)>c^{\prime}$ for all $n$. By Proposition 4 , at least a $c^{\prime} \varphi(p-1)>0$ fraction of the matrices in $M_{n}\left(\mathbb{Z}_{p}\right)$ have determinants which are primitive roots of unity modulo $p$.

| $p$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\varphi(p-1)$ | 1 | 1 | 2 | 2 | 4 | 4 | 8 | 6 |
| $c(p)$ | 0.289 | 0.280 | 0.380 | 0.279 | 0.360 | 0.306 | 0.469 | 0.315 |

Table 1: Approximate values of $c(p)$

Remark 3. (i) While the exact values of $c(p)=\lim _{n \rightarrow \infty} f_{p}(n) \varphi(p-1)$ appear difficult to determine, we have calculated some approximate values of $c(p)$ as shown in Table 1.
(ii) It might seem from Table 1 that $c(p)$ does not vary much with $p$. Nevertheless, it might in fact be the case that $\inf _{p \text { prime }} c(p)=0$. A primorial prime is a prime of the form $p=\prod_{i=1}^{k} p_{i}+1$, where $p_{1}<p_{2}<\cdots<p_{k}$ are the first $k$ primes. For such a prime $p=\prod_{i=1}^{k} p_{i}+1$,

$$
c(p)=\frac{\varphi(p-1)}{p-1} \prod_{j=1}^{\infty} \frac{p^{j}-1}{p^{j}}<\frac{\varphi(p-1)}{p-1}=\prod_{j=1}^{k} \frac{p_{j}-1}{p_{j}} .
$$

It is an open problem whether or not there are infinitely many primorial primes, but heuristic arguments suggest that this should be the case. Suppose there actually are infinitely many such primes. Then, since

$$
\sum_{j=1}^{\infty}\left(1-\frac{p_{j}-1}{p_{j}}\right)=\sum_{j=1}^{\infty} \frac{1}{p_{j}}
$$

diverges [104], $c(p)$ diverges to 0 as $k \rightarrow \infty$, which shows that $\inf _{p} c(p)=0$.
We are now ready to define PRIM-LWE. First, we define:

$$
M_{n}^{p r i m}\left(\mathbb{Z}_{p}\right)=\left\{\mathbf{M} \in M_{n}\left(\mathbb{Z}_{p}\right): \operatorname{det}(\mathbf{M}) \text { is a generator of } \mathbb{Z}_{p}^{*}\right\}
$$

Recall that for a vector $\mathbf{s} \in \mathbb{Z}_{p}^{n}$ and a noise distribution $\chi$ over $\mathbb{Z}_{p}$, LWE distribution $D_{n, p, \mathbf{s}, \chi}^{\mathrm{LWE}}$ is defined as the distribution over $\mathbb{Z}_{p}^{n} \times \mathbb{Z}_{p}$ that is obtained by choosing $\mathbf{a} \stackrel{\$}{\leftarrow} \mathbb{Z}_{p}^{n}, e \stackrel{\Phi}{\leftarrow} \chi$, and outputting $(\mathbf{a},\langle\mathbf{a}, \mathbf{s}\rangle+e)$.

Definition 17 (PRIM-LWE). Let $n \geq 1$ and $p \geq 2$. Then, for a matrix $S \in M_{n}^{p r i m}\left(\mathbb{Z}_{p}\right)$ and a noise distribution $\chi$ over $M_{n}\left(\mathbb{Z}_{p}\right)$, define the PRIM-LWE distribution $D_{n, p, \mathbf{S}, \chi}^{\mathrm{PRIME}}$, to be the distribution over $M_{n}\left(\mathbb{Z}_{p}\right) \times M_{n}\left(\mathbb{Z}_{p}\right)$ obtained by choosing a matrix $\mathbf{A} \stackrel{\$}{\leftarrow} M_{n}\left(\mathbb{Z}_{p}\right)$ uniformly at random, $\mathbf{E} \stackrel{\$}{\leftarrow} \chi$, and outputting $(\mathbf{A}, \mathbf{A S}+\mathbf{E})$.

For distributions $\psi$ over $\mathbb{Z}_{p}^{n}$ and $\chi$ over $\mathbb{Z}_{p}$, the decision-LWE ${ }_{n, p, \psi, \chi}$ problem is to distinguish between $(a, b) \leftarrow D_{n, p, \mathbf{s}, \chi}^{\mathrm{LWE}}$ and a sample drawn uniformly from $\mathbb{Z}_{p}^{n} \times \mathbb{Z}_{p}$, where $\mathbf{s} \leftarrow \psi$. Similarly, for distributions $\psi$ over $M_{n}^{p r i m}\left(\mathbb{Z}_{p}\right)$ and $\chi$ over $M_{n}\left(\mathbb{Z}_{p}\right)$, the decision-PRIM-LWE ${ }_{n, p, \psi, \chi}$ problem is to distinguish between $(\mathbf{A}, \mathbf{B}) \leftarrow D_{n, p, \mathbf{S}, \chi}^{\mathrm{PRIM}}, \mathrm{LWE}$ and a sample drawn uniformly from $M_{n}\left(\mathbb{Z}_{p}\right) \times M_{n}\left(\mathbb{Z}_{p}\right)$, where $\mathbf{S} \leftarrow \psi$.

Theorem 10. Let $\psi$ and $\psi^{\prime}$ be the uniform distributions over $M_{n}^{p r i m}\left(\mathbb{Z}_{p}\right)$ and $\mathbb{Z}_{p}^{n}$ respectively. Suppose $\chi$ is the distribution over $M_{n}\left(\mathbb{Z}_{p}\right)$ obtained by selecting each entry of the matrix independently from the distribution $\chi^{\prime}$ over $\mathbb{Z}_{p}$. Then, solving decision-PRIM-LWE $E_{n, p, \psi, \chi}$ is at least as hard as decision-LWE $E_{n, p, \psi^{\prime}, \chi^{\prime}}$, up to an $O\left(n^{2}\right)$ factor.

Proof. Let $\varepsilon$ be the advantage of an adversary in solving decision-LWE ${ }_{n, p, \psi^{\prime}, \chi^{\prime}}$. By a standard hybrid argument, the advantage of distinguishing $(\mathbf{A}, \mathbf{A S}+\mathbf{E})$ from a sample uniformly drawn from $M_{n}\left(\mathbb{Z}_{p}\right) \times \mathbb{Z}_{p}^{n}$ is at most $n \varepsilon$.

A sample $(\mathbf{A}, \mathbf{A S}+\mathbf{E})$ where $\mathbf{A}, \mathbf{S} \stackrel{\$}{\leftrightarrows} M_{n}\left(\mathbb{Z}_{p}\right)$ is the same as $n$ samples $\left(\mathbf{A}, \mathbf{A} \mathbf{s}_{i}+\mathbf{E}_{i}\right)$, with $n$ different secrets $\mathbf{s}_{i}(i \in[n])$. Hence, the advantage of an adversary in distinguishing $(\mathbf{A}, \mathbf{A S}+\mathbf{E})$ from uniformly random is at most $n^{2} \varepsilon$.

By Proposition 5: $c=\inf _{n \geq 1}\left|M_{n}^{p r i m}\left(\mathbb{Z}_{p}\right)\right| /\left|M_{n}\left(\mathbb{Z}_{p}\right)\right|>0$. Given $m=\lceil 1 / c\rceil \operatorname{samples}\left(\mathbf{A}_{i}, \mathbf{A}_{i} \mathbf{S}_{i}+\mathbf{E}_{i}\right)$, where $\mathbf{S}_{i} \stackrel{\$}{\leftarrow} M_{n}\left(\mathbb{Z}_{p}\right)$,

$$
\operatorname{Pr}\left[\mathbf{S}_{i} \in M_{n}^{p r i m}\left(\mathbb{Z}_{p}\right) \text { for some } i\right] \geq 1-(1-c)^{m} \geq 1-e^{-c m} \geq 1-e^{-1}
$$

Therefore, if $\mathbf{S} \stackrel{\$}{\leftarrow} M_{n}^{\text {prim }}\left(\mathbb{Z}_{p}\right)$, then the advantage of an adversary in distinguishing $(\mathbf{A}, \mathbf{A S}+\mathbf{E})$ from uniformly random $(\mathbf{A}, \mathbf{B})$ is at most:

$$
\frac{m n^{2}}{1-e^{-1}} \varepsilon=O\left(n^{2}\right) \varepsilon
$$

## 8 Graph-Based Access Structure Hiding Verifiable Secret Sharing

In this section, we present the first graph-based access structure hiding verifiable secret sharing scheme, which is also the first LWE-based secret sharing scheme for general access structures.

Note 2. The (loose) description that follows gives a high-level overview of the actual scheme, and its only purpose is to facilitate better understanding of the full scheme. For the sake of simplicity, we assume that the access structure tokens are generated as inner products and unions (as described in Note 1).

Unlike the regular definition of STCON in the context of secret sharing, wherein parties are represented by edges in the graph, we denote parties by nodes in the graph.

### 8.1 High-Level Overview

Based on a minimal authorized set $\Omega \in \Gamma_{0}$, the dealer generates a connected DAG, $G=(V, E)$, where $|V|=\ell$, such that $G$ contains a source node $s$ and a sink node $t$. Each vertex/node in $G$ houses exactly one party with the STCON housing the parties in $\Omega$, i.e., the number of nodes in the STCON is $|\Omega|$. Figure 3 gives an example graph wherein node $v_{2}$ denotes $s$ and node $v_{6}$ represents $t$, with $\Omega=\left\{P_{1}, P_{2}, P_{3}, P_{4}\right\}$.


Fig. 3: Example DAG with STCON representing $\Omega=\left\{P_{1}, P_{2}, P_{3}, P_{4}\right\}$.

A unique matrix $\mathbf{A}_{v}$ along with the corresponding 'trapdoor information' $\tau_{i}$ is associated with each node $v \in V$ (i.e., party $P_{v} \in \mathcal{P}$ ), and "encodings" in the scheme are defined relative to the directed paths in $G$. Let $k$ be the secret to be shares. A "small" matrix $\mathbf{S}$, such that $\operatorname{det}(\mathbf{S})=k$, is encoded with respect to a path $u \rightsquigarrow v$ via another "small" matrix $\mathbf{D}_{u}$ such that $\mathbf{D}_{u} \cdot \mathbf{A}_{u} \approx \mathbf{A}_{v} \cdot \mathbf{S}_{u}$, where $\mathbf{S}_{u}=\mathbf{S}^{\mho_{u}^{(\Gamma)}}$. Access structure token for party $P_{u}$ with respect to access structure $\Gamma$ is denoted by $\mho_{u}^{(\Gamma)}$, and generated by following the procedures given in Section 6 and Note 1. For the sake of simplicity, we assume that the access structure tokens are generated as inner products and unions (as described in Note 1). For one randomly selected party $P_{j} \in \Omega$, the share is generated as: $\mathbf{S}_{j}=\mathbf{S}^{\mho_{j}^{(T)}+1}$. Given 'trapdoor information' $\tau_{u}$ for $\mathbf{A}_{u}$, encoding $\mathbf{D}_{u}$ for share $\mathbf{S}_{u}$ with respect to $\operatorname{sink} v$ is generated such that:

$$
\mathbf{D}_{u} \cdot \mathbf{A}_{u}=\mathbf{A}_{v} \cdot \mathbf{S}_{u}+\mathbf{E}_{u}
$$

where $\mathbf{E}_{u}$ is a small LWE error matrix. It is easy to see that the LWE instance $\left\{\mathbf{A}_{u}, \mathbf{B}_{u}\left(=\mathbf{A}_{v} \cdot \mathbf{S}_{u}+\mathbf{E}_{u}\right)\right\}$ remains hard for appropriate parameters and dimensions. Encodings relative to paths $v \rightsquigarrow w$ and $u \rightsquigarrow v$ can be multiplied to get an encoding relative to path $u \rightsquigarrow w$. Namely, given:

$$
\mathbf{D}_{v} \cdot \mathbf{A}_{v}=\mathbf{A}_{w} \cdot \mathbf{S}_{v}+\mathbf{E}_{v} \quad \text { and } \quad \mathbf{D}_{u} \cdot \mathbf{A}_{u}=\mathbf{A}_{v} \cdot \mathbf{S}_{u}+\mathbf{E}_{u}
$$

we obtain:

$$
\mathbf{D}_{v} \cdot \mathbf{D}_{u} \cdot \mathbf{A}_{u}=\mathbf{D}_{v} \cdot\left(\mathbf{A}_{v} \cdot \mathbf{S}_{u}+\mathbf{E}_{u}\right)=\mathbf{A}_{w} \cdot \mathbf{S}_{v} \cdot \mathbf{S}_{u}+\mathbf{E}^{\prime}
$$

such that the matrices, $\mathbf{D}_{v} \cdot \mathbf{D}_{u}, \mathbf{S}_{v} \cdot \mathbf{S}_{u}$ and $\mathbf{E}^{\prime}$ remain small. Note that our procedures for generating and multiplying the encodings are similar to that of Gentry et al. [122], but apart from that two schemes are completely different; their aim was to develop a multilinear map scheme and their tools do not include extremal set theory.

The final encoding for any set of parties is generated by combining their respective encodings according to the source-sink order of the nodes that house them. For example, in Figure 3, the final encoding for the path from $s$ to $t$ is given by $\mathbf{D}=\mathbf{D}_{2} \mathbf{D}_{3} \mathbf{D}_{4} \mathbf{D}_{5}$. Specifically, if $\mathbf{A}_{1}$ is the matrix of party $P_{1}$, housed by $s$, and $\mathbf{A}_{6}$ is the matrix assigned to party $P_{6}$, housed by $t$, then we can compute:

$$
\begin{aligned}
\mathbf{D} \cdot \mathbf{A}_{1} \bmod N & =\mathbf{A}_{6} \cdot \mathbf{S}^{1+\sum_{i \in \Omega} \mho_{i}^{(\Gamma)}}+\mathbf{E}^{\prime} \bmod N \\
& =\mathbf{A}_{6} \cdot \mathbf{S}^{1+c m}+\mathbf{E}^{\prime} \bmod N \quad \text { (using Lemma 2) } \\
& =\mathbf{A}_{6} \cdot \mathbf{S}^{1+c \varphi(N)}+\mathbf{E}^{\prime} \bmod N,
\end{aligned}
$$

where $N$ is some composite integer, $\varphi$ denotes Euler's totient function (see Theorem 4) and $m=\varphi(N)$ is the modulo with respect to which the set-system (as described in Theorem 1) is defined. The scheme ensures that any authorized subset of parties $\mathcal{A} \supseteq \Omega$ possesses the 'trapdoor information' required to invert $\mathbf{A}_{6} \cdot \mathbf{S}^{1+c \varphi(N)}+\mathbf{E}^{\prime} \bmod N$, and recover the secret $k=\operatorname{det}(\mathbf{S})$.

### 8.2 Detailed Scheme: Share Generation

Let $m=\prod_{i=1}^{r} p_{i}\left(\min \left(p_{i}\right)=3\right)$ be a positive integer with $r>1$ odd prime divisors such that $2 m+1$ is prime. Recall from Dirichlet's Theorem (see Theorem 2) that there are infinitely many odd integers $m$ such that $2 m+1$ is prime. As described by Theorem 1 , define a set-system $\mathcal{H}$ modulo $m$ over a universe of $h$ elements such that for all $H_{1}, H_{2} \in \mathcal{H}$, it holds that exactly one of the following three conditions is true

- $\left|H_{1}\right|=\left|H_{2}\right|=\eta m$, where $\eta$ is some even integer,
$-\left|H_{1}\right|=l\left|H_{2}\right|$,
$-\left|H_{2}\right|=l\left|H_{1}\right|$,
where $l=2$.
Let $m^{\prime}=m p_{r^{\prime}}$ be a positive integer, where $p_{r^{\prime}}$ is an odd prime such that for all $i \in[r]$, it holds that: $p_{r^{\prime}} \neq p_{i}$. According to Theorem 1, define a set-system $\mathcal{H}^{\prime}$ modulo $m^{\prime}$ over a universe of $h$ elements. Since $m$ is a factor of $m^{\prime}$, the following holds for all $H \in \mathcal{H}^{\prime}$ :

$$
|H|=0 \bmod m^{\prime}=0 \bmod m .
$$

Note that for appropriate choice of the underlying set-system $\mathcal{G}$ (see Proposition 1), it holds that $\left|\mathcal{H} \cap \mathcal{H}^{\prime}\right|>0$. Hence, we pick a set $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$ to generate access structure tokens. We know that the following holds for some $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$ :

- $H$ is a proper subset of exactly $s^{l-1}$ sets and not a proper superset of any sets in $\mathcal{H}^{\prime}$,
- $H$ is a proper superset of exactly $l$ sets and not a proper subset of any sets in $\mathcal{H}$,
where $s \geq \exp \left(c \frac{(\log h)^{r}}{(\log \log h)^{r-1}}\right)$.
Note 3 (Encoding $l+1$ monotone access structures via two moduli). Let us examine the benefits of using two moduli and two set-systems. We know from Section 5 that access structure tokens operate over a fixed set $H$ and its $s^{l-1}$ proper supersets. Also recall that $H$ does not exactly represent the minimal authorized subset $\Omega$, instead it is a randomly sampled set, picked to enforce the desired access structure $\Gamma=\operatorname{cl}(\Omega)$. Having a set $H$ with $s^{l-1}$ proper supersets in $\mathcal{H}^{\prime}$ and $l$ proper subsets in $\mathcal{H}$ enables us to use carefully generated access structure tokens to capture the $l$ minimal authorized subsets that are represented by subsets of $H$ in $\mathcal{H}$. Let $\widetilde{\mathcal{H}} \in \mathcal{H}$ denote the collection of $l$ proper subsets of $H$, and $\widehat{\mathcal{H}} \in \mathcal{H}^{\prime}$ denote the $s^{l-1}$ proper supersets of $H$. Update these as: $\widetilde{\mathcal{H}}=\widetilde{\mathcal{H}} \cup H$ and $\widehat{\mathcal{H}}=\widehat{\mathcal{H}} \cup H$ to denote the collections of $l+1$ subsets and $s^{l-1}+1$ supersets of $H$, respectively. Further, let $\wp=\left\{\Gamma_{1}, \ldots, \Gamma_{l+1}\right\}$ denote the family of monotone access structures that originate from the family of minimal authorized subsets $\left\{\Omega_{i}\right\}_{i=1}^{l+1}$, where $\Gamma_{i}=\operatorname{cl}\left(\Omega_{i}\right)$ for $i \in[l+1]$. Let $\left\{\mho_{i}^{(\wp)}\right\}_{i=1}^{\ell}$ denote the access structure tokens that capture the $l+1$ access structures in $\wp$. Then, for an access structure token combining function $f$, it follows that the following holds for all subsets of parties $\mathcal{A} \in \wp:$

$$
f\left(\left\{\mho^{(\wp)}\right\}_{i \in \mathcal{A}}\right)=|H \cap \tilde{H}|=0 \bmod m \quad \text { OR } \quad f\left(\left\{\mho^{(\wp)}\right\}_{i \in \mathcal{A}}\right)=|H \cap \widehat{H}|=0 \bmod m^{\prime},
$$

where $\widehat{H} \in \widehat{\mathcal{H}}$ and $\widetilde{H} \in \widetilde{\mathcal{H}}$. Since $\bmod m$ and $\bmod m^{\prime}$ correspond to the set-systems $\mathcal{H}$ and $\mathcal{H}^{\prime}$, respectively, we need two moduli to realize this functionality. Note that the access structure token generation procedure discussed in Section 5 is not suitable to achieve this goal but any procedure that operates over the sets in a manner that guarantees that the outputs of unions remain inside the collection of some fixed set-systems can be used to harness the power of two moduli to achieve significant improvements over the current known upper bound of $2^{.637 \ell+o(\ell)}$ on the share size for secret sharing for general (monotone) access structures.

We are now ready to present our detailed access structure hiding verifiable secret sharing scheme that works with two moduli. Since we do not have an access structure encoding procedure that satisfies the properties outlined in Note 3, we use our access structure encoding procedures from Section 6. As explained in Section 8.1, we arrange the $\ell$ parties as nodes in a DAG $G$. Without loss of generality, we assume that the parties lie on a single directed path, as shown in Figure 4. Each party $P_{i} \in \mathcal{P}$ operates in:
$-\mathbb{Z}_{q}$ if $i=1 \bmod 2$,
$-\mathbb{Z}_{q^{\prime}}$ if $i=0 \bmod 2$.


Fig. 4: Parties $\mathcal{P}=\left\{P_{1}, \ldots, P_{\ell}\right\}$ arranged as a simple DAG: a generalization of Figure 3.

For a prime $p=2 m+1$, let $q=p c$ and $q^{\prime}=p c^{\prime}$, such that $p \nmid c, c^{\prime}$ and $\left|q^{\prime}\right|=|q|+\epsilon(|q|)$, where $\epsilon$ is a negligible function (see Definition 4). We ensure that $q, q^{\prime}=(d \lambda)^{\Theta(d)}$ such that the following holds:

$$
q<q^{\prime}, \quad p \leq \sqrt{\log q} \quad \text { and } \quad \frac{2 p-1}{2 p}<\frac{q}{q^{\prime}}<\frac{2 p}{2 p+1} .
$$

The other parameters are chosen as: $n=\Theta(d \lambda \log (d \lambda))$ and $w=\Theta(n \log q)=\Theta\left(d^{2} \lambda \log ^{2}(d \lambda)\right)$.
The secret $k(\neq 0) \in \mathbb{Z}_{p}$ is a primitive root modulo $p$, and gets encoded using a $n \times n$ matrix $\mathbf{S}$ such that $\|\mathbf{S}\|<p$ and $\operatorname{det}(\mathbf{S})=k$. Given a minimal authorized subset $\Omega \in \Gamma_{0}$, the dealer uses $H \in \mathcal{H} \cap \mathcal{H}^{\prime}$ to generate $\ell$ access structure tokens $\left\{\mho_{i}^{(\Gamma)}\right\}_{i=1}^{\ell} \in \mathbb{Z}_{m} \cup \mathbb{Z}_{m^{\prime}} \backslash\{0\}$ (as defined by Note 1 in Section 6) that capture the access structure $\Gamma=\operatorname{cl}(\Omega)$. For $s=\sqrt{n}, \sigma=\Theta(\sqrt{n \log q})=\Theta\left(\sqrt{n \log q^{\prime}}\right)$, and security parameter $\lambda$, the dealer generates following for each party $P_{i} \in \mathcal{P}$ :

- sample a $w \times n$ matrix $\mathbf{A}_{i}$ such that $\left\|\mathbf{A}_{i}\right\|<p$,
- compute the 'trapdoor information' $\tau_{i}$ for $\mathbf{A}_{i}$ using the lattice-trapdoor generation algorithm from [207] and a fixed generator matrix $G$,
- sample a $w \times n$ matrix $\mathbf{E}_{i}$ from the discrete Gaussian distribution $\chi=D_{\mathbb{Z}, s}$ subject to the restriction that $\left\|\mathbf{E}_{i}\right\|<s \sqrt{\lambda}$,
- except for a randomly picked party $P_{j}$, compute the share for party $P_{i}$ as: $\mathbf{S}_{i}=\mathbf{S}^{\gamma_{i}^{(\Gamma)}} \bmod p$. The share for party $P_{j}$ is generated as: $\mathbf{S}_{j}=\mathbf{S}^{\mho_{j}^{(\Gamma)}+1} \bmod p$,
- use $\tau_{i}$ to compute a $w \times w$ encoding $\mathbf{D}_{i}$ of $P_{i}$ 's share $\mathbf{S}_{i}$ such that the following relations hold (source-sink; see Figure 4):

$$
\begin{aligned}
\mathbf{D}_{1} \mathbf{A}_{1} & =\mathbf{A}_{2} \mathbf{S}_{1}+\mathbf{E}_{1} \\
\mathbf{D}_{2} \mathbf{A}_{2} & =\mathbf{A}_{3} \mathbf{S}_{2}+\mathbf{E}_{2} \\
& \vdots \\
\mathbf{D}_{\ell-1} \mathbf{A}_{\ell-1} & =\mathbf{A}_{\ell} \mathbf{S}_{\ell-1}+\mathbf{E}_{\ell-1},
\end{aligned}
$$

where $\left\|\mathbf{D}_{i}\right\|<\sigma \sqrt{\lambda}$.
Since the entries of $\mathbf{A}_{i}$ are bounded by $p$, the following follows from our selection of $q$ and $q^{\prime}$ :

$$
\left\lfloor\frac{q^{\prime}}{q} \mathbf{A}_{i}\right\rceil=\mathbf{A}_{i} \text { for odd } i, \quad \text { and } \quad\left\lfloor\frac{q}{q^{\prime}} \mathbf{A}_{i}\right\rceil=\mathbf{A}_{i} \text { for even } i .
$$

This means that we may naturally interpret the entries of $\mathbf{A}_{i}$ 's as being in both $\mathbb{Z}_{q}$ and $\mathbb{Z}_{q^{\prime}}$.
Notations: The following notations are used frequently throughout the rest of this section.

- Without loss of generality, let $v_{i}$ be the node housing the party $P_{i}$ for all $i \in[r]$.
- We use $\vec{\Pi}$ to denote a product that is computed in the order that is defined by the relative positions of the nodes present in the given directed path, from source to sink. We call such products in-order. For instance, the following denotes the in-order product of the 'trapdoor information' of all parties in the DAG depicted in Figure 3:

$$
\overrightarrow{\prod_{i \in \mathcal{P}}} \tau_{i}=\tau_{5} \tau_{1} \tau_{2} \tau_{3} \tau_{4} \tau_{6} \tau_{7}
$$

- Similarly, if the multiplications are performed in the opposite order to what is defined by the given directed path; i.e., the multiplications are performed from sink to source, then we call it reverse-order product and denote it as $\Pi$. For example, the reverse-order product of the 'trapdoor information' of all parties in the DAG depicted in Figure 3 is: $\overleftrightarrow{\prod_{i \in \mathcal{P}}} \tau_{i}=\tau_{7} \tau_{6} \tau_{4} \tau_{3} \tau_{2} \tau_{1} \tau_{5}$.
- For a subset of parties $\mathcal{A} \subseteq \mathcal{P}$, which forms a directed path $\mathfrak{P}$ in the DAG $G$, let $P_{\triangleright}^{(\mathcal{A})}$ denote the party that is housed by the node that is at the beginning of $\mathfrak{P}$. Similarly, let $P_{\triangleleft}^{(\mathcal{A})}$ denote the party that is housed by the node that is at the end of $\mathfrak{P}$.
Let $\tau_{\triangleleft}^{(\mathcal{A})}$ denote the 'trapdoor information' corresponding to the matrix $\mathbf{A}_{\triangleleft}^{(\mathcal{A})}$ of party $P_{\triangleleft}^{(\mathcal{A})}$. Each party $P_{i} \in \mathcal{P}$ receives its share as: $\left\{\mho_{i}^{(\Gamma)}, \Psi_{i}^{(k)}\right\}$, where $\Psi_{i}^{(k)}=\left\{\mathbf{A}_{i}, \tilde{\tau}_{i}, \mathbf{D}_{i}\right\}$ and $\tilde{\tau}_{i}$ is randomly sampled such that it holds for all subsets of parties $\mathcal{A} \subseteq \mathcal{P}$ that: $\tau_{\triangleleft}^{(\mathcal{A})}=\overrightarrow{\prod_{i \in \mathcal{A}}} \tilde{\tau}_{i}$ (in $\mathbb{Z}_{q}$ or $\mathbb{Z}_{q^{\prime}}$, depending on the value of $i \bmod 2$ ) if and only if $\mathcal{A} \supseteq \Omega$, i.e., $\mathcal{A} \in \Gamma$.


### 8.3 Secret Reconstruction and Correctness

In order to reconstruct the secret, any subset of parties $\mathcal{A} \subseteq \mathcal{P}$ first combine their access structure tokens $\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{A}}$ and verify that:

$$
\sum_{i \in \mathcal{A}} \mho_{i}^{(\Gamma)}=0 \bmod m \quad \text { OR } \quad \sum_{i \in \mathcal{A}} \mho_{i}^{(\Gamma)}=0 \bmod m^{\prime}
$$

It follows from Section 6 that the access structure tokens can be generate such that above condition holds for any authorized subset of parties $\mathcal{A} \in \Gamma$, while for all unauthorized subsets $\mathcal{B} \notin \Gamma$, it holds that:

$$
\sum_{i \in \mathcal{B}} \mho_{i}^{(\Gamma)} \neq 0 \bmod m \quad \text { AND } \quad \sum_{i \in \mathcal{B}} \mho_{i}^{(\Gamma)} \neq 0 \bmod m^{\prime}
$$

Once it is established that $\mathcal{A} \in \Gamma$, then the parties combine their encodings $\left\{\mathbf{D}_{i}\right\}_{i \in \mathcal{A}}$, in the correct order as:

$$
\begin{equation*}
\prod_{i \in \mathcal{A}} \mathbf{D}_{i} \mathbf{A}_{\triangleright}^{(\mathcal{A})}=\mathbf{D} \mathbf{A}_{\triangleright}^{(\mathcal{A})}=\mathbf{A}_{\triangleleft}^{(\mathcal{A})} \prod_{i \in \mathcal{A}} \mathbf{S}^{\mho_{i}^{(\Gamma)}}+\mathbf{E}^{\prime}=\mathbf{A}_{\triangleleft}^{(\mathcal{A})} \mathbf{S}^{\sum_{i \in \mathcal{A}}} \mho_{i}^{(\Gamma)}+1+\mathbf{E}^{\prime} \tag{8.1}
\end{equation*}
$$

Recall that $\mathbf{A}_{\triangleright}^{(\mathcal{A})}$ and $\mathbf{A}_{\triangleleft}^{(\mathcal{A})}$ respectively denote the matrices of the parties housed by the first and final nodes in the directed path formed by the nodes housing the parties in $\mathcal{A}$. Depending on the value of $i \bmod 2$, each party $P_{i} \in \mathcal{P}$ operates within its respective $\bmod q$ or $\bmod q^{\prime}$ world. Without loss of generality, let $P_{\triangleleft}^{(\mathcal{A})}$ operate in modulo $q$ world. Recall that for $\mathcal{A} \in \Gamma$, it holds that:

$$
\sum_{i \in \mathcal{A}} \mho_{i}^{(\Gamma)}=0 \bmod m \quad \text { OR } \quad \sum_{i \in \mathcal{A}} \mho_{i}^{(\Gamma)}=0 \bmod m^{\prime}
$$

i.e., it holds that:

$$
\sum_{i \in \mathcal{A}} \mho_{i}^{(\Gamma)}=c(p-1)
$$

where $c \geq 1$ is an integer. Recall that the size of all sets in $\mathcal{H}$ and $\mathcal{H}^{\prime}$ is an even multiple of $m$ and $p=2 m+1$. Therefore, sizes of the intersections between any subset-superset pairs must also be even multiples of $m$. Hence, for all $\vartheta=0 \bmod m$ and/or $\vartheta=0 \bmod m^{\prime}$, it holds that $\vartheta=c(p-1)$, where $c \geq 1$ is an integer.

Recall that $q=p c$ and $q^{\prime}=p c^{\prime}$, where $p \nmid c, c^{\prime}$. Hence, for authorized subsets of parties $\mathcal{A} \in \Gamma$, Equation (8.1) equates to:

$$
\begin{equation*}
\mathbf{A}_{\triangleleft}^{(\mathcal{A})} \mathbf{S}^{\left\langle\mathbf{v}, \sum_{i \in \mathcal{A}} \mathbf{v}_{i}\right\rangle+1}+\mathbf{E}^{\prime}=\mathbf{A}_{\triangleleft}^{(\mathcal{A})} \mathbf{S}^{c(p-1)+1}+\mathbf{E}^{\prime} . \tag{8.2}
\end{equation*}
$$

We know that only authorized subset of parties $\mathcal{A} \in \Gamma$ can combine their trapdoor shares $\left\{\tilde{\tau}_{i}\right\}_{i \in \mathcal{A}}$ to generate the trapdoor $\tau_{\triangleleft}^{(\mathcal{A})}$ required to invert $\mathbf{A}_{\mathcal{A}}$. Hence, it follows from Equations (8.1) and (8.2) that for small $\mathbf{E}^{\prime}$, the LWE inversion algorithm from [207] can be used with $\tau_{\triangleleft}^{(\mathcal{A})}$ to compute matrix $\mathbf{S}^{c(p-1)+1}$. We know from Fermat's little theorem (See Theorem 3) that:

$$
\operatorname{det}(\mathbf{S})^{c(p-1)+1}=\operatorname{det}(\mathbf{S})^{0+1} \bmod p .
$$

Hence, the secret can be recovered as $\operatorname{det}(\mathbf{S})^{c(p-1)+1}=\operatorname{det}(\mathbf{S}) \bmod p=k$. Next, we prove that $\mathbf{E}^{\prime}$ is indeed small for a bounded number of parties.
Lemma 4. It holds that the largest $\mathbf{E}^{\prime}$, computed by combining all encodings $\mathbf{D}_{i}$ as:

$$
\mathbf{D}_{\ell-1} \mathbf{D}_{\ell-2} \cdots \mathbf{D}_{1} \mathbf{A}_{1}=\mathbf{A}_{\ell} \mathbf{S}_{\ell-1} \mathbf{S}_{\ell-2} \cdots \mathbf{S}_{1}+\mathbf{E}^{\prime}
$$

has entries bounded by $O\left(\sqrt{d^{6 \ell-11} \lambda^{4 \ell-5} \log ^{6 \ell-11}(d \lambda)}\right)$.
Proof. In the setting depicted in Figure 4, if we combine the shares from parties $P_{1}$ and $P_{2}$, we obtain:

$$
\begin{aligned}
\mathbf{D}_{2} \mathbf{D}_{1} \mathbf{A}_{1} & =\mathbf{D}_{2} \mathbf{A}_{2} \mathbf{S}_{1}+\mathbf{D}_{2} \mathbf{E}_{1} \\
& =\left(\mathbf{A}_{3} \mathbf{S}_{2}+\mathbf{E}_{2}\right) \mathbf{S}_{1}+\mathbf{D}_{2} \mathbf{E}_{1} \\
& =\mathbf{A}_{3} \mathbf{S}_{2} \mathbf{S}_{1}+\mathbf{E}_{2}^{\prime},
\end{aligned}
$$

where $\mathbf{E}_{2}^{\prime}=\mathbf{E}_{2} \mathbf{S}_{1}+\mathbf{D}_{2} \mathbf{E}_{1}$. Hence, it follows that:

$$
\begin{aligned}
\left\|\mathbf{E}_{2}^{\prime}\right\| & <n \cdot\left\|\mathbf{E}_{2}\right\| \cdot\left\|\mathbf{S}_{1}\right\|+m \cdot\left\|\mathbf{D}_{2}\right\| \cdot\left\|\mathbf{E}_{1}\right\| \\
& =O\left(\sqrt{d^{7} \lambda^{7} \log ^{7}(d \lambda)}\right) .
\end{aligned}
$$

If we now combine this with the share from party $P_{3}$, we obtain:

$$
\begin{aligned}
\mathbf{D}_{3} \mathbf{D}_{2} \mathbf{D}_{1} \mathbf{A}_{1} & =\mathbf{D}_{3} \mathbf{A}_{3} \mathbf{S}_{2} \mathbf{S}_{1}+\mathbf{D}_{3} \mathbf{E}_{2}^{\prime} \\
& =\left(\mathbf{A}_{4} \mathbf{S}_{3}+\mathbf{E}_{3}\right) \mathbf{S}_{2} \mathbf{S}_{1}+\mathbf{D}_{3} \mathbf{E}_{2}^{\prime} \\
& =\mathbf{A}_{4} \mathbf{S}_{3} \mathbf{S}_{2} \mathbf{S}_{1}+\mathbf{E}_{3}^{\prime},
\end{aligned}
$$

where $\mathbf{E}_{3}^{\prime}=\mathbf{E}_{3} \mathbf{S}_{2} \mathbf{S}_{1}+\mathbf{D}_{3} \mathbf{E}_{2}^{\prime}$. Then,

$$
\begin{aligned}
\left\|\mathbf{E}_{3}^{\prime}\right\| & <n^{2} \cdot\left\|\mathbf{E}_{3}\right\| \cdot\left\|\mathbf{S}_{2}\right\| \cdot\left\|\mathbf{S}_{1}\right\|+m \cdot\left\|\mathbf{D}_{3}\right\| \cdot\left\|\mathbf{E}_{2}^{\prime}\right\| \\
& =O\left(\sqrt{d^{13} \lambda^{11} \log ^{13}(d \lambda)}\right) .
\end{aligned}
$$

Therefore, by induction, it follows for any $\left\|\mathbf{E}^{\prime}\right\|$ that:

$$
\begin{aligned}
\left\|\mathbf{E}^{\prime}\right\| \leq\left\|\mathbf{E}_{\ell-1}^{\prime}\right\| & =O\left(\sqrt{d^{6(\ell-1)-5} \lambda^{4(\ell-1)-1} \log ^{6(\ell-1)-5}(d \lambda)}\right) \\
& =O\left(\sqrt{d^{6 \ell-11} \lambda^{4 \ell-5} \log ^{6 \ell-11}(d \lambda)}\right)
\end{aligned}
$$

### 8.4 Secret and Share Verification

After a successful secret reconstruction, any honest party $P_{i} \in \mathcal{A}$ in any authorized subset $\mathcal{A} \in \Gamma$ can verify the correctness of all shares $\left\{\mathbf{S}_{i}\right\}_{i \in \mathcal{A}}$ from the reconstructed secret $k$. The verification is performed by removing $P_{i}$ from the directed path to $P_{\triangleleft}^{(\mathcal{A})}$ and then using $\tau_{\triangleleft}^{(\mathcal{A})}$ to invert the resulting PRIM-LWE instance. For example, if the directed path formed by the nodes housing the parties in $\mathcal{A}$ contains $P_{i} \longrightarrow P_{j} \longrightarrow P_{t}$ at the end, where $t=1 \bmod 2$, then party $P_{i}$ 's share can be verified by computing:

$$
\mathbf{D}_{t} \mathbf{D}_{j} \mathbf{A}_{j}=\mathbf{A}_{t} \mathbf{S}^{\mho_{j}^{(\Gamma)}+v_{t}^{(\Gamma)}}+\mathbf{E}^{\prime}
$$

and inverting the output by using 'trapdoor information' $\tau_{t}$ to find $\mathbf{S}^{\mho_{j}^{(\Gamma)}+\mho_{t}^{(\Gamma)}}$, and use $\mho_{j}^{(\Gamma)}, \mho_{t}^{(\Gamma)}$ to verify its consistency with $k=\operatorname{det}(\mathbf{S}) \bmod p$.

Note that with our verification procedure, there can be a non-negligible probability of an inconsistent share to pass as valid due to random chance. For a setting with $\lceil\ell / 2\rceil$ malicious parties, the probability of an inconsistent share passing the verification of all honest parties is: $(1 / p-1)^{\left\lfloor\frac{\ell}{2}\right\rfloor}$. Hence, larger values of $p$ lead to smaller probabilities of verification failures by all honest parties. Note that unlike traditional VSS schemes, our scheme does not guarantee that all honest parties recover a consistent secret. Instead, it allows detection of malicious behavior without requiring any additional communication or cryptographic subroutines.

### 8.5 Maximum Share Size

Since our access structure hiding verifiable secret sharing scheme works with minimal authorized subsets, its maximum share size is achieved when the access structure contains the largest possible number of minimal authorized subsets. For a set of $\ell$ parties, the maximum number of unique minimal authorized subsets in any access structure is $\binom{\ell}{\ell / 2}$. Recall that the secret $k=\operatorname{det}(\mathbf{S})$ belongs to $\mathbb{Z}_{p}$. Hence, for $|q| \approx\left|q^{\prime}\right|$, it holds that $|k| \approx \sqrt{q}$. For each minimal authorized subset, every party $P_{i} \in \mathcal{P}$ receives a share of size $q(2 m n+1)$. Since $q=\operatorname{poly}(n)$ and the size of each access structure token is $\Theta(h)$ (see Section 6), the maximum share size is (using results from [88]):

$$
\begin{aligned}
\max \left(\Psi_{i}^{(k)}\right) & \leq\binom{\ell}{\ell / 2}(\sqrt{q}(2 m n+1)+\Theta(h)) \\
& =(1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(\sqrt{q}\left(2 q^{\varrho}+1\right)+\Theta(h)\right) \\
& =(1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+\sqrt{q}+\Theta(h)\right)
\end{aligned}
$$

where $\varrho \leq 1$ is a constant and $h$ is the number of elements over which our set-systems are defined.
Possible Improvements: If one is able to realize the access structure encoding that is described in Note 3, then the maximum share size drops by a factor of $l \geq 2$ by using that procedure instead of the one that we used in our scheme. Hence, the maximum share size (with respect to the secret size) of the resulting scheme
would be:

$$
\begin{aligned}
\max \left(\Psi_{i}^{(k)}\right) & \leq \frac{1}{l+1}\binom{\ell}{\ell / 2} \sqrt{q}(2 m n+2) \\
& =\frac{1}{l+1}\left((1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}} \sqrt{q}\left(2 q^{\varrho}+2\right)\right) \\
& =\frac{1}{l+1}\left((1+o(1)) \frac{2^{\ell}}{\sqrt{\pi \ell / 2}}\left(2 q^{\varrho+0.5}+2 \sqrt{q}\right)\right),
\end{aligned}
$$

where $\varrho \leq 1$ is a constant and $l \geq 2$ is as defined by Theorem 1 .

### 8.6 Secrecy and Privacy

Lemmas 2 and 3 establish perfect completeness and perfect soundness of our scheme, respectively (see Definition 15). We argued about perfect correctness while explaining the secret reconstruction procedure. Hence, we are left with proving computational secrecy and statistical hiding.

Theorem 11 (Statistical Hiding). Every unauthorized subset $\mathcal{B} \notin \Gamma$ can identify itself to be outside $\Gamma$ by using its set of access structure tokens, $\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}$. Given that the decision-LWE problem is hard, the following holds for all unauthorized subsets $\mathcal{B} \notin \Gamma$ and all access structures $\Gamma^{\prime} \subseteq 2^{\mathcal{P}}$, where $\Gamma \neq \Gamma^{\prime}$ and $\mathcal{B} \notin \Gamma^{\prime}:$

$$
\left|\operatorname{Pr}\left[\Gamma \mid\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}\right]-\operatorname{Pr}\left[\Gamma^{\prime} \mid\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}\right]\right|=2^{-\omega}
$$

where $\omega=|\mathcal{P} \backslash \mathcal{B}|$ is the security parameter.
Proof. It follows from Lemma 3 that the following holds for all unauthorized subsets of parties $\mathcal{B} \notin \Gamma$ :

$$
\sum_{i \in \mathcal{B}} \mho_{i}^{(\Gamma)} \neq 0 \bmod m,
$$

i.e., any unauthorized subset $\mathcal{B} \notin \Gamma$ can use its access structure tokens to identify itself as outside of the access structure $\Gamma$. The security parameter $\omega=|\mathcal{P} \backslash \mathcal{B}|$ accounts for this minimum information that is available to any unauthorized subset $\mathcal{B} \notin \Gamma$.

We know that the set $H \stackrel{\$}{\leftarrow} \mathcal{H}$ is randomly sampled. Furthermore, the access structure tokens $\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}$ given to the parties are permuted according to a random permutation $\gamma$. Hence, it follows from the randomness of $H$ and $\gamma$ that:

$$
\left|\operatorname{Pr}\left[\Gamma \mid\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}\right]-\operatorname{Pr}\left[\Gamma^{\prime} \mid\left\{\mho_{i}^{(\Gamma)}\right\}_{i \in \mathcal{B}}\right]\right|=2^{-\omega} .
$$

Theorem 12 (Computational Secrecy). Given that decision-LWE problem is hard, it holds for every unauthorized subset $\mathcal{B} \notin \Gamma$ and all different secrets $k_{1}, k_{2} \in \mathcal{K}$ that the distributions $\left\{\Psi^{\left(k_{1}\right)}\right\}_{i \in \mathcal{B}}$ and $\left\{\Psi^{\left(k_{2}\right)}\right\}_{i \in \mathcal{B}}$ are computationally indistinguishable with respect to the security parameter $\varepsilon \cdot|\mathcal{B}|$, where $\varepsilon$ denotes the advantage of a polynomial-time adversary against a PRIM-LWE instance.
Proof. Recall that $\Psi_{i}^{(k)}=\left\{\mathbf{A}_{i}, \tilde{\tau}_{i}, \mathbf{D}_{i}\right\}$. We know that the 'trapdoor shares' $\left\{\tilde{\tau}_{i}\right\}_{i \in[\ell]}$ are generated randomly such that the following holds only for authorized subsets of parties $\mathcal{A} \in \Gamma$ :

$$
\tau_{\triangleleft}^{(\mathcal{A})}=\overrightarrow{\prod_{i \in \mathcal{A}}} \tilde{\tau}_{i} .
$$

Hence, it follows from one-time pad that $\left\{\tilde{\tau}_{i}\right\}_{i \in \mathcal{B}}$ leaks no information about the trapdoors of any matrix $\left\{\mathbf{A}_{i}\right\}_{i \in \mathcal{B}}$. It follows from the hardness of PRIM-LWE that the pairs $\left(\mathbf{A}_{i}, \mathbf{D}_{i}\right)_{i \in \mathcal{B}}$ do not leak any non-negligible information to any unauthorized subset of parties $\mathcal{B} \notin \Gamma$ because it cannot reconstruct the correct trapdoor $\tau_{\triangleleft}^{(\mathcal{B})}$. Hence, it follows that the distributions $\left\{\Psi^{\left(k_{1}\right)}\right\}_{i \in \mathcal{B}}$ and $\left\{\Psi^{\left(k_{2}\right)}\right\}_{i \in \mathcal{B}}$ are computationally indistinguishable with respect to the security parameter $\varepsilon \cdot|\mathcal{B}|$, where $\varepsilon$ denotes the advantage of a polynomial-time adversary against a PRIM-LWE instance.

## 9 Conclusion

Secret sharing is a foundational and versatile tool with direct applications to many useful cryptographic protocols. Its applications include multiple privacy-preserving techniques, but the privacy-preserving guarantees of secret sharing itself have not received adequate attention. In this paper, we bolstered the privacy-preserving guarantees and verifiability of secret sharing by extending a recent work of Sehrawat and Desmedt [266] wherein they introduced hidden access structures that remain unknown until some authorized subset of parties assembles. Unlike the solution from [266], our scheme tolerates malicious parties and supports all possible monotone access structures. We introduced an approach to combine the learning with errors (LWE) problem with our novel superpolynomial sized set-systems to realize secret sharing for all monotone hidden access structures. Our scheme is the first secret sharing solution to support malicious behavior identification and share verifiability in malicious-majority settings. It is also the first LWE-based secret sharing scheme for general access structures. As the building blocks of our scheme, we constructed a novel set-system with restricted intersections and introduced a new variant of the LWE problem, called PRIM-LWE, wherein the secret matrix is sampled from the set matrices whose determinants are generators of $\mathbb{Z}_{q}^{*}$, where $q$ is the LWE modulus. We also gave concrete directions for future work that will reduce our scheme's share size to be smaller than the best known upper bound for secret sharing over general (i.e., all monotone) access structures.

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[^1]:    ${ }^{\top}$ member sets do not all have equal size

[^2]:    || all member sets have equal size

[^3]:    ** member sets do not all have equal size

