# Computational complexity of problems for deterministic presentations of sofic shifts 

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#### Abstract

Sofic shifts are symbolic dynamical systems defined by the set of bi-infinite sequences on an edge-labeled directed graph, called a presentation. We study the computational complexity of an array of natural decision problems about presentations of sofic shifts, such as whether a given graph presents a shift of finite type, or an irreducible shift; whether one graph presents a subshift of another; and whether a given presentation is minimal, or has a synchronizing word. Leveraging connections to automata theory, we first observe that these problems are all decidable in polynomial time when the given presentation is irreducible (strongly connected), via algorithms both known and novel to this work. For the general (reducible) case, however, we show they are all PSPACE-complete. All but one of these problems (subshift) remain polynomial-time solvable when restricting to synchronizing deterministic presentations. We also study the size of synchronizing words and synchronizing deterministic presentations.


Keywords: sofic shifts; symbolic dynamics; computational complexity; automata theory

## 1 Introduction

Symbolic dynamics in dimension one is the study of shift spaces, which are topological dynamical systems given by "shifting" bi-infinite sequences of symbols. Sofic shifts are shift spaces whose points are given by the label sequences for bi-infinite walks in a labeled graph, called a presentation. As they characterize the factors of subshifts of finite type (SFTs), sofic shifts have fundamental importance in symbolic dynamics. They also have an array of applications both within and outside of dynamical systems, including billiards, ergodic theory, continuous dynamics, and information theory, automata theory, and matrix theory [19]. In particular, one motivation for the present work is the set of computational problems that arise in application to continuous maps via Conley index theory $[8,15,16]$.

| Problem | Input | Decision |
| :--- | :--- | :--- |
| Irreducibility | $G$ | Is $X_{G}$ irreducible? |
| Equality | $G, H$ | Is $X_{G}=X_{H}$ ? |
| SubShift | $G, H$ | Is $X_{G} \subseteq X_{H} ?$ |
| SFT | $G$ | ${\text { Is } X_{G} \text { an SFT? }}^{\exists \text { SDP }}$ |
| Minimality | $G$ | ${\text { Does } X_{G} \text { have an SDP? }}^{G, k}$ |
| Does $X_{G}$ have a $k$-vertex deterministic presentation? |  |  |
| SyncWord | $G$ | Does $G$ have a synchronizing word? |

Table 1: Natural decision problems for sofic shifts. We show that all are PSPACE-hard in the general case; see Table 2 for an overview of our results. For the inputs to the problems, $G, H$ are deterministic presentations, and $k$ is a positive integer.

Despite their fundamental importance, however, many basic questions about the computational complexity of sofic shifts remain open. In Table 1, we give seven natural decision problems, many of which also arise frequently in applications. For example, given a labeled graph, does it present an SFT? Does a given reducible presentation actually present an irreducible sofic shift? Do two given labeled graphs present the same shift? For special cases, such when the given presentations are irreducible, some of these problems are known to be in P , i.e., they admit a polynomial-time algorithm. For the general case, however, only the complexity of SYncWord is known: it is PSPACE-complete to determine whether a given deterministic presentation has a synchronizing word [4].

In this work, we resolve the complexity of the remaining six problems, showing that they are all PSPACE-complete in the general case. We also study two special cases, sofic shifts given by deterministic presentations which are either irreducible or synchronizing. For these cases, the problems are generally in P. In fact, the only exception is SUBSHIFT for synchronizing deterministic presentations, which is again PSPACE-complete. Our reductions also shed light on the size of the smallest synchronizing word and minimal synchronizing deterministic presentation, namely that both can be exponentially large in the given presentation. These results are significant for understanding sofic shifts in their own right, as well as relevant for applications.

### 1.1 Relation to the literature

Conjugacy Absent from our list of problems is arguably the most important: deciding whether two sofic shifts are isomorphic, or conjugate. In general, the decidability of this conjugacy problem is open, even when restricting to the class of SFTs. Verifying the natural certificates of conjugacy, known as sliding block codes, is computable in polynomial time for SFTs given by vertex shifts, and deciding if there is a certificate of a fixed size is GI-hard, meaning there is a polynomial-time reduction from the graph isomorphism problem to that problem [22]. One can partially decide nonconjugacy via conjugacy invariants, i.e., properties which isomorphic objects share. For dynamical systems, a set of
invariants related to the connectivity of the state space are being (topologically) transitive, mixing, and nonwandering. For shift spaces, topological transitivity is also known as irreducibility. The class of SFTs is contained with the class of sofic shifts, and the class of SFTs is closed under conjugacy; thus, for sofic shifts, being an SFT is a conjugacy invariant. Conen and Paul [6] show that all the above invariants are decidable. We show that for sofic shifts given by deterministic presentations, deciding if they are irreducible or an SFT is PSPACE-complete. The PSPACE-hardness of being mixing or nondwandering also follows immediately from our reduction. Interestingly, it also follows from our reduction that deciding conjugacy of sofic shifts is at least PSPACE-hard.

Automata theory Sofic shifts have a close relationship with automata theory. A finite automaton (FA) roughly corresponds to an edge-labeled directed graph with a set of initial and accepting states. ${ }^{1}$ The language of an FA is the set of words labeled by a path starting at an initial state and ending at an accepting state. The languages described by finite automata are known as regular languages. Similarly, we define the language of a shift space to be the set of finite words appearing in points of the shift space.

A basic result about shift spaces says that shift spaces are determined by their languages: two shift spaces are equal if and only if their languages are equal. Furthermore, the languages of sofic shifts are regular in the above sense. ${ }^{2}$ More specifically, by interpreting a presentation of a sofic shift as an FA where every state is both initial and final, then the language of a presentation (interpreted as an FA) is the same as the language of the sofic shift it presents. These connections allows us to use automata-theoretic tools to study sofic shifts.

A deterministic finite automaton (DFA) over an alphabet $\Sigma$ is an FA with a single initial state, such that at each state $q$ and for each $a \in \Sigma$, there is exactly one edge leaving $q$ labeled $a$. A deterministic presentation of a sofic shift, when thought of as an FA, has a similar definition: a presentation is deterministic if for each $a \in \Sigma$, there is at most one edge leaving that state labeled $a$. Comparing the languages of two DFAs (i.e. whether their languages are equal, or if one is a subset of the other) is computable in polynomial time. However, for FAs in general, the same problem is PSPACE-complete [18]. Comparing the languages of presentations in general is also PSPACE-complete, as a corollary of results from Czeizler and Kari [7]. The question that remains is therefore the complexity of comparing languages of deterministic presentations. It is likely known that comparing languages of irreducible (i.e. strongly connected) deterministic presentations is computable in polynomial time; in Section 3.2, we give an algorithm. For deterministic presentations in general, we show in Section 4.1 that comparing languages (and thus comparing the shift spaces they present) of this type of FA is PSPACE-complete.

[^0]Minimization of presentations We say two FAs are equivalent if they have the same language. Algorithms for minimizing a DFA, i.e. finding an equivalent DFA with fewer states, have been well-studied. This minimization problem has nice properties: every DFA has an unique minimal equivalent DFA which can be computed in polynomial time. For FAs in general, minimization is not as nice: FAs do not necessarily have unique minimal equivalent FA [2], and deciding if there is an FA with fewer states than a given FA is PSPACE-complete [18].

As a class of FAs, irreducible deterministic presentations of sofic shifts have similar minimality properties to DFAs: every irreducible deterministic presentation has a unique minimal equivalent irreducible deterministic presentation which is computable in polynomial time [17]. (Furthermore, the property characterizing a minimal irreducible deterministic presentation in a sense is exactly the same property as a minimal DFA.) This observation leaves the question: do reducible deterministic presentations share the minimzation properties of the class of DFAs or of the class of FAs? The class of FAs arguably closest to general deterministic presentations are the multiple-entry DFAs (mDFAs), which are essentially DFAs with one source of nondeterminism: multiple initial states. A deterministic presentation can be made into an equivalent mDFA by by adding a sink state, thus "fully determinizing" every state, and then interpreting every state but the sink state as an initial and final state (c.f. sink vertex graph in Section 3.1). As mDFAs share the minimization properties of the class of FAs [10, 18], and general deterministic presentations do not have unique minimal equivalent deterministic presentations [12], one may suspect that minimization of general deterministic presentations is PSPACE-complete. Indeed, we show this result in Section 4.2.

Synchronizing words Another equivalent way of defining a DFA is by specifying a transition function; i.e., a function $\delta: Q \times \Sigma \rightarrow Q$ for some set of states $Q$ and finite alphabet $\Sigma$. The transition function then naturally extends to a function $\delta: Q \times \Sigma^{*} \rightarrow Q$ from the states and words over an alphabet. A synchronizing word (also called a reset word) for a DFA is a word that transitions every state to a single state: $w$ is synchronizing if $\delta(p, w)=\delta(q, w)$ for all states $p$ and $q$. (Equivalently, the function $q \mapsto \delta(q, w)$ is a constant function.) A deterministic presentation can be seen as a DFA with a partially defined transition function (a function whose domain is a subset of $Q \times \Sigma$ ); call a DFA with a partial transition function a partial DFA. There are multiple ways to generalize the notion of a synchronizing word to partial DFAs, for example, a carefully synchronizing word [24] and a exact synchronizing word [23,25]. The former is a word whose transition is defined at all states and sends all states to a single state; the latter is a word whose transition is defined at least one state and sends every state (where it is defined) to the same state. Interpreting a deterministic presentation as a partial DFA, a synchronizing word for a deterministic presentation of a sofic shift is defined as an exact synchronizing word. Note that a DFA might be called synchronizing or synchronized if it has a synchronizing word; for presentations of sofic shifts, our usage of a synchronizing presentation corresponds to that of Jonoska [12]: for every state $q \in Q$, there is a synchronizing word
that sends every state to $q$.
For DFAs, one can decide if a synchronizing word exists (and find one) in polynomial time via Eppstein's algorithm [9]. Independently, Travers and Crutchfield [25] gave a similar algorithm which can be used to determine if a synchronizing word exists in an irreducible deterministic presentation. In Section 3.1, we describe an algorithm to find a synchronizing word in an irreducible deterministic presentations (Algorithm 1) which combines the techniques of the previously two mentioned algorithms. We extend these techniques to subshift testing for irreducible deterministic presentations (Algorithm 2) and then use synchronizing word algorithm and subshift testing algorithm together as subprocedures for testing if a deterministic presentation is synchronizing. For deterministic presentations in general, deciding if a synchronizing word exists is PSPACE-complete, and the size of a minimum length synchronizing word may be exponentially large with respect to the number of states. While both of these facts were already implied by Berlinkov [4], for completeness we provide proofs in Sections 4.3 and 5.1, respectively.

Synchronizing deterministic presentations Jonoska [12] introduced synchronizing deterministic presentations, as defined above: for every state $q \in Q$, there is a synchronizing word that sends every state to $q$. The shift spaces given by synchronizing deterministic presentations slightly generalize those given by irreducible deterministic presentations while retaining serveral nice properties. For example, synchronizing deterministic presentations share the minimization properties of the class of DFAs: every synchronizing deterministic presentation has a unique minimal equivalent synchronizing deterministic presentation that is computable in polynomial time. For irreducible deterministic presentations, it is known that Equality and SFT are in P. We show that the algorithms for the irreducible case generalize cleanly to the synchronizing case, implying that EQUALITY and SFT are in $P$ for synchronizing determinstic presentations (Sections 3.5 and 3.4). Interestingly, although SUBSHIFT is in P for irreducible deterministic presentations, SUBSHIFT is PSPACE-complete for synchronizing deterministic presentations (Remark 4.12).

Not all sofic shifts have synchronizing deterministic presentations. In fact, we show that the problem of deciding whether a sofic shift has a synchronizing deterministic presentation, $\exists$ SDP, is PSPACE-complete (Section 4.1). For irreducible sofic shifts, minimal synchronizing deterministic presentations and minimal deterministic presentations are the same. For reducible sofic shifts, however, these minimal presentations are not necessarily the same. Indeed, we show that a minimal synchronizing deterministic one can be exponentially larger than a minimal deterministic one (Section 5.2).

## 2 Background and Setting

### 2.1 Shift spaces and presentations

Here, we introduce basic notions about shift spaces, sofic shifts, and presentations. Definitions and notation follow Lind and Marcus [17].

Let $\Sigma$ be a finite set. We refer to a finite sequence as a word, and we denote by $\Sigma^{*}$ the set of words over $\Sigma$. A subset of $\Sigma^{*}$ is called a language. For $w \in \Sigma^{*}$, we denote the length of $w$ as $|w|$. We denote the empty word as $\epsilon$, and note that $|\epsilon|=0$ and $\epsilon \in \Sigma^{*}$. The full $\Sigma$-shift is the set $\Sigma^{\mathbb{Z}}$ of bi-infinite sequences over $\Sigma$. Let $x \in \Sigma^{\mathbb{Z}}$. For $i \leq j$, we denote $x_{[i, j]} \triangleq x_{i} x_{i+1} \ldots x_{j}$. We say a word $w$ appears in $x$ if there are $i$ and $j$ with $x_{[i, j]}=w$. For a collection of words $\mathcal{F} \subseteq \Sigma^{*}$, we define $\mathrm{X}_{\mathcal{F}} \triangleq\left\{x \in \Sigma^{\mathbb{Z}}\right.$ : no word in $\mathcal{F}$ appears in $\left.x\right\}$. A shift space is a subset $X \subseteq \Sigma^{\mathbb{Z}}$ of the full $\Sigma$-shift such that there is a collection of words $\mathcal{F}$ with $X=\mathrm{X}_{\mathcal{F}}$. A shift of finite type (SFT) is a shift space $X=\mathrm{X}_{\mathcal{F}}$ for some finite set $\mathcal{F}$.

For a subset $X \subseteq \Sigma^{\mathbb{Z}}$ of the full $\Sigma$-shift, we define the language $\mathcal{B}(X)$ of $X$ to be the set of words that appear in some $x \in X$, i.e., $\mathcal{B}(X) \triangleq\left\{x_{[i, j]}: x \in X, i \leq j\right\}$. Shift spaces are characterized by their languages: for every shift space $X \subseteq \Sigma^{\mathbb{Z}}$, one has that $X=\mathrm{X}_{\Sigma^{*} \mid \mathcal{B}(X)}$. Thus, for shift spaces $X$ and $Y$, if $\mathcal{B}(X)=\mathcal{B}(Y)$, then $X=Y$ [17, Proposition 1.3.4]. Additionally, one can easily show inclusion is also respected: $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$ if and only if $X \subseteq Y$. Finally, for $u \in \Sigma^{*}$, we define the follower set of $u$ as the set $F_{X}(u) \triangleq\left\{w \in \Sigma^{*}\right.$ : $u w \in \mathcal{B}(X)\}$.

To define sofic shifts, we will work with edge-labeled, directed multigraphs, where self loops and multiple edges between vertices are permitted. Formally, a labeled graph $G$ consists of a finite set $Q$ of vertices (or states), a finite set $E$ of edges, functions $i: E \rightarrow Q$ and $t: E \rightarrow Q$, assigning each edge an initial and terminal vertex, and a function $\mathcal{L}: E \rightarrow \Sigma$, assigning each edge a label. For a given graph $G$, the symbols $Q_{G}, E_{G}, i_{G}, t_{G}$, and $\mathcal{L}_{G}$ will refer to the above sets and functions for the graph $G$. Additionally, we define the alphabet of $G$ as the set $\mathcal{A}_{G}$ of labels appearing on edges in $G$ (i.e. $\mathcal{A}_{G} \triangleq \mathcal{L}_{G}\left(E_{G}\right)$ ). When the labels are irrelevant, we will sometimes call a labeled graph a graph.

If $G$ is a labeled graph and $P \subseteq Q$ is a subset of vertices, then the subgraph induced by $P$ (in $G$ ) is the labeled graph $H$ given by $Q_{H}, E_{H}, i_{H}, t_{H}, \mathcal{L}_{H}$, where: $Q_{H} \triangleq P ; E_{H} \triangleq\{e \in$ $\left.E_{G}: i_{G}(e) \in P, t_{G}(e) \in P\right\} ; i_{H}, t_{H}$ are $i_{G}, t_{G}$ restricted to $E_{H} ;$ and $\mathcal{L}_{H}$ is $\mathcal{L}_{G}$ restricted to $Q_{H}$.

Let $G$ be a labeled graph. A path in $G$ is a finite sequence $\pi=e_{1} \ldots e_{n}$ of edges with $t_{G}\left(e_{i}\right)=i_{G}\left(e_{i+1}\right)$ for $i=1, \ldots, n-1$. We assign $i_{G}(\pi) \triangleq e_{1}$ and $t_{G}(\pi) \triangleq e_{n}$, and say $\pi$ starts at $i_{G}(\pi)$ and ends at $t_{G}(\pi)$. Additionally, we assign $\mathcal{L}_{G}(\pi) \triangleq \mathcal{L}_{G}\left(e_{1}\right) \ldots \mathcal{L}_{G}\left(e_{n}\right)$, and say $\mathcal{L}_{G}(\pi)$ is the label of $\pi$. Similarly, a bi-infinite path in $G$ is a bi-infinite sequence $x \in E_{G}^{\mathbb{Z}}$ of edges with $t_{G}\left(x_{i}\right)=i_{G}\left(x_{i+1}\right)$ for all $i \in \mathbb{Z}$. For a bi-infinite path $x$ in $G$, we assign the label of $x$ as the bi-infinite sequence $\mathcal{L}_{G}(x) \in \mathcal{A}_{G}^{\mathbb{Z}}$ with $\mathcal{L}_{G}(x)_{i} \triangleq \mathcal{L}_{G}\left(x_{i}\right)$. For a vertex, $q$ we define the follower set of $q$ in $G$ as the set $F_{G}(q) \triangleq\left\{\mathcal{L}_{G}(\pi): \pi \text { is a path in } G \text { starting at } q\right\}^{3}$

We now have the necessary definitions to define sofic shifts. For a labeled graph $G$, we assign it the shift space

$$
X_{G} \triangleq\left\{\mathcal{L}_{G}(x): x \text { is a bi-infinite path in } G\right\}
$$

A sofic shift is a shift space $X$ such that $X=X_{G}$ for some labeled graph $G$, and we say $G$ is a presentation of $X$ and that $X$ is the sofic shift presented by $G$. For a proof that $X_{G}$ is actually a shift space, see Lind and Marcus [17, Theorem 3.1.4].

[^1]Let $G$ be a labeled graph. For a given path $\pi$ in $G$, it could be the case that $\mathcal{L}_{G}(\pi)$ is not in the language of $\mathrm{X}_{G}$, as $\pi$ might not appear in a bi-infinite path. We say a vertex $q$ in $G$ is stranded if there is no edge starting at $q$ or if there is no edge ending at $q$. If no vertex is stranded, then we say $G$ is essential. When $G$ is essential, every path appears in a biinfinite path, so $\mathcal{L}_{G}(\pi)$ is always in the language of $X_{G}$. If one removes a stranded vertex from a presentation, the sofic shift presented by the resulting presentation is the same as the one presented by the original presentation. Thus, every sofic shift has an essential presentation, which can be obtained by iteratively removing stranded vertices until no more exist [17, Proposition 2.2.10]. We therefore make the following convention: a presentation refers to an essential labeled graph. We will still refer to labeled graphs as being potentially nonessential; the distinction is needed for the algorithms in Section 3, where we may call our algorithms on nonessential graphs (line 7 of Algorithm 3).

Let $G$ be a labeled graph. We say $G$ is deterministic (also called right-resolving) if for every vertex $q$ and every $a \in \mathcal{A}_{G}$, there is at most one edge labeled $a$ starting at $q$. If for every vertex $q$ and $a \in \mathcal{A}_{G}$ there is exactly one edge labeled $a$ starting at $q$, we say $G$ is fully deterministic. If $G$ is deterministic, one can show by induction that for every vertex $q$ and word $w$, if $\pi$ is a path starting at a vertex $q$ and $\mathcal{L}_{G}(\pi)=w$, then $\pi$ is the unique path starting at $q$ with $\mathcal{L}_{G}(\pi)=w$. This observation motivates the following definition: if there is some path $\pi$ starting at $q$ with $\mathcal{L}_{G}(\pi)=w$, we define $q \cdot w \triangleq t(\pi)$, otherwise, if there is no such $\pi$, we leave $q \cdot w$ undefined. We call • the transition action. Because of determinism, the transition action is a well-defined partial operation between the vertices of $G$ and words over the alphabet of $G$, and $q \cdot w$ is defined if and only if $w \in F_{G}(q)$. The transition action satisfies the following useful properties, for any state $q \in Q_{G}$ and words $u, v \in \mathcal{A}_{G}^{*}:$
(i) $u v \in F_{G}(q)$ if and only if $u \in F_{G}(q)$ and $v \in F_{G}(q \cdot u)$;
(ii) if $u v \in F_{G}(q)$, then $q \cdot u v=(q \cdot u) \cdot v$.

When $G$ is fully deterministic, $q \cdot w$ is defined for all $q$ and $w \in \mathcal{A}_{G}^{*}$, as then $F_{G}(q)=\mathcal{A}_{G}^{*}$.
The transition action naturally extends to a total operation between subsets of vertices of $G$ and words over the alphabet of $G$ : for each subset $S \subseteq Q_{G}$ of vertices and word $w$, we set $S \cdot w \triangleq\left\{q \cdot w: q \in S, w \in F_{G}(q)\right\}$. Every sofic shift therefore has a deterministic presentation [17, Theorem 3.3.2], using the same idea as the subset construction from automata theory [14]. By definition, the transition action on subsets is monotonic and distributes over union: $S \subseteq T$ implies $S \cdot w \subseteq T \cdot w$ and $(S \cup T) \cdot w=(S \cdot w) \cup(T \cdot w)$ for all $S, T$, and $w$.

### 2.2 Types of presentations

Let $G$ be a deterministic labeled graph and let $w$ be a word. We say $w$ is synchronizing for $G$ if $Q_{G} \cdot w=\{r\}$ for some vertex $r \in Q_{G}$. In this case, we say $w$ synchronizes to $r$ (in $G$ ). We say a vertex $q$ is synchronizing if there is a word that synchronizes to $q$. We say $G$ is synchronizing if every vertex in $G$ is synchronizing. Let $X$ be a shift space. An intrinsically
synchronizing word $w$ for $X$ is a word $w \in \mathcal{B}(X)$ such that whenever $u w, w v \in \mathcal{B}(X)$, then $u w v \in \mathcal{B}(X)$. If $w$ is synchronizing for $G$, then $w$ is intrinsically synchronizing for $X_{G}$, but the converse need not hold; see Lemma 2.2.

Let $X$ be a shift space. We say $X$ is irreducible if for every $u, v \in \mathcal{B}(X)$, there is a word $w$ such that $u w v \in \mathcal{B}(X)$; if $X$ is not irreducible, then we say $X$ is reducible. For a graph $G$, we say $G$ is irreducible (or strongly connected) if for every pair of vertices $p$ and $q$, there is a path starting at $p$ and ending at $q$. If $G$ is not irreducible, we say $G$ is reducible. One can easily show that if $G$ is irreducible, then $\mathrm{X}_{G}$ is irreducible. However, $\mathrm{X}_{G}$ may be irreducible even if $G$ is reducible. (See Figure 1.)

Let $G$ be a graph, and let $p$ and $q$ be vertices in $G$. We say $q$ is reachable from $p$ if there is a path starting at $p$ and ending at $q$. Under the equivalence relation where $p \approx q$ when $q$ is reachable from $p$ and $q$ is reachable from $q$, the equivalence classes are called irreducible components as the subgraphs induced by them are irreducible. We say an irreducible component $C$ is initial if whenever $q$ is reachable from $p$ and $q \in C$, then $p \in C$. Dually, we say a irreducible component $C$ is terminal if whenever if $q$ is reachable from $p$ and $p \in C$, then $q \in C$.

Let $G$ be a labeled graph. We say two vertices $p$ and $q$ in $G$ are follower-equivalent if $F_{G}(p)=F_{G}(q)$, an equivalence relation $\sim$. We say $G$ is follower-separated if no distinct pair of vertices are follower equivalent. Given a labeled graph $G$, the follower-separation of $G$ is the the labeled graph $G / \sim$ whose vertices are the follower-equivalence classes of $G$ and with exactly one edge labeled $a$ between two classes $C_{1}$ and $C_{2}$ if and only if there is an edge labeled $a$ in $G$ from a vertex in $C_{1}$ to a vertex in $C_{2}$. Informally, the follower-separation of $G$ collapses vertices in a given follower-equivalence class into a single vertex. The follower-separation of $G$ enjoys the following properties: we have $G / \sim$ is follower-separated and $X_{G}=X_{G / \sim}$; if $G$ is deterministic, then $G / \sim$ is deterministic; if $G$ is essential, then $G / \sim$ is essential [17, Lemma 3.3.8]; if $G$ is synchronizing, then $G / \sim$ is synchronizing [12, Proposition 4.3]. In particular, every sofic shift has a followerseparated, deterministic presentation.

The notion of follower-equivalence is similar to the notion of equivalent states in deterministic finite automata (DFA; see Sections 1.1 and 4). In fact, one may reduce the problem of computing follower-equivalence to computing equivalent states in DFA, as follows. Add a "sink" state to $G$, and edges to the sink state to make $G$ fully deterministic (c.f. Section 3.1). Now consider the resulting graph as a DFA, with an arbitrary initial state, and where every state but the sink state is an accepting state. One can show that two states in $G$ are follower-equivalent if and only if they are equivalent as states in the constructed DFA. Therefore, one can use Hopcroft's algorithm for state equivalence in DFAs to compute follower-equivalences in polynomial time [11].

### 2.3 Basic results

In this section, we discuss several useful facts which we use throughout the paper. To begin, the following statements about a deterministic presentation $G$ establish basic re-
lationships between its transition action, follower sets $F_{G}$ and $F_{X_{G}}$, the language $\mathcal{B}\left(\mathrm{X}_{G}\right)$, and synchronizing words for $G .{ }^{4}$ The statements follow immediatly from the definitions.

Proposition 2.1. Let $G$ be a deterministic presentation. Then, we have
(i) $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ if and only if $Q_{G} \cdot w \neq \varnothing$;
(ii) $\mathcal{B}\left(\mathrm{X}_{G}\right)=\bigcup_{q \in Q} F_{G}(q)$;
(iii) $F_{X_{G}}(w)=\bigcup_{q \in Q \cdot w} F_{G}(q)$;
(iv) if $w$ is synchronizes to $r$ in $G$, then $F_{X_{G}}(w)=F_{G}(r)$;
(v) if $w$ is intrinsically synchronizing for $\mathrm{X}_{G}$ and $w \in F_{\mathrm{X}_{G}}(u)$, then $F_{\mathrm{X}_{G}}(u w)=F_{\mathrm{X}_{G}}(w)$.

Next, we review results of Jonoska [12] about synchronizing deterministic presentations. First, we state a useful result about the correspondence between synchronizing and intrinsically synchronizing words in synchronizing deterministic presentations. For deterministic presentations in general, only the forward implication of this result holds. The following is essentially Proposition 9.5 of Jonoska [12].

Lemma 2.2. Let $G$ be a follower-separated, synchronizing deterministic presentation. Then, $w$ is synchronizing for $G$ if and only if $w$ is intrinsically synchronizing for $X_{G} .{ }^{5}$

Proof. Suppose $w$ is synchronizing for $G$, and let $u w, w v \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. As $w$ is synchronizing for $G$, by Proposition 2.1(iv), it follows that there is a vertex $r$ such that $F_{X_{G}}(u w)=F_{G}(r)$ and $v \in F_{G}(r)$. Thus, we have $v \in F_{G}(u w)$ so $u w v \in \mathcal{B}\left(\mathrm{X}_{G}\right)$.

Conversely, suppose $w$ is intrinsically synchronizing for $\mathrm{X}_{G}$. Let $p$ and $q$ be vertices in $G$ with $w \in F_{G}(p)$ and $w \in F_{G}(q)$. As $G$ is synchronizing, let $u_{p}$ and $u_{q}$ be words synchronizing to $p$ and $q$ in $G$, respectively. We next show that $F_{G}(p \cdot w) \subseteq F_{G}(q \cdot w)$. Let $v \in F_{G}(p \cdot w)$, so that $w v \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. As $u_{q}$ synchronizes to $q$ in $G$ and $w \in F_{G}(q)$, we have $u_{q} w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. As $w$ is intrinsically synchronizing for $\mathrm{X}_{G}$, we have $u_{q} w v \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, i.e. $Q_{G} \cdot u_{q} w v \neq \varnothing$. But as $Q_{G} \cdot u_{q} w=\{q \cdot w\}$, we have $v \in F_{G}(p \cdot w)$. Thus, $F_{G}(p \cdot w) \subseteq F_{G}(q \cdot w)$; moreover, the same argument swapping the roles of $p$ and $q$ gives $F_{G}(q \cdot w) \subseteq F_{G}(p \cdot w)$ and therefore $F_{G}(p \cdot w)=F_{G}(q \cdot w)$. As $G$ is follower-separated, we conclude $p \cdot w=q \cdot w$, implying that $w$ is synchronizing for $G$.

The following characterization of when a sofic shift has a synchronizing deterministic presentation is slightly modified from Theorem 8.13 and Corollary 9.6 in Jonoska [12].

Theorem 2.3. Let $X \subseteq \Sigma^{\mathbb{Z}}$ be a sofic shift. Then, $X$ has a synchronizing deterministic presentation if and only if for every $u \in \mathcal{B}(X)$ there is an intrinsically synchronizing word $w$ for $X$ such that $u \in F_{X}(w)$.

[^2]Proof. Let $G$ be a synchronizing deterministic presentation for $X$, and let $w \in B(X)$. By Proposition 2.1(ii), there is a vertex $q$ such that $w \in F_{G}(q)$. As $G$ is synchronizing, there is a word $u$ that synchronizes to $q$. By Proposition 2.1(iv), we have $F_{X}(u)=F_{G}(q)$, so $w \in F_{X}(u)$. By Lemma 2.2, we also have that $u$ is intrinsically synchronizing for $X$.

Conversely, suppose for every $u \in \mathcal{B}(X)$ there is an intrinsically synchronizing word $w$ for $X$ such that $u \in F_{X}(w)$. Let $\mathcal{C}$ be the collection of the follower sets of intrinsically synchronizing words for $X$, i.e.

$$
\mathcal{C} \triangleq\left\{F_{X}(w): w \text { is intrinsically synchronizing for } X\right\} .
$$

This collection is finite since the collection of all follower sets of a sofic shift is finite [17, Theorem 3.2.10]. We will construct a synchronizing deterministic presentation $G$ whose vertex set is $\mathcal{C}$. For each $a \in \Sigma$ and $F_{X}(w) \in \mathcal{C}$, if $a \in F_{X}(w)$, add an edge labeled $a$ from $F_{X}(w)$ to $F_{X}(w a)$. This definition is well-defined, i.e., does not depend on the choice of $w$, by the following two facts, both assuming $a \in F_{X}(w)$ : if $F_{X}(w)=F_{X}\left(w^{\prime}\right)$, then $F_{X}(w a)=F_{X}\left(w^{\prime} a\right)$, and $w a$ is intrinsically synchronizing (so that $F_{X}(w a) \in \mathcal{C}$ ). By construction, $G$ is deterministic. One can also establish the following properties of $G$ : for $F_{X}(u) \in \mathcal{C}$, we have $F_{G}\left(F_{X}(u)\right)=F_{X}(u)$, and if $w \in F_{X}(u)$, then $F_{X}(u) \cdot w=F_{X}(u w)$.

We next show that $G$ is synchronizing. Let $F_{X}(w) \in \mathcal{C}$, so that $w$ is intrinsicaly synchronizing for $X$. We will show that $w$ synchronizes to $F_{X}(w)$ in $G$. Let $F_{X}(u) \in \mathcal{C}$, and suppose $w \in F_{G}\left(F_{X}(u)\right)$. As $w$ is intrinsically synchronizing and $w \in F_{X}(u)$, by Proposition 2.1, we have that $F_{X}(u w)=F_{X}(w)$. This implies that $F_{X}(u) \cdot w=F_{X}(u w)=F_{X}(w)$. Thus, for any $F_{X}(u) \in \mathcal{C}$ with $w \in F_{G}\left(F_{X}(u)\right)$, we have $F_{X}(u) \cdot w=F_{X}(w)$, so $w$ synchronizes to $F_{X}(w)$ in $G$.

It remains to show $X=X_{G}$. By construction, the follower set of a vertex in $G$ is a follower set of a word in $X$, so $\mathcal{B}\left(\mathrm{X}_{G}\right) \subseteq \mathcal{B}(X)$. Conversely, let $u \in \mathcal{B}(X)$. By our initial assumption, there is an intrinsically synchronizing word $w$ for $X$ with $u \in F_{X}(w)$. As $F_{G}\left(F_{X}(w)\right)=F_{X}(w)$, we have $u \in F_{G}\left(F_{X}(w)\right)$, i.e., there is a vertex $q$ in $G$ such that $u \in F_{G}(q) \subseteq \mathcal{B}\left(\mathrm{X}_{G}\right)$. Thus, we have $\mathcal{B}(X) \subseteq \mathcal{B}\left(\mathrm{X}_{G}\right)$ and consequently $\mathcal{B}(X)=\mathcal{B}\left(\mathrm{X}_{G}\right)$.

The next result says when the sofic shift presented by a (possibly reducible) deterministic presentation is irreducible. The forward implication of this result follows from Lemma 6.4 of Jonoska [12], and the reverse implication follows immediately from the irreducibility of $H$.

Theorem 2.4. Let $G$ be follower-separated, deterministic presentation. Let $H$ be the subgraph induced by the synchronizing vertices of $G$. Then, $X_{G}$ is irreducible if and only if $X_{G}=X_{H}$ and $H$ is induced by a terminal irreducible component.

Finally, we state some facts about SFTs. The first is a characterization of when a shift space is an SFT, and the second is a sufficient condition for when $X_{G}$ is an SFT for a presentation $G$. Respectively, these correspond to Theorem 2.1.8 and Proposition 2.2.6 of Lind and Marcus [17].
Theorem 2.5. A shift space $X$ is an SFT if and only if there exists an integer $M \geq 0$ such that every word $w \in \mathcal{B}(X)$ with $|w| \geq M$ is intrinsically synchronizing for $X$.

Lemma 2.6. Let $G$ be a presentation. If every edge in $G$ is labeled uniquely, then $X_{G}$ is an SFT.

## 3 Complexity Upper Bounds and Algorithms

In this section we detail polynomial-time algorithms for some problems in Table 1. In particular, we give polynomial-time algorithms for SFT and EQUALITY for synchronizing deterministic presentations, and additionally for SYNCWORD and SUBSHIFT for irreducible presentations. We also give a polynomial-time algorithm to test whether a given deterministic presentation is synchronizing. Some algorithms follow from known results, whereas others, to our knowledge, are novel to this work.

### 3.1 Finding synchronizing words

Eppstein [9] gives a polynomial-time algorithm for finding synchronizing words in fully deterministic presentations. Here, we show the algorithm can be extended to irreducible deterministic presentations, implying that SYNCWORD is in $P$ for such presentations. As we show in Theorem 4.16, SyncWord is PSPACE-complete for general presentations.

Theorem 3.1. Given an irreducible deterministic labeled graph $G$, Algorithm 1 returns a synchronizing word for $G$ if one exists, and nil otherwise.

To prove this result, we first introduce the notion of pair-synchronizing words. Let $p$ and $q$ be vertices in a deterministic graph $G$. We say a word $w$ is pair-synchronizing for $p$ and $q$ if $|\{p, q\} \cdot w|=1$, i.e., if there exists a vertex $r \in Q_{G}$ such that $\{p, q\} \cdot w=\{r\}$. This condition breaks into the following three cases:
(i) $w \in F_{G}(p)$ and $w \notin F_{G}(q)$;
(ii) $w \notin F_{G}(p)$ and $w \in F_{G}(q)$;
(iii) $w \in F_{G}(p) \cap F_{G}(q)$ and $p \cdot w=q \cdot w$.

If $X \subseteq Q_{G}$ is a subset of vertices with $|X| \geq 2$ and $w$ is pair-synchronizing for distinct $p, q \in X$, then we have $|X|>|X \cdot w| \geq 1$. This property motivates Algorithm 1. The algorithm operates by iteratively building a word $u$ and tracking a subset $X$ of vertices, maintaining the invariants that $Q_{G} \cdot u=X$ and $|X| \geq 1$. On each iteration of the main loop, the algorithm searches for a pair-synchronizing word $w$ for some pair of distinct vertices in $X$, and if one is found, then updates $u$ to $u w$ and $X$ to $X \cdot w$. The property above ensures that the invariants of $X$ and $u$ are maintained. Since $|X|$ must decrease by at least 1 in each iteration, the algorithm returns after at most $\left|Q_{G}\right|$ iterations.

Proof of Theorem 3.1. If Algorithm 1 returns a non-nil value, it must have exited at line 10, which implies $|X| \leq 1$. As the invariant that $|X| \geq 1$ was maintained throughout the algorithm, we must have $\left|Q_{G} \cdot u\right|=1$, so the word $u$ that was returned is a synchronizing word for $G$.

Conversely, if Algorithm 1 returned nil, it must have exited at line 9, which implies that there are two distinct vertices such that there is no pair-synchronizing word for them. Yet, as we show next, if $G$ has a synchronizing word, then every pair of distinct vertices has a pair-synchronizing word. Thus, $G$ must not have a synchronizing word.

Let $p$ and $q$ be distinct vertices in $G$, and suppose $w$ is a synchronizing word for $G$. As $w$ is synchronizing, there is some vertex $s$ with $w \in F_{G}(s)$. As $G$ is irreducible, there is a word $u$ such that $p \cdot u=s$, which implies that $u w \in F_{G}(p)$. If $u w \notin F_{G}(q)$, then $u w$ is a pair-synchronizing for $p$ and $q$ under case (i) above. Otherwise, we have $u w \in F_{G}(q)$. As $w$ is synchronizing, then $p \cdot u w=q \cdot u w$, so $u w$ is still pair-synchronizing for $p$ and $q$, under case (iii).

```
Algorithm 1 Finding synchronizing words
Require: \(G\) is a deterministic graph
    procedure SYNCHRONIZING-WORD \((G)\)
        \(X \leftarrow Q_{G} ; \quad u \leftarrow \epsilon\)
        while \(|X| \geq 2\) do
            choose distinct \(p, q \in X\)
            find a word \(w\) that is pair-synchronizing for \(p\) and \(q\)
            if \(w\) exists then
                \(X \leftarrow X \cdot w ; \quad u \leftarrow u w\)
            else
                return nil
            return u
```

To implement this Algorithm 1 in polynomial time (with respect to the size of its input $G$ ), we need a method to compute a pair-synchronizing word for a given pair of vertices. We give such a method using two auxillary graphs, the first of which encodes what words are not within a follower set of a vertex, and the second of which encodes pairs of paths sharing the same label.

If $G$ is a labeled graph and $\Gamma$ is an alphabet, the sink vertex graph of $G$ with alphabet $\Gamma$ is the graph $G^{0}$ constructed as follows. Start with the graph $G$, and add a new vertex 0 to $G^{0}$. For every vertex $q$ in $G^{0}$ and $\ell \in \Gamma$, add an edge labeled $\ell$ from $q$ to 0 if there is no edge labeled $\ell$ starting at $q$. One can show that for $w \in \Gamma^{*}$ and $q \in Q_{G}$, we have $w \notin F_{G}(q)$ if and only if there is a path labeled $w$ from $q$ to 0 in $G^{0}$.

If $G$ and $H$ are labeled graphs, then the label product graph of $G$ and $H$ is the graph $G * H$ whose vertices are $Q_{G} \times Q_{H}$ and with an edge between $\left(p_{1}, p_{2}\right)$ and $\left(q_{1}, q_{2}\right)$ labeled $\ell$ if and only if there is an edge labeled $\ell$ from $p_{1}$ to $q_{1}$ in $G$ and an edge labeled $\ell$ from $p_{2}$ to $q_{2}$ in $H$. One can show that for $w \in\left(\mathcal{A}_{G} \cup \mathcal{A}_{H}\right)^{*}$, there is a path labeled $w$ from $p_{1}$ to $q_{1}$ in $G$ and a path labeled $w$ from $p_{2}$ to $q_{2}$ in $G$ if and only if there is a path labeled $w$ from $\left(p_{1}, p_{2}\right)$ to ( $q_{1}, q_{2}$ ) in $G * H$.

Let $G$ be a labeled graph, let $G^{0}$ be the sink vertex graph of $G$ with alphabet $\mathcal{A}_{G}$ and let $G^{0} * G^{0}$ be the label product graph of $G^{0}$ and $G^{0}$. With the properties of the auxillary
graphs, one can show that the following conditions are equivalent to cases (i)-(iii) from the pair-synchronizing definition.
(I) there is a path in $G^{0} * G^{0}$ from $(p, q)$ to $(r, 0)$ for some vertex $r$ in $G$;
(II) there is a path in $G^{0} * G^{0}$ from $(p, q)$ to $(0, r)$ for some vertex $r$ in $G$;
(III) there is a path in $G^{0} * G^{0}$ from $(p, q)$ to $(r, r)$ for some vertex $r$ in $G$.

Using, say, a depth-first search, one can determine if there is a pair-synchronizing word for a given pair of vertices by testing for the existence of a path satisfying one of (I)-(III). The size of $G^{0} * G^{0}$ is $O\left(\left|Q_{G}\right|^{2} \cdot\left|\mathcal{A}_{G}\right|\right)$, so one can construct the graph and query the existence of such path in polynomial time. Each iteration of Algorithm 1 therefore takes polynomial time, and furthermore, as there are at most $\left|Q_{G}\right|$ iterations, in total, the algorithm will take polynomial time.

### 3.2 Testing for subshift

We now turn to the SUBSHIFT problem for deterministic presentations $G$ and $H$ where $G$ is irreducible. The key idea behind the algorithm is to try to find a word exhibiting the fact that $\mathrm{X}_{G} \nsubseteq \mathrm{X}_{H}$. We say that $w$ separates $G$ from $H$ if $Q_{G} \cdot w \neq \varnothing$ while $Q_{H} \cdot w=$ $\varnothing$. When $G$ and $H$ are essential, the existence of such a word is equivalent to $X_{G} \nsubseteq$ $\mathrm{X}_{H}$. Algorithm 2 adapts the algorithm for synchronizing words to find such separating words, thus showing Subshift for $G$ and $H$ is in P when $G$ is irreducible. We show in Theorem 4.4 that the general problem is PSPACE-complete.

We state the correctness of Algorithm 2 for the more general case of labeled graphs, which need not be essential, since we rely on that case for Theorem 3.3.

Theorem 3.2. Given deterministic labeled graphs $G$ and $H$, where $G$ is irreducible, Algorithm 2 returns a word separating $G$ from $H$ if one exists, and returns nil otherwise.

Like in Algorithm 1, Algorithm 2 operates by iteratively building a word $u$. In addition, the algorithm fixes a vertex $p_{0} \in Q_{G}$, and maintains a vertex $p \in Q_{G}$ and subset $X \subseteq Q_{H}$ satisfying the invariants $u \in F_{G}\left(p_{0}\right), p_{0} \cdot u=p$, and $Q_{H} \cdot u=X$. In each iteration of the main loop, the algorithm searches for a word $w$ such that $w \in F_{G}(p)$ while $w \notin F_{H}(q)$ for some $q \in X$. If one is found, the algorithm updates $u$ to $u w, p$ to $p \cdot w$, and $X$ to $X \cdot w$, which maintains the invariants. As $w \notin F_{H}(q)$ and $q \in X$, we have $|X|>|X \cdot w|$, so the algorithm again terminates in at most $\left|Q_{H}\right|$ iterations.

Proof of Theorem 3.2. If Algorithm 2 returns a non-nil value, it must have exited at line 10, so $|X|=0$. The invariants give $Q_{H} \cdot u=X=\varnothing$ and $u \in F_{G}\left(p_{0}\right)$, meaning $Q_{H} \cdot u=\varnothing$ and $Q_{G} \cdot u \neq \varnothing$. Thus, $u$ separates $G$ from $H$.

Conversely, if Algorithm 2 returns nil, it must have exited at line 9, which implies there exist $p \in Q_{G}$ and $q \in Q_{H}$ such that there is no word $w$ with $w \in F_{G}(p)$ and $w \notin F_{H}(q)$. We show below that, if some word separates $G$ from $H$, then for every $p^{\prime} \in Q_{G}$ and $q^{\prime} \in Q_{H}$,
there is a word $w$ with $w \in F_{G}\left(p^{\prime}\right)$ and $w \notin F_{H}\left(q^{\prime}\right)$. By contraposition, therefore, no word separates $G$ from $H$.

Suppose there is a word $w$ separating $G$ from $H$, so that we have $Q_{G} \cdot w \neq \varnothing$ while $Q_{H} \cdot w=\varnothing$. Then, there is some vertex $p^{*} \in Q_{G}$ such that $w \in F_{G}\left(p^{*}\right)$ and $w \notin F_{H}\left(q^{\prime}\right)$ for every $q^{\prime} \in Q_{H}$. Let $p \in Q_{G}$ and $q \in Q_{H}$. As $G$ is irreducible, there is some $u \in F_{G}(p)$ such that $p \cdot u=p^{*}$, giving $u w \in F_{G}(p)$. Let $q^{\prime} \triangleq q \cdot u$. By the above, $w \notin F_{H}\left(q^{\prime}\right)$, and thus $u w \notin F_{H}(q)$.

```
Algorithm 2 Subshift testing
Require: \(G\) is an irreducible deterministic labeled graph
Require: \(H\) is a deterministic labeled graph
    procedure SEPARATING-WORD ( \(G, H\) )
        \(p_{0} \leftarrow\) any element in \(Q_{G} ; \quad p \leftarrow p_{0} ; \quad X \leftarrow Q_{H} ; \quad u \leftarrow \epsilon\)
        while \(|X|>0\) do
                \(q \leftarrow\) any element in \(X\)
                find a word \(w\) such that \(w \in F_{G}(p)\) and \(w \notin F_{H}(q)\)
                if \(w\) exists then
                    \(p \leftarrow p \cdot w ; \quad X \leftarrow X \cdot w ; \quad u \leftarrow u w\)
            else
                    return nil
        return u
```

Analagously to Algorithm 1, we can implement Algorithm 2 in polynomial time by noticing that the existence of a word $w$ such that $w \in F_{G}(p)$ and $w \notin F_{H}(p)$ is equivalent to the existence of a path in $G * H^{0}$ from $(p, q)$ to $(r, 0)$ for some vertex $r$ in $G$, where $H^{0}$ is the sink vertex graph of $H$ with alphabet $\mathcal{A}_{G} \cup \mathcal{A}_{H}$ and $G * H^{0}$ is the label product graph of $G$ and $H^{0}$.

### 3.3 Testing for synchronizing presentations

With Algorithm 1 and Algorithm 2, we can now establish a polynomial-time algorithm for checking if a given deterministic graph is synchronizing, given by Algorithm 3. The correctness of the algorithm is implied by the following characterization of a synchronizing presentation.

Theorem 3.3. Let $G$ be a deterministic labeled graph with vertex set $Q$. Then, $G$ is synchronizing if and only if for each initial irreducible component $C$, there exists (i) a synchronizing word for the subgraph induced by $C$ and (ii) a word separating the subgraph induced by $C$ from the subgraph induced by $Q \backslash C$.

Proof. Suppose $G$ is synchronizing. Let $C$ be an initial component of $G$, and fix $r \in C$. As $G$ is synchronizing, let $w$ be a word that synchronizes to $r$ in $G$. As $w$ is synchronizing for $G$, there is some vertex $p \in Q$ such that $p \cdot w=r$. Since $r \in C$ and $C$ is initial, we must
have $p \in C$. Thus $C \cdot w=\{r\}$, establishing (i). As $C$ is initial and $r \in C$, we cannot have $q \cdot w=r$ for any $q \notin C$. We conclude $(Q \backslash C) \cdot w=\varnothing$. As $C \cdot w=\{r\} \neq \varnothing$, we have (ii).

Conversely, suppose for each initial irreducible component $C$, (i) there is a synchronizing word $u_{C}$ for the subgraph induced by $C$ and (ii) there is a word $w_{C}$ separating the subgraph induced by $C$ from the subgraph induced by $Q \backslash C$. Let $r$ be any vertex in $G$. Let $C$ be an initial irreducible component such that $r$ is reachable from every vertex in $C$. Condition (i) gives $C \cdot u_{C}=\{p\}$ for some $p \in C$. Condition (ii) gives some vertex $q \in C$ with $w_{C} \in F_{G}(q)$ such that $q \cdot w_{C} \in C$ and $(Q \backslash C) \cdot w_{C}=\varnothing$. As $C$ is an irreducible component and $p, q \in C$, then there is some word $x$ such that $p \cdot x=q$. As $q \cdot w_{C} \in C$ and $r$ is reachable from every vertex in $C$, there is some word $y$ such that $\left(q \cdot w_{C}\right) \cdot y=r$. Combining the above with a straightforward calculation for $Q \backslash C$, we have

$$
\begin{aligned}
C \cdot u_{C} x w_{C} y & =\{p\} \cdot x w_{C} y=\{q\} \cdot w_{C} y=\left\{q \cdot w_{C}\right\} \cdot y=\{r\} \\
(Q \backslash C) \cdot u_{C} x w_{C} y & =\left((Q \backslash C) \cdot u_{C} x\right) \cdot w_{C} y \subseteq(Q \backslash C) \cdot w_{C} y=\varnothing \cdot y=\varnothing
\end{aligned}
$$

Thus, $Q \cdot u_{C} x w_{C} y=\left(C \cdot u_{C} x w_{C} y\right) \cup\left((Q \backslash C) \cdot u_{C} x w_{C} y\right)=\{r\}$. As $r$ was arbitrary, $G$ is synchronizing.

```
Algorithm 3 Recognizing synchronizing presentations
Require: \(G\) is a deterministic labeled graph
    procedure IS-SYNCHRONIZING \((G)\)
        \(\mathcal{C} \leftarrow\) initial irreducible components of \(G\)
        for \(C \in \mathcal{C}\) do
            \(G[C] \leftarrow\) subgraph induced by \(C\)
            \(G[\bar{C}] \leftarrow\) subgraph induced by \(Q_{G} \backslash C\)
            \(u \leftarrow\) SYNCHRONIZING-WORD \((G[C])\)
            \(v \leftarrow \operatorname{SEPARATING-WORD}(G[C], G[\bar{C}])\)
            if \(u\) is nil or \(v\) is nil then
                return false
        return true
```


### 3.4 SFT testing for synchronizing deterministic presentations

The proof of Theorem 3.4.17 of Lind and Marcus [17] implicitly describes a polynomialtime algorithm to test whether an irreducible sofic shift, given as an irreducible deterministic presentation, is an SFT, and Schrock [21] gives a similar algorithm explicitly. We extend these algorithms to synchronizing deterministic presentations.

For a deterministic labeled graph $G$, we define the labeled graph $\hat{G}$ as the label product graph $G * G$ (see Section 3.1) with the diagonal vertices removed, i.e., those of the form $(q, q)$. Given a follower-separated synchronizing deterministic presentation $G$, the algorithm to recognize if $X_{G}$ is an SFT is to simply test if the graph $\hat{G}$ is acyclic. (A cycle is a nonempty path that starts and ends at the same vertex, and we say a graph is acyclic
if it has no cycle.) This algorithm runs in polynomial time, as the size of $\hat{G}$ is quadratic with respect to the size of $G$, and it is well-known that one can test whether a directed graph is acyclic in linear time.

To show the correctness of this algorithm, we first show that $\hat{G}$ characterizes the nonsynchronizing words of $G$.

Lemma 3.4. Let $G$ be a deterministic presentation and let $w \in \mathcal{B}\left(X_{G}\right)$. Then, there is a path in $\hat{G}$ labeled $w$ if and only if $w$ is not synchronizing for $G$.
Proof. Suppose there is a path $\pi$ in $\hat{G}$ labeled $w$, from $(p, q)$ to $\left(p^{\prime}, q^{\prime}\right)$. Then, we have $p \neq q$ and $p^{\prime} \neq q^{\prime}$, and $p \cdot w=p^{\prime}$ and $q \cdot w=q^{\prime}$. Thus, $Q_{G} \cdot w \supseteq\{p, q\} \cdot w=\left\{p^{\prime}, q^{\prime}\right\}$. As $p^{\prime} \neq q^{\prime}$, we have $\left|Q_{G} \cdot w\right| \geq 2$ so $w$ is not synchronizing for $G$. Conversely, if $w$ was not synchronizing for $G$, then $\left|Q_{G} \cdot w\right| \geq 2$. Let $p^{\prime}, q^{\prime} \in Q_{G} \cdot w$ be distinct. Let $p, q \in Q_{G}$ such that $p \cdot w=p^{\prime}$ and $q \cdot w=q^{\prime}$. If for some factoring $w=u v$ we had $p \cdot u=q \cdot u$, then $p^{\prime}=p \cdot w=(p \cdot u) \cdot v=(q \cdot u) \cdot v=q \cdot w=q^{\prime}$, a contradiction to $p^{\prime}$ and $q^{\prime}$ being distinct. Thus, there is a path labeled $w$ in $G * G$ from $(p, q)$ to ( $p^{\prime}, q^{\prime}$ ) which does not pass through any diagonal vertices, meaning it is a labeled path in $\hat{G}$.

Because of the correspondence of synchronizing and intrinsically synchronizing words in follower-separated synchronizing deterministic presentations, we can use $\hat{G}$ to characterize when $X_{G}$ is an SFT.

Theorem 3.5. Let $G$ be a follower-separated synchronizing deterministic presentation. Then, $X_{G}$ is an SFT if and only if $\hat{G}$ is acyclic.
Proof. Suppose $\hat{G}$ had a cycle. By Theorem 2.5, to show that $X_{G}$ is not an SFT, it suffices to show that for every $M \geq 0$, there exists a word $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ with $|w| \geq M$ that is not intrinsically synchronizing for $X_{G}$. Let $M \geq 0$. Since $\hat{G}$ has a cycle, in particular it has a path of any length. Let $\pi$ be a path in $\hat{G}$ of length at least $M$, and let $w$ be its label. We have $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ and $|w| \geq M$. By Lemma 3.4, $w$ is not synchronizing for $G$, and by Lemma 2.2, $w$ is therefore not intrinsically synchronizing for $X_{G}$.

Conversely, suppose $\hat{G}$ is acyclic, and let $M \triangleq\left|Q_{\hat{G}}\right|$. By Theorem 2.5, to show $X_{G}$ is an SFT, it suffices to show that every word $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ with $|w| \geq M$ is intrinsically synchronizing for $\mathrm{X}_{G}$. Let $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ and suppose $|w| \geq M$. Suppose for a contradiction that $w$ is not intrinsically synchronizing for $X_{G}$. By Lemmas 2.2 and 3.4 once again, there is a path in $\hat{G}$ labeled $w$, of length $|w| \geq M$. As $G$ is acyclic, however, every path in $\hat{G}$ must have length strictly less than $\left|Q_{\hat{G}}\right|=M$, a contradiction. Thus, $w$ must be intrinsically synchronizing for $X_{G}$.

Remark 3.6. Let $X$ be a shift space and $M \geq 0$. Say $X$ is $M$-step if every word $w \in \mathcal{B}(X)$ with $w \geq M$ is intrinsically synchronizing for $X$. With this definition and rephrasing Theorem 2.5, a shift space is an SFT if and only if it is $M$-step for some $M \geq 0$. The converse direction then implies that if $\hat{G}$ is acyclic, then it must be $\left(\left|Q_{G}\right|^{2}-\left|Q_{G}\right|\right)$-step, as $\left|Q_{\hat{G}}\right|=\left|Q_{G}\right|^{2}-\left|Q_{G}\right|$. Thus, if $G$ is a follower-separated synchronizing deterministic presentation, then $\mathrm{X}_{G}$ is an SFT if and only if it is $\left(\left|Q_{G}\right|^{2}-\left|Q_{G}\right|\right)$-step. (Cf. [17, Theorem 3.4.17].)

### 3.5 Isomorphism and equality

A homomorphism between deterministic labeled graphs $G$ and $H$ is a mapping $\varphi: Q_{G} \rightarrow$ $Q_{H}$ that preserves the transition action: we have $F_{G}(q)=F_{H}(\varphi(q))$ and $\varphi(q \cdot w)=\varphi(q) \cdot w$ for all $q \in Q_{G}$ and $w \in F_{G}(q)$. An isomorphism is a bijective homomorphism. In general, the problem of deciding isomorphism between deterministic labeled graphs is GIcomplete, meaning it has a polynomial-time many-one reduction to and from the graph isomorphism problem on unlabeled graphs [5]. For follower-separated graphs, however, the problem is in P . To show this, we need the following lemma, which states that preserving the follower set of a vertex is sufficient for being a homomorphism onto a follower-separated graph.

Lemma 3.7. Let $G$ and $H$ be deterministic labeled graphs, and $\varphi: Q_{G} \rightarrow Q_{H}$ a map between their vertices. If $H$ is follower-separated, then $\varphi$ is a homomorphism if and only if $F_{G}(q)=F_{H}(\varphi(q))$ for all $q \in Q_{G}$.

Proof. That homomorphisms preserve follower sets follows directly from the definition. For the converse, suppose $F_{G}(q)=F_{H}(\varphi(q))$ for all $q \in Q_{G}$. Let $q \in Q_{G}$ and $w \in F_{G}(q)$. As $H$ is follower-separated, it suffices to show that $F_{H}(\varphi(q \cdot w))=F_{H}(\varphi(q) \cdot w)$ to show that $\varphi(q \cdot w)=\varphi(q) \cdot w$. For any $u$, we have

$$
\begin{aligned}
& u \in F_{H}(\varphi(q \cdot w)) \\
\Longleftrightarrow & u \in F_{G}(q \cdot w) \\
\Longleftrightarrow & w u \in F_{G}(q) \\
\Longleftrightarrow & w u \in F_{H}(\varphi(q)) \\
\Longleftrightarrow & u \in F_{H}(\varphi(q) \cdot w) .
\end{aligned}
$$

Thus $F_{H}(\varphi(q \cdot w))=F_{H}(\varphi(q) \cdot w)$.
Now, given two follower-separated deterministic labeled graphs $G$ and $H$, we can test if they are isomorphic by taking the disjoint union graph $G+H$, computing the followerequivalences of $G+H$, and testing if all the follower-equivalence classes are pairs (i.e. sets of size 2). As $G$ and $H$ are follower-separated, if two distinct vertices in $G+H$ are follower-equivalent, then one of them must be a vertex from $G$ and the other from $H$. Thus, if all the follower-equivalence classes are pairs, then a bijective map $\varphi: Q_{G} \rightarrow Q_{H}$ that preserves the follower set of a vertex can be read off from the pairs. By Lemma 3.7, this map is an isomorphism. Conversely, if $\varphi: Q_{G} \rightarrow Q_{H}$ is an isomorphism, then for $p \in Q_{G}$ and $q \in Q_{H}$ with $F_{G}(p)=F_{H}(q)$, then $F_{H}(\varphi(p))=F_{H}(q)$ and so $\varphi(p)=q$. In other words, for any $p \in Q_{G}$ and any $q \in Q_{H}$ follower-equivalent to $p$, then $\varphi(p)=q$. This implies that all the follower-equivalence classes of $G+H$ are pairs.

Since the follower set of a vertex is preserved under an isomorphism, if $G$ and $H$ are isomorphic deterministic presentations, then $X_{G}=X_{H}$. However, even for followerseparated presentations, the converse is not necessarily true. (See Figure 1.) But Jonoska [12, Corollary 5.4] proved that any two follower-separated synchronizing deterministic


Figure 1: A reducible, follower-separated, deterministic presentation $G$. Let $H$ be the subgraph induced by $q_{2}$ and $q_{3}$, which is irreducible and follower-separated. Then, $\mathrm{X}_{G}=$ $X_{H}$, but $G$ and $H$ are not isomorphic.
presentations of the same sofic shift are isomorphic, which implies that recognizing isomorphism is sufficient for recognizing if $X_{G}=X_{H}$ when $G$ and $H$ are follower-separated synchronizing deterministic presentations. Furthermore, this implies that EQuAlity is in P for synchronizing deterministic presentations, as given synchronizing deterministic presentations $G$ and $H$, to determine if $X_{G}=X_{H}$, one can test if $G / \sim$ and $H / \sim$, the follower-separations of $G$ and $H$, are isomorphic.

## 4 Complexity Lower Bounds

In the following sections, we show all the decision problems in Table 1 are PSPACE-hard. In Appendix A, we show all those decision problems are in PSPACE. As a result, we have the following.

Theorem 4.1. Every problem in Table 1 is PSPACE-complete.
To establish the hardness of these decision problems, we will leverage hardness results from the automata theory literature. To relate automata to sofic shifts, we will treat automata as a type of labeled graph. Formally, we define a deterministic finite automaton (DFA) to be a fully deterministic labeled graph $M$ with a designated initial state $s \in Q_{M}$ and set of accepting states $F \subseteq Q_{M}$. For DFAs, following convention from the automata literature, we will write the transition action as a function $\delta(q, w) \triangleq q \cdot w$. The language of $M$ is the set $L(M) \triangleq\left\{w \in \mathcal{A}_{M}^{*}: \delta(s, w) \in F\right\}$. Note that $L(M)$ may differ from $\mathcal{B}\left(\mathrm{X}_{M}\right)$, the language of the sofic shift presented by $M$. In fact, as DFAs are fully deterministic, we always have $\mathcal{B}\left(\mathrm{X}_{M}\right)=\mathcal{A}_{M}^{*}$, meaning $\mathrm{X}_{M}$ is always the full shift.

We will reduce from the DFA intersection nonemptiness problem (DFAInt) and DFA union universality problem (DFAUnion), both of which are PSPACE-complete. The DFAInt problem asks whether, given $n$ DFAs $M_{1}, \ldots, M_{n}$ over a common input alphabet $\Sigma$, is $\bigcap_{i=1}^{n} L\left(M_{i}\right) \neq \varnothing$ ? Similarly, the DFAUnion problem asks whether, given $n$ DFAs $M_{1}, \ldots, M_{n}$ over a common input alphabet $\Sigma$, is $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ ? Kozen [13] showed that DFAINT is PSPACE-complete; one can see that DFAUnion is PSPACE-complete from the following two facts: (i) the complement of DFAInT is PSPACE-complete, and (ii) $\bigcap_{i=1}^{n} L\left(M_{i}\right)=\varnothing$ if and only if $\bigcup_{i=1}^{n} L\left(\overline{M_{i}}\right)=\Sigma^{*}$, where $\overline{M_{i}}$ is $M_{i}$ with the accepting states being the complement of the accepting states of $M_{i}$. Within our reductions, for an instance $M_{1}, \ldots, M_{n}$ of

DFAUnion or DFAInt, we will let $Q_{i}, \delta_{i}, s_{i}$, and $F_{i}$ denote the set of states, transition function, initial state, and set of accepting states for $M_{i}$. (The $Q_{i}$ are assumed to be pairwise disjoint.)

### 4.1 Hardness of equality, containment, irreducibility, and SDP existence

In this section, we give a single polynomial-time reduction, which reduces DfaUnion simultaneously to Subshift, Equality, Irreducibility, and $\exists$ SDP, giving the following.

Theorem 4.2. Subshift, EQuAlity, Irreducibility, and $\exists$ Sdp are PSPACE-hard.
The idea behind the reduction is to create pre-initial states $p_{i}$ for each DFA $M_{i}$, and chain these together in a loop, with special symbols $\triangleright$ into and $\triangleleft$ out of each DFA. We then add a special state $p^{*}$ in its own initial irreducible component, whose follower set contains $\left\{\triangleright w \triangleleft: w \in \Sigma^{*}\right\}$. (See Figure 2 for a visualization.) Letting $H$ be the whole graph minus the special state $p^{*}$, we can therefore test whether the DFA languages union to $\Sigma^{*}$ by asking whether $X_{G}=X_{H}$, i.e., whether $p^{*}$ was needed to cover all possible strings $w \in \Sigma^{*}$ between $\triangleright$ and $\triangleleft$. Equivalently, we could test $X_{G} \subseteq X_{H}$, since the reverse inclusion is immediate. As $H$ is an irreducible presentation, we could also test whether $X_{G}$ is irreducible. Finally, the reduction to $\exists$ SDP follows for the following reasons: first, $H$ is a synchronizing determinstic presentation (as $\triangleleft \ell^{i-1}$ synchronizes to $p_{i}$ for all $i$ ), so when the langauges of the DFAs union to $\Sigma^{*}, H$ is a synchronizing deterministic presentation for $\mathrm{X}_{G}$; second, when there is a word $w \in \Sigma^{*}$ not in the language of any of the DFAs, one can show $X_{G}$ does not have a synchronizing deterministic presentation by invoking Theorem 2.3 and showing that any $u$ such that $u \triangleright w \triangleleft \in \mathcal{B}\left(X_{G}\right)$ is not intrinisically synchronizing.

Reduction A. Let $M_{1}, \ldots, M_{n}$ be an instance to the DfaUnion problem. Construct the deterministic presentation $G$ as follows. For each $i=1, \ldots, n$,

1. add a state $p_{i}$ (the $i$ th pre-initial state) to $G$;
2. embed $M_{i}$ into $G$;
3. add a self loop labeled $*$ on $p_{i}$;
4. add an edge labeled $\triangleright$ from $p_{i}$ to the corresponding initial state $s_{i}$;
5. for each accepting state $q \in F_{i}$, add an edge labeled $\triangleleft$ from $q$ to $p_{1}$;
6. for each state $q \in Q_{i}$, add an edge labeled $\ell$ from $q$ to $s_{i}$.

Then, add two states $p^{*}$ and $s^{*}$, add a self loop labeled $*$ on $p^{*}$, add an edge labeled $\triangleright$ from $p^{*}$ to $s^{*}$, and add an edge labeled $\triangleleft$ from $s^{*}$ to $p_{1}$. For each $a \in \Sigma$, add a self loop labeled $a$ on $s^{*}$. For each $i=1, \ldots, n-1$, add an edge labeled $\ell$ from $p_{i}$ to $p_{i+1}$. Finally, add an edge labeled $\ell$ from $p_{n}$ to $s^{*}$. (See Figure 2.)

Let $G$ be the deterministic presentation obtained from Reduction A on an instance $M_{1}, \ldots, M_{n}$. Without loss of generality, for each $i$, we may assume that (I) $F_{i} \neq \varnothing$, as otherwise, if $F_{i}=\varnothing$, then $L\left(M_{i}\right)=\varnothing$ so thus $L\left(M_{i}\right)$ does not contribute to the union;


Figure 2: Schematic of Reduction A. The edges labeled $\ell$ from each state $q \in Q_{i}$ to $s_{i}$ are not pictured.
and (II) every state $q \in Q_{i}$ is reachable from $s_{i}$, as when modifying $M_{i}$ to $M_{i}^{\prime}$ by removing those unreachable states, we have $L\left(M_{i}\right)=L\left(M_{i}^{\prime}\right)$. Let $H$ be the subgraph in $G$ induced by every vertex but $p^{*}$. The following lemma summarizes several useful properties of the reduction.

Lemma 4.3. The following hold of Reduction A.
(i) $H$ is synchronizing and irreducible, and $G$ is essential;
(ii) $\Sigma^{*} \subseteq F_{G}(q)$ for all $q \in \bigcup_{i=1}^{n} Q_{i}$;
(iii) $\triangleright w \triangleleft \in F\left(p_{i}\right)$ if and only if $w \in L\left(M_{i}\right)$;
(iv) $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{H}\right)$ if and only if $w \in \bigcup_{i=1}^{n} L\left(M_{i}\right)$.
(v) If $\bigcup_{i=1}^{n} L\left(M_{i}\right) \neq \Sigma^{*}$, there exists $w \in \Sigma^{*}$ with $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right) \backslash \mathcal{B}\left(\mathrm{X}_{H}\right)$.

Proof. For (i), by assumption (I), there exists a state in $Q_{i}$ with an edge labeled $\triangleleft$ to $p_{1}$. By assumption (II), every state is reachable from $s_{i}$, so there exists a path from $s_{i}$ to $p_{1}$. Thus for any state in $Q_{i}$, one can always find a way to $p_{1}$ by returning to $s_{i}$ via an $\ell$ edge, and then finding a way to $p_{1}$. As $p_{1}$ can reach any other vertex in $H$, any state in $Q_{i}$ can
reach any other vertex in $H$. From this, we can see that $H$ is irreducible, and it follows that $G$ is essential. We have that $H$ is synchronizing as $Q_{G} \cdot \triangleleft=\left\{p_{1}\right\}$ and every vertex in $H$ is reachable from $p_{1}$.

For the other statements, first note that each of the $M_{i}$ are emulated by the transition action of $G$ in the following way: for $q \in Q_{i}$ and $w \in \Sigma^{*}$, we have $w \in F_{G}(q)$ and $q \cdot w=$ $\delta_{i}(q, w)$ and $q \in F_{i}$ if and only if $\triangleleft \in F_{G}(q)$. Thus, (ii) follows. For (iii), note that $p_{i} \cdot \triangleright=s_{i}$ and $w \triangleleft \in F_{G}\left(s_{i}\right)$ if and only if $w \in L\left(M_{i}\right)$; thus, $\triangleright w \triangleleft \in F_{G}\left(p_{i}\right)$ if and only if $w \in L\left(M_{i}\right)$. For (iv), note that $Q_{H} \cdot \triangleright=\left\{s_{1}, \ldots, s_{n}\right\}$; thus, by the previous observations, $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{H}\right)$ if and only if $w \in \bigcup_{i=1}^{n} L\left(M_{i}\right)$. Finally (v) follows from (iv) and the fact that $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ for all $w \in \Sigma^{*}$.

With these properties, we can establish the correctness of Reduction A. The first theorem shows that it reduces DfaUnion to Subshift.

Theorem 4.4. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{G} \subseteq X_{H}$.
Proof. Suppose $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$. To establish $X_{G} \subseteq X_{H}$, we only need to show $F_{G}\left(p^{*}\right) \subseteq$ $\mathcal{B}\left(\mathrm{X}_{H}\right)$. Let $u \in F_{G}\left(p^{*}\right)$. If $p^{*} \cdot u=p^{*}$, then by construction, we have $u=*^{m}$ for some $m \geq 0$, and as $u \in F_{G}\left(p_{1}\right)$, then $u \in \mathcal{B}\left(X_{H}\right)$. Otherwise, if $p^{*} \cdot u=s^{*}$, then we can factor $u$ into $u=*^{m} \triangleright w$, where $m \geq 0$ and $w \in \Sigma^{*}$. Similarly, by Lemma 4.3(ii), we can find $u \in F_{G}\left(p_{1}\right)$, so $u \in \mathcal{B}\left(X_{H}\right)$. Finally, if $p^{*} \cdot u \notin\left\{p^{*}, s^{*}\right\}$, then we can factor $u$ into $u=u_{1} u_{2}$, where $u_{1}=*^{m} \triangleright w \triangleleft$ for some $m \geq 0$ and $w \in \Sigma^{*}, p^{*} \cdot u_{1}=p_{1}$, and $u_{2} \in F_{G}\left(p_{1}\right)$. As $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, Lemma 4.3(iii) implies $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{H}\right)$, and in particular, there is some $i$ such that $\triangleright w \triangleleft \in F_{G}\left(p_{i}\right)$. As $p_{i} \cdot \triangleright w \triangleleft=p_{i} \cdot *^{m} \triangleright w \triangleleft=p_{i} \cdot u_{1}=p_{1}$, we have $u_{1} \in F_{G}\left(p_{i}\right)$ and $u_{2} \in F_{G}\left(p_{i} \cdot u_{1}\right)$. Thus $u_{1} u_{2}=u \in F_{G}\left(p_{i}\right)$, and $u \in \mathcal{B}\left(X_{H}\right)$.

Conversely, Lemma 4.3(v) gives some $w \in \Sigma^{*}$ with $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right) \backslash \mathcal{B}\left(\mathrm{X}_{H}\right)$. Hence, $\mathcal{B}\left(\mathrm{X}_{G}\right) \nsubseteq \mathcal{B}\left(\mathrm{X}_{H}\right)$, and thus $\mathrm{X}_{G} \nsubseteq \mathrm{X}_{H}$.

Immediately, as $\mathrm{X}_{H} \subseteq \mathrm{X}_{G}$, we have that DfaUnion reduces to Equality.
Corollary 4.5. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $\mathrm{X}_{G}=\mathrm{X}_{H}$.
The reduction to Irreducibility now follows as well.
Theorem 4.6. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{G}$ is irreducible.
Proof. If $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, Corollary 4.5 implies $X_{G}=X_{H}$; as $X_{H}$ is irreducible by Lemma 4.3(i), so is $X_{G}$. Conversely, Lemma 4.3(v) gives some $w \in \Sigma^{*}$ with $\triangleright w \triangleleft \epsilon$ $\mathcal{B}\left(\mathrm{X}_{G}\right) \backslash \mathcal{B}\left(\mathrm{X}_{H}\right)$. As $\triangleleft$ synchronizes to $p_{1}$, for every $u \in F_{\mathrm{X}_{G}}(\triangleright w \triangleleft)$, we have $F_{\mathrm{X}_{G}}(\triangleright w \triangleleft u)=$ $F_{G}\left(p_{1} \cdot u\right)$. Furthermore, as $p_{1} \cdot u \in Q_{H}$, we have $F_{G}\left(p_{1} \cdot u\right) \subseteq \mathcal{B}\left(X_{H}\right)$. Yet $\triangleright w \triangleleft \notin \mathcal{B}\left(\mathrm{X}_{H}\right)$, so we must have $\triangleright w \triangleleft \notin F_{G}\left(p_{1} \cdot u\right)$. Thus, for every $u \in F_{X_{G}}(\triangleright w \triangleleft)$, we have $\triangleright w \triangleleft \notin$ $F_{G}\left(p_{1} \cdot u\right)=F_{X_{G}}(\triangleright w \triangleleft u)$. In other words, $\triangleright w \triangleleft \in \mathcal{B}\left(X_{G}\right)$ but there is no word $u$ such that $\triangleright w \triangleleft u \triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$.

Remark 4.7. Let $X$ be a shift space. Say $X$ is mixing if for every $u, v \in \mathcal{B}(X)$, there is an $N$ such that for every $n \geq N$, there is a word $w$ with $|w|=n$ and $u w v \in \mathcal{B}(X)$. Mixing
implies irreducibility, so if $X_{G}$ is mixing, then $X_{G}$ is irreducible. Note that $X_{H}$ is mixing, as given $u, v \in \mathcal{B}\left(\mathrm{X}_{H}\right)$, one can find words $w_{1}, w_{2}$ such that $u w_{1}$ synchronizes to $p_{1}$ and $v \in F_{G}\left(p_{1} \cdot w_{2}\right)$, and as $p_{1} \cdot *^{m}=p_{1}$ for every $m$, we have that $u w_{1} *^{m} w_{2} v \in \mathcal{B}\left(\mathrm{X}_{H}\right)$ for every $m$. Thus, $X_{G}$ is irreducible if and only if it is mixing, so deciding if the sofic shift presented by a deterministic presentation is mixing is PSPACE-hard.

Similarly, say $X$ is nonwandering if for every $u \in \mathcal{B}\left(X_{G}\right)$, there is a word $w$ such that $u w u \in \mathcal{B}(X)$. Irreducibility implies nonwandering, so if $X_{G}$ is irreducible, then $X_{G}$ is nonwandering. Note that the proof of Theorem 4.6 shows that if $\bigcup_{i=1}^{n} L\left(M_{i}\right) \neq \Sigma^{*}$, then $\mathrm{X}_{G}$ is not nonwandering. Thus, $\mathrm{X}_{G}$ is irreducible if and only if it is nonwandering, so deciding if the sofic shift presented by a deterministic presentation is nonwandering is PSPACE-hard.

Finally, we show that Reduction A also reduces DfaUnion to $\exists$ Sdp.
Theorem 4.8. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{G}$ has a synchronizing deterministic presentation.

Proof. If $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, then Corollary 4.5 implies $X_{G}=X_{H}$, and as Lemma 4.3(i) implies $H$ is synchronizing, then $X_{G}$ has a synchronizing deterministic presentation. Conversely, Lemma 4.3(v) gives some $w \in \Sigma^{*}$ with $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right) \backslash \mathcal{B}\left(\mathrm{X}_{H}\right)$. Suppose for a contradiction that $X_{G}$ has a synchronizing deterministic presentation. By Theorem 2.3, there must be some $u \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ such that $u$ is intrinsically synchronizing for $\mathrm{X}_{G}$ and $\triangleright w \triangleleft \in F_{X_{G}}(u)$. As $\triangleright w \triangleleft \notin \mathcal{B}\left(X_{H}\right)$, the only vertex in $G$ with $\triangleright w \triangleleft$ in its follower set is $p^{*}$. Therefore, $\triangleright w \triangleleft \in F_{X_{G}}(u)$ implies $p^{*} \in Q_{G} \cdot u$. By construction, the only such $u$ take the form $u=*^{k}$ for some $k \geq 0$. However, $*^{k}$ is not intrinsically synchronizing for $X_{G}$ : we have $\triangleleft *^{k} \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ and $*^{k} \triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ but as $\triangleright w \triangleleft \notin \mathcal{B}\left(\mathrm{X}_{H}\right)$ and $\triangleleft *^{k}$ synchronizes to a vertex in $H$, it must be the case that $\triangleleft *^{k} \triangleright w \triangleleft \notin \mathcal{B}\left(\mathrm{X}_{G}\right)$. Thus, $u=*^{k}$ is not intrinsically synchronizing, a contradiction. We conclude that $\mathrm{X}_{G}$ does not have a synchronizing deterministic presentation.

### 4.2 Hardness of SFT Testing and Minimization

We now give a similar polynomial-time reduction, which reduces DfaUnion simultaneously to SFt and Minimality, giving the following.

Theorem 4.9. The problems Sft and Minimality are PSPACE-hard.
The reduction is similar in spirit to Reduction A. We still add edges labeled $\triangleleft$ out of each DFA into a terminal state, but instead of adding edges labeled $\triangleright$ into the DFAs from new pre-initial states, we instead add these edges from within the DFAs to their corresponding initial states. We also add self loops on each DFA state labeled $\ell$. We then add a special state $s^{*}$ in its own initial component, whose follower set contains $\left\{w \triangleleft: w \in(\Sigma \cup\{\ell\})^{*}\right\}$. See Figure 3 for a visualization. The first observation we make is that, if and only if the DFA languages union to $\Sigma^{*}$, the shift $X_{G}$ is presented by the graph $H$ in Figure 4. Since $H$ presents an SFT, to show that we reduce to the SFT problem, we


Figure 3: Schematic of Reduction B.


Figure 4: The graph $H$. As in Figure 3, the $\Sigma$ above the self loop on $q_{2}$ represents a self loop labeled $a$ for each $a \in \Sigma$.
need only argue that $X_{G}$ is not an SFT when there is some word $w$ not in the language of any DFA. Because the $\ell$ self loops arbitrarily delay the DFA decision to accept or reject, they prevent $\mathrm{X}_{G}$ from having a finite list of forbidden words, or equivalently, from being $M$-step for any finite $M$. Finally, to show we reduce to Minimality, we show that $\mathrm{X}_{G}$ does not have a 2 -vertex presentation when the DFA languages do not union to $\Sigma^{*}$.
Reduction B. Let $M_{1}, \ldots, M_{n}$ be an instance to the DfaUnion problem. Construct the deterministic presentation $G$ as follows. Add a state $t$ (the terminal state), and add a self loop labeled $\star$ on $t$. Add a state $s^{*}$, and add self loops on $s^{*}$ labeled by each symbol in $\Sigma \cup\{\ell\}$. Add an edge labeled $\triangleleft$ from $s^{*}$ to $t$. Finally, for each $i=1, \ldots, n$,

1. embed $M_{i}$ into $G$;
2. for each state $q \in Q_{i}$, add an edge labeled $\triangleright$ from $q$ to $s_{i}$
3. for each accepting state $q \in F_{i}$, add an edge labeled $\triangleleft$ from $q$ to $t$;
4. for each state $q \in Q_{i}$, add a self loop labeled $\ell$ on $q$.

See Figure 3 for a visualization.
We once again summarize the salient properties of the reduction. First, we define a notation that will be used multiple times: for a word $w$, we let $h_{\ell}(w)$ denote $w$ with all the $\ell$ 's removed. (That is, $h_{\ell}$ is the string homomorphism such that $h_{\ell}(\ell)=\epsilon$ and $h(a)=a$ for $a \neq \ell$.)

Lemma 4.10. The following hold of Reduction B and the graph $H$ from Figure 4.
(i) $X_{G} \subseteq X_{H}$;
(ii) $(\Sigma \cup\{\ell, \triangleright\})^{*}, \subseteq F_{G}(q)$ for all $q \in \bigcup_{i=1}^{n} Q_{i}$;
(iii) for $w \in \Sigma^{*}, w \triangleleft \in F\left(s_{i}\right)$ if and only if $w \in L\left(M_{i}\right)$;
(iv) for $w \in \Sigma^{*}, \triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ if and only if $w \in \bigcup_{i=1}^{n} L\left(M_{i}\right)$.
(v) for $w \in(\Sigma \cup\{\ell\})^{*}, \triangleright h_{\ell}(w) \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ if and only if $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$;

Proof. For (i), let $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. If $w$ does not contain $\triangleleft$, then either $w \in(\Sigma \cup\{\ell, \triangleright\})^{*}$ or $w=\star^{m}$ for some $m \geq 0$; if $w \in(\Sigma \cup\{\ell, \triangleright\})^{*}$, then $w \in F_{H}\left(q_{1}\right)$; if $w=\star^{m}$ for some $m \geq 0$, then $w \in F_{H}\left(q_{2}\right)$. Otherwise, if $w$ contains $\triangleleft$, then we can factor $w$ into $w=u \triangleleft \star^{m}$ where $u \in(\Sigma \cup\{\ell, \triangleright\})^{*}$ and $m \geq 0$, for which it follows that $w \in F_{H}\left(q_{1}\right)$. Thus, for every $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, we have $w \in \mathcal{B}\left(\mathrm{X}_{H}\right)$.

For the other statements, first note that each of the $M_{i}$ are emulated by the transition action of $G$ in the following way: for $q \in Q_{i}$ and $w \in \Sigma^{*}$, we have $w \in F_{G}(q)$ and $q$. $w=\delta_{i}(q, w)$ and $q \in F_{i}$ if and only if $\triangleleft \in F_{G}(q)$. Thus, (ii) follows from the emulation observation and the fact that $\ell \in F_{G}(q)$ and $q \cdot \ell=q$ and $\triangleright \in F_{G}(q)$ and $q \cdot \triangleright=s_{i}$ for all $q \in Q_{i}$. Statement (iii) follows immediately from the emulation observation as well. Note that $Q_{G} \cdot \triangleright=\left\{s_{1}, \ldots, s_{n}\right\}$, so (iv) follows from (iii). Finally, for (v), as $q \cdot \ell=q$ for $q \in \bigcup_{i=1}^{n} Q_{i}$, by induction on the number of $\ell$ 's in $w$, one can show that $Q_{G} \cdot \triangleright w=$ $Q_{G} \cdot \triangleright h_{\ell}(w)$; thus, we have $Q_{G} \cdot \triangleright h_{\ell}(w) \triangleleft=Q_{G} \cdot \triangleright w \triangleleft$, which implies $Q_{G} \cdot \triangleright h_{\ell}(w) \triangleleft \neq \varnothing$ if and only if $Q_{G} \cdot \triangleright w \triangleleft \neq \varnothing$.

To show the correctness of Reduction A, we first give an alternate reduction to SuBSHIFT.

Theorem 4.11. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{H} \subseteq X_{G}$.
Proof. First suppose $\mathcal{B}\left(\mathrm{X}_{H}\right) \subseteq \mathcal{B}\left(\mathrm{X}_{G}\right)$. For every word $w \in \Sigma^{*}$, we have $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{H}\right)$, giving $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. Lemma 4.10(iv) now implies $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$. For the converse, suppose $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, and let $u \in \mathcal{B}\left(X_{H}\right)$. There are two cases: either $u \in F_{H}\left(q_{1}\right)$ or $u \in F_{H}\left(q_{2}\right)$. If $u \in F_{H}\left(q_{2}\right)$, then $u=\star^{m}$ for some $m \geq 0$, which implies that $u \in F_{G}(t)$, and so $u \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. Thus, to complete the proof we need to show that if $u \in F_{H}\left(q_{1}\right)$, then $u \in \mathcal{B}\left(\mathrm{X}_{G}\right)$.

Suppose $u \in F_{H}\left(q_{1}\right)$. We further break this case into the possible values of $q_{1} \cdot u$. If $q_{1} \cdot u=q_{1}$, then $u \in(\Sigma \cup\{\ell, \triangleright\})^{*}$, so $u \in F_{G}\left(s_{1}\right)$ and thus $u \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. If $q_{1} \cdot u=q_{2}$, then $u=v \triangleleft \star^{m}$ for some $m \geq 0$ and $v \in(\Sigma \cup\{\ell, \triangleright\})^{*}$. If $v$ does not contain the symbol $\triangleright$, then $v \in(\Sigma \cup\{\ell\})^{*}$, which implies $v \triangleleft \star^{m}=u \in F_{G}\left(s^{*}\right)$ and thus $u \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. Otherwise,
if $v$ contains the symbol $\triangleright$, then we can factor $v$ into $v=x \triangleright w$ where $w$ contains no $\triangleright$; i.e. $w \in(\Sigma \cup\{\ell\})^{*}$. Then, we have that $h_{\ell}(w) \in \Sigma^{*}$, and as $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, we have $\triangleright h_{\ell}(w) \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. By Lemma 4.10(v), we have $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, and by Lemma 4.10(iii), we have $w \triangleleft \in F_{G}\left(s_{i}\right)$ for some $s_{i}$. Collecting facts, we have $x \triangleright \in F_{G}\left(s_{i}\right)$ and $s_{i} \cdot x \triangleright=s_{i}$ and $s_{i} \cdot w \triangleleft=t$ and $*^{m} \in F_{G}(t)$. Combining those facts gives us that $x \triangleright w \triangleleft \star^{m}=u \in F_{G}\left(s_{i}\right)$, so $u \in \mathcal{B}\left(\mathrm{X}_{G}\right)$.

Remark 4.12. Interestingly, Theorem 4.11 gives us another proof that Subshift is PSPACEhard. However, we can easily extend Theorem 4.11 to a stronger hardness result. ${ }^{6}$ Specifically, we can show that SUBSHIFT is PSPACE-hard even when both input instances are synchronizing deterministic presentations. Note that $H$ is synchronizing while $G$ is not. Construct the presentation $G^{\prime}$ as follows: construct $G$, and let $S \triangleq\left\{s_{1}, \ldots, s_{n}, s^{*}\right\}$. For each $q \in S$, add a self loop labeled $\ell_{q}$ on $q$.

For each vertex in $q \in S$, we have that $\ell_{q}$ synchronizes to $q$ in $G^{\prime}$. Note that every vertex is reachable from a vertex in $S$, so this implies that $G^{\prime}$ is synchronizing. Here, we note that $X_{H} \subseteq X_{G}$ if and only if $X_{H} \subseteq X_{G^{\prime}}$ : the forward direction follows from the fact that $X_{G} \subseteq X_{G^{\prime}}$, and the reverse direction follows from the fact that if $w \in \mathcal{B}\left(X_{H}\right)$ and $w \in \mathcal{B}\left(\mathrm{X}_{G^{\prime}}\right)$, then $w$ does not contain the new labels $\left\{\ell_{s_{1}}, \ldots, \ell_{s_{n}}, \ell_{s^{*}}\right\}$ added in $G^{\prime}$, so it must be the case that $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. This establishes the claim that Subshift is PSPACE-hard even when both instances are synchronizing deterministic presentations.

As $X_{G} \subseteq X_{H}$ by Lemma 4.10(i), we have the following.
Corollary 4.13. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{H}=X_{G}$.
We new show that Reduction B reduces DfaUnion to SFT.
Theorem 4.14. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{G}$ is an SFT.
Proof. The edges in $H$ are labeled uniquely, so by Lemma 2.6, $\mathrm{X}_{H}$ is an SFT. Thus, if we have $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, then by Corollary 4.13, $X_{G}=X_{H}$ is an SFT.

Conversely, suppose $X_{G}$ is an SFT. By Theorem 2.5, there is an $M$ such that whenever $u v, v w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ and $|v| \geq M$, then $u v w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. Let $w \in \Sigma^{*}$. We can find $\triangleright w \ell^{M} \in F_{G}\left(s_{1}\right)$ and $w \ell^{M} \triangleleft \in F_{G}\left(s^{*}\right)$, so we have $\triangleright w \ell^{M}, w \ell^{M} \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ and and thus $\triangleright w \ell^{M} \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. As $h_{\ell}\left(w \ell^{M}\right)=w$, Lemma 4.10(v) implies that $\triangleright w \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. It follows that $w \in \bigcup_{i=1}^{n} L\left(M_{i}\right)$ by Lemma 4.10(iv).

Along with Corollary 4.13, the following shows that Reduction B also reduces DFAUnion to Minimality.

Theorem 4.15. $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$ if and only if $X_{G}$ has a deterministic presentation with 2 vertices.

[^3]Proof. If $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$, then $X_{H}=X_{G}$ by Corollary 4.13, so $H$ is a 2-vertex presentration of $X_{G}$. Conversely, suppose $H^{\prime}$ is a deterministic presentation of $X_{G}$ with 2 vertices. Here, we'll show that $H^{\prime}$ is isomorphic to $H$, which implies $\mathrm{X}_{H^{\prime}}=\mathrm{X}_{H}$ and thus $\mathrm{X}_{H}=\mathrm{X}_{G}$, so by Corollary 4.13, we have $\bigcup_{i=1}^{n} L\left(M_{i}\right)=\Sigma^{*}$.

As $\triangleleft \in F_{G}\left(s^{*}\right)$, there must be an edge $e_{\triangleleft}$ labeled $\triangleleft$ in $H^{\prime}$. If $e_{\triangleleft}$ were a self loop, then $\triangleleft \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, a contradiction. Thus, the vertices $q_{1}^{\prime} \triangleq i\left(e_{\triangleleft}\right)$ and $q_{2}^{\prime} \triangleq t\left(e_{\triangleleft}\right)$ must be distinct, and $Q_{H^{\prime}}=\left\{q_{1}^{\prime}, q_{2}^{\prime}\right\}$. Moreover, $e_{\triangleleft}$ must be the unique edge labeled $\triangleleft$, as in all other cases $H^{\prime}$ either fails to be deterministic or we again have $\triangleleft \triangleleft \in \mathcal{B}\left(X_{G}\right)$, a contradiction.

As $a \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, for each $a \in \Sigma \cup\{\ell\}$, we have an edge $e_{a}$ labeled $a$ ending at $q_{1}^{\prime}$. For $a \in \Sigma \cup\{\ell\}$, any edge labeled $a$ must start at $q_{1}^{\prime}$ : if such an edge started at $q_{2}^{\prime}$, then $\triangleleft a \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, a contradiction. Thus, $e_{a}$ is a self loop and by determinism, $e_{a}$ is the unique edge labeled $a$ in $H^{\prime}$.

As $\triangleleft \star \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, there must be an edge $e_{\star}$ labeled $\star$ starting at $q_{2}^{\prime}$. If $e_{\star}$ ends at $q_{1}^{\prime}$, then $\star \triangleleft \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, a contradiction, so $e_{\star}$ is a self loop. Any edge labeled $\star$ must start at $q_{2}^{\prime}$ : if such an edge started at $q_{1}^{\prime}$, then $\ell \star \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, a contradiction. Thus, by determinism, $e_{\star}$ is the unique edge labeled $\star$ in $H^{\prime}$.

Finally, as $\triangleright \ell \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, there must be some edge $e_{\triangleright}$ labeled $\triangleright$ ending at $q_{1}^{\prime}$. Any edge labeled $\triangleright$ must start at $q_{1}^{\prime}$ : if such an edge start at $q_{2}^{\prime}$, then $\triangleleft \triangleright \in \mathcal{B}\left(\mathrm{X}_{G}\right)$, a contradiction. Thus, $e_{\triangleright}$ is a self loop and is the unique edge labeled $\triangleright$ in $H^{\prime}$. All of the above implies that the map $q_{i}^{\prime} \mapsto q_{i}$ is an isomorphism between $H^{\prime}$ and $H$.

### 4.3 Hardness of Existence of Synchronizing Words

Berlinkov [4] showed SyncWord was PSPACE-hard via reduction from the PSPACEcomplete problem of subset synchronizability: given a DFA $M$ and a subset $S \subseteq Q_{M}$, is there a word $w$ such that $|S \cdot w|=1$ ? For completeness, we show the hardness of SyncWord via Reduction C from the "complement" of DfaUnion, DFAInt.

Theorem 4.16. SyncWord is PSPACE-hard.
For the reduction, we again create pre-initial states $p_{i}$ for each DFA $M_{i}$, with special symbols $\triangleright$ into and $\triangleleft$ out of each DFA, and include them in parallel as in Reduction B. The edges out of accepting states all go to the same shared succes state $t$. We also add edges labeled $\triangleleft$ from each nonaccepting state in $M_{i}$ to an individual fail state $r_{i}$. By completing this construction appropriately, we ensure that a word is synchronizing if and only if it is synchronizing to $t$, i.e., if and only if every DFA accepts the subword between $\triangleright$ and $\triangleleft$.
Reduction C. Let $M_{1}, \ldots, M_{n}$ be an instance to the DfAInt problem, and without loss of generality, assume $n \geq 2$. We will construct a essential, deterministic presentation $G$ as follows. First, add a state $t$ (the success state), and add a self loop labeled $\triangleleft$ on $t$. Then, for each $i=1, \ldots, n$,

1. add a state $p_{i}$ (the $i$ th pre-initial state);


Figure 5: Schematic of Reduction C.
2. add a state $r_{i}$ (the $i$ th fail state);
3. add self loops labeled $\triangleleft$ on $p_{i}$ and $r_{i}$;
4. for each $a \in \Sigma$, add a self loop labeled $a$ on $p_{i}$;
5. embed $M_{i}$ into $G$;
6. add an edge from $p_{i}$ labeled $\triangleright$ to the corresponding initial state $s_{i}$ of $M_{i}$;
7. for each accepting state $q$ in $M_{i}$, add an edge labeled $\triangleleft$ from $q$ to $t$;
8. for each nonaccepting state $q$ in $M_{i}$, add an edge labeled $\triangleleft$ from $q$ to $r_{i}$.

See Figure 5 for a visualization.
To show the correctness of the reduction, we characterize the synchronizing words of $G$.

Theorem 4.17. Let $G$ be the deterministic presentation obtained from Reduction $C$ on an instance $M_{1}, \ldots, M_{n}$. A word $u \in(\Sigma \cup\{\triangleright, \triangleleft\})^{*}$ is synchronizing for $G$ if and only if there is some $v \in(\Sigma \cup\{\triangleleft\})^{*}, k \geq 1$, and $w \in \bigcap_{i=1}^{n} L\left(M_{i}\right)$ such that $u=v \triangleright w \triangleleft^{k}$.

Proof. As usual, the transition action of $G$ emulates the behavior of the $M_{i}$ : for $w \in \Sigma^{*}$, we have $w \in L\left(M_{i}\right)$ if and only if $p_{i} \cdot \triangleright w \triangleleft=t$, and $w \notin L\left(M_{i}\right)$ if and only if $p_{i} \cdot \triangleright w \triangleleft=r_{i}$.

Suppose $u$ is a synchronizing word for $G$. Then, $u$ must contain at least one $\triangleright$; otherwise $u \in(\Sigma \cup\{\triangleleft\})^{*}$, and thus $p_{i} \cdot u=p_{i}$ for each $i$, giving $\left|Q_{G} \cdot u\right| \geq n \geq 2$. We can therefore write $u=u_{1} \triangleright u_{2}$. By construction, $u$ contains at most one $\triangleright$, so $u_{1}, u_{2} \in(\Sigma \cup\{\triangleleft\})^{*}$. Moreover, we must have $u_{2}=w \triangleleft^{k}$ for some $w \in \Sigma^{*}$ and $k \geq 0$. Since $Q_{G} \cdot u_{1} \triangleright w=\left\{p_{1} \cdot \triangleright w, \ldots, p_{n} \cdot \triangleright w\right\}$ and the $Q_{i}$ are pairwise disjoint, we have $\left|Q_{G} \cdot u_{1} \triangleright w\right|=n \geq 2$. Since $u$ is synchronizing, we therefore must have $k \geq 1$. Now since $\triangleleft \in F_{G}(q)$ for $q \in \bigcup_{i=1}^{n} Q_{i}$, if there are $i \neq j$ such that $p_{i} \cdot \triangleright w \triangleleft$ and $p_{j} \cdot \triangleright w \triangleleft$ are
not both $t$, then $\left|Q_{G} \cdot u\right| \geq 2$. As $u$ is synchronizing, we conclude $p_{i} \cdot \triangleright w \triangleleft=t$ for all $i$. By the above, $w \in \bigcap_{i=1}^{n} L\left(M_{i}\right)$. Hence, we have $u=u_{1} \triangleright w \triangleleft^{k}$ with $u_{1} \in(\Sigma \cup\{\triangleleft\})^{*}, k \geq 1$, and $w \in \bigcap_{i=1}^{n} L\left(M_{i}\right)$.

Conversely, let $v \in(\Sigma \cup\{\triangleleft\})^{*}, k \geq 1$, and $w \in \bigcap_{i=1}^{n} L\left(M_{i}\right)$. Thus, $p_{i} \cdot \triangleright w \triangleleft=t$ for all $i$, which implies $Q_{G} \cdot v \triangleright w \triangleleft^{k}=\{t\}$.

Corollary 4.18. $\bigcap_{i=1}^{n} L\left(M_{i}\right) \neq \varnothing$ if and only if $G$ has a synchronizing word.
As Reduction C therefore reduces Dfalnt to SyncWord, Theorem 4.16 follows.

## 5 Size of Synchronizing Words and SDPs

Our reductions also shed light on the size of synchronizing words and presentations. In particular, given a presentation with $n$ vertices, the size of its smallest synchronizing word can be exponentially large in $n$. Similarly, the size of the smallest synchronizing deterministic presentation can also be exponentially large.

### 5.1 Shortest synchronizing word size

If NP $\neq$ PSPACE, there cannot be a polynomial upper bound with respect to the number of vertices for the length of the shortest synchronizing word, as SYNCWORD is PSPACEhard. Berlinkov [4] show an unconditional exponential lower bound on maximum length of the shortest synchronizing word, which implies there cannot be a polynomial upper bound for the length of the shortest synchronizing word. Here, we give a simpler construction that achieves roughly the same bound.

First we observe the following property of Reduction B.
Lemma 5.1. Let $G$ be a presentation obtained from Reduction $B$ on some input $M_{1}, \ldots, M_{n}$. If $\bigcap_{i=1}^{n} L\left(M_{i}\right) \neq \varnothing$, then the minimum length of a synchronizing word for $G$ is 2 more than the minimum length of a word in $\bigcap_{i=1}^{n} L\left(M_{i}\right)$.

Proof. From Theorem 4.17, a word $u$ is synchronizing for $G$ if and only if it has the form $u=v \triangleright w \triangleleft^{k}$ for any $v \in(\Sigma \cup\{\triangleleft\})^{*}, w \in \bigcap_{i=1}^{n} L\left(M_{i}\right)$, and $k \geq 1$. A minimum-length synchronizing word $u^{*}$ for $G$ therefore has $v=\epsilon$ and $k=1$, and takes $w=w^{*}$ to be a word of minimum length in $\bigcap_{i=1}^{n} L\left(M_{i}\right)$. Thus $\left|u^{*}\right|=\left|\triangleright w^{*} \triangleleft\right|=2+\left|w^{*}\right|$.

Therefore, to find a presentation of a sofic shift with a large shortest synchronizing word, it suffices to apply Reduction B to DFAs that have a large shortest word in the intersection of their languages. In Appendix B, we adapt a construction from Ang [1] of a family of DFAs $M_{i, k}$ such that each $M_{i, k}$ has 3 states and and the shortest word in $\bigcap_{i=0}^{k} L\left(M_{i, k}\right)$ is $2^{k}$. Using this family of DFAs, if we let $G_{k}$ denote Reduction B applied to $M_{0, k}, M_{1, k}, \ldots M_{k, k}$, then by Lemma 5.1, the shortest synchronizing word for $G_{k}$ has length $2^{k}+2$. The number of vertices in $G_{k}$ is $2(k+1)+1$ auxillary vertices plus $3(k+1)$ from the DFAs, giving use $5 k+6$ total vertices. We may then define a family of graphs $G^{(n)}$ on
$n$ vertices, which take $G_{k}$ where $k \triangleq\left\lfloor\frac{n-6}{5}\right\rfloor$ and add $n-k$ vertices without affecting the shortest synchronizing word (e.g., by adding a self loops labeled with every $a \in \Sigma \cup\{\triangleright\}$ and adding an edge labeled $\triangleleft$ to $t)$. As $k=\Omega(n)$, this family exhibits the following lower bound.

Theorem 5.2. There is a family of deterministic presentaions $\left\{G^{(n)}\right\}$ such that for sufficiently large $n$, each $G^{(n)}$ has $n$ vertices and the minimum length of a synchronizing word for $G^{(n)}$ is $2^{\Omega(n)}$.

Remark 5.3. Černý's conjecture states that if a $n$-state DFA has a synchronizing word, then there is one of length at most $(n-1)^{2}$ [26]. The previous theorem is a counterexample to the generalization of the Černýs conjecture to deterministic presentations of sofic shifts.

### 5.2 Minimal Synchronizing Deterministic Presentation Size

Throughout this subsection, we will abbreviate "synchronizing deterministic presentation" to SDP. Let $X$ be a sofic shift. A minimal SDP of $X$ is a SDP of $X$ possessing the fewest number of vertices among all SDPs of $X$. Similarly, a minimal deterministic presentation of $X$ is a deterministic presentation of $X$ possessing the fewest number of vertices among all deterministic presentations of $X$. For a given sofic shift $X$, minimal SDPs of $X$ are unique up to isomorphism, while minimal deterministic presentations are not neccessarily unique if $X$ is reducible [12]. For irreducible sofic shifts, minimal SDPs are minimal derterministic presentations and vice versa. For reducible sofic shifts, minimal SDPs are minimal deterministic presentations are not necessarily the same.

In fact, we show that the minimal SDP can be exponential larger than a minimal deterministic presentation. The proof relies on multiple-entry DFAs, which are FAs whose only nondeterminism is the fact that there are multiple possible initial states. Formally, a $k$-entry DFA $N$ is a fully deterministic labeled graph along with $k$ states $s_{1}, \ldots, s_{k} \in Q_{N}$ and a set of final states $F \subseteq Q_{N}$. The language of $N$ is defined as $L(N) \triangleq \bigcup_{i=1}^{k}\left\{w \in \mathcal{A}_{N}^{*}\right.$ : $\left.\delta\left(s_{i}, w\right) \in F\right\}$. By Holzer et al. [10, Lemma 3], there exists a family $\left\{C_{k}\right\}$ of multiple-entry DFAs, where $C_{k}$ is a $k$-entry DFA with $k$ states, and and the minimal DFA for $L\left(C_{k}\right)$ has $\sum_{i=1}^{k}\binom{k}{i}=2^{k}-1$ states. In other words, even if automata have deterministic transition relations, passing from multiple start states to a single start state can incur an exponential increase in size. We emulate this construction when passing from a nonsynchronizing presentation to a synchronizing presentation.

Given a $k$-entry DFA $N$, we construct a sofic shift $\mathrm{X}_{G}$ with deterministic presentation $G$ as follows. First, embed $N$ into $G$, add a state $t$ (the terminal state), and add a self loop labeled $\star$ on $t$. Then, for each $i=1, \ldots, k$,

1. add a state $p_{i}$ (the $i$ th pre-initial state);
2. add a self loop labeled $*$ on $p_{i}$;
3. add an edge from $p_{i}$ labeled $\triangleright$ to the corresponding state $s_{i}$ of $N$;
4. for each accepting state $q \in F$, add an edge labeled $\triangleleft$ from $q$ to $t$


Figure 6: Schematic of the construction from Theorem 5.4.

See Figure 6 for a visualization.
Theorem 5.4. Let $N$ be a $k$-entry DFA, and let $G$ be the deterministic presentation obtained from the above construction applied to $N$. Let $M$ be the minimal DFA of $L(N)$. Interpreting $M$ as a 1-entry DFA, let $H$ be the deterministic presentation obtained from the construction applied to $M$. Then, $H$ is the minimal SDP for $\mathrm{X}_{G}$.

Proof. The proof that $X_{G}=X_{H}$ is similar to the proof of Theorem 4.11. To show $H$ is the minimal SDP for $X_{G}$, by Jonoska [12, Theorem 5.5], it suffices to show $H$ is followerseparated. For distinct $p, q \in Q_{M}$, as $M$ is a minimal DFA, either there is some word $\delta(p, w) \in F$ with $\delta(q, w) \notin F$ or there is some word $\delta(p, w) \notin F$ with $\delta(q, w) \in F$. Without loss of generality, we may assume the former, as we can swap the roles of $p$ and $q$ for the latter case. Then, $w \triangleleft \in F_{H}(p)$ while $w \triangleleft \notin F_{H}(q)$, so $p$ and $q$ have distinct follower sets. For distinct $p, q \in Q_{H}$ where one of $p$ or $q$ is the pre-initial state or the terminal state, follower-separation follows from the presence of $*$ or $\star$.

Thus, the size of the minimal SDP for $X_{G}$ is determined by the size of the minimal DFA for $L(N)$. Applying this construction to $C_{k}$ gives us a deterministic presentation with $2 k+1$ vertices whose minimal synchronizing deterministic presentation with $\left(2^{k}-1\right)+2=$ $2^{k}+1$ vertices. The following theorem follows easily from this observation.

Theorem 5.5. There is a family of sofic shifts $\left\{X_{n}\right\}$ such that for sufficiently large $n$, the minimal deterministic presentation of $X_{n}$ has at most $n$ vertices and the minimal synchronizing deterministic presentation of $X_{n}$ has $2^{\Omega(n)}$ vertices.

Proof. Let $n$ be sufficiently large. If we apply the construction to $C_{n^{\prime}}$ where $n^{\prime} \triangleq\left\lfloor\frac{n-1}{2}\right\rfloor$, we get a presentation $G$ with at most $n$ vertices such that the minimal SDP for $X_{G}$ has $2^{n^{\prime}}+1$

| Problem | Automata |  | Sofic Shifts |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | DFA | FA | IDP | SDP | DP | P |
| Universality | P | PSP-c | P | P | P | PSP-c [7] |
| EQUALITY | P | PSP-c | P | P | PSP-c | PSP-c |
| SUBSHIFT | P | PSP-c | P | PSP-c | PSP-c | PSP-c |
| Minimality | P | PSP-c | P [17] | P [12] | PSP-c | PSP-c |
| SyncWord | P [9] |  | P |  | PSP-c [4] | PSP-c |
| IRREDUCIBILITY |  |  |  | P | PSP-c | PSP-c |
| SFT |  |  | P [17] | P | PSP-c | PSP-c |
| $\exists \mathrm{SDP}$ |  |  |  |  | PSP-c | PSP-c |

Table 2: An overview of our results and comparison to related automata theory results. The classes sofic shifts are as follows: IDP = irreducible deterministic presentation, SDP $=$ synchronizing deterministic presentation, $\mathrm{DP}=$ (general) deterministic presentation, and $\mathrm{P}=$ (general) presentation. The complexity classes are $\mathrm{P}=$ solvable in polynomial time and PSP-c $=$ PSPACE-complete. For FAs, SubShift means "is $L(M) \subseteq L(N)$ ?" Entries of the table corresponding to results we prove (or re-prove) have hyperlinks to their respective proofs.
vertices. Thus, since $G$ has at most $n$ vertices, then a minimal deterministic presentation for $\mathrm{X}_{G}$ must have at most $n$ vertices, and as $n^{\prime}=\Omega(n)$, the minimal SDP for $\mathrm{X}_{G}$ has $2^{\Omega(n)}$ vertices.

## 6 Discussion

We first overview our results, together with a discussion of related problems and a comparison to results from automata theory. We conclude with open problems.

### 6.1 Overview of results

We summarize our results in Table 2. The table includes complexity results from the automata theory literature for analagous problems for DFAs and FAs. One conclusion from this table is that, from a computational complexity standpoint, irreducible presentations behave like DFAs, as do synchronizing deterministic presentations with the exception of SUBSHIFT. On the other hand, non-deterministic presentations behave like NFAs, as do general deterministic presentations, with the exception of Universality.

The remainder of this subsection is devoted to entries of the table which were not discussed in the previous sections. To begin, the problem of Universality asks whether a given deterministic presentation $G$ satisfies $X_{G}=\mathcal{A}_{G}^{\mathbb{Z}}$. Clearly, we have $X_{G} \subseteq \mathcal{A}_{G}^{\mathbb{Z}}$ for any $G$, so the problem reduces to deciding whether $\mathcal{A}_{G}^{\mathbb{Z}} \subseteq \mathrm{X}_{G}$. This condition can be decided in polynomial time, as there is an irreducible deterministic presentation of $\mathcal{A}_{G}^{\mathbb{Z}}$ which is
a single vertex and a self loop on that vertex labeled $a$ for each $a \in \mathcal{A}_{G}$, so one may use Algorithm 2 to decide whether $\mathcal{A}_{G}^{\mathbb{Z}} \subseteq \mathrm{X}_{G}$. However, for nondeterministic presentations, universality is PSPACE-complete [7]. Universality is equivalent to Minimality for $k=1$, i.e. deciding if the sofic shift presented by a determinstic presentation has a 1vertex presentation, as $X_{G}=\mathcal{A}_{G}^{\mathbb{Z}}$ exactly when $X_{G}$ has a 1-vertex presentation. Thus, Minimality for $k=1$ is in P , and our results show that Minimality for $k \geq 2$ is PSPACEcomplete. (Our reduction shows hardness for $k=2$; simple modifications give $k>2$.)

Given a synchronizing determinstic presentation, SyncWord is trivial, as a synchronizing word always exists. To actually find a synchronizing word in this case, however, Algorithm 1 is not sufficient: for reducible presentations, the algorithm can fail when there are two vertices with the same follower set but no word sending them to the same vertex (e.g. $r_{1}$ and $r_{2}$ in Reduction C). Fortunately, the proof of Theorem 3.3 gives a method of constructing a word that synchronizes to any vertex: find a synchronizing word for an initial irreducible component, a word that separates it from the rest of the graph, and finally a word leading to the desired vertex. This procedure can be implemented using Algorithm 1 and Algorithm 2 using only polynomial time.

Similarly, Irreducibility is trivial for irreducible deterministic presentations, as the shift is guaranteed to be irreducible. For a synchronizing deterministic presentation $G$, Irreducibility can be decided by testing whether $G$ is irreducible. In particular, if $X_{G}$ is irreducible, then by Theorem 2.4, the subgraph induced by all the synchronizing vertices is irreducible; as every vertex is synchronizing, $G$ is therefore irreducible.

The problem $\exists$ SDP is also trivial for irreducible deterministic presentations, since every irreducible sofic shift has a synchronizing deterministic presentation by Theorem 2.4. One can compute this synchronizing deterministic presentation in polynomial time by simply computing the follower-separation $G / \sim$. The presentation $G / \sim$ is irreducible as $G$ is, so every vertex in $G / \sim$ is reachable from every other vertex. As every followerseparated deterministic presentation has a synchronizing word, by irreducibility, one can extend this word to one that synchronizes to any other vertex.

For deterministic presentations in general, all the problems in Table 2, with the exception of SYNCWORD, remain PSPACE-complete when restricted to follower-separated instances. The reason is those problems ask a question about the sofic shift a given input presents, and follower-separation of an input is a polynomial-time operation that preserves the sofic shift it presents. For example, given presentations $G$ and $H$, deciding if $\mathrm{X}_{G}=\mathrm{X}_{H}$ is equivalent to deciding if $\mathrm{X}_{G / \sim}=\mathrm{X}_{H / \sim}$, as $\mathrm{X}_{G}=\mathrm{X}_{G / \sim}$ and $\mathrm{X}_{H}=\mathrm{X}_{H / \sim}$ 。

### 6.2 Open problems

Aside from long-standing open problems like the decidability of conjugacy for sofic shifts, our work suggests several interesting open questions pertaining to the size of various objects. For deterministic presentations in general, the shortest synchronizing word in a presentation can be exponentially large. However, for follower-separated determinsitic presentations, the shortest synchronizing word has at most cubic length with respect
to the number of vertices; for a follower-separated input to Algorithm 1, the algorithm always finds a synchronizing word, and one can easily see that the word returned must be at most cubic length. Actually, by Exercise 3.4.10 of Lind and Marcus [17], one can see that upper bound can be improved to $n(n-1)$, where $n$ is the number of vertices in the presentation. To our knowledge, it is open whether this bound is tight.

For a shift space $X$, define the minimum step of $X$ to be the minimum $M$ such that $X$ is $M$-step. By Jonoska [12], every SFT $X$ has a synchronizing deterministic presentation. Let $s(X)$ denote the number of vertices the minimal synchronizing deterministic presentation of $X$. What is the relationship between $s(X)$ and the minimum step of $X$ ? By Remark 3.6, we know that the minimum step of an SFT $X$ is $O\left(s(X)^{2}\right)$. To our knowledge, it is also open whether this bound it tight. One lower bound arises from the family of run-length limited shifts $\left\{X_{n}\right\}$, which have minimum step $\Omega\left(s\left(X_{n}\right)\right)$. We can repeat the same question for the size of a minimal deterministic presentation. Let $s_{d}(X)$ denote the number of vertices in a minimal deterministic presentation of $X$. What is the relationship between $s_{d}(X)$ and the minimum step of $X$ ? In Appendix A, we generalize Remark 3.6 to deterministic presentations in general as Proposition A.1: for a determinstic presentation $G, \mathrm{X}_{G}$ is an SFT if and only if it is $2^{2\left|Q_{G}\right| \text {-step, which implies that the minimum step of an }}$ SFT $X$ is $2^{O\left(s_{d}(X)\right)}$. Again, it is open whether this bound is tight.

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## Appendix A Problems in PSPACE

Here, we show that all the problems in Table 1 are in PSPACE. We will rely heavily on Savitch's theorem: if there is a nondeterministic polynomial-space algorithm for a decision problem, then there is a (deterministic) polynomial-space algorithm for it as well [3].

Let $G$ be a deterministic presentation. In general, for the decision problems we work with, given a word $w$, we usually want to know the value of $Q_{G} \cdot w$. In particular, we have the correspondence $Q_{G} \cdot w \neq \varnothing$ if and only if $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$. In fact, $q \in Q_{G} \cdot w$ if and only if there is path labeled $w$ ending at that $q$. Note the asymmetry of information here: the set of vertices in $G$ such that there is a path labeled $w$ starting at that vertex is not encoded in $Q_{G} \cdot w$. A situation arises because of this asymmetry when designing polynomial-space algorithms for $\exists$ SDP and SFT when only using the transition action: one must deduce $Q_{G} \cdot u w$ given only $Q_{G} \cdot w$ and $u$, but not $w$, a problem which is generally ill-posed.

To fix this asymmetry, we introduce the action of a word. The action of a word $w$ in $G$ is a binary relation $\llbracket w \rrbracket_{G}$ on the vertices of $G$ such that $(p, q) \in \llbracket w \rrbracket_{G}$ if and only if there is a path labeled $w$ from $p$ to $q$. In other words,

$$
\llbracket w \rrbracket_{G} \triangleq\left\{(p, q) \in Q^{2}: w \in F_{G}(p) \text { and } p \cdot w=q\right\} .
$$

Note that $Q_{G} \cdot w=\left\{q \in Q: \exists p \in Q,(p, q) \in \llbracket w \rrbracket_{G}\right\}$, so the action of a word still encodes the transition action, but also includes more information. In particular, there is a path labled $w$ ending at $q$ if and only if there is a vertex $p$ with $(p, q) \in \llbracket w \rrbracket_{G}$, and there is a path labeled $w$ starting at $p$ if and only if there is a vertex $q$ with $(p, q) \in \llbracket w \rrbracket_{G}$. In fact, we have $\llbracket w \rrbracket_{G} \neq \varnothing$ if and only if $w \in \mathcal{B}\left(X_{G}\right)$. Observe that $\llbracket \epsilon \rrbracket_{G}=\left\{(q, q): q \in Q_{G}\right\}$.

Just as the transition action had a nice algebraic behvaior via the equation $S \cdot u v=$ $(S \cdot u) \cdot v$, there is an analagous equation for actions involving the relational composition operation. For binary relations $R, S \subseteq Q^{2}$, define the relational composition $R$; $S$ (pronounced $R$ then $S$ ) as

$$
R ; S \triangleq\left\{(p, r) \in Q^{2}: \exists q \in Q,(p, q) \in R \text { and }(q, r) \in S\right\}
$$

One can verify that $(p, q) \in \llbracket u \rrbracket_{G} ; \llbracket v \rrbracket_{G}$ if and only if there is a path labeled $u v$ from $p$ to $q$. This implies that for all $u, v \in \mathcal{A}_{G}^{*}$, we have $\llbracket u v \rrbracket_{G}=\llbracket u \rrbracket_{G} ; \llbracket v \rrbracket_{G}$. For example, we can deduce that $\llbracket \epsilon \rrbracket_{G}$ acts as an identity: $\llbracket w \rrbracket_{G} ; \llbracket \epsilon \rrbracket_{G}=\llbracket w \epsilon \rrbracket_{G}=\llbracket w \rrbracket_{G}$. Algebraically, we can summarize that $\llbracket \cdot \rrbracket_{G}$ is a semigroup morphism from $\mathcal{A}_{G}^{*}$ that recognizes $\mathcal{B}\left(\mathrm{X}_{G}\right)$ [20].

We first show that SUBSHIFT is PSPACE by giving a nondeterministic polynomial-time algorithm for the complement. Given a deterministic presentations $G$ and $H$, deciding $\mathrm{X}_{G} \nsubseteq \mathrm{X}_{H}$ is equivalent to deciding if there exists a word $w$ with $w \in \mathcal{B}\left(\mathrm{X}_{G}\right)$ and $w \notin$ $\mathcal{B}\left(\mathrm{X}_{H}\right)$. Using $\llbracket \cdot \rrbracket_{G}$, this is equivalent to deciding if there exists a word $w$ with $\llbracket w \rrbracket_{G} \neq \varnothing$ and $\llbracket w \rrbracket_{H}=\varnothing$. The following nondeterministic algorithm decides the latter predicate: initialize a relation $R$ to $\llbracket \epsilon \rrbracket_{G}$ and a relation $S$ to $\llbracket \epsilon \rrbracket_{H}$, and then repeat the following forever: if $R \neq \varnothing$ and $S=\varnothing$, then return true; otherwise, nondeterministically choose some $a \in \mathcal{A}_{G} \cup \mathcal{A}_{H}$ and update $R$ to $R ; \llbracket a \rrbracket_{G}$ and $S$ to $S ; \llbracket a \rrbracket_{H}$.

The size of $R$ and $S$ are polynomial with respect to the size of $G$, so the algorithm is a nondeterminstic polynomial-space algorithm. ${ }^{7}$ Thus, by Savitch's theorem, SUBSHIFT is in PSPACE. Membership of Equality in PSPACE follows, by testing both $\mathrm{X}_{G} \subseteq \mathrm{X}_{H}$ and $\mathrm{X}_{H} \subseteq \mathrm{X}_{G}$, using the polynomial-space algorithm for SUBSHIFT twice.

Next, we have that Minimality is in PSPACE: given a deterministic presentation $G$ and a positive integer $k$, when $\left|Q_{G}\right| \leq k$, we can always admit a presentation of $X_{G}$ with $k$ vertices by adding superfluous vertices to $G$. In the case of $\left|Q_{G}\right|>k$, we can nondeterministically guess a presentation with $k$ vertices (whose size is polynomially bounded as $\left|Q_{G}\right|>k$ ) and use the polynomial-space algorithm for EQUALITY to determine if our guess is a presentation of $X_{G}$.

We also have that Irreducibility is in PSPACE: given a follower-separated deterministic presentation $G$, using the polynomial-space algorithm for EQUALITY, we can test if any terminal irreducible component presents $X_{G}$. By Theorem 2.4, such a terminal irreducible component exists if and only if $X_{G}$ is irreducible.

The argument for $\exists$ SDP is more complex. We will break the algorithm into nondeterministic subprocedures, which each perform a particular test. We can determinize all three with Savitch's theorem, allowing us to use them in further subprocedures. Let $G$ be a deterministic presentation, and let $R \subseteq Q_{G}^{2}$ be a binary relation.

- We say $R$ is an action if there is a word $w$ with $\llbracket w \rrbracket_{G}=R$. We denote the set of actions as $\llbracket \mathcal{A}_{G}^{*} \rrbracket$. A simple nondeterministic polynomial-space procedure to test if $R$ is an action can be implemented by initializing a relation $S$ as $\llbracket \epsilon \rrbracket_{G}$, and in a loop forever: if $S=R$, then return true; otherwise, nondeterminically choose $a \in \mathcal{A}_{G}$ and update $S$ to $S ; \llbracket a \rrbracket_{G}$.
- We say $R$ is intrinsically synchronizing if for every $S, T \in \llbracket \mathcal{A}_{G}^{*} \rrbracket, S ; R \neq \varnothing$ and $R ; T \neq \varnothing$ imply $S ; R ; T \neq \varnothing$. One can verify that for a word $w \in \mathcal{B}\left(\mathrm{X}_{G}\right), w$ is intrinsically synchronizing for $\mathrm{X}_{G}$ if and only if $\llbracket w \rrbracket_{G}$ is intrinsically synchronizing. A nondeterministic polynomial-space procedure to test if $R$ is not intrinsically synchronizing can be implemented by nondeterministically choosing $S, T \subseteq Q^{2}$, and

[^4]testing the following four predicates: (i) $S$ and $T$ are actions, (ii) $S ; R \neq \varnothing$, (iii) $R ; T \neq \varnothing$, and (iv) $S ; R ; T=\varnothing$. If all the tests were true, then return true.

- We say $R$ is preceded by an intrinsically synchronizing word if there is some $S \in$ $\llbracket \mathcal{A}_{G}^{*} \rrbracket$ that is intrinsically synchronizing and $S ; R \neq \varnothing$. One can verify that for a word $w$, there is a word $u$ that is intrinsically synchronizing for $\mathrm{X}_{G}$ with $w \in F_{\mathrm{X}_{G}}(u)$ if and only if $\llbracket w \rrbracket_{G}$ is preceded by an intrinsically synchronizing word. A nondeterministic polynomial-space procedure to test if $R$ is preceded by an intrinsically synchronizing word can be implemented by nondeterministically choose $S \subseteq Q^{2}$, and testing three predicates: (i) $S$ is an action, (ii) $S$ is intrinsically synchronizing, and (iii) $S ; R \neq \varnothing$. If all the tests were true, then return true.

With these definitions and by Theorem 2.3, one can verify that $X_{G}$ has a synchronizing determinsitic presentation if and only if for every $R \in \llbracket \mathcal{A}_{G}^{*} \rrbracket$ with $R \neq \varnothing, R$ is preceded by an intrinsically synchronizing word. Using our subprocedures, a nondeterministic polynomial-space procedure to test if $X_{G}$ does not have a synchronizing deterministic presentation can be implemented by nondeterministically choosing $R \subseteq Q^{2}$ and testing whether (i) $R$ is an action, (ii) $R \neq \varnothing$, and (iii) $R$ is not preceded by an intrinsically synchronizing word.

Finally, we show SFt is in PSPACE. Recall from Remark 3.6 that for a followerseparated synchronizing deterministic presentation $G, \mathrm{X}_{G}$ is an SFT if and only if it is $\left(\left|Q_{G}\right|^{2}-\left|Q_{G}\right|\right)$-step. To decide SFT in polynomial space, we first generalize this characterization to when $G$ is not necessarily follower-separated and synchronizing.

Proposition A.1. Let $G$ be a deterministic presentation. Then, $X_{G}$ is an SFT if and only if it is $2^{2\left|Q_{G}\right|}$-step.

Proof. If $X_{G}$ is $2^{2\left|Q_{G}\right|}$-step, then it is an SFT. Conversely, suppose $X_{G}$ is an SFT. Then, by Jonoska [12, Corollary 5.4, Proposition 6.2], $\mathrm{X}_{G}$ has a follower-separated synchronizing deterministic presentation, and it has at most $2^{\left|Q_{G}\right|}$ vertices. Thus, by Remark 3.6, $\mathrm{X}_{G}$ is $M$-step for some $M \leq\left(2^{\left|Q_{G}\right|}\right)^{2}-\left(2^{\left|Q_{G}\right|}\right)$. Since any shift space that is $M$-step is also $M^{\prime}$-step for every $M^{\prime} \geq M$, then $X_{G}$ must be $2^{2\left|Q_{G}\right|}$-step.

A nondeterministic polynomial-space procedure to test if $X_{G}$ is not $2^{2\left|Q_{G}\right|}$-step can be implemented by initializing a counter with $2\left|Q_{G}\right|$ bits to 0 and initializing a relation $R$ to $\llbracket \epsilon \rrbracket_{G}$. Then, use the counter to repeat the following $2^{2\left|Q_{G}\right|}$ times: nondeterministically choose $a \in \mathcal{A}_{G}$ and update $R$ to $R ; \llbracket a \rrbracket_{G}$. After the loop, nondeterministically choose $S \subseteq Q^{2}$ and test the following three predicates: (i) $S$ is an action, (ii) $R$; $S \neq \varnothing$, and (iii) $R ; S$ is not intrinsically synchronizing. If all the tests were true, then return true.

## Appendix B Intersection Construction

To prove Theorem 5.2, we adapt the construction from Ang [1]. We construct a family of DFAs $M_{i, k}$ and show that they satisfy the following.

Theorem B.1. For every $k \geq 0$, the language $\bigcap_{i=0}^{k} L\left(M_{i, k}\right)$ is nonempty and the minimum length of a word in $\bigcap_{i=0}^{k} L\left(M_{i, k}\right)$ is $2^{k}$.

We will define $M_{i, k}$ for $i \geq 0$ and $k \geq 0$. Each $M_{i, k}$ has the state set $Q \triangleq\left\{q_{0}, q_{1}, q^{*}\right\}$ and are defined over the alphabet $\{0,1, \ldots, k\}$. The transition function of $M_{i, k}$ is denoted $\delta_{i, k}$ and is defined as $\delta_{i}$ restricted to the domain $Q \times\{0,1, \ldots, k\}$, where $\delta_{i}: Q \times \mathbb{N} \rightarrow Q$ is defined as follows. For $j \geq 0$, we define

$$
\delta_{0}\left(q_{0}, j\right) \triangleq\left\{\begin{array} { l l } 
{ q _ { 1 } } & { \text { if } j = 0 } \\
{ q ^ { * } } & { \text { otherwise } }
\end{array} \quad \delta _ { 0 } ( q _ { 1 } , j ) \triangleq \left\{\begin{array}{ll}
q_{1} & \text { if } j \neq 0 \\
q^{*} & \text { otherwise }
\end{array} \quad \delta_{0}\left(q^{*}, j\right) \triangleq q^{*}\right.\right.
$$

For $i \geq 1$ and $j \geq 0$, we define

$$
\delta_{i}\left(q_{0}, j\right) \triangleq\left\{\begin{array} { l l } 
{ q _ { 0 } } & { \text { if } j > i } \\
{ q _ { 1 } } & { \text { if } j < i } \\
{ q ^ { * } } & { \text { otherwise } }
\end{array} \quad \delta _ { i } ( q _ { 1 } , j ) \triangleq \left\{\begin{array}{ll}
q_{0} & \text { if } j=i \\
q_{1} & \text { if } j>i \\
q^{*} & \text { otherwise }
\end{array} \quad \delta_{i}\left(q^{*}, j\right) \triangleq q^{*}\right.\right.
$$

We set the initial state of $M_{i, k}$ to $q_{0}$ for every $i \geq 0$ and $k \geq 0$. We set the final state of $M_{0, k}$ to just $q_{1}$, and for $i \geq 1$, we set the final state of $M_{i, k}$ to just $q_{0}$. Figure 7 depicts an example of the construction. Essentially, for $i \geq 1$, we have that $\delta_{i}$ acts on $Q$ in the following way: $q^{*}$ is a sink state, so no word that under that action of $\delta_{i}$ that visits the sink state will be in the language of $M_{i, k}$; for $j>i$, we have that $q \mapsto \delta_{i}(q, j)$ is the identity function; for $j \leq i$, the only transition out of $q_{0}$ that does not lead to $q^{*}$ is when $j \neq i$, and conversely, the only transition out of $q_{1}$ that does not lead to $q^{*}$ is when $j=i$.

Define $h_{k}:\{0,1, \ldots, k\}^{*} \rightarrow\{0,1, \ldots, k\}^{*}$ to be the the string homomorphism $h_{k}: j \mapsto$ $j k$ that inserts the symbol $k$ after every symbol from the input. Define $w_{0} \triangleq 0$ and $w_{k+1} \triangleq h_{k+1}\left(w_{k}\right)$ for $k \geq 0$. Once can show by induction for $k \geq 0$ that $w_{k} \in \bigcap_{i=0}^{k} L\left(M_{i, k}\right)$ and that $\left|w_{k}\right|=2^{k}$. Thus, the minimum length of a word in $\bigcap_{i=0}^{k} L\left(M_{i, k}\right)$ is at most $2^{k}$. In fact, as we show next, any word in $\bigcap_{i=0}^{k} L\left(M_{i, k}\right)$ must have length at least $2^{k}$, so $w_{k}$ achieves the minimal length.
Proposition B.2. For all $k \geq 0$, if $w \in \bigcap_{i=0}^{k} L\left(M_{i, k}\right)$, then $|w| \geq 2^{k}$.
Proof. We show this by induction. For $k=0$, any word in $L\left(M_{0,0}\right)$ must have length at least $1=2^{0}$, so the proposition holds for $k=0$.

Now, suppose the proposition holds at a given $k \geq 0$. Let $w \in \bigcap_{i=1}^{k+1} L\left(M_{i, k+1}\right)$. Note that when $i \leq k$, then $\delta_{i, k+1}(q, k+1)=q$ for all $q \in Q$. Thus, if we let $w^{\prime}$ denote the word with every occurance of $k+1$ in $w$ removed, then for $i \in\{0,1, \ldots, k\}$, we have $\delta_{i, k+1}\left(q, w^{\prime}\right)=\delta_{i, k+1}(q, w)$, which implies that $w^{\prime} \in \bigcap_{i=1}^{k} L\left(M_{i, k+1}\right)$. In fact, as $w^{\prime} \in\{0,1, \ldots, k\}^{*}$ and as $\delta_{i, k+1}(q, j)=\delta_{i, k}(q, j)$ when $j \in\{0,1, \ldots, k\}$, we have $w^{\prime} \in$ $\bigcap_{i=0}^{k} L\left(M_{i, k}\right)$. By our induction hypothesis, we have that $\left|w^{\prime}\right| \geq 2^{k}$.

Note that as $w \in L\left(M_{k+1, k+1}\right)$, then $w$ must alternate between some symbol in $\{0,1, \ldots, k\}$ and $k+1$, which implies number of $k+1$ 's in $w$ must equal the number of $\{0,1, \ldots, k\}$ 's. Thus, as $\left|w^{\prime}\right|$ measures the number of $\{0,1, \ldots, k\}$ 's in $w$, the number of $k+1$ 's in $w$ must


Figure 7: $M_{i, k}$ for $i \in\{0,1,2,3\}$ and $k=3$. The transitions to $q^{*}$ are not depicted.
be at least $2^{k}$. Putting this together, we have $|w| \geq 2 \cdot 2^{k}=2^{k+1}$, so the proposition holds at $k+1$.


[^0]:    ${ }^{1}$ We define an FA the same way some authors define a nondeterminisc finite automaton (NFA); we use "FA" to avoid confusion when we consider deterministic finite automata (DFAs) as a subset of FAs.
    ${ }^{2}$ The languages of sofic shifts are exactly the languages which are factorial, prolongable, and regular. [17, Proposition 1.3.4]

[^1]:    ${ }^{3}$ Under this definition, the empty path $\epsilon$, i.e., the empty sequence of edges, is a valid path in $G$ but one where $i_{G}$ and $t_{G}$ are undefined. To rectify this omission, for every vertex $q \in Q_{G}$, we declare $\epsilon_{q}$ to be a path such that the length of $\epsilon_{q}$ is $0, \epsilon_{q}$ starts and ends at $q$, and $\mathcal{L}\left(\epsilon_{q}\right)=\epsilon$.

[^2]:    ${ }^{4}$ Recall that a presentation refers to an essential labeled graph, a necessary condition here, as these statements do not necessarily hold if $G$ is nonessential.
    ${ }^{5}$ We remind the reader that our notion of synchronization is different from "careful" sychronization from automata theory; see "Synchronizing words" in § 1.1.

[^3]:    ${ }^{6}$ In fact, there is another hardness result proved by the previous theorem: SUBSHIFT is PSPACE-hard even when the first argument is fixed. That is, for each deterministic presentation $H$, it is PSPACE-hard to decide when given a deterministic presentation $G$ whether $X_{H} \subseteq X_{G}$.

[^4]:    ${ }^{7}$ This algorithm does not halt, but in principle, any space-bounded algorithm can be modified to halt with a logarithmic overhead.

