Computational complexity of problems for deterministic presentations of sofic shifts

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Abstract

Sofic shifts are symbolic dynamical systems defined by the set of bi-infinite sequences on an edge-labeled directed graph, called a presentation. We study the computational complexity of an array of natural decision problems about presentations of sofic shifts, such as whether a given graph presents a shift of finite type, or an irreducible shift; whether one graph presents a subshift of another; and whether a given presentation is minimal, or has a synchronizing word. Leveraging connections to automata theory, we first observe that these problems are all decidable in polynomial time when the given presentation is irreducible (strongly connected), via algorithms both known and novel to this work. For the general (reducible) case, however, we show they are all PSPACE-complete. All but one of these problems (subshift) remain polynomial-time solvable when restricting to synchronizing deterministic presentations. We also study the size of synchronizing words and synchronizing deterministic presentations.

Keywords: sofic shifts; symbolic dynamics; computational complexity; automata theory

1 Introduction

Symbolic dynamics in dimension one is the study of shift spaces, which are topological dynamical systems given by "shifting" bi-infinite sequences of symbols. Sofic shifts are shift spaces whose points are given by the label sequences for bi-infinite walks in a labeled graph, called a presentation. As they characterize the factors of subshifts of finite type (SFTs), sofic shifts have fundamental importance in symbolic dynamics. They also have an array of applications both within and outside of dynamical systems, including billiards, ergodic theory, continuous dynamics, and information theory, automata theory, and matrix theory [19]. In particular, one motivation for the present work is the set of computational problems that arise in application to continuous maps via Conley index theory [8, 15, 16].

Problem	Input	Decision
IRREDUCIBILITY	G	Is X_G irreducible?
Equality	G,H	Is $X_G = X_H$?
Subshift	G,H	Is $X_G \subseteq X_H$?
Sft	G	Is X_G an SFT?
ЗSdp	G	Does X_G have an SDP?
MINIMALITY	G, k	Does X_G have a <i>k</i> -vertex deterministic presentation?
SyncWord	G	Does G have a synchronizing word?

Table 1: Natural decision problems for sofic shifts. We show that all are PSPACE-hard in the general case; see Table 2 for an overview of our results. For the inputs to the problems, G, H are deterministic presentations, and k is a positive integer.

Despite their fundamental importance, however, many basic questions about the computational complexity of sofic shifts remain open. In Table 1, we give seven natural decision problems, many of which also arise frequently in applications. For example, given a labeled graph, does it present an SFT? Does a given reducible presentation actually present an irreducible sofic shift? Do two given labeled graphs present the same shift? For special cases, such when the given presentations are irreducible, some of these problems are known to be in P, i.e., they admit a polynomial-time algorithm. For the general case, however, only the complexity of SYNCWORD is known: it is PSPACE-complete to determine whether a given deterministic presentation has a synchronizing word [4].

In this work, we resolve the complexity of the remaining six problems, showing that they are all PSPACE-complete in the general case. We also study two special cases, sofic shifts given by deterministic presentations which are either irreducible or synchronizing. For these cases, the problems are generally in P. In fact, the only exception is SUBSHIFT for synchronizing deterministic presentations, which is again PSPACE-complete. Our reductions also shed light on the size of the smallest synchronizing word and minimal synchronizing deterministic presentation, namely that both can be exponentially large in the given presentation. These results are significant for understanding sofic shifts in their own right, as well as relevant for applications.

1.1 Relation to the literature

Conjugacy Absent from our list of problems is arguably the most important: deciding whether two sofic shifts are isomorphic, or *conjugate*. In general, the decidability of this conjugacy problem is open, even when restricting to the class of SFTs. Verifying the natural certificates of conjugacy, known as sliding block codes, is computable in polynomial time for SFTs given by vertex shifts, and deciding if there is a certificate of a fixed size is Gl-hard, meaning there is a polynomial-time reduction from the graph isomorphism problem to that problem [22]. One can partially decide nonconjugacy via conjugacy invariants, i.e., properties which isomorphic objects share. For dynamical systems, a set of

invariants related to the connectivity of the state space are being (topologically) transitive, mixing, and nonwandering. For shift spaces, topological transitivity is also known as irreducibility. The class of SFTs is contained with the class of sofic shifts, and the class of SFTs is closed under conjugacy; thus, for sofic shifts, being an SFT is a conjugacy invariant. Conen and Paul [6] show that all the above invariants are decidable. We show that for sofic shifts given by deterministic presentations, deciding if they are irreducible or an SFT is PSPACE-complete. The PSPACE-hardness of being mixing or nondwandering also follows immediately from our reduction. Interestingly, it also follows from our reduction that deciding conjugacy of sofic shifts is at least PSPACE-hard.

Automata theory Sofic shifts have a close relationship with automata theory. A finite automaton (FA) roughly corresponds to an edge-labeled directed graph with a set of initial and accepting states. ¹ The language of an FA is the set of words labeled by a path starting at an initial state and ending at an accepting state. The languages described by finite automata are known as *regular* languages. Similarly, we define the language of a shift space to be the set of finite words appearing in points of the shift space.

A basic result about shift spaces says that shift spaces are determined by their languages: two shift spaces are equal if and only if their languages are equal. Furthermore, the languages of sofic shifts are regular in the above sense.² More specifically, by interpreting a presentation of a sofic shift as an FA where every state is both initial and final, then the language of a presentation (interpreted as an FA) is the same as the language of the sofic shift it presents. These connections allows us to use automata-theoretic tools to study sofic shifts.

A deterministic finite automaton (DFA) over an alphabet Σ is an FA with a single initial state, such that at each state q and for each $a \in \Sigma$, there is exactly one edge leaving q labeled a. A deterministic presentation of a sofic shift, when thought of as an FA, has a similar definition: a presentation is *deterministic* if for each $a \in \Sigma$, there is *at most* one edge leaving that state labeled a. Comparing the languages of two DFAs (i.e. whether their languages are equal, or if one is a subset of the other) is computable in polynomial time. However, for FAs in general, the same problem is PSPACE-complete [18]. Comparing the languages of presentations in general is also PSPACE-complete, as a corollary of results from Czeizler and Kari [7]. The question that remains is therefore the complexity of comparing languages of deterministic presentations. It is likely known that comparing languages of irreducible (i.e. strongly connected) deterministic presentations is computable in polynomial time; in Section 3.2, we give an algorithm. For deterministic presentations in general, the scheme and general is also PSPACE-complete.

¹We define an FA the same way some authors define a *nondeterminisc finite automaton (NFA)*; we use "FA" to avoid confusion when we consider deterministic finite automata (DFAs) as a subset of FAs.

²The languages of sofic shifts are exactly the languages which are *factorial*, *prolongable*, and regular. [17, Proposition 1.3.4]

Minimization of presentations We say two FAs are *equivalent* if they have the same language. Algorithms for *minimizing* a DFA, i.e. finding an equivalent DFA with fewer states, have been well-studied. This minimization problem has nice properties: every DFA has an unique minimal equivalent DFA which can be computed in polynomial time. For FAs in general, minimization is not as nice: FAs do not necessarily have unique minimal equivalent FA [2], and deciding if there is an FA with fewer states than a given FA is PSPACE-complete [18].

As a class of FAs, irreducible deterministic presentations of sofic shifts have similar minimality properties to DFAs: every irreducible deterministic presentation has a unique minimal equivalent irreducible deterministic presentation which is computable in polynomial time [17]. (Furthermore, the property characterizing a minimal irreducible deterministic presentation in a sense is exactly the same property as a minimal DFA.) This observation leaves the question: do reducible deterministic presentations share the minimzation properties of the class of DFAs or of the class of FAs? The class of FAs arguably closest to general deterministic presentations are the *multiple-entry DFAs* (mDFAs), which are essentially DFAs with one source of nondeterminism: multiple initial states. A deterministic presentation can be made into an equivalent mDFA by by adding a sink state, thus "fully determinizing" every state, and then interpreting every state but the sink state as an initial and final state (c.f. sink vertex graph in Section 3.1). As mDFAs share the minimization properties of the class of FAs [10, 18], and general deterministic presentations do not have unique minimal equivalent deterministic presentations [12], one may suspect that minimization of general deterministic presentations is PSPACE-complete. Indeed, we show this result in Section 4.2.

Synchronizing words Another equivalent way of defining a DFA is by specifying a transition function; i.e., a function $\delta: Q \times \Sigma \rightarrow Q$ for some set of states Q and finite alphabet Σ . The transition function then naturally extends to a function $\delta: Q \times \Sigma^* \to Q$ from the states and words over an alphabet. A synchronizing word (also called a reset word) for a DFA is a word that transitions every state to a single state: w is synchronizing if $\delta(p,w) = \delta(q,w)$ for all states p and q. (Equivalently, the function $q \mapsto \delta(q,w)$ is a constant function.) A deterministic presentation can be seen as a DFA with a partially defined transition function (a function whose domain is a subset of $Q \times \Sigma$); call a DFA with a partial transition function a *partial* DFA. There are multiple ways to generalize the notion of a synchronizing word to partial DFAs, for example, a *carefully* synchronizing word [24] and a *exact* synchronizing word [23, 25]. The former is a word whose transition is defined at all states and sends all states to a single state; the latter is a word whose transition is defined at least one state and sends every state (where it is defined) to the same state. Interpreting a deterministic presentation as a partial DFA, a synchronizing word for a deterministic presentation of a sofic shift is defined as an exact synchronizing word. Note that a DFA might be called synchronizing or synchronized if it has a synchronizing word; for presentations of sofic shifts, our usage of a synchronizing presentation corresponds to that of Jonoska [12]: for every state $q \in Q$, there is a synchronizing word

that sends every state to q.

For DFAs, one can decide if a synchronizing word exists (and find one) in polynomial time via Eppstein's algorithm [9]. Independently, Travers and Crutchfield [25] gave a similar algorithm which can be used to determine if a synchronizing word exists in an irreducible deterministic presentation. In Section 3.1, we describe an algorithm to find a synchronizing word in an irreducible deterministic presentations (Algorithm 1) which combines the techniques of the previously two mentioned algorithms. We extend these techniques to subshift testing for irreducible deterministic presentations (Algorithm 2) and then use synchronizing word algorithm and subshift testing algorithm together as subprocedures for testing if a deterministic presentation is synchronizing. For deterministic presentations in general, deciding if a synchronizing word exists is PSPACE-complete, and the size of a minimum length synchronizing word may be exponentially large with respect to the number of states. While both of these facts were already implied by Berlinkov [4], for completeness we provide proofs in Sections 4.3 and 5.1, respectively.

Synchronizing deterministic presentations Jonoska [12] introduced synchronizing deterministic presentations, as defined above: for every state $q \in Q$, there is a synchronizing word that sends every state to q. The shift spaces given by synchronizing deterministic presentations slightly generalize those given by irreducible deterministic presentations while retaining serveral nice properties. For example, synchronizing deterministic presentations share the minimization properties of the class of DFAs: every synchronizing deterministic presentation that is computable in polynomial time. For irreducible deterministic presentations, it is known that EQUALITY and SFT are in P. We show that the algorithms for the irreducible case generalize cleanly to the synchronizing case, implying that EQUALITY and SFT are in P for synchronizing deterministic presentations (Sections 3.5 and 3.4). Interestingly, although SUBSHIFT is in P for irreducible deterministic presentations, SUBSHIFT is PSPACE-complete for synchronizing deterministic presentations (Remark 4.12).

Not all sofic shifts have synchronizing deterministic presentations. In fact, we show that the problem of deciding whether a sofic shift has a synchronizing deterministic presentation, \exists SDP, is PSPACE-complete (Section 4.1). For irreducible sofic shifts, minimal synchronizing deterministic presentations and minimal deterministic presentations are the same. For reducible sofic shifts, however, these minimal presentations are not necessarily the same. Indeed, we show that a minimal synchronizing deterministic one can be exponentially larger than a minimal deterministic one (Section 5.2).

2 Background and Setting

2.1 Shift spaces and presentations

Here, we introduce basic notions about shift spaces, sofic shifts, and presentations. Definitions and notation follow Lind and Marcus [17].

Let Σ be a finite set. We refer to a finite sequence as a *word*, and we denote by Σ^* the set of words over Σ . A subset of Σ^* is called a *language*. For $w \in \Sigma^*$, we denote the length of w as |w|. We denote the empty word as ϵ , and note that $|\epsilon| = 0$ and $\epsilon \in \Sigma^*$. The *full* Σ -*shift* is the set $\Sigma^{\mathbb{Z}}$ of bi-infinite sequences over Σ . Let $x \in \Sigma^{\mathbb{Z}}$. For $i \leq j$, we denote $x_{[i,j]} \triangleq x_i x_{i+1} \dots x_j$. We say a word w appears in x if there are i and j with $x_{[i,j]} = w$. For a collection of words $\mathcal{F} \subseteq \Sigma^*$, we define $X_{\mathcal{F}} \triangleq \{x \in \Sigma^{\mathbb{Z}} : \text{no word in } \mathcal{F} \text{ appears in } x\}$. A *shift space* is a subset $X \subseteq \Sigma^{\mathbb{Z}}$ of the full Σ -shift such that there is a collection of words \mathcal{F} with $X = X_{\mathcal{F}}$. A *shift of finite type (SFT)* is a shift space $X = X_{\mathcal{F}}$ for some finite set \mathcal{F} .

For a subset $X \subseteq \Sigma^{\mathbb{Z}}$ of the full Σ -shift, we define the *language* $\mathcal{B}(X)$ of X to be the set of words that appear in some $x \in X$, i.e., $\mathcal{B}(X) \triangleq \{x_{[i,j]} : x \in X, i \leq j\}$. Shift spaces are characterized by their languages: for every shift space $X \subseteq \Sigma^{\mathbb{Z}}$, one has that $X = X_{\Sigma^* \setminus \mathcal{B}(X)}$. Thus, for shift spaces X and Y, if $\mathcal{B}(X) = \mathcal{B}(Y)$, then X = Y [17, Proposition 1.3.4]. Additionally, one can easily show inclusion is also respected: $\mathcal{B}(X) \subseteq \mathcal{B}(Y)$ if and only if $X \subseteq Y$. Finally, for $u \in \Sigma^*$, we define the *follower set* of u as the set $F_X(u) \triangleq \{w \in \Sigma^* : uw \in \mathcal{B}(X)\}$.

To define sofic shifts, we will work with edge-labeled, directed multigraphs, where self loops and multiple edges between vertices are permitted. Formally, a *labeled graph G* consists of a finite set Q of *vertices* (or *states*), a finite set E of *edges*, functions $i: E \to Q$ and $t: E \to Q$, assigning each edge an *initial* and *terminal* vertex, and a function $\mathcal{L}: E \to \Sigma$, assigning each edge a *label*. For a given graph G, the symbols Q_G , E_G , i_G , t_G , and \mathcal{L}_G will refer to the above sets and functions for the graph G. Additionally, we define the *alphabet of G* as the set \mathcal{A}_G of labels appearing on edges in G (i.e. $\mathcal{A}_G \triangleq \mathcal{L}_G(E_G)$). When the labels are irrelevant, we will sometimes call a labeled graph a graph.

If *G* is a labeled graph and $P \subseteq Q$ is a subset of vertices, then the *subgraph induced by P* (in *G*) is the labeled graph *H* given by $Q_H, E_H, i_H, t_H, \mathcal{L}_H$, where: $Q_H \triangleq P$; $E_H \triangleq \{e \in E_G : i_G(e) \in P, t_G(e) \in P\}$; i_H, t_H are i_G, t_G restricted to E_H ; and \mathcal{L}_H is \mathcal{L}_G restricted to Q_H .

Let *G* be a labeled graph. A *path* in *G* is a finite sequence $\pi = e_1 \dots e_n$ of edges with $t_G(e_i) = i_G(e_{i+1})$ for $i = 1, \dots, n-1$. We assign $i_G(\pi) \triangleq e_1$ and $t_G(\pi) \triangleq e_n$, and say π starts at $i_G(\pi)$ and ends at $t_G(\pi)$. Additionally, we assign $\mathcal{L}_G(\pi) \triangleq \mathcal{L}_G(e_1) \dots \mathcal{L}_G(e_n)$, and say $\mathcal{L}_G(\pi)$ is the *label* of π . Similarly, a *bi-infinite path* in *G* is a bi-infinite sequence $x \in E_G^{\mathbb{Z}}$ of edges with $t_G(x_i) = i_G(x_{i+1})$ for all $i \in \mathbb{Z}$. For a bi-infinite path x in *G*, we assign the *label* of x as the bi-infinite sequence $\mathcal{L}_G(x) \in \mathcal{A}_G^{\mathbb{Z}}$ with $\mathcal{L}_G(x)_i \triangleq \mathcal{L}_G(x_i)$. For a vertex, q we define the *follower set* of q in *G* as the set $F_G(q) \triangleq \{\mathcal{L}_G(\pi) : \pi$ is a path in *G* starting at $q\}$.³

We now have the necessary definitions to define sofic shifts. For a labeled graph G, we assign it the shift space

 $X_G \triangleq \{ \mathcal{L}_G(x) : x \text{ is a bi-infinite path in } G \}.$

A *sofic shift* is a shift space X such that $X = X_G$ for some labeled graph G, and we say G is a *presentation* of X and that X is the sofic shift *presented* by G. For a proof that X_G is actually a shift space, see Lind and Marcus [17, Theorem 3.1.4].

³Under this definition, the empty path ϵ , i.e., the empty sequence of edges, is a valid path in *G* but one where i_G and t_G are undefined. To rectify this omission, for every vertex $q \in Q_G$, we declare ϵ_q to be a path such that the length of ϵ_q is 0, ϵ_q starts and ends at q, and $\mathcal{L}(\epsilon_q) = \epsilon$.

Let *G* be a labeled graph. For a given path π in *G*, it could be the case that $\mathcal{L}_G(\pi)$ is not in the language of X_G , as π might not appear in a bi-infinite path. We say a vertex *q* in *G* is *stranded* if there is no edge starting at *q* or if there is no edge ending at *q*. If no vertex is stranded, then we say *G* is *essential*. When *G* is essential, every path appears in a biinfinite path, so $\mathcal{L}_G(\pi)$ is always in the language of X_G . If one removes a stranded vertex from a presentation, the sofic shift presented by the resulting presentation is the same as the one presented by the original presentation. Thus, every sofic shift has an essential presentation, which can be obtained by iteratively removing stranded vertices until no more exist [17, Proposition 2.2.10]. We therefore make the following convention: a *presentation* refers to an essential labeled graph. We will still refer to labeled graphs as being potentially nonessential; the distinction is needed for the algorithms in Section 3, where we may call our algorithms on nonessential graphs (line 7 of Algorithm 3).

Let *G* be a labeled graph. We say *G* is *deterministic* (also called *right-resolving*) if for every vertex *q* and every $a \in A_G$, there is at most one edge labeled *a* starting at *q*. If for every vertex *q* and $a \in A_G$ there is exactly one edge labeled *a* starting at *q*, we say *G* is *fully deterministic*. If *G* is deterministic, one can show by induction that for every vertex *q* and word *w*, if π is a path starting at a vertex *q* and $\mathcal{L}_G(\pi) = w$, then π is the unique path starting at *q* with $\mathcal{L}_G(\pi) = w$. This observation motivates the following definition: if there is some path π starting at *q* with $\mathcal{L}_G(\pi) = w$, we define $q \cdot w \triangleq t(\pi)$, otherwise, if there is no such π , we leave $q \cdot w$ undefined. We call \cdot the *transition action*. Because of determinism, the transition action is a well-defined partial operation between the vertices of *G* and words over the alphabet of *G*, and $q \cdot w$ is defined if and only if $w \in F_G(q)$. The transition action satisfies the following useful properties, for any state $q \in Q_G$ and words $u, v \in \mathcal{A}_G^*$:

- (i) $uv \in F_G(q)$ if and only if $u \in F_G(q)$ and $v \in F_G(q \cdot u)$;
- (ii) if $uv \in F_G(q)$, then $q \cdot uv = (q \cdot u) \cdot v$.

When *G* is fully deterministic, $q \cdot w$ is defined for all q and $w \in \mathcal{A}_{G}^{*}$, as then $F_{G}(q) = \mathcal{A}_{G}^{*}$.

The transition action naturally extends to a total operation between subsets of vertices of *G* and words over the alphabet of *G*: for each subset $S \subseteq Q_G$ of vertices and word *w*, we set $S \cdot w \triangleq \{q \cdot w : q \in S, w \in F_G(q)\}$. Every sofic shift therefore has a deterministic presentation [17, Theorem 3.3.2], using the same idea as the subset construction from automata theory [14]. By definition, the transition action on subsets is monotonic and distributes over union: $S \subseteq T$ implies $S \cdot w \subseteq T \cdot w$ and $(S \cup T) \cdot w = (S \cdot w) \cup (T \cdot w)$ for all *S*, *T*, and *w*.

2.2 Types of presentations

Let *G* be a deterministic labeled graph and let *w* be a word. We say *w* is *synchronizing* for *G* if $Q_G \cdot w = \{r\}$ for some vertex $r \in Q_G$. In this case, we say *w* synchronizes to *r* (in *G*). We say a vertex *q* is synchronizing if there is a word that synchronizes to *q*. We say *G* is synchronizing if every vertex in *G* is synchronizing. Let *X* be a shift space. An *intrinsically*

synchronizing word w for X is a word $w \in \mathcal{B}(X)$ such that whenever $uw, wv \in \mathcal{B}(X)$, then $uwv \in \mathcal{B}(X)$. If w is synchronizing for G, then w is intrinsically synchronizing for X_G, but the converse need not hold; see Lemma 2.2.

Let *X* be a shift space. We say *X* is *irreducible* if for every $u, v \in \mathcal{B}(X)$, there is a word *w* such that $uwv \in \mathcal{B}(X)$; if *X* is not irreducible, then we say *X* is *reducible*. For a graph *G*, we say *G* is *irreducible* (or *strongly connected*) if for every pair of vertices *p* and *q*, there is a path starting at *p* and ending at *q*. If *G* is not irreducible, we say *G* is reducible. One can easily show that if *G* is irreducible, then X_G is irreducible. However, X_G may be irreducible even if *G* is reducible. (See Figure 1.)

Let *G* be a graph, and let *p* and *q* be vertices in *G*. We say *q* is *reachable* from *p* if there is a path starting at *p* and ending at *q*. Under the equivalence relation where $p \approx q$ when *q* is reachable from *p* and *q* is reachable from *q*, the equivalence classes are called *irreducible components* as the subgraphs induced by them are irreducible. We say an irreducible component *C* is *initial* if whenever *q* is reachable from *p* and *q* \in *C*, then $p \in C$. Dually, we say a irreducible component *C* is *terminal* if whenever if *q* is reachable from *p* and $p \in C$, then $q \in C$.

Let *G* be a labeled graph. We say two vertices *p* and *q* in *G* are *follower-equivalent* if $F_G(p) = F_G(q)$, an equivalence relation \sim . We say *G* is *follower-separated* if no distinct pair of vertices are follower equivalent. Given a labeled graph *G*, the *follower-separation* of *G* is the the labeled graph G/\sim whose vertices are the follower-equivalence classes of *G* and with exactly one edge labeled *a* between two classes C_1 and C_2 if and only if there is an edge labeled *a* in *G* from a vertex in C_1 to a vertex in C_2 . Informally, the follower-separation of *G* collapses vertices in a given follower-equivalence class into a single vertex. The follower-separation of *G* enjoys the following properties: we have G/\sim is follower-separated and $X_G = X_{G/\sim}$; if *G* is deterministic, then G/\sim is deterministic; if *G* is essential, then G/\sim is essential [17, Lemma 3.3.8]; if *G* is synchronizing, then G/\sim is synchronizing [12, Proposition 4.3]. In particular, every sofic shift has a follower-separated, deterministic presentation.

The notion of follower-equivalence is similar to the notion of equivalent states in deterministic finite automata (DFA; see Sections 1.1 and 4). In fact, one may reduce the problem of computing follower-equivalence to computing equivalent states in DFA, as follows. Add a "sink" state to G, and edges to the sink state to make G fully deterministic (c.f. Section 3.1). Now consider the resulting graph as a DFA, with an arbitrary initial state, and where every state but the sink state is an accepting state. One can show that two states in G are follower-equivalent if and only if they are equivalent as states in the constructed DFA. Therefore, one can use Hopcroft's algorithm for state equivalence in DFAs to compute follower-equivalences in polynomial time [11].

2.3 Basic results

In this section, we discuss several useful facts which we use throughout the paper. To begin, the following statements about a deterministic presentation G establish basic re-

lationships between its transition action, follower sets F_G and F_{X_G} , the language $\mathcal{B}(X_G)$, and synchronizing words for G.⁴ The statements follow immediatly from the definitions.

Proposition 2.1. Let G be a deterministic presentation. Then, we have

- (i) $w \in \mathcal{B}(X_G)$ if and only if $Q_G \cdot w \neq \emptyset$;
- (ii) $\mathcal{B}(\mathsf{X}_G) = \bigcup_{q \in Q} F_G(q);$
- (iii) $F_{\mathsf{X}_G}(w) = \bigcup_{q \in Q \cdot w} F_G(q);$
- (iv) if *w* is synchronizes to *r* in *G*, then $F_{X_G}(w) = F_G(r)$;
- (v) if w is intrinsically synchronizing for X_G and $w \in F_{X_G}(u)$, then $F_{X_G}(uw) = F_{X_G}(w)$.

Next, we review results of Jonoska [12] about synchronizing deterministic presentations. First, we state a useful result about the correspondence between synchronizing and intrinsically synchronizing words in synchronizing deterministic presentations. For deterministic presentations in general, only the forward implication of this result holds. The following is essentially Proposition 9.5 of Jonoska [12].

Lemma 2.2. Let *G* be a follower-separated, synchronizing deterministic presentation. Then, *w* is synchronizing for *G* if and only if *w* is intrinsically synchronizing for X_G .⁵

Proof. Suppose *w* is synchronizing for *G*, and let $uw, wv \in \mathcal{B}(X_G)$. As *w* is synchronizing for *G*, by Proposition 2.1(iv), it follows that there is a vertex *r* such that $F_{X_G}(uw) = F_G(r)$ and $v \in F_G(r)$. Thus, we have $v \in F_G(uw)$ so $uwv \in \mathcal{B}(X_G)$.

Conversely, suppose *w* is intrinsically synchronizing for X_G . Let *p* and *q* be vertices in *G* with $w \in F_G(p)$ and $w \in F_G(q)$. As *G* is synchronizing, let u_p and u_q be words synchronizing to *p* and *q* in *G*, respectively. We next show that $F_G(p \cdot w) \subseteq F_G(q \cdot w)$. Let $v \in F_G(p \cdot w)$, so that $wv \in \mathcal{B}(X_G)$. As u_q synchronizes to *q* in *G* and $w \in F_G(q)$, we have $u_qw \in \mathcal{B}(X_G)$. As *w* is intrinsically synchronizing for X_G , we have $u_qwv \in \mathcal{B}(X_G)$, i.e. $Q_G \cdot u_qwv \neq \emptyset$. But as $Q_G \cdot u_qw = \{q \cdot w\}$, we have $v \in F_G(p \cdot w)$. Thus, $F_G(p \cdot w) \subseteq F_G(q \cdot w)$; moreover, the same argument swapping the roles of *p* and *q* gives $F_G(q \cdot w) \subseteq F_G(p \cdot w)$ and therefore $F_G(p \cdot w) = F_G(q \cdot w)$. As *G* is follower-separated, we conclude $p \cdot w = q \cdot w$, implying that *w* is synchronizing for *G*.

The following characterization of when a sofic shift has a synchronizing deterministic presentation is slightly modified from Theorem 8.13 and Corollary 9.6 in Jonoska [12].

Theorem 2.3. Let $X \subseteq \Sigma^{\mathbb{Z}}$ be a sofic shift. Then, *X* has a synchronizing deterministic presentation if and only if for every $u \in \mathcal{B}(X)$ there is an intrinsically synchronizing word *w* for *X* such that $u \in F_X(w)$.

⁴Recall that a presentation refers to an *essential* labeled graph, a necessary condition here, as these statements do not necessarily hold if G is nonessential.

⁵We remind the reader that our notion of synchronization is different from "careful" sychronization from automata theory; see "Synchronizing words" in § 1.1.

Proof. Let *G* be a synchronizing deterministic presentation for *X*, and let $w \in B(X)$. By Proposition 2.1(ii), there is a vertex *q* such that $w \in F_G(q)$. As *G* is synchronizing, there is a word *u* that synchronizes to *q*. By Proposition 2.1(iv), we have $F_X(u) = F_G(q)$, so $w \in F_X(u)$. By Lemma 2.2, we also have that *u* is intrinsically synchronizing for *X*.

Conversely, suppose for every $u \in \mathcal{B}(X)$ there is an intrinsically synchronizing word w for X such that $u \in F_X(w)$. Let \mathcal{C} be the collection of the follower sets of intrinsically synchronizing words for X, i.e.

$$C \triangleq \{F_X(w) : w \text{ is intrinsically synchronizing for } X\}.$$

This collection is finite since the collection of all follower sets of a sofic shift is finite [17, Theorem 3.2.10]. We will construct a synchronizing deterministic presentation G whose vertex set is C. For each $a \in \Sigma$ and $F_X(w) \in C$, if $a \in F_X(w)$, add an edge labeled a from $F_X(w)$ to $F_X(wa)$. This definition is well-defined, i.e., does not depend on the choice of w, by the following two facts, both assuming $a \in F_X(w)$: if $F_X(w) = F_X(w')$, then $F_X(wa) = F_X(w'a)$, and wa is intrinsically synchronizing (so that $F_X(wa) \in C$). By construction, G is deterministic. One can also establish the following properties of G: for $F_X(u) \in C$, we have $F_G(F_X(u)) = F_X(u)$, and if $w \in F_X(u)$, then $F_X(u) \cdot w = F_X(uw)$.

We next show that *G* is synchronizing. Let $F_X(w) \in C$, so that *w* is intrinsically synchronizing for *X*. We will show that *w* synchronizes to $F_X(w)$ in *G*. Let $F_X(u) \in C$, and suppose $w \in F_G(F_X(u))$. As *w* is intrinsically synchronizing and $w \in F_X(u)$, by Proposition 2.1, we have that $F_X(uw) = F_X(w)$. This implies that $F_X(u) \cdot w = F_X(uw) = F_X(w)$. Thus, for any $F_X(u) \in C$ with $w \in F_G(F_X(u))$, we have $F_X(u) \cdot w = F_X(w)$, so *w* synchronizes to $F_X(w)$ in *G*.

It remains to show $X = X_G$. By construction, the follower set of a vertex in *G* is a follower set of a word in *X*, so $\mathcal{B}(X_G) \subseteq \mathcal{B}(X)$. Conversely, let $u \in \mathcal{B}(X)$. By our initial assumption, there is an intrinsically synchronizing word *w* for *X* with $u \in F_X(w)$. As $F_G(F_X(w)) = F_X(w)$, we have $u \in F_G(F_X(w))$, i.e., there is a vertex *q* in *G* such that $u \in F_G(q) \subseteq \mathcal{B}(X_G)$. Thus, we have $\mathcal{B}(X) \subseteq \mathcal{B}(X_G)$ and consequently $\mathcal{B}(X) = \mathcal{B}(X_G)$. \Box

The next result says when the sofic shift presented by a (possibly reducible) deterministic presentation is irreducible. The forward implication of this result follows from Lemma 6.4 of Jonoska [12], and the reverse implication follows immediately from the irreducibility of H.

Theorem 2.4. Let *G* be follower-separated, deterministic presentation. Let *H* be the subgraph induced by the synchronizing vertices of *G*. Then, X_G is irreducible if and only if $X_G = X_H$ and *H* is induced by a terminal irreducible component.

Finally, we state some facts about SFTs. The first is a characterization of when a shift space is an SFT, and the second is a sufficient condition for when X_G is an SFT for a presentation *G*. Respectively, these correspond to Theorem 2.1.8 and Proposition 2.2.6 of Lind and Marcus [17].

Theorem 2.5. A shift space *X* is an SFT if and only if there exists an integer $M \ge 0$ such that every word $w \in \mathcal{B}(X)$ with $|w| \ge M$ is intrinsically synchronizing for *X*.

Lemma 2.6. Let *G* be a presentation. If every edge in *G* is labeled uniquely, then X_G is an SFT.

3 Complexity Upper Bounds and Algorithms

In this section we detail polynomial-time algorithms for some problems in Table 1. In particular, we give polynomial-time algorithms for SFT and EQUALITY for synchronizing deterministic presentations, and additionally for SYNCWORD and SUBSHIFT for irreducible presentations. We also give a polynomial-time algorithm to test whether a given deterministic presentation is synchronizing. Some algorithms follow from known results, whereas others, to our knowledge, are novel to this work.

3.1 Finding synchronizing words

Eppstein [9] gives a polynomial-time algorithm for finding synchronizing words in fully deterministic presentations. Here, we show the algorithm can be extended to irreducible deterministic presentations, implying that SYNCWORD is in P for such presentations. As we show in Theorem 4.16, SYNCWORD is PSPACE-complete for general presentations.

Theorem 3.1. Given an irreducible deterministic labeled graph *G*, Algorithm 1 returns a synchronizing word for *G* if one exists, and nil otherwise.

To prove this result, we first introduce the notion of pair-synchronizing words. Let p and q be vertices in a deterministic graph G. We say a word w is *pair-synchronizing* for p and q if $|\{p,q\} \cdot w| = 1$, i.e., if there exists a vertex $r \in Q_G$ such that $\{p,q\} \cdot w = \{r\}$. This condition breaks into the following three cases:

- (i) $w \in F_G(p)$ and $w \notin F_G(q)$;
- (ii) $w \notin F_G(p)$ and $w \in F_G(q)$;
- (iii) $w \in F_G(p) \cap F_G(q)$ and $p \cdot w = q \cdot w$.

If $X \subseteq Q_G$ is a subset of vertices with $|X| \ge 2$ and w is pair-synchronizing for distinct $p, q \in X$, then we have $|X| > |X \cdot w| \ge 1$. This property motivates Algorithm 1. The algorithm operates by iteratively building a word u and tracking a subset X of vertices, maintaining the invariants that $Q_G \cdot u = X$ and $|X| \ge 1$. On each iteration of the main loop, the algorithm searches for a pair-synchronizing word w for some pair of distinct vertices in X, and if one is found, then updates u to uw and X to $X \cdot w$. The property above ensures that the invariants of X and u are maintained. Since |X| must decrease by at least 1 in each iteration, the algorithm returns after at most $|Q_G|$ iterations.

Proof of Theorem 3.1. If Algorithm 1 returns a non-nil value, it must have exited at line 10, which implies $|X| \le 1$. As the invariant that $|X| \ge 1$ was maintained throughout the algorithm, we must have $|Q_G \cdot u| = 1$, so the word u that was returned is a synchronizing word for G.

Conversely, if Algorithm 1 returned nil, it must have exited at line 9, which implies that there are two distinct vertices such that there is no pair-synchronizing word for them. Yet, as we show next, if G has a synchronizing word, then every pair of distinct vertices has a pair-synchronizing word. Thus, G must not have a synchronizing word.

Let p and q be distinct vertices in G, and suppose w is a synchronizing word for G. As w is synchronizing, there is some vertex s with $w \in F_G(s)$. As G is irreducible, there is a word u such that $p \cdot u = s$, which implies that $uw \in F_G(p)$. If $uw \notin F_G(q)$, then uw is a pair-synchronizing for p and q under case (i) above. Otherwise, we have $uw \in F_G(q)$. As w is synchronizing, then $p \cdot uw = q \cdot uw$, so uw is still pair-synchronizing for p and q, under case (iii).

Algo	Algorithm 1 Finding synchronizing words				
Requ	Require: <i>G</i> is a deterministic graph				
1: p	1: procedure SYNCHRONIZING-WORD(<i>G</i>)				
2:	$X \leftarrow Q_G; u \leftarrow \epsilon$				
3:	while $ X \ge 2$ do				
4:	choose distinct $p, q \in X$				
5:	find a word w that is pair-synchronizing for p and q				
6:	if w exists then				
7:	$X \leftarrow X \cdot w; u \leftarrow uw$				
8:	else				
9:	return nil				
10:	return u				

To implement this Algorithm 1 in polynomial time (with respect to the size of its input G), we need a method to compute a pair-synchronizing word for a given pair of vertices. We give such a method using two auxillary graphs, the first of which encodes what words are not within a follower set of a vertex, and the second of which encodes pairs of paths sharing the same label.

If *G* is a labeled graph and Γ is an alphabet, the *sink vertex graph of G with alphabet* Γ is the graph G^0 constructed as follows. Start with the graph *G*, and add a new vertex 0 to G^0 . For every vertex *q* in G^0 and $\ell \in \Gamma$, add an edge labeled ℓ from *q* to 0 if there is no edge labeled ℓ starting at *q*. One can show that for $w \in \Gamma^*$ and $q \in Q_G$, we have $w \notin F_G(q)$ if and only if there is a path labeled *w* from *q* to 0 in G^0 .

If *G* and *H* are labeled graphs, then the *label product graph of G and H* is the graph G * H whose vertices are $Q_G \times Q_H$ and with an edge between (p_1, p_2) and (q_1, q_2) labeled ℓ if and only if there is an edge labeled ℓ from p_1 to q_1 in *G* and an edge labeled ℓ from p_2 to q_2 in *H*. One can show that for $w \in (\mathcal{A}_G \cup \mathcal{A}_H)^*$, there is a path labeled *w* from p_1 to q_1 in *G* and a path labeled *w* from p_2 to q_2 in *G* if and only if there is a path labeled *w* from p_1 to q_1 in *G* and a path labeled *w* from p_2 to q_2 in *G* if and only if there is a path labeled *w* from (p_1, p_2) to (q_1, q_2) in G * H.

Let *G* be a labeled graph, let G^0 be the sink vertex graph of *G* with alphabet \mathcal{A}_G and let $G^0 * G^0$ be the label product graph of G^0 and G^0 . With the properties of the auxillary

graphs, one can show that the following conditions are equivalent to cases (i)-(iii) from the pair-synchronizing definition.

- (I) there is a path in $G^0 * G^0$ from (p,q) to (r,0) for some vertex r in G;
- (II) there is a path in $G^0 * G^0$ from (p,q) to (0,r) for some vertex r in G;
- (III) there is a path in $G^0 * G^0$ from (p,q) to (r,r) for some vertex r in G.

Using, say, a depth-first search, one can determine if there is a pair-synchronizing word for a given pair of vertices by testing for the existence of a path satisfying one of (I)-(III). The size of $G^0 * G^0$ is $O(|Q_G|^2 \cdot |A_G|)$, so one can construct the graph and query the existence of such path in polynomial time. Each iteration of Algorithm 1 therefore takes polynomial time, and furthermore, as there are at most $|Q_G|$ iterations, in total, the algorithm will take polynomial time.

3.2 Testing for subshift

We now turn to the SUBSHIFT problem for deterministic presentations *G* and *H* where *G* is irreducible. The key idea behind the algorithm is to try to find a word exhibiting the fact that $X_G \notin X_H$. We say that *w* separates *G* from *H* if $Q_G \cdot w \neq \emptyset$ while $Q_H \cdot w = \emptyset$. When *G* and *H* are essential, the existence of such a word is equivalent to $X_G \notin X_H$. Algorithm 2 adapts the algorithm for synchronizing words to find such separating words, thus showing SUBSHIFT for *G* and *H* is in P when *G* is irreducible. We show in Theorem 4.4 that the general problem is PSPACE-complete.

We state the correctness of Algorithm 2 for the more general case of labeled graphs, which need not be essential, since we rely on that case for Theorem 3.3.

Theorem 3.2. Given deterministic labeled graphs *G* and *H*, where *G* is irreducible, Algorithm 2 returns a word separating *G* from *H* if one exists, and returns nil otherwise.

Like in Algorithm 1, Algorithm 2 operates by iteratively building a word u. In addition, the algorithm fixes a vertex $p_0 \in Q_G$, and maintains a vertex $p \in Q_G$ and subset $X \subseteq Q_H$ satisfying the invariants $u \in F_G(p_0)$, $p_0 \cdot u = p$, and $Q_H \cdot u = X$. In each iteration of the main loop, the algorithm searches for a word w such that $w \in F_G(p)$ while $w \notin F_H(q)$ for some $q \in X$. If one is found, the algorithm updates u to uw, p to $p \cdot w$, and X to $X \cdot w$, which maintains the invariants. As $w \notin F_H(q)$ and $q \in X$, we have $|X| > |X \cdot w|$, so the algorithm again terminates in at most $|Q_H|$ iterations.

Proof of Theorem 3.2. If Algorithm 2 returns a non-nil value, it must have exited at line 10, so |X| = 0. The invariants give $Q_H \cdot u = X = \emptyset$ and $u \in F_G(p_0)$, meaning $Q_H \cdot u = \emptyset$ and $Q_G \cdot u \neq \emptyset$. Thus, u separates G from H.

Conversely, if Algorithm 2 returns nil, it must have exited at line 9, which implies there exist $p \in Q_G$ and $q \in Q_H$ such that there is no word w with $w \in F_G(p)$ and $w \notin F_H(q)$. We show below that, if some word separates G from H, then for every $p' \in Q_G$ and $q' \in Q_H$, there is a word *w* with $w \in F_G(p')$ and $w \notin F_H(q')$. By contraposition, therefore, no word separates *G* from *H*.

Suppose there is a word *w* separating *G* from *H*, so that we have $Q_G \cdot w \neq \emptyset$ while $Q_H \cdot w = \emptyset$. Then, there is some vertex $p^* \in Q_G$ such that $w \in F_G(p^*)$ and $w \notin F_H(q')$ for every $q' \in Q_H$. Let $p \in Q_G$ and $q \in Q_H$. As *G* is irreducible, there is some $u \in F_G(p)$ such that $p \cdot u = p^*$, giving $uw \in F_G(p)$. Let $q' \triangleq q \cdot u$. By the above, $w \notin F_H(q')$, and thus $uw \notin F_H(q)$.

Algorithm 2 Subshift testing					
Require: <i>G</i> is an irreducible deterministic labeled graph					
Require: <i>H</i> is a deterministic labeled graph					
1: procedure SEPARATING-WORD(<i>G</i> , <i>H</i>)					
2: $p_0 \leftarrow \text{any element in } Q_G; p \leftarrow p_0; X \leftarrow Q_H; u \leftarrow \epsilon$					
3: while $ X > 0$ do					
4: $q \leftarrow \text{any element in } X$					
5: find a word w such that $w \in F_G(p)$ and $w \notin F_H(q)$					
6: if <i>w</i> exists then					
7: $p \leftarrow p \cdot w; X \leftarrow X \cdot w; u \leftarrow uw$					
8: else					
9: return nil					
10: return u					

Analogously to Algorithm 1, we can implement Algorithm 2 in polynomial time by noticing that the existence of a word w such that $w \in F_G(p)$ and $w \notin F_H(p)$ is equivalent to the existence of a path in $G * H^0$ from (p,q) to (r,0) for some vertex r in G, where H^0 is the sink vertex graph of H with alphabet $\mathcal{A}_G \cup \mathcal{A}_H$ and $G * H^0$ is the label product graph of G and H^0 .

3.3 Testing for synchronizing presentations

With Algorithm 1 and Algorithm 2, we can now establish a polynomial-time algorithm for checking if a given deterministic graph is synchronizing, given by Algorithm 3. The correctness of the algorithm is implied by the following characterization of a synchronizing presentation.

Theorem 3.3. Let *G* be a deterministic labeled graph with vertex set *Q*. Then, *G* is synchronizing if and only if for each initial irreducible component *C*, there exists (i) a synchronizing word for the subgraph induced by *C* and (ii) a word separating the subgraph induced by *C* from the subgraph induced by $Q \setminus C$.

Proof. Suppose *G* is synchronizing. Let *C* be an initial component of *G*, and fix $r \in C$. As *G* is synchronizing, let *w* be a word that synchronizes to *r* in *G*. As *w* is synchronizing for *G*, there is some vertex $p \in Q$ such that $p \cdot w = r$. Since $r \in C$ and *C* is initial, we must

have $p \in C$. Thus $C \cdot w = \{r\}$, establishing (i). As *C* is initial and $r \in C$, we cannot have $q \cdot w = r$ for any $q \notin C$. We conclude $(Q \setminus C) \cdot w = \emptyset$. As $C \cdot w = \{r\} \neq \emptyset$, we have (ii).

Conversely, suppose for each initial irreducible component *C*, (i) there is a synchronizing word u_C for the subgraph induced by *C* and (ii) there is a word w_C separating the subgraph induced by *C* from the subgraph induced by $Q \setminus C$. Let *r* be any vertex in *G*. Let *C* be an initial irreducible component such that *r* is reachable from every vertex in *C*. Condition (i) gives $C \cdot u_C = \{p\}$ for some $p \in C$. Condition (ii) gives some vertex $q \in C$ with $w_C \in F_G(q)$ such that $q \cdot w_C \in C$ and $(Q \setminus C) \cdot w_C = \emptyset$. As *C* is an irreducible component and $p, q \in C$, then there is some word *x* such that $p \cdot x = q$. As $q \cdot w_C \in C$ and *r* is reachable from every vertex in *C*, there is some word *y* such that $(q \cdot w_C) \cdot y = r$. Combining the above with a straightforward calculation for $Q \setminus C$, we have

$$C \cdot u_C x w_C y = \{p\} \cdot x w_C y = \{q\} \cdot w_C y = \{q \cdot w_C\} \cdot y = \{r\},\$$
$$(Q \setminus C) \cdot u_C x w_C y = ((Q \setminus C) \cdot u_C x) \cdot w_C y \subseteq (Q \setminus C) \cdot w_C y = \emptyset \cdot y = \emptyset.$$

Thus, $Q \cdot u_C x w_C y = (C \cdot u_C x w_C y) \cup ((Q \setminus C) \cdot u_C x w_C y) = \{r\}$. As r was arbitrary, G is synchronizing.

Algor	Algorithm 3 Recognizing synchronizing presentations				
Requ	Require: <i>G</i> is a deterministic labeled graph				
1: procedure IS-SYNCHRONIZING(<i>G</i>)					
2:	$\mathcal{C} \leftarrow \text{initial irreducible components of } G$				
3:	for $C \in \mathcal{C}$ do				
4:	$G[C] \leftarrow$ subgraph induced by C				
5:	$G[\overline{C}] \leftarrow$ subgraph induced by $Q_G \setminus C$				
6:	$u \leftarrow \text{SYNCHRONIZING-WORD}(G[C])$				
7:	$v \leftarrow \text{Separating-Word}(G[C], G[\overline{C}])$				
8:	if <i>u</i> is nil or <i>v</i> is nil then				
9:	return false				
10:	return true				

3.4 SFT testing for synchronizing deterministic presentations

The proof of Theorem 3.4.17 of Lind and Marcus [17] implicitly describes a polynomialtime algorithm to test whether an irreducible sofic shift, given as an irreducible deterministic presentation, is an SFT, and Schrock [21] gives a similar algorithm explicitly. We extend these algorithms to synchronizing deterministic presentations.

For a deterministic labeled graph G, we define the labeled graph \hat{G} as the label product graph G * G (see Section 3.1) with the diagonal vertices removed, i.e., those of the form (q,q). Given a follower-separated synchronizing deterministic presentation G, the algorithm to recognize if X_G is an SFT is to simply test if the graph \hat{G} is acyclic. (A *cycle* is a nonempty path that starts and ends at the same vertex, and we say a graph is *acyclic* if it has no cycle.) This algorithm runs in polynomial time, as the size of \hat{G} is quadratic with respect to the size of G, and it is well-known that one can test whether a directed graph is acyclic in linear time.

To show the correctness of this algorithm, we first show that \hat{G} characterizes the nonsynchronizing words of *G*.

Lemma 3.4. Let *G* be a deterministic presentation and let $w \in \mathcal{B}(X_G)$. Then, there is a path in \hat{G} labeled *w* if and only if *w* is not synchronizing for *G*.

Proof. Suppose there is a path π in \hat{G} labeled w, from (p,q) to (p',q'). Then, we have $p \neq q$ and $p' \neq q'$, and $p \cdot w = p'$ and $q \cdot w = q'$. Thus, $Q_G \cdot w \supseteq \{p,q\} \cdot w = \{p',q'\}$. As $p' \neq q'$, we have $|Q_G \cdot w| \ge 2$ so w is not synchronizing for G. Conversely, if w was not synchronizing for G, then $|Q_G \cdot w| \ge 2$. Let $p',q' \in Q_G \cdot w$ be distinct. Let $p,q \in Q_G$ such that $p \cdot w = p'$ and $q \cdot w = q'$. If for some factoring w = uv we had $p \cdot u = q \cdot u$, then $p' = p \cdot w = (p \cdot u) \cdot v = (q \cdot u) \cdot v = q \cdot w = q'$, a contradiction to p' and q' being distinct. Thus, there is a path labeled w in G * G from (p,q) to (p',q') which does not pass through any diagonal vertices, meaning it is a labeled path in \hat{G} .

Because of the correspondence of synchronizing and intrinsically synchronizing words in follower-separated synchronizing deterministic presentations, we can use \hat{G} to characterize when X_G is an SFT.

Theorem 3.5. Let *G* be a follower-separated synchronizing deterministic presentation. Then, X_G is an SFT if and only if \hat{G} is acyclic.

Proof. Suppose \hat{G} had a cycle. By Theorem 2.5, to show that X_G is not an SFT, it suffices to show that for every $M \ge 0$, there exists a word $w \in \mathcal{B}(X_G)$ with $|w| \ge M$ that is not intrinsically synchronizing for X_G . Let $M \ge 0$. Since \hat{G} has a cycle, in particular it has a path of any length. Let π be a path in \hat{G} of length at least M, and let w be its label. We have $w \in \mathcal{B}(X_G)$ and $|w| \ge M$. By Lemma 3.4, w is not synchronizing for G, and by Lemma 2.2, w is therefore not intrinsically synchronizing for X_G .

Conversely, suppose \hat{G} is acyclic, and let $M \triangleq |Q_{\hat{G}}|$. By Theorem 2.5, to show X_G is an SFT, it suffices to show that every word $w \in \mathcal{B}(X_G)$ with $|w| \ge M$ is intrinsically synchronizing for X_G . Let $w \in \mathcal{B}(X_G)$ and suppose $|w| \ge M$. Suppose for a contradiction that w is not intrinsically synchronizing for X_G . By Lemmas 2.2 and 3.4 once again, there is a path in \hat{G} labeled w, of length $|w| \ge M$. As G is acyclic, however, every path in \hat{G} must have length strictly less than $|Q_{\hat{G}}| = M$, a contradiction. Thus, w must be intrinsically synchronizing for X_G .

Remark 3.6. Let X be a shift space and $M \ge 0$. Say X is *M*-step if every word $w \in \mathcal{B}(X)$ with $w \ge M$ is intrinsically synchronizing for X. With this definition and rephrasing Theorem 2.5, a shift space is an SFT if and only if it is *M*-step for some $M \ge 0$. The converse direction then implies that if \hat{G} is acyclic, then it must be $(|Q_G|^2 - |Q_G|)$ -step, as $|Q_{\hat{G}}| = |Q_G|^2 - |Q_G|$. Thus, if G is a follower-separated synchronizing deterministic presentation, then X_G is an SFT if and only if it is $(|Q_G|^2 - |Q_G|)$ -step. (Cf. [17, Theorem 3.4.17].)

3.5 Isomorphism and equality

A homomorphism between deterministic labeled graphs G and H is a mapping $\varphi : Q_G \to Q_H$ that preserves the transition action: we have $F_G(q) = F_H(\varphi(q))$ and $\varphi(q \cdot w) = \varphi(q) \cdot w$ for all $q \in Q_G$ and $w \in F_G(q)$. An *isomorphism* is a bijective homomorphism. In general, the problem of deciding isomorphism between deterministic labeled graphs is Gl-complete, meaning it has a polynomial-time many-one reduction to and from the graph isomorphism problem on unlabeled graphs [5]. For follower-separated graphs, however, the problem is in P. To show this, we need the following lemma, which states that preserving the follower set of a vertex is sufficient for being a homomorphism onto a follower-separated graph.

Lemma 3.7. Let *G* and *H* be deterministic labeled graphs, and $\varphi : Q_G \to Q_H$ a map between their vertices. If *H* is follower-separated, then φ is a homomorphism if and only if $F_G(q) = F_H(\varphi(q))$ for all $q \in Q_G$.

Proof. That homomorphisms preserve follower sets follows directly from the definition. For the converse, suppose $F_G(q) = F_H(\varphi(q))$ for all $q \in Q_G$. Let $q \in Q_G$ and $w \in F_G(q)$. As H is follower-separated, it suffices to show that $F_H(\varphi(q \cdot w)) = F_H(\varphi(q) \cdot w)$ to show that $\varphi(q \cdot w) = \varphi(q) \cdot w$. For any u, we have

$$u \in F_{H}(\varphi(q \cdot w))$$

$$\iff u \in F_{G}(q \cdot w)$$

$$\iff wu \in F_{G}(q)$$

$$\iff wu \in F_{H}(\varphi(q))$$

$$\iff u \in F_{H}(\varphi(q) \cdot w).$$

Thus $F_H(\varphi(q \cdot w)) = F_H(\varphi(q) \cdot w)$.

Now, given two follower-separated deterministic labeled graphs *G* and *H*, we can test if they are isomorphic by taking the disjoint union graph G + H, computing the followerequivalences of G + H, and testing if all the follower-equivalence classes are pairs (i.e. sets of size 2). As *G* and *H* are follower-separated, if two distinct vertices in G + H are follower-equivalent, then one of them must be a vertex from *G* and the other from *H*. Thus, if all the follower-equivalence classes are pairs, then a bijective map $\varphi : Q_G \to Q_H$ that preserves the follower set of a vertex can be read off from the pairs. By Lemma 3.7, this map is an isomorphism. Conversely, if $\varphi : Q_G \to Q_H$ is an isomorphism, then for $p \in Q_G$ and $q \in Q_H$ with $F_G(p) = F_H(q)$, then $F_H(\varphi(p)) = F_H(q)$ and so $\varphi(p) = q$. In other words, for any $p \in Q_G$ and any $q \in Q_H$ follower-equivalent to *p*, then $\varphi(p) = q$. This implies that all the follower-equivalence classes of G + H are pairs.

Since the follower set of a vertex is preserved under an isomorphism, if *G* and *H* are isomorphic deterministic presentations, then $X_G = X_H$. However, even for follower-separated presentations, the converse is not necessarily true. (See Figure 1.) But Jonoska [12, Corollary 5.4] proved that any two follower-separated synchronizing deterministic

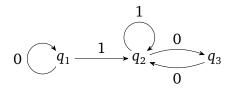


Figure 1: A reducible, follower-separated, deterministic presentation *G*. Let *H* be the subgraph induced by q_2 and q_3 , which is irreducible and follower-separated. Then, $X_G = X_H$, but *G* and *H* are not isomorphic.

presentations of the same sofic shift are isomorphic, which implies that recognizing isomorphism is sufficient for recognizing if $X_G = X_H$ when *G* and *H* are follower-separated synchronizing deterministic presentations. Furthermore, this implies that EQUALITY is in P for synchronizing deterministic presentations, as given synchronizing deterministic presentations *G* and *H*, to determine if $X_G = X_H$, one can test if G/\sim and H/\sim , the follower-separations of *G* and *H*, are isomorphic.

4 Complexity Lower Bounds

In the following sections, we show all the decision problems in Table 1 are PSPACE-hard. In Appendix A, we show all those decision problems are in PSPACE. As a result, we have the following.

Theorem 4.1. Every problem in Table 1 is PSPACE-complete.

To establish the hardness of these decision problems, we will leverage hardness results from the automata theory literature. To relate automata to sofic shifts, we will treat automata as a type of labeled graph. Formally, we define a *deterministic finite automaton* (*DFA*) to be a fully deterministic labeled graph *M* with a designated initial state $s \in Q_M$ and set of accepting states $F \subseteq Q_M$. For DFAs, following convention from the automata literature, we will write the transition action as a function $\delta(q, w) \triangleq q \cdot w$. The *language* of *M* is the set $L(M) \triangleq \{w \in A_M^* : \delta(s, w) \in F\}$. Note that L(M) may differ from $\mathcal{B}(X_M)$, the language of the sofic shift presented by *M*. In fact, as DFAs are fully deterministic, we always have $\mathcal{B}(X_M) = A_M^*$, meaning X_M is always the full shift.

We will reduce from the *DFA* intersection nonemptiness problem (DFAINT) and *DFA* union universality problem (DFAUNION), both of which are PSPACE-complete. The DFAINT problem asks whether, given *n* DFAs M_1, \ldots, M_n over a common input alphabet Σ , is $\bigcap_{i=1}^n L(M_i) \neq \emptyset$? Similarly, the DFAUNION problem asks whether, given *n* DFAs M_1, \ldots, M_n over a common input alphabet Σ , is $\bigcup_{i=1}^n L(M_i) = \Sigma^*$? Kozen [13] showed that DFAINT is PSPACE-complete; one can see that DFAUNION is PSPACE-complete from the following two facts: (i) the complement of DFAINT is PSPACE-complete, and (ii) $\bigcap_{i=1}^n L(M_i) = \emptyset$ if and only if $\bigcup_{i=1}^n L(\overline{M_i}) = \Sigma^*$, where $\overline{M_i}$ is M_i with the accepting states being the complement of the accepting states of M_i . Within our reductions, for an instance M_1, \ldots, M_n of DFAUNION or DFAINT, we will let Q_i , δ_i , s_i , and F_i denote the set of states, transition function, initial state, and set of accepting states for M_i . (The Q_i are assumed to be pairwise disjoint.)

4.1 Hardness of equality, containment, irreducibility, and SDP existence

In this section, we give a single polynomial-time reduction, which reduces DFAUNION simultaneously to SUBSHIFT, EQUALITY, IRREDUCIBILITY, and \exists SDP, giving the following.

Theorem 4.2. SUBSHIFT, EQUALITY, IRREDUCIBILITY, and **BSDP** are PSPACE-hard.

The idea behind the reduction is to create pre-initial states p_i for each DFA M_i , and chain these together in a loop, with special symbols \triangleright into and \triangleleft out of each DFA. We then add a special state p^* in its own initial irreducible component, whose follower set contains { $\triangleright w \triangleleft : w \in \Sigma^*$ }. (See Figure 2 for a visualization.) Letting *H* be the whole graph minus the special state p^* , we can therefore test whether the DFA languages union to Σ^* by asking whether $X_G = X_H$, i.e., whether p^* was needed to cover all possible strings $w \in \Sigma^*$ between \triangleright and \triangleleft . Equivalently, we could test $X_G \subseteq X_H$, since the reverse inclusion is immediate. As H is an irreducible presentation, we could also test whether X_G is irreducible. Finally, the reduction to \exists SDP follows for the following reasons: first, H is a synchronizing deterministic presentation (as $\triangleleft \ell^{i-1}$ synchronizes to p_i for all i), so when the langauges of the DFAs union to Σ^* , *H* is a synchronizing deterministic presentation for X_G ; second, when there is a word $w \in \Sigma^*$ not in the language of any of the DFAs, one can show X_G does not have a synchronizing deterministic presentation by invoking Theorem 2.3 and showing that any *u* such that $u \triangleright w \triangleleft \in \mathcal{B}(X_G)$ is not intrinisically synchronizing. **Reduction A.** Let M_1, \ldots, M_n be an instance to the DFAUNION problem. Construct the deterministic presentation *G* as follows. For each i = 1, ..., n,

- 1. add a state p_i (the *i*th *pre-initial* state) to *G*;
- 2. embed M_i into G;
- 3. add a self loop labeled * on p_i ;
- 4. add an edge labeled \triangleright from p_i to the corresponding initial state s_i ;
- 5. for each accepting state $q \in F_i$, add an edge labeled \triangleleft from q to p_1 ;
- 6. for each state $q \in Q_i$, add an edge labeled ℓ from q to s_i .

Then, add two states p^* and s^* , add a self loop labeled * on p^* , add an edge labeled \triangleright from p^* to s^* , and add an edge labeled \triangleleft from s^* to p_1 . For each $a \in \Sigma$, add a self loop labeled a on s^* . For each i = 1, ..., n - 1, add an edge labeled ℓ from p_i to p_{i+1} . Finally, add an edge labeled ℓ from p_n to s^* . (See Figure 2.)

Let *G* be the deterministic presentation obtained from Reduction A on an instance M_1, \ldots, M_n . Without loss of generality, for each *i*, we may assume that (I) $F_i \neq \emptyset$, as otherwise, if $F_i = \emptyset$, then $L(M_i) = \emptyset$ so thus $L(M_i)$ does not contribute to the union;

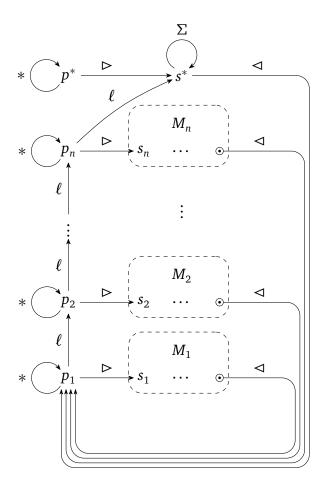


Figure 2: Schematic of Reduction A. The edges labeled ℓ from each state $q \in Q_i$ to s_i are not pictured.

and (II) every state $q \in Q_i$ is reachable from s_i , as when modifying M_i to M'_i by removing those unreachable states, we have $L(M_i) = L(M'_i)$. Let H be the subgraph in G induced by every vertex but p^* . The following lemma summarizes several useful properties of the reduction.

Lemma 4.3. The following hold of Reduction A.

- (i) *H* is synchronizing and irreducible, and *G* is essential;
- (ii) $\Sigma^* \subseteq F_G(q)$ for all $q \in \bigcup_{i=1}^n Q_i$;
- (iii) $\triangleright w \triangleleft \in F(p_i)$ if and only if $w \in L(M_i)$;
- (iv) $\triangleright w \triangleleft \in \mathcal{B}(X_H)$ if and only if $w \in \bigcup_{i=1}^n L(M_i)$.
- (v) If $\bigcup_{i=1}^{n} L(M_i) \neq \Sigma^*$, there exists $w \in \Sigma^*$ with $\triangleright w \triangleleft \in \mathcal{B}(X_G) \setminus \mathcal{B}(X_H)$.

Proof. For (i), by assumption (I), there exists a state in Q_i with an edge labeled \triangleleft to p_1 . By assumption (II), every state is reachable from s_i , so there exists a path from s_i to p_1 . Thus for any state in Q_i , one can always find a way to p_1 by returning to s_i via an ℓ edge, and then finding a way to p_1 . As p_1 can reach any other vertex in H, any state in Q_i can reach any other vertex in *H*. From this, we can see that *H* is irreducible, and it follows that *G* is essential. We have that *H* is synchronizing as $Q_G \cdot \triangleleft = \{p_1\}$ and every vertex in *H* is reachable from p_1 .

For the other statements, first note that each of the M_i are emulated by the transition action of G in the following way: for $q \in Q_i$ and $w \in \Sigma^*$, we have $w \in F_G(q)$ and $q \cdot w = \delta_i(q, w)$ and $q \in F_i$ if and only if $\triangleleft \in F_G(q)$. Thus, (ii) follows. For (iii), note that $p_i \cdot \triangleright = s_i$ and $w \triangleleft \in F_G(s_i)$ if and only if $w \in L(M_i)$; thus, $\triangleright w \triangleleft \in F_G(p_i)$ if and only if $w \in L(M_i)$. For (iv), note that $Q_H \cdot \triangleright = \{s_1, \dots, s_n\}$; thus, by the previous observations, $\triangleright w \triangleleft \in \mathcal{B}(X_H)$ if and only if $w \in \bigcup_{i=1}^n L(M_i)$. Finally (v) follows from (iv) and the fact that $\triangleright w \triangleleft \in \mathcal{B}(X_G)$ for all $w \in \Sigma^*$.

With these properties, we can establish the correctness of Reduction A. The first theorem shows that it reduces DFAUNION to SUBSHIFT.

Theorem 4.4. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if $X_G \subseteq X_H$.

Proof. Suppose $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$. To establish $X_G \subseteq X_H$, we only need to show $F_G(p^*) \subseteq \mathcal{B}(X_H)$. Let $u \in F_G(p^*)$. If $p^* \cdot u = p^*$, then by construction, we have $u = *^m$ for some $m \ge 0$, and as $u \in F_G(p_1)$, then $u \in \mathcal{B}(X_H)$. Otherwise, if $p^* \cdot u = s^*$, then we can factor u into $u = *^m \triangleright w$, where $m \ge 0$ and $w \in \Sigma^*$. Similarly, by Lemma 4.3(ii), we can find $u \in F_G(p_1)$, so $u \in \mathcal{B}(X_H)$. Finally, if $p^* \cdot u \notin \{p^*, s^*\}$, then we can factor u into $u = u_1u_2$, where $u_1 = *^m \triangleright w \triangleleft$ for some $m \ge 0$ and $w \in \Sigma^*$, $p^* \cdot u_1 = p_1$, and $u_2 \in F_G(p_1)$. As $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$, Lemma 4.3(ii) implies $\triangleright w \triangleleft \in \mathcal{B}(X_H)$, and in particular, there is some i such that $\triangleright w \triangleleft \in F_G(p_i)$. As $p_i \cdot \triangleright w \triangleleft = p_i \cdot *^m \triangleright w \triangleleft = p_i \cdot u_1 = p_1$, we have $u_1 \in F_G(p_i)$ and $u_2 \in F_G(p_i \cdot u_1)$. Thus $u_1u_2 = u \in F_G(p_i)$, and $u \in \mathcal{B}(X_H)$.

Conversely, Lemma 4.3(v) gives some $w \in \Sigma^*$ with $\triangleright w \triangleleft \in \mathcal{B}(X_G) \setminus \mathcal{B}(X_H)$. Hence, $\mathcal{B}(X_G) \notin \mathcal{B}(X_H)$, and thus $X_G \notin X_H$.

Immediately, as $X_H \subseteq X_G$, we have that DFAUNION reduces to EQUALITY.

Corollary 4.5. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if $X_G = X_H$.

The reduction to IRREDUCIBILITY now follows as well.

Theorem 4.6. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if X_G is irreducible.

Proof. If $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$, Corollary 4.5 implies $X_G = X_H$; as X_H is irreducible by Lemma 4.3(i), so is X_G . Conversely, Lemma 4.3(v) gives some $w \in \Sigma^*$ with $\triangleright w \triangleleft \in \mathcal{B}(X_G) \setminus \mathcal{B}(X_H)$. As \triangleleft synchronizes to p_1 , for every $u \in F_{X_G}(\triangleright w \triangleleft)$, we have $F_{X_G}(\triangleright w \triangleleft u) = F_G(p_1 \cdot u)$. Furthermore, as $p_1 \cdot u \in Q_H$, we have $F_G(p_1 \cdot u) \subseteq \mathcal{B}(X_H)$. Yet $\triangleright w \triangleleft \notin \mathcal{B}(X_H)$, so we must have $\triangleright w \triangleleft \notin F_G(p_1 \cdot u)$. Thus, for every $u \in F_{X_G}(\triangleright w \triangleleft)$, we have $\triangleright w \triangleleft \notin F_G(p_1 \cdot u)$. In other words, $\triangleright w \triangleleft \in \mathcal{B}(X_G)$ but there is no word u such that $\triangleright w \triangleleft u \triangleright w \triangleleft \in \mathcal{B}(X_G)$. □

Remark 4.7. Let *X* be a shift space. Say *X* is mixing if for every $u, v \in \mathcal{B}(X)$, there is an *N* such that for every $n \ge N$, there is a word *w* with |w| = n and $uwv \in \mathcal{B}(X)$. Mixing

implies irreducibility, so if X_G is mixing, then X_G is irreducible. Note that X_H is mixing, as given $u, v \in \mathcal{B}(X_H)$, one can find words w_1, w_2 such that uw_1 synchronizes to p_1 and $v \in F_G(p_1 \cdot w_2)$, and as $p_1 \cdot *^m = p_1$ for every m, we have that $uw_1 *^m w_2 v \in \mathcal{B}(X_H)$ for every m. Thus, X_G is irreducible if and only if it is mixing, so deciding if the sofic shift presented by a deterministic presentation is mixing is PSPACE-hard.

Similarly, say X is nonwandering if for every $u \in \mathcal{B}(X_G)$, there is a word w such that $uwu \in \mathcal{B}(X)$. Irreducibility implies nonwandering, so if X_G is irreducible, then X_G is nonwandering. Note that the proof of Theorem 4.6 shows that if $\bigcup_{i=1}^{n} L(M_i) \neq \Sigma^*$, then X_G is not nonwandering. Thus, X_G is irreducible if and only if it is nonwandering, so deciding if the sofic shift presented by a deterministic presentation is nonwandering is PSPACE-hard.

Finally, we show that Reduction A also reduces DFAUNION to \exists SDP.

Theorem 4.8. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if X_G has a synchronizing deterministic presentation.

Proof. If $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$, then Corollary 4.5 implies $X_G = X_H$, and as Lemma 4.3(i) implies *H* is synchronizing, then X_G has a synchronizing deterministic presentation. Conversely, Lemma 4.3(v) gives some $w \in \Sigma^*$ with $\triangleright w \triangleleft \in \mathcal{B}(X_G) \setminus \mathcal{B}(X_H)$. Suppose for a contradiction that X_G has a synchronizing deterministic presentation. By Theorem 2.3, there must be some $u \in \mathcal{B}(X_G)$ such that *u* is intrinsically synchronizing for X_G and $\triangleright w \triangleleft \in F_{X_G}(u)$. As $\triangleright w \triangleleft \notin \mathcal{B}(X_H)$, the only vertex in *G* with $\triangleright w \triangleleft$ in its follower set is *p*^{*}. Therefore, $\triangleright w \triangleleft \notin \mathcal{B}(X_H)$, the only vertex in *G* with $\triangleright w \triangleleft$ in its follower set the form $u = *^k$ for some $k \ge 0$. However, $*^k$ is not intrinsically synchronizing for X_G : we have $\triangleleft *^k \in \mathcal{B}(X_G)$ and $*^k \triangleright w \triangleleft \in \mathcal{B}(X_G)$ but as $\triangleright w \triangleleft \notin \mathcal{B}(X_H)$ and $\triangleleft *^k$ synchronizes to a vertex in *H*, it must be the case that $\triangleleft *^k \triangleright w \triangleleft \notin \mathcal{B}(X_G)$. Thus, $u = *^k$ is not intrinsically synchronizing, a contradiction. We conclude that X_G does not have a synchronizing deterministic presentation. □

4.2 Hardness of SFT Testing and Minimization

We now give a similar polynomial-time reduction, which reduces DFAUNION simultaneously to SFT and MINIMALITY, giving the following.

Theorem 4.9. The problems SFT and MINIMALITY are PSPACE-hard.

The reduction is similar in spirit to Reduction A. We still add edges labeled \triangleleft out of each DFA into a terminal state, but instead of adding edges labeled \triangleright into the DFAs from new pre-initial states, we instead add these edges from within the DFAs to their corresponding initial states. We also add self loops on each DFA state labeled ℓ . We then add a special state s^* in its own initial component, whose follower set contains $\{w \triangleleft : w \in (\Sigma \cup \{\ell\})^*\}$. See Figure 3 for a visualization. The first observation we make is that, if and only if the DFA languages union to Σ^* , the shift X_G is presented by the graph H in Figure 4. Since H presents an SFT, to show that we reduce to the SFT problem, we

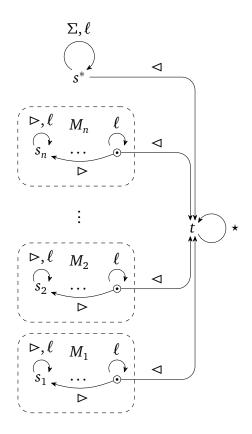


Figure 3: Schematic of Reduction B.

$$\triangleright, \ell, \Sigma \bigcirc q_1 \longrightarrow q_2 \bigcirc \star$$

Figure 4: The graph *H*. As in Figure 3, the Σ above the self loop on q_2 represents a self loop labeled *a* for each $a \in \Sigma$.

need only argue that X_G is not an SFT when there is some word w not in the language of any DFA. Because the ℓ self loops arbitrarily delay the DFA decision to accept or reject, they prevent X_G from having a finite list of forbidden words, or equivalently, from being M-step for any finite M. Finally, to show we reduce to MINIMALITY, we show that X_G does not have a 2-vertex presentation when the DFA languages do not union to Σ^* .

Reduction B. Let M_1, \ldots, M_n be an instance to the DFAUNION problem. Construct the deterministic presentation *G* as follows. Add a state *t* (the *terminal* state), and add a self loop labeled \star on *t*. Add a state *s*^{*}, and add self loops on *s*^{*} labeled by each symbol in $\Sigma \cup \{\ell\}$. Add an edge labeled \triangleleft from *s*^{*} to *t*. Finally, for each *i* = 1,...,*n*,

- 1. embed M_i into G;
- 2. for each state $q \in Q_i$, add an edge labeled \triangleright from q to s_i
- 3. for each accepting state $q \in F_i$, add an edge labeled \triangleleft from q to t;
- 4. for each state $q \in Q_i$, add a self loop labeled ℓ on q.

See Figure 3 for a visualization.

We once again summarize the salient properties of the reduction. First, we define a notation that will be used multiple times: for a word w, we let $h_{\ell}(w)$ denote w with all the ℓ 's removed. (That is, h_{ℓ} is the string homomorphism such that $h_{\ell}(\ell) = \epsilon$ and h(a) = a for $a \neq \ell$.)

Lemma 4.10. The following hold of Reduction B and the graph H from Figure 4.

- (i) $X_G \subseteq X_H$;
- (ii) $(\Sigma \cup \{\ell, \triangleright\})^*, \subseteq F_G(q)$ for all $q \in \bigcup_{i=1}^n Q_i$;
- (iii) for $w \in \Sigma^*$, $w \triangleleft \in F(s_i)$ if and only if $w \in L(M_i)$;
- (iv) for $w \in \Sigma^*$, $\triangleright w \triangleleft \in \mathcal{B}(X_G)$ if and only if $w \in \bigcup_{i=1}^n L(M_i)$.
- (v) for $w \in (\Sigma \cup \{\ell\})^*$, $\triangleright h_\ell(w) \triangleleft \in \mathcal{B}(X_G)$ if and only if $\triangleright w \triangleleft \in \mathcal{B}(X_G)$;

Proof. For (i), let $w \in \mathcal{B}(X_G)$. If *w* does not contain ⊲, then either $w \in (\Sigma \cup \{\ell, \triangleright\})^*$ or $w = \star^m$ for some $m \ge 0$; if $w \in (\Sigma \cup \{\ell, \triangleright\})^*$, then $w \in F_H(q_1)$; if $w = \star^m$ for some $m \ge 0$, then $w \in F_H(q_2)$. Otherwise, if *w* contains ⊲, then we can factor *w* into $w = u \lhd \star^m$ where $u \in (\Sigma \cup \{\ell, \triangleright\})^*$ and $m \ge 0$, for which it follows that $w \in F_H(q_1)$. Thus, for every $w \in \mathcal{B}(X_G)$, we have $w \in \mathcal{B}(X_H)$.

For the other statements, first note that each of the M_i are emulated by the transition action of G in the following way: for $q \in Q_i$ and $w \in \Sigma^*$, we have $w \in F_G(q)$ and $q \cdot w = \delta_i(q, w)$ and $q \in F_i$ if and only if $\lhd \in F_G(q)$. Thus, (ii) follows from the emulation observation and the fact that $\ell \in F_G(q)$ and $q \cdot \ell = q$ and $\triangleright \in F_G(q)$ and $q \cdot \triangleright = s_i$ for all $q \in Q_i$. Statement (iii) follows immediately from the emulation observation as well. Note that $Q_G \cdot \triangleright = \{s_1, \dots, s_n\}$, so (iv) follows from (iii). Finally, for (v), as $q \cdot \ell = q$ for $q \in \bigcup_{i=1}^n Q_i$, by induction on the number of ℓ 's in w, one can show that $Q_G \cdot \triangleright w =$ $Q_G \cdot \triangleright h_\ell(w)$; thus, we have $Q_G \cdot \triangleright h_\ell(w) \lhd = Q_G \cdot \triangleright w \lhd$, which implies $Q_G \cdot \triangleright h_\ell(w) \lhd \neq \emptyset$ if and only if $Q_G \cdot \triangleright w \lhd \neq \emptyset$.

To show the correctness of Reduction A, we first give an alternate reduction to SUB-SHIFT.

Theorem 4.11. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if $X_H \subseteq X_G$.

Proof. First suppose $\mathcal{B}(X_H) \subseteq \mathcal{B}(X_G)$. For every word $w \in \Sigma^*$, we have $\triangleright w \triangleleft \in \mathcal{B}(X_H)$, giving $\triangleright w \triangleleft \in \mathcal{B}(X_G)$. Lemma 4.10(iv) now implies $\bigcup_{i=1}^n L(M_i) = \Sigma^*$. For the converse, suppose $\bigcup_{i=1}^n L(M_i) = \Sigma^*$, and let $u \in \mathcal{B}(X_H)$. There are two cases: either $u \in F_H(q_1)$ or $u \in F_H(q_2)$. If $u \in F_H(q_2)$, then $u = \star^m$ for some $m \ge 0$, which implies that $u \in F_G(t)$, and so $u \in \mathcal{B}(X_G)$. Thus, to complete the proof we need to show that if $u \in F_H(q_1)$, then $u \in \mathcal{B}(X_G)$.

Suppose $u \in F_H(q_1)$. We further break this case into the possible values of $q_1 \cdot u$. If $q_1 \cdot u = q_1$, then $u \in (\Sigma \cup \{\ell, \triangleright\})^*$, so $u \in F_G(s_1)$ and thus $u \in \mathcal{B}(X_G)$. If $q_1 \cdot u = q_2$, then $u = v \triangleleft \star^m$ for some $m \ge 0$ and $v \in (\Sigma \cup \{\ell, \triangleright\})^*$. If v does not contain the symbol \triangleright , then $v \in (\Sigma \cup \{\ell\})^*$, which implies $v \triangleleft \star^m = u \in F_G(s^*)$ and thus $u \in \mathcal{B}(X_G)$. Otherwise,

if *v* contains the symbol \triangleright , then we can factor *v* into $v = x \triangleright w$ where *w* contains no \triangleright ; i.e. $w \in (\Sigma \cup \{\ell\})^*$. Then, we have that $h_{\ell}(w) \in \Sigma^*$, and as $\bigcup_{i=1}^n L(M_i) = \Sigma^*$, we have $\triangleright h_{\ell}(w) \triangleleft \in \mathcal{B}(X_G)$. By Lemma 4.10(v), we have $\triangleright w \triangleleft \in \mathcal{B}(X_G)$, and by Lemma 4.10(iii), we have $w \triangleleft \in F_G(s_i)$ for some s_i . Collecting facts, we have $x \triangleright \in F_G(s_i)$ and $s_i \cdot x \triangleright = s_i$ and $s_i \cdot w \triangleleft = t$ and $*^m \in F_G(t)$. Combining those facts gives us that $x \triangleright w \triangleleft *^m = u \in F_G(s_i)$, so $u \in \mathcal{B}(X_G)$.

Remark 4.12. Interestingly, Theorem 4.11 gives us another proof that SUBSHIFT is PSPACEhard. However, we can easily extend Theorem 4.11 to a stronger hardness result.⁶ Specifically, we can show that SUBSHIFT is PSPACE-hard even when both input instances are synchronizing deterministic presentations. Note that *H* is synchronizing while *G* is not. Construct the presentation *G'* as follows: construct *G*, and let $S \triangleq \{s_1, \ldots, s_n, s^*\}$. For each $q \in S$, add a self loop labeled ℓ_q on q.

For each vertex in $q \in S$, we have that ℓ_q synchronizes to q in G'. Note that every vertex is reachable from a vertex in S, so this implies that G' is synchronizing. Here, we note that $X_H \subseteq X_G$ if and only if $X_H \subseteq X_{G'}$: the forward direction follows from the fact that $X_G \subseteq X_{G'}$, and the reverse direction follows from the fact that if $w \in \mathcal{B}(X_H)$ and $w \in \mathcal{B}(X_{G'})$, then w does not contain the new labels $\{\ell_{s_1}, \ldots, \ell_{s_n}, \ell_{s^*}\}$ added in G', so it must be the case that $w \in \mathcal{B}(X_G)$. This establishes the claim that SUBSHIFT is PSPACE-hard even when both instances are synchronizing deterministic presentations.

As $X_G \subseteq X_H$ by Lemma 4.10(i), we have the following.

Corollary 4.13. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if $X_H = X_G$.

We new show that Reduction B reduces DFAUNION to SFT.

Theorem 4.14. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if X_G is an SFT.

Proof. The edges in *H* are labeled uniquely, so by Lemma 2.6, X_H is an SFT. Thus, if we have $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$, then by Corollary 4.13, $X_G = X_H$ is an SFT.

Conversely, suppose X_G is an SFT. By Theorem 2.5, there is an M such that whenever $uv, vw \in \mathcal{B}(X_G)$ and $|v| \ge M$, then $uvw \in \mathcal{B}(X_G)$. Let $w \in \Sigma^*$. We can find $\triangleright w\ell^M \in F_G(s_1)$ and $w\ell^M \triangleleft \in F_G(s^*)$, so we have $\triangleright w\ell^M$, $w\ell^M \triangleleft \in \mathcal{B}(X_G)$ and and thus $\triangleright w\ell^M \triangleleft \in \mathcal{B}(X_G)$. As $h_\ell(w\ell^M) = w$, Lemma 4.10(v) implies that $\triangleright w \triangleleft \in \mathcal{B}(X_G)$. It follows that $w \in \bigcup_{i=1}^n L(M_i)$ by Lemma 4.10(iv).

Along with Corollary 4.13, the following shows that Reduction B also reduces DFAU-NION to MINIMALITY.

Theorem 4.15. $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$ if and only if X_G has a deterministic presentation with 2 vertices.

⁶In fact, there is another hardness result proved by the previous theorem: SUBSHIFT is PSPACE-hard even when the first argument is fixed. That is, for each deterministic presentation *H*, it is PSPACE-hard to decide when given a deterministic presentation *G* whether $X_H \subseteq X_G$.

Proof. If $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$, then $X_H = X_G$ by Corollary 4.13, so H is a 2-vertex presentration of X_G . Conversely, suppose H' is a deterministic presentation of X_G with 2 vertices. Here, we'll show that H' is isomorphic to H, which implies $X_{H'} = X_H$ and thus $X_H = X_G$, so by Corollary 4.13, we have $\bigcup_{i=1}^{n} L(M_i) = \Sigma^*$.

As $\triangleleft \in F_G(s^*)$, there must be an edge e_{\triangleleft} labeled \triangleleft in H'. If e_{\triangleleft} were a self loop, then $\triangleleft \triangleleft \in \mathcal{B}(X_G)$, a contradiction. Thus, the vertices $q'_1 \triangleq i(e_{\triangleleft})$ and $q'_2 \triangleq t(e_{\triangleleft})$ must be distinct, and $Q_{H'} = \{q'_1, q'_2\}$. Moreover, e_{\triangleleft} must be the unique edge labeled \triangleleft , as in all other cases H' either fails to be deterministic or we again have $\triangleleft \triangleleft \in \mathcal{B}(X_G)$, a contradiction.

As $a \triangleleft \in \mathcal{B}(X_G)$, for each $a \in \Sigma \cup \{\ell\}$, we have an edge e_a labeled a ending at q'_1 . For $a \in \Sigma \cup \{\ell\}$, any edge labeled a must start at q'_1 : if such an edge started at q'_2 , then $\neg a \in \mathcal{B}(X_G)$, a contradiction. Thus, e_a is a self loop and by determinism, e_a is the unique edge labeled a in H'.

As $\triangleleft \star \in \mathcal{B}(X_G)$, there must be an edge e_{\star} labeled \star starting at q'_2 . If e_{\star} ends at q'_1 , then $\star \triangleleft \in \mathcal{B}(X_G)$, a contradiction, so e_{\star} is a self loop. Any edge labeled \star must start at q'_2 : if such an edge started at q'_1 , then $\ell \star \in \mathcal{B}(X_G)$, a contradiction. Thus, by determinism, e_{\star} is the unique edge labeled \star in H'.

Finally, as $\triangleright \ell \in \mathcal{B}(X_G)$, there must be some edge e_{\triangleright} labeled \triangleright ending at q'_1 . Any edge labeled \triangleright must start at q'_1 : if such an edge start at q'_2 , then $\triangleleft \triangleright \in \mathcal{B}(X_G)$, a contradiction. Thus, e_{\triangleright} is a self loop and is the unique edge labeled \triangleright in H'. All of the above implies that the map $q'_i \mapsto q_i$ is an isomorphism between H' and H.

4.3 Hardness of Existence of Synchronizing Words

Berlinkov [4] showed SYNCWORD was PSPACE-hard via reduction from the PSPACEcomplete problem of *subset synchronizability*: given a DFA M and a subset $S \subseteq Q_M$, is there a word w such that $|S \cdot w| = 1$? For completeness, we show the hardness of SYNCWORD via Reduction C from the "complement" of DFAUNION, DFAINT.

Theorem 4.16. SYNCWORD is PSPACE-hard.

For the reduction, we again create pre-initial states p_i for each DFA M_i , with special symbols \triangleright into and \triangleleft out of each DFA, and include them in parallel as in Reduction B. The edges out of accepting states all go to the same shared succes state t. We also add edges labeled \triangleleft from each nonaccepting state in M_i to an individual fail state r_i . By completing this construction appropriately, we ensure that a word is synchronizing if and only if it is synchronizing to t, i.e., if and only if every DFA accepts the subword between \triangleright and \triangleleft .

Reduction C. Let M_1, \ldots, M_n be an instance to the DFAINT problem, and without loss of generality, assume $n \ge 2$. We will construct a essential, deterministic presentation *G* as follows. First, add a state *t* (the *success* state), and add a self loop labeled \triangleleft on *t*. Then, for each $i = 1, \ldots, n$,

1. add a state p_i (the *i*th *pre-initial* state);

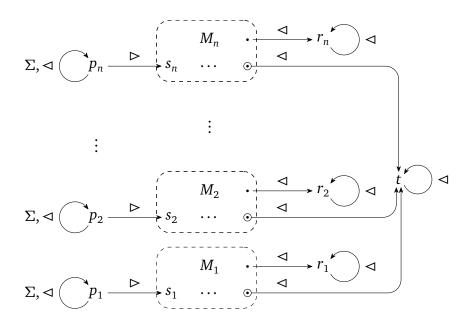


Figure 5: Schematic of Reduction C.

- 2. add a state r_i (the *i*th *fail* state);
- 3. add self loops labeled \triangleleft on p_i and r_i ;
- 4. for each $a \in \Sigma$, add a self loop labeled a on p_i ;
- 5. embed M_i into G;
- 6. add an edge from p_i labeled \triangleright to the corresponding initial state s_i of M_i ;
- 7. for each accepting state *q* in M_i , add an edge labeled \triangleleft from *q* to *t*;
- 8. for each nonaccepting state q in M_i , add an edge labeled \triangleleft from q to r_i .

See Figure 5 for a visualization.

To show the correctness of the reduction, we characterize the synchronizing words of G.

Theorem 4.17. Let *G* be the deterministic presentation obtained from Reduction C on an instance M_1, \ldots, M_n . A word $u \in (\Sigma \cup \{\triangleright, \triangleleft\})^*$ is synchronizing for *G* if and only if there is some $v \in (\Sigma \cup \{\triangleleft\})^*$, $k \ge 1$, and $w \in \bigcap_{i=1}^n L(M_i)$ such that $u = v \triangleright w \triangleleft^k$.

Proof. As usual, the transition action of *G* emulates the behavior of the M_i : for $w \in \Sigma^*$, we have $w \in L(M_i)$ if and only if $p_i \cdot \triangleright w \triangleleft = t$, and $w \notin L(M_i)$ if and only if $p_i \cdot \triangleright w \triangleleft = r_i$.

Suppose *u* is a synchronizing word for *G*. Then, *u* must contain at least one \triangleright ; otherwise $u \in (\Sigma \cup \{ \triangleleft \})^*$, and thus $p_i \cdot u = p_i$ for each *i*, giving $|Q_G \cdot u| \ge n \ge 2$. We can therefore write $u = u_1 \triangleright u_2$. By construction, *u* contains at most one \triangleright , so $u_1, u_2 \in (\Sigma \cup \{ \triangleleft \})^*$. Moreover, we must have $u_2 = w \triangleleft^k$ for some $w \in \Sigma^*$ and $k \ge 0$. Since $Q_G \cdot u_1 \triangleright w = \{p_1 \cdot \triangleright w, \dots, p_n \cdot \triangleright w\}$ and the Q_i are pairwise disjoint, we have $|Q_G \cdot u_1 \triangleright w| = n \ge 2$. Since *u* is synchronizing, we therefore must have $k \ge 1$. Now since $\triangleleft \in F_G(q)$ for $q \in \bigcup_{i=1}^n Q_i$, if there are $i \ne j$ such that $p_i \cdot \triangleright w \triangleleft$ and $p_j \cdot \triangleright w \triangleleft$ are

not both *t*, then $|Q_G \cdot u| \ge 2$. As *u* is synchronizing, we conclude $p_i \cdot \triangleright w \triangleleft = t$ for all *i*. By the above, $w \in \bigcap_{i=1}^n L(M_i)$. Hence, we have $u = u_1 \triangleright w \triangleleft^k$ with $u_1 \in (\Sigma \cup \{\triangleleft\})^*$, $k \ge 1$, and $w \in \bigcap_{i=1}^n L(M_i)$.

Conversely, let $v \in (\Sigma \cup \{ \triangleleft \})^*$, $k \ge 1$, and $w \in \bigcap_{i=1}^n L(M_i)$. Thus, $p_i \cdot \triangleright w \triangleleft = t$ for all i, which implies $Q_G \cdot v \triangleright w \triangleleft^k = \{t\}$.

Corollary 4.18. $\bigcap_{i=1}^{n} L(M_i) \neq \emptyset$ if and only if *G* has a synchronizing word.

As Reduction C therefore reduces DFAINT to SYNCWORD, Theorem 4.16 follows.

5 Size of Synchronizing Words and SDPs

Our reductions also shed light on the size of synchronizing words and presentations. In particular, given a presentation with n vertices, the size of its smallest synchronizing word can be exponentially large in n. Similarly, the size of the smallest synchronizing deterministic presentation can also be exponentially large.

5.1 Shortest synchronizing word size

If NP \neq PSPACE, there cannot be a polynomial upper bound with respect to the number of vertices for the length of the shortest synchronizing word, as SYNCWORD is PSPACEhard. Berlinkov [4] show an unconditional exponential lower bound on maximum length of the shortest synchronizing word, which implies there cannot be a polynomial upper bound for the length of the shortest synchronizing word. Here, we give a simpler construction that achieves roughly the same bound.

First we observe the following property of Reduction B.

Lemma 5.1. Let *G* be a presentation obtained from Reduction B on some input M_1, \ldots, M_n . If $\bigcap_{i=1}^n L(M_i) \neq \emptyset$, then the minimum length of a synchronizing word for *G* is 2 more than the minimum length of a word in $\bigcap_{i=1}^n L(M_i)$.

Proof. From Theorem 4.17, a word *u* is synchronizing for *G* if and only if it has the form $u = v \triangleright w \triangleleft^k$ for any $v \in (\Sigma \cup \{\triangleleft\})^*$, $w \in \bigcap_{i=1}^n L(M_i)$, and $k \ge 1$. A minimum-length synchronizing word u^* for *G* therefore has $v = \epsilon$ and k = 1, and takes $w = w^*$ to be a word of minimum length in $\bigcap_{i=1}^n L(M_i)$. Thus $|u^*| = |\triangleright w^* \triangleleft| = 2 + |w^*|$.

Therefore, to find a presentation of a sofic shift with a large shortest synchronizing word, it suffices to apply Reduction B to DFAs that have a large shortest word in the intersection of their languages. In Appendix B, we adapt a construction from Ang [1] of a family of DFAs $M_{i,k}$ such that each $M_{i,k}$ has 3 states and and the shortest word in $\bigcap_{i=0}^{k} L(M_{i,k})$ is 2^{k} . Using this family of DFAs, if we let G_{k} denote Reduction B applied to $M_{0,k}, M_{1,k}, \ldots M_{k,k}$, then by Lemma 5.1, the shortest synchronizing word for G_{k} has length $2^{k} + 2$. The number of vertices in G_{k} is 2(k+1) + 1 auxillary vertices plus 3(k+1) from the DFAs, giving use 5k + 6 total vertices. We may then define a family of graphs $G^{(n)}$ on

n vertices, which take G_k where $k \triangleq \lfloor \frac{n-6}{5} \rfloor$ and add n-k vertices without affecting the shortest synchronizing word (e.g., by adding a self loops labeled with every $a \in \Sigma \cup \{ \triangleright \}$ and adding an edge labeled \triangleleft to *t*). As $k = \Omega(n)$, this family exhibits the following lower bound.

Theorem 5.2. There is a family of deterministic presentaions $\{G^{(n)}\}$ such that for sufficiently large *n*, each $G^{(n)}$ has *n* vertices and the minimum length of a synchronizing word for $G^{(n)}$ is $2^{\Omega(n)}$.

Remark 5.3. Černý's conjecture states that if a *n*-state DFA has a synchronizing word, then there is one of length at most $(n-1)^2$ [26]. The previous theorem is a counterexample to the generalization of the Černý's conjecture to deterministic presentations of sofic shifts.

5.2 Minimal Synchronizing Deterministic Presentation Size

Throughout this subsection, we will abbreviate "synchronizing deterministic presentation" to SDP. Let X be a sofic shift. A *minimal SDP* of X is a SDP of X possessing the fewest number of vertices among all SDPs of X. Similarly, a *minimal deterministic presentation* of X is a deterministic presentation of X possessing the fewest number of vertices among all deterministic presentations of X. For a given sofic shift X, minimal SDPs of Xare unique up to isomorphism, while minimal deterministic presentations are not neccessarily unique if X is reducible [12]. For irreducible sofic shifts, minimal SDPs are minimal derterministic presentations and vice versa. For reducible sofic shifts, minimal SDPs are minimal deterministic presentations are not necessarily the same.

In fact, we show that the minimal SDP can be exponential larger than a minimal deterministic presentation. The proof relies on multiple-entry DFAs, which are FAs whose only nondeterminism is the fact that there are multiple possible initial states. Formally, a *k*-entry DFA N is a fully deterministic labeled graph along with k states $s_1, \ldots, s_k \in Q_N$ and a set of final states $F \subseteq Q_N$. The language of N is defined as $L(N) \triangleq \bigcup_{i=1}^k \{w \in \mathcal{A}_N^* : \delta(s_i, w) \in F\}$. By Holzer et al. [10, Lemma 3], there exists a family $\{C_k\}$ of multiple-entry DFAs, where C_k is a *k*-entry DFA with *k* states, and and the minimal DFA for $L(C_k)$ has $\sum_{i=1}^k {k \choose i} = 2^k - 1$ states. In other words, even if automata have deterministic transition relations, passing from multiple start states to a single start state can incur an exponential increase in size. We emulate this construction when passing from a nonsynchronizing presentation.

Given a *k*-entry DFA *N*, we construct a sofic shift X_G with deterministic presentation *G* as follows. First, embed *N* into *G*, add a state *t* (the *terminal* state), and add a self loop labeled \star on *t*. Then, for each i = 1, ..., k,

- 1. add a state p_i (the *i*th *pre-initial* state);
- 2. add a self loop labeled * on p_i ;
- 3. add an edge from p_i labeled \triangleright to the corresponding state s_i of N;
- 4. for each accepting state $q \in F$, add an edge labeled \triangleleft from q to t

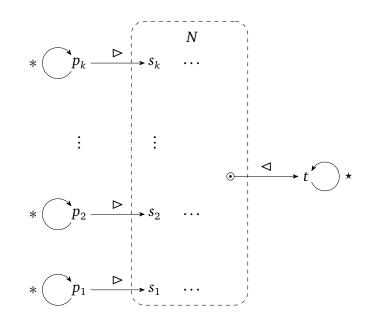


Figure 6: Schematic of the construction from Theorem 5.4.

See Figure 6 for a visualization.

Theorem 5.4. Let *N* be a *k*-entry DFA, and let *G* be the deterministic presentation obtained from the above construction applied to *N*. Let *M* be the minimal DFA of L(N). Interpreting *M* as a 1-entry DFA, let *H* be the deterministic presentation obtained from the construction applied to *M*. Then, *H* is the minimal SDP for X_G .

Proof. The proof that $X_G = X_H$ is similar to the proof of Theorem 4.11. To show *H* is the minimal SDP for X_G , by Jonoska [12, Theorem 5.5], it suffices to show *H* is follower-separated. For distinct $p, q \in Q_M$, as *M* is a minimal DFA, either there is some word $\delta(p, w) \notin F$ with $\delta(q, w) \notin F$ or there is some word $\delta(p, w) \notin F$ with $\delta(q, w) \notin F$. Without loss of generality, we may assume the former, as we can swap the roles of *p* and *q* for the latter case. Then, $w \triangleleft \notin F_H(p)$ while $w \triangleleft \notin F_H(q)$, so *p* and *q* have distinct follower sets. For distinct $p, q \in Q_H$ where one of *p* or *q* is the pre-initial state or the terminal state, follower-separation follows from the presence of * or *.

Thus, the size of the minimal SDP for X_G is determined by the size of the minimal DFA for L(N). Applying this construction to C_k gives us a deterministic presentation with 2k+1 vertices whose minimal synchronizing deterministic presentation with $(2^k-1)+2 = 2^k + 1$ vertices. The following theorem follows easily from this observation.

Theorem 5.5. There is a family of sofic shifts $\{X_n\}$ such that for sufficiently large n, the minimal deterministic presentation of X_n has at most n vertices and the minimal synchronizing deterministic presentation of X_n has $2^{\Omega(n)}$ vertices.

Proof. Let *n* be sufficiently large. If we apply the construction to $C_{n'}$ where $n' \triangleq \lfloor \frac{n-1}{2} \rfloor$, we get a presentation *G* with at most *n* vertices such that the minimal SDP for X_G has $2^{n'} + 1$

	Aut	omata	Sofic Shifts			
Problem	DFA	FA	IDP	SDP	DP	Р
UNIVERSALITY	Р	PSP-c	Р	Р	Р	PSP-c [7]
Equality	Р	PSP-c	Р	Р	PSP-c	PSP-c
Subshift	Р	PSP-c	Р	PSP-c	PSP-c	PSP-c
MINIMALITY	Р	PSP-c	P [17]	P [12]	PSP-c	PSP-c
SyncWord	P [9]		Р		PSP-c [4]	PSP-c
IRREDUCIBILITY				Р	PSP-c	PSP-c
Sft			P [17]	Р	PSP-c	PSP-c
ЗSDP					PSP-c	PSP-c

Table 2: An overview of our results and comparison to related automata theory results. The classes sofic shifts are as follows: IDP = irreducible deterministic presentation, SDP = synchronizing deterministic presentation, DP = (general) deterministic presentation, and P = (general) presentation. The complexity classes are P = solvable in polynomial time and PSP-c = PSPACE-complete. For FAs, SUBSHIFT means "is $L(M) \subseteq L(N)$?" Entries of the table corresponding to results we prove (or re-prove) have hyperlinks to their respective proofs.

vertices. Thus, since *G* has at most *n* vertices, then a minimal deterministic presentation for X_G must have at most *n* vertices, and as $n' = \Omega(n)$, the minimal SDP for X_G has $2^{\Omega(n)}$ vertices.

6 Discussion

We first overview our results, together with a discussion of related problems and a comparison to results from automata theory. We conclude with open problems.

6.1 Overview of results

We summarize our results in Table 2. The table includes complexity results from the automata theory literature for analagous problems for DFAs and FAs. One conclusion from this table is that, from a computational complexity standpoint, irreducible presentations behave like DFAs, as do synchronizing deterministic presentations with the exception of SUBSHIFT. On the other hand, non-deterministic presentations behave like NFAs, as do general deterministic presentations, with the exception of UNIVERSALITY.

The remainder of this subsection is devoted to entries of the table which were not discussed in the previous sections. To begin, the problem of UNIVERSALITY asks whether a given deterministic presentation *G* satisfies $X_G = \mathcal{A}_G^{\mathbb{Z}}$. Clearly, we have $X_G \subseteq \mathcal{A}_G^{\mathbb{Z}}$ for any *G*, so the problem reduces to deciding whether $\mathcal{A}_G^{\mathbb{Z}} \subseteq X_G$. This condition can be decided in polynomial time, as there is an irreducible deterministic presentation of $\mathcal{A}_G^{\mathbb{Z}}$ which is

a single vertex and a self loop on that vertex labeled *a* for each $a \in A_G$, so one may use Algorithm 2 to decide whether $A_G^{\mathbb{Z}} \subseteq X_G$. However, for nondeterministic presentations, universality is PSPACE-complete [7]. UNIVERSALITY is equivalent to MINIMALITY for k = 1, i.e. deciding if the sofic shift presented by a deterministic presentation has a 1vertex presentation, as $X_G = A_G^{\mathbb{Z}}$ exactly when X_G has a 1-vertex presentation. Thus, MINIMALITY for k = 1 is in P, and our results show that MINIMALITY for $k \ge 2$ is PSPACEcomplete. (Our reduction shows hardness for k = 2; simple modifications give k > 2.)

Given a synchronizing determinstic presentation, SYNCWORD is trivial, as a synchronizing word always exists. To actually find a synchronizing word in this case, however, Algorithm 1 is not sufficient: for reducible presentations, the algorithm can fail when there are two vertices with the same follower set but no word sending them to the same vertex (e.g. r_1 and r_2 in Reduction C). Fortunately, the proof of Theorem 3.3 gives a method of constructing a word that synchronizes to any vertex: find a synchronizing word for an initial irreducible component, a word that separates it from the rest of the graph, and finally a word leading to the desired vertex. This procedure can be implemented using Algorithm 1 and Algorithm 2 using only polynomial time.

Similarly, IRREDUCIBILITY is trivial for irreducible deterministic presentations, as the shift is guaranteed to be irreducible. For a synchronizing deterministic presentation G, IRREDUCIBILITY can be decided by testing whether G is irreducible. In particular, if X_G is irreducible, then by Theorem 2.4, the subgraph induced by all the synchronizing vertices is irreducible; as every vertex is synchronizing, G is therefore irreducible.

The problem \exists SDP is also trivial for irreducible deterministic presentations, since every irreducible sofic shift has a synchronizing deterministic presentation by Theorem 2.4. One can compute this synchronizing deterministic presentation in polynomial time by simply computing the follower-separation G/\sim . The presentation G/\sim is irreducible as G is, so every vertex in G/\sim is reachable from every other vertex. As every follower-separated deterministic presentation has a synchronizing word, by irreducibility, one can extend this word to one that synchronizes to any other vertex.

For deterministic presentations in general, all the problems in Table 2, with the exception of SYNCWORD, remain PSPACE-complete when restricted to follower-separated instances. The reason is those problems ask a question about the sofic shift a given input presents, and follower-separation of an input is a polynomial-time operation that preserves the sofic shift it presents. For example, given presentations *G* and *H*, deciding if $X_{G} = X_{H}$ is equivalent to deciding if $X_{G/\sim} = X_{H/\sim}$, as $X_{G} = X_{G/\sim}$ and $X_{H} = X_{H/\sim}$.

6.2 Open problems

Aside from long-standing open problems like the decidability of conjugacy for sofic shifts, our work suggests several interesting open questions pertaining to the size of various objects. For deterministic presentations in general, the shortest synchronizing word in a presentation can be exponentially large. However, for follower-separated deterministic presentations, the shortest synchronizing word has at most cubic length with respect to the number of vertices; for a follower-separated input to Algorithm 1, the algorithm always finds a synchronizing word, and one can easily see that the word returned must be at most cubic length. Actually, by Exercise 3.4.10 of Lind and Marcus [17], one can see that upper bound can be improved to n(n-1), where n is the number of vertices in the presentation. To our knowledge, it is open whether this bound is tight.

For a shift space X, define the *minimum step* of X to be the minimum M such that X is M-step. By Jonoska [12], every SFT X has a synchronizing deterministic presentation. Let s(X) denote the number of vertices the minimal synchronizing deterministic presentation of X. What is the relationship between s(X) and the minimum step of X? By Remark 3.6, we know that the minimum step of an SFT X is $O(s(X)^2)$. To our knowledge, it is also open whether this bound it tight. One lower bound arises from the family of *run-length limited shifts* $\{X_n\}$, which have minimum step $\Omega(s(X_n))$. We can repeat the same question for the size of a minimal deterministic presentation. Let $s_d(X)$ denote the number of vertices in a minimal deterministic presentation of X. What is the relationship between $s_d(X)$ and the minimum step of X? In Appendix A, we generalize Remark 3.6 to deterministic presentations in general as Proposition A.1: for a deterministic presentation G, X_G is an SFT if and only if it is $2^{2|Q_G|}$ -step, which implies that the minimum step of an SFT X is $2^{O(s_d(X))}$. Again, it is open whether this bound is tight.

References

- [1] Thomas Ang. Problems related to shortest strings in formal languages. Master's thesis, University of Waterloo, 2010.
- [2] André Arnold, Anne Dicky, and Maurice Nivat. A note about minimal nondeterministic automata. *Bulletin of the EATCS*, 47:166–169, 1992.
- [3] Sanjeev Arora and Boaz Barak. *Computational complexity: a modern approach*. Cambridge University Press, 2009.
- [4] Mikhail V Berlinkov. On two algorithmic problems about synchronizing automata. In International Conference on Developments in Language Theory, pages 61–67. Springer, 2014.
- [5] Kellogg S Booth. Isomorphism testing for graphs, semigroups, and finite automata are polynomially equivalent problems. *SIAM Journal on Computing*, 7(3):273–279, 1978.
- [6] Ethan M Conen and Michael E Paul. Finite procedures for sofic systems. *Monatshefte für Mathematik*, 83(4):265–278, 1977.
- [7] Eugen Czeizler and Jarkko Kari. On testing the equality of sofic systems. *Internal Proceedings of the XIth MONS DAYS Of Theoretical Computer Science*, 2006.

- [8] Sarah Day and Rafael Frongillo. Sofic shifts via conley index theory: Computing lower bounds on recurrent dynamics for maps. *SIAM Journal on Applied Dynamical Systems*, 18(3):1610–1642, 2019.
- [9] David Eppstein. Reset sequences for monotonic automata. *SIAM Journal on Computing*, 19(3):500–510, 1990.
- [10] Markus Holzer, Kai Salomaa, and Sheng Yu. On the state complexity of k-entry deterministic finite automata. *Journal of Automata, Languages and Combinatorics*, 6(4):453–466, 2001.
- [11] John Hopcroft. An n log n algorithm for minimizing states in a finite automaton. In *Theory of machines and computations*, pages 189–196. Elsevier, 1971.
- [12] Nataša Jonoska. Sofic shifts with synchronizing presentations. *Theoretical Computer Science*, 158(1-2):81–115, 1996.
- [13] Dexter Kozen. Lower bounds for natural proof systems. In 18th Annual Symposium on Foundations of Computer Science (sfcs 1977), pages 254–266. IEEE, 1977.
- [14] Dexter C Kozen. *Automata and computability*. Springer Science & Business Media, 2012.
- [15] Jaroslaw Kwapisz. Cocyclic subshifts. *Mathematische Zeitschrift*, 234(2):255–290, 2000.
- [16] Jaroslaw Kwapisz. Transfer operator, topological entropy and maximal measure for cocyclic subshifts. *Ergodic Theory and Dynamical Systems*, 24(4):1173–1197, 2004.
- [17] Douglas Lind and Brian Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 2nd edition, 2021.
- [18] Andreas Malcher. Minimizing finite automata is computationally hard. *Theoretical Computer Science*, 327(3):375–390, 2004.
- [19] B. Marcus and S. Williams. Symbolic dynamics. *Scholarpedia*, 3(11):2923, 2008.
 doi: 10.4249/scholarpedia.2923. revision #143845.
- [20] Jean-Éric Pin. Mathematical foundations of automata theory. *Lecture notes LIAFA, Université Paris*, 7, 2010.
- [21] Tyler Schrock. *On the complexity of isomorphism in finite group theory and symbolic dynamics*. PhD thesis, University of Colorado at Boulder, 2019.
- [22] Tyler Schrock and Rafael Frongillo. Computational complexity of k-block conjugacy. *Theoretical Computer Science*, 856:21–40, 2021.

- [23] H Shabana. Exact synchronization in partial deterministic automata. In *Journal of Physics: Conference Series*, volume 1352, page 012047. IOP Publishing, 2019.
- [24] Hanan Shabana and Mikhail Volkov. Careful synchronization of partial deterministic finite automata. *arXiv preprint arXiv:2002.01045*, 2020.
- [25] Nicholas F Travers and James P Crutchfield. Exact synchronization for finite-state sources. *Journal of Statistical Physics*, 145(5):1181–1201, 2011.
- [26] Mikhail V Volkov. Synchronizing automata and the Černý conjecture. In International conference on language and automata theory and applications, pages 11–27. Springer, 2008.

Appendix A Problems in PSPACE

Here, we show that all the problems in Table 1 are in PSPACE. We will rely heavily on Savitch's theorem: if there is a nondeterministic polynomial-space algorithm for a decision problem, then there is a (deterministic) polynomial-space algorithm for it as well [3].

Let *G* be a deterministic presentation. In general, for the decision problems we work with, given a word *w*, we usually want to know the value of $Q_G \cdot w$. In particular, we have the correspondence $Q_G \cdot w \neq \emptyset$ if and only if $w \in \mathcal{B}(X_G)$. In fact, $q \in Q_G \cdot w$ if and only if there is path labeled *w* ending at that *q*. Note the asymmetry of information here: the set of vertices in *G* such that there is a path labeled *w* starting at that vertex is not encoded in $Q_G \cdot w$. A situation arises because of this asymmetry when designing polynomial-space algorithms for \exists SDP and SFT when only using the transition action: one must deduce $Q_G \cdot uw$ given only $Q_G \cdot w$ and *u*, but not *w*, a problem which is generally ill-posed.

To fix this asymmetry, we introduce the *action* of a word. The action of a word w in G is a binary relation $\llbracket w \rrbracket_G$ on the vertices of G such that $(p,q) \in \llbracket w \rrbracket_G$ if and only if there is a path labeled w from p to q. In other words,

$$\llbracket w \rrbracket_G \triangleq \{ (p,q) \in Q^2 : w \in F_G(p) \text{ and } p \cdot w = q \}.$$

Note that $Q_G \cdot w = \{q \in Q : \exists p \in Q, (p,q) \in \llbracket w \rrbracket_G\}$, so the action of a word still encodes the transition action, but also includes more information. In particular, there is a path labled *w* ending at *q* if and only if there is a vertex *p* with $(p,q) \in \llbracket w \rrbracket_G$, and there is a path labeled *w* starting at *p* if and only if there is a vertex *q* with $(p,q) \in \llbracket w \rrbracket_G$. In fact, we have $\llbracket w \rrbracket_G \neq \emptyset$ if and only if $w \in \mathcal{B}(X_G)$. Observe that $\llbracket \varepsilon \rrbracket_G = \{(q,q) : q \in Q_G\}$.

Just as the transition action had a nice algebraic behvaior via the equation $S \cdot uv = (S \cdot u) \cdot v$, there is an analagous equation for actions involving the *relational composition* operation. For binary relations $R, S \subseteq Q^2$, define the relational composition R; S (pronounced R *then* S) as

$$R; S \triangleq \{(p,r) \in Q^2 : \exists q \in Q, (p,q) \in R \text{ and } (q,r) \in S \}.$$

One can verify that $(p,q) \in \llbracket u \rrbracket_G ; \llbracket v \rrbracket_G$ if and only if there is a path labeled uv from p to q. This implies that for all $u, v \in \mathcal{A}_G^*$, we have $\llbracket uv \rrbracket_G = \llbracket u \rrbracket_G ; \llbracket v \rrbracket_G$. For example, we can deduce that $\llbracket \epsilon \rrbracket_G$ acts as an identity: $\llbracket w \rrbracket_G ; \llbracket \epsilon \rrbracket_G = \llbracket w \epsilon \rrbracket_G = \llbracket w \rrbracket_G$. Algebraically, we can summarize that $\llbracket \cdot \rrbracket_G$ is a semigroup morphism from \mathcal{A}_G^* that recognizes $\mathcal{B}(X_G)$ [20].

We first show that SUBSHIFT is PSPACE by giving a nondeterministic polynomial-time algorithm for the complement. Given a deterministic presentations *G* and *H*, deciding $X_G \notin X_H$ is equivalent to deciding if there exists a word *w* with $w \in \mathcal{B}(X_G)$ and $w \notin \mathcal{B}(X_H)$. Using $\llbracket \cdot \rrbracket_G$, this is equivalent to deciding if there exists a word *w* with $\llbracket w \rrbracket_G \neq \emptyset$ and $\llbracket w \rrbracket_H = \emptyset$. The following nondeterministic algorithm decides the latter predicate: initialize a relation *R* to $\llbracket \epsilon \rrbracket_G$ and a relation *S* to $\llbracket \epsilon \rrbracket_H$, and then repeat the following forever: if $R \neq \emptyset$ and $S = \emptyset$, then return true; otherwise, nondeterministically choose some $a \in \mathcal{A}_G \cup \mathcal{A}_H$ and update *R* to *R*; $\llbracket a \rrbracket_G$ and *S* to *S*; $\llbracket a \rrbracket_H$.

The size of *R* and *S* are polynomial with respect to the size of *G*, so the algorithm is a nondeterministic polynomial-space algorithm. ⁷ Thus, by Savitch's theorem, SUBSHIFT is in PSPACE. Membership of EQUALITY in PSPACE follows, by testing both $X_G \subseteq X_H$ and $X_H \subseteq X_G$, using the polynomial-space algorithm for SUBSHIFT twice.

Next, we have that MINIMALITY is in PSPACE: given a deterministic presentation *G* and a positive integer *k*, when $|Q_G| \le k$, we can always admit a presentation of X_G with *k* vertices by adding superfluous vertices to *G*. In the case of $|Q_G| > k$, we can nondeterministically guess a presentation with *k* vertices (whose size is polynomially bounded as $|Q_G| > k$) and use the polynomial-space algorithm for EQUALITY to determine if our guess is a presentation of X_G .

We also have that IRREDUCIBILITY is in PSPACE: given a follower-separated deterministic presentation *G*, using the polynomial-space algorithm for EQUALITY, we can test if any terminal irreducible component presents X_G . By Theorem 2.4, such a terminal irreducible component exists if and only if X_G is irreducible.

The argument for \exists SDP is more complex. We will break the algorithm into nondeterministic subprocedures, which each perform a particular test. We can determinize all three with Savitch's theorem, allowing us to use them in further subprocedures. Let *G* be a deterministic presentation, and let $R \subseteq Q_G^2$ be a binary relation.

- We say *R* is an action if there is a word *w* with [[*w*]]_G = *R*. We denote the set of actions as [[A_G^{*}]]. A simple nondeterministic polynomial-space procedure to test if *R* is an action can be implemented by initializing a relation *S* as [[*ϵ*]]_G, and in a loop forever: if *S* = *R*, then return true; otherwise, nondeterminically choose *a* ∈ A_G and update *S* to *S*; [[*a*]]_G.
- We say *R* is intrinsically synchronizing if for every $S, T \in \llbracket \mathcal{A}_G^* \rrbracket$, $S ; R \neq \emptyset$ and $R ; T \neq \emptyset$ imply $S ; R ; T \neq \emptyset$. One can verify that for a word $w \in \mathcal{B}(X_G)$, *w* is intrinsically synchronizing for X_G if and only if $\llbracket w \rrbracket_G$ is intrinsically synchronizing. A nondeterministic polynomial-space procedure to test if *R* is not intrinsically synchronizing can be implemented by nondeterministically choosing $S, T \subseteq Q^2$, and

⁷This algorithm does not halt, but in principle, any space-bounded algorithm can be modified to halt with a logarithmic overhead.

testing the following four predicates: (i) *S* and *T* are actions, (ii) *S* ; $R \neq \emptyset$, (iii) *R* ; $T \neq \emptyset$, and (iv) *S* ; *R* ; $T = \emptyset$. If all the tests were true, then return true.

• We say *R* is preceded by an intrinsically synchronizing word if there is some $S \in [A_G^*]$ that is intrinsically synchronizing and $S; R \neq \emptyset$. One can verify that for a word *w*, there is a word *u* that is intrinsically synchronizing for X_G with $w \in F_{X_G}(u)$ if and only if $[w]_G$ is preceded by an intrinsically synchronizing word. A nondeterministic polynomial-space procedure to test if *R* is preceded by an intrinsically synchronizing word can be implemented by nondeterministically choose $S \subseteq Q^2$, and testing three predicates: (i) *S* is an action, (ii) *S* is intrinsically synchronizing, and (iii) $S; R \neq \emptyset$. If all the tests were true, then return true.

With these definitions and by Theorem 2.3, one can verify that X_G has a synchronizing deterministic presentation if and only if for every $R \in [\![\mathcal{A}_G^*]\!]$ with $R \neq \emptyset$, R is preceded by an intrinsically synchronizing word. Using our subprocedures, a nondeterministic polynomial-space procedure to test if X_G does not have a synchronizing deterministic presentation can be implemented by nondeterministically choosing $R \subseteq Q^2$ and testing whether (i) R is an action, (ii) $R \neq \emptyset$, and (iii) R is not preceded by an intrinsically synchronizing word.

Finally, we show SFT is in PSPACE. Recall from Remark 3.6 that for a followerseparated synchronizing deterministic presentation *G*, X_G is an SFT if and only if it is $(|Q_G|^2 - |Q_G|)$ -step. To decide SFT in polynomial space, we first generalize this characterization to when *G* is not necessarily follower-separated and synchronizing.

Proposition A.1. Let *G* be a deterministic presentation. Then, X_G is an SFT if and only if it is $2^{2|Q_G|}$ -step.

Proof. If X_G is $2^{2|Q_G|}$ -step, then it is an SFT. Conversely, suppose X_G is an SFT. Then, by Jonoska [12, Corollary 5.4, Proposition 6.2], X_G has a follower-separated synchronizing deterministic presentation, and it has at most $2^{|Q_G|}$ vertices. Thus, by Remark 3.6, X_G is M-step for some $M \le (2^{|Q_G|})^2 - (2^{|Q_G|})$. Since any shift space that is M-step is also M'-step for every $M' \ge M$, then X_G must be $2^{2|Q_G|}$ -step.

A nondeterministic polynomial-space procedure to test if X_G is not $2^{2|Q_G|}$ -step can be implemented by initializing a counter with $2|Q_G|$ bits to 0 and initializing a relation R to $\llbracket \epsilon \rrbracket_G$. Then, use the counter to repeat the following $2^{2|Q_G|}$ times: nondeterministically choose $a \in \mathcal{A}_G$ and update R to R; $\llbracket a \rrbracket_G$. After the loop, nondeterministically choose $S \subseteq Q^2$ and test the following three predicates: (i) S is an action, (ii) R; $S \neq \emptyset$, and (iii) R; S is not intrinsically synchronizing. If all the tests were true, then return true.

Appendix B Intersection Construction

To prove Theorem 5.2, we adapt the construction from Ang [1]. We construct a family of DFAs $M_{i,k}$ and show that they satisfy the following.

Theorem B.1. For every $k \ge 0$, the language $\bigcap_{i=0}^{k} L(M_{i,k})$ is nonempty and the minimum length of a word in $\bigcap_{i=0}^{k} L(M_{i,k})$ is 2^{k} .

We will define $M_{i,k}$ for $i \ge 0$ and $k \ge 0$. Each $M_{i,k}$ has the state set $Q \triangleq \{q_0, q_1, q^*\}$ and are defined over the alphabet $\{0, 1, \ldots, k\}$. The transition function of $M_{i,k}$ is denoted $\delta_{i,k}$ and is defined as δ_i restricted to the domain $Q \times \{0, 1, \ldots, k\}$, where $\delta_i : Q \times \mathbb{N} \to Q$ is defined as follows. For $j \ge 0$, we define

$$\delta_0(q_0, j) \triangleq \begin{cases} q_1 & \text{if } j = 0 \\ q^* & \text{otherwise} \end{cases} \qquad \delta_0(q_1, j) \triangleq \begin{cases} q_1 & \text{if } j \neq 0 \\ q^* & \text{otherwise} \end{cases} \qquad \delta_0(q^*, j) \triangleq q^*$$

For $i \ge 1$ and $j \ge 0$, we define

$$\delta_{i}(q_{0}, j) \triangleq \begin{cases} q_{0} & \text{if } j > i \\ q_{1} & \text{if } j < i \\ q^{*} & \text{otherwise} \end{cases} \quad \delta_{i}(q_{1}, j) \triangleq \begin{cases} q_{0} & \text{if } j = i \\ q_{1} & \text{if } j > i \\ q^{*} & \text{otherwise} \end{cases} \quad \delta_{i}(q^{*}, j) \triangleq q^{*}$$

We set the initial state of $M_{i,k}$ to q_0 for every $i \ge 0$ and $k \ge 0$. We set the final state of $M_{0,k}$ to just q_1 , and for $i \ge 1$, we set the final state of $M_{i,k}$ to just q_0 . Figure 7 depicts an example of the construction. Essentially, for $i \ge 1$, we have that δ_i acts on Q in the following way: q^* is a sink state, so no word that under that action of δ_i that visits the sink state will be in the language of $M_{i,k}$; for j > i, we have that $q \mapsto \delta_i(q, j)$ is the identity function; for $j \le i$, the only transition out of q_0 that does not lead to q^* is when $j \ne i$, and conversely, the only transition out of q_1 that does not lead to q^* is when j = i.

Define $h_k: \{0, 1, ..., k\}^* \to \{0, 1, ..., k\}^*$ to be the the string homomorphism $h_k: j \mapsto jk$ that inserts the symbol k after every symbol from the input. Define $w_0 \triangleq 0$ and $w_{k+1} \triangleq h_{k+1}(w_k)$ for $k \ge 0$. Once can show by induction for $k \ge 0$ that $w_k \in \bigcap_{i=0}^k L(M_{i,k})$ and that $|w_k| = 2^k$. Thus, the minimum length of a word in $\bigcap_{i=0}^k L(M_{i,k})$ is at most 2^k . In fact, as we show next, any word in $\bigcap_{i=0}^k L(M_{i,k})$ must have length at least 2^k , so w_k achieves the minimal length.

Proposition B.2. For all $k \ge 0$, if $w \in \bigcap_{i=0}^{k} L(M_{i,k})$, then $|w| \ge 2^{k}$.

Proof. We show this by induction. For k = 0, any word in $L(M_{0,0})$ must have length at least $1 = 2^0$, so the proposition holds for k = 0.

Now, suppose the proposition holds at a given $k \ge 0$. Let $w \in \bigcap_{i=1}^{k+1} L(M_{i,k+1})$. Note that when $i \le k$, then $\delta_{i,k+1}(q,k+1) = q$ for all $q \in Q$. Thus, if we let w' denote the word with every occurance of k + 1 in w removed, then for $i \in \{0, 1, \ldots, k\}$, we have $\delta_{i,k+1}(q,w') = \delta_{i,k+1}(q,w)$, which implies that $w' \in \bigcap_{i=1}^{k} L(M_{i,k+1})$. In fact, as $w' \in \{0, 1, \ldots, k\}^*$ and as $\delta_{i,k+1}(q,j) = \delta_{i,k}(q,j)$ when $j \in \{0, 1, \ldots, k\}$, we have $w' \in \bigcap_{i=0}^{k} L(M_{i,k})$. By our induction hypothesis, we have that $|w'| \ge 2^k$.

Note that as $w \in L(M_{k+1,k+1})$, then *w* must alternate between some symbol in $\{0, 1, ..., k\}$ and k + 1, which implies number of k + 1's in *w* must equal the number of $\{0, 1, ..., k\}$'s. Thus, as |w'| measures the number of $\{0, 1, ..., k\}$'s in *w*, the number of k + 1's in *w* must

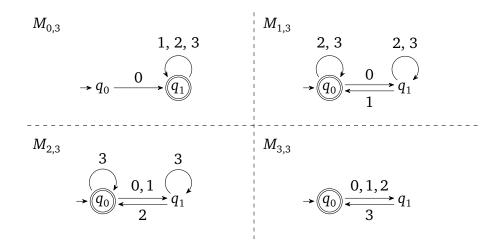


Figure 7: $M_{i,k}$ for $i \in \{0, 1, 2, 3\}$ and k = 3. The transitions to q^* are not depicted.

be at least 2^k . Putting this together, we have $|w| \ge 2 \cdot 2^k = 2^{k+1}$, so the proposition holds at k + 1.