On Mixed Linear Layouts of Series-Parallel Graphs

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Abstract. A mixed s-stack q-queue layout of a graph consists of a linear order of its vertices and of a partition of its edges into s stacks and q queues, such that no two edges in the same stack cross and no two edges in the same queue nest. In 1992, Heath and Rosenberg conjectured that every planar graph admits a mixed 1-stack 1-queue layout. Recently, Pupyrev disproved this conjectured by demonstrating a planar partial 3-tree that does not admit a 1-stack 1-queue layout. In this note, we strengthen Pupyrev's result by showing that the conjecture does not hold even for 2-trees, also known as series-parallel graphs.

Keywords: mixed linear layouts, queue layouts, book embeddings, seriesparallel graphs

1 Introduction

Over the years, linear layouts of graphs have been a fruitful subject of intense research, which has resulted in several remarkable results both of combinatorial and of algorithmic nature; see, e.g., [6,13,18,20,26,28]. A linear layout of graph is defined by a total order of its vertex-set and by a partition of its edge-set into a number of subsets, called *pages*. By imposing different constraints on the edges that may reside in the same page, one obtains different types of linear layouts; see [1,7,20,24,28]. The most notable ones are arguably the stack and the queue layouts (the former are commonly referred to as *book embeddings* in the literature), as is evident from the numerous papers that have been published over the years; see [14] for a short introduction.

In a stack (queue) layout of a graph, no two indepedent edges of the same page, called stack (queue) in this context, are allowed to cross (nest, resp.) with respect to the underlying linear order; see [6] and [20]. In other words, the endpoints of the edges assigned to the same stack follow the last-in-first-out model in the underlying linear order, while the endpoints of the edges assigned

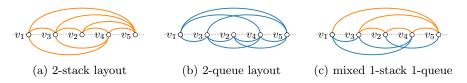


Fig. 1: Illustration of different linear layouts of the complete graph on five vertices v_1, \ldots, v_5 minus the edge (v_1, v_2) .

to the same queue follow the first-in-first-out model; see Fig. 1. The minimum number of stacks (queues) required by any of the stack (queue) layouts of a graph is commonly referred to as its *stack-number* (*queue-number*, resp.). Accordingly, the stack-number (queue-number) of a class of graphs is the maximum stack-number (queue-number, resp.) over all its members.

Known Results. A large body of the literature is devoted to the study of bounds on the stack- and the queue-number of different classes of graphs.

For stack layouts, the most remarkable result is due to Yannakakis, who back in 1986 showed that every planar graph admits a 4-stack layout [27,28]. Recently, Bekos et al. [5] and Yannakakis [29] independently established that the stacknumber of the class of planar graphs is 4, by demonstrating planar graphs that do not admit 3-stack layouts. Certain subclasses of planar graphs, however, allow for layouts with fewer than four stacks, e.g., 4-connected planar graphs [23], series-parallel graphs [25], planar 3-trees [18], and others [4,8,15,16,17,21,22].

For queue layouts, Dujmović et al. [13] recently showed that every planar graph admits a 49-queue layout, improving over previously known logarithmic bounds [3,10,11,12]. However, the exact queue-number of the class of planar graphs is not yet known, as the currently best-known lower bound is 4 [2]. Again, several subclasses of planar graphs allow for layouts with significantly fewer than 49 queues, e.g., outerplanar graphs [19], series-parallel graphs [25] and planar 3-trees [2].

Motivation. Back in 1992, Heath and Rosenberg [20] proposed a natural generalization of stack and queue layouts, called $mixed\ s$ -stack q-queue layout, that supports s stack-pages and q queue-pages. In their seminal paper [20], they conjectured that every planar graph admits a mixed 1-stack 1-queue layout. However, Pupyrev [24] recently showed that the conjecture does not hold even for partial planar 3-trees. This negative result naturally raises the question whether the conjecture holds for other subclasses of planar graphs. To this end, Pupyrev conjectured that bipartite planar graphs admit mixed 1-stack 1-queue layouts.

Our contribution. We make a step forward in understanding which subclasses of planar graphs admit mixed 1-stack 1-queue layouts by providing a negative certificate for the class of 2-trees (also known as maximal series-parallel graphs). This improves upon the partial planar 3-tree negative example by Pupyrev [24]. Note that 2-trees admit both 2-stack layouts and 3-queue layouts [25].

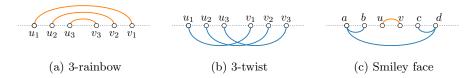


Fig. 2: Illustration of: (a) a 3-rainbow, (b) a 3-twist, and (c) a smiley face.

Preliminaries. A linear order \prec of a graph G is a total order of its vertices. Let $F = \{(u_i, v_i); i = 1, \ldots, k\}$ be a set of $k \geq 2$ independent edges such that $u_i \prec v_i$, for all $1 \leq i \leq k$. If the order is $[u_1, \ldots, u_k, v_k, \ldots, v_1]$, then we say that the edges of F form a k-rainbow, while if the order is $[u_1, \ldots, u_k, v_1, \ldots, v_k]$, then the edges of F form a k-twist. Two edges that form a 2-twist (2-rainbow) are referred to as crossing (nested, resp.). A stack (queue) is a set of pairwise non-crossing (non-nested, resp.) edges. A mixed s-stack q-queue layout $\mathcal L$ of G consists of a linear order \prec of G and a partition of the edges of G into s stacks and g queues; for short, we refer to $\mathcal L$ as mixed layout when s = q = 1. An edge in a stack (queue) in $\mathcal L$ is called a stack-edge (queue-edge, resp.).

The operation of attaching a vertex u to an edge (v, w) of a graph G consists of adding to G vertex u and edges (u, v) and (u, w). Vertex u is said to be attached or being an attachment of (v, w). A 2-tree is a graph obtained from an edge by repeatedly attaching a vertex to an edge. Consider a mixed s-stack q-queue layout \mathcal{L} of a 2-tree. We say that a vertex u attached to an edge (v, w) is a stack-attachment (queue-attachment) of (v, w) if both (u, v) and (u, w) are stack-edges (queue-edges, resp.) in \mathcal{L} . Vertex u is a mixed-attachment of (v, w) if one of (u, v) and (u, w) is a queue-edge and the other is a stack-edge in \mathcal{L} .

2 The Main Result

In this section, we define a family $\{G(k,\ell);\ k,\ell\in\mathbb{N}^+\}$ of 2-trees, and we prove that infinitely many members of it do not admit mixed layouts. For $\ell\geq 1$, $G(1,\ell)$ is an edge; for k>1, $G(k,\ell)$ is obtained from $G(k-1,\ell)$ by attaching ℓ vertices to each edge of it. For convenience, we let $\overline{G}(k,\ell)$ be the graph $G(k,\ell)\setminus G(k-1,\ell)$, that is, the graph induced by the edges that belong to $G(k,\ell)$ but not to $G(k-1,\ell)$. In the following lemmas, we study properties of a mixed layout of graph $G(k,\ell)$.

Lemma 1. Let \mathcal{L} be a mixed layout of $G(k,\ell)$ with $k > 1, \ell > 2$. Then, every edge of $G(k-1,\ell)$ has at most two stack-attachments in \mathcal{L} .

Proof. Let (a, b) be an edge of $G(k-1, \ell)$ and assume to the contrary that there exist three stack-attachments u, v and w of $\overline{G}(k, \ell)$ attached to (a, b) in \mathcal{L} . Neglecting edge (a, b), vertices a, b, u, v and w induce a $K_{2,3}$ in $G(k, \ell)$, whose edges are all stack-edges in \mathcal{L} . This is a contradiction, since the subgraph induced by the stack-edges of $G(k, \ell)$ must be outerplanar [6], while $K_{2,3}$ is not.

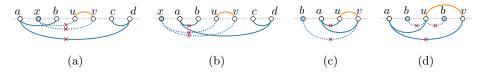


Fig. 3: Illustrations for the proofs (a-b) of Lemma 2, and (c-d) of Lemma 3.

A smiley face $\langle a, b, u, v, c, d \rangle$ in a mixed layout consists of six vertices $a \prec b \prec u \prec v \prec c \prec d$ and four edges (a, b), (c, d), (a, d), and (u, v), such that (a, b), (c, d), and (a, d) are queue-edges, and thus (u, v) is a stack-edge; see Fig. 2c.

Lemma 2. Let \mathcal{L} be a mixed layout of $G(k,\ell)$ with $k > 1, \ell > 2$. Then, a smiley face cannot be formed by the vertices of $G(k-1,\ell)$ in \mathcal{L} .

Proof. Assume to the contrary that a smiley face $\langle a,b,u,v,c,d \rangle$ is formed in \mathcal{L} by vertices of $G(k-1,\ell)$. Consider any vertex x of $\overline{G}(k,\ell)$ attached to the stackedge (u,v). If $a \prec x \prec d$, then the queue-edge (a,d) forms a 2-rainbow both with (u,x) and with (v,x); see Fig. 3a. If $x \prec a$, then the queue-edge (a,b) forms a 2-rainbow both with (u,x) and with (v,x); see Fig. 3b. If $d \prec x$, then the queue-edge (c,d) forms a 2-rainbow both with (u,x) and with (v,x). Hence, neither (u,x) nor (v,x) is a queue-edge, so x is a stack-attachment. Since $\ell > 2$, (u,v) has more than two stack-attachments in \mathcal{L} , contradicting Lemma 1.

Lemma 3. Let \mathcal{L} be a mixed layout of $G(k,\ell)$ with $k > 1, \ell > 2$. Let a,b,c be queue-attachments of an edge (u,v) of $G(k-1,\ell)$ with $u \prec v$. Then $u \prec a,b,c \prec v$.

Proof. Assume to the contrary that $a \prec u$ (the case $v \prec a$ is symmetric). We first prove that $a \prec u$ implies $v \prec b, c$. Indeed, if $b \prec a$, then the queue-edges (b,v) and (a,u) form a 2-rainbow; see Fig. 3c. If $a \prec b \prec v$, then the queue-edges (a,v) and (b,u) form a 2-rainbow; see Fig. 3d. Thus, $v \prec b$ and analogously $v \prec c$. Symmetrically, $v \prec c$ implies $b \prec u$. Hence, $b \prec u \prec v \prec b$; a contradiction. \Box

Lemma 4. Let \mathcal{L} be a mixed layout of $G(k,\ell)$ with $k > 4, \ell > 6$. Then, every queue-edge of $G(k-3,\ell)$ has at most six queue-attachments in \mathcal{L} .

Proof. Assume for a contradiction that there is a queue-edge (u, v) in $G(k-3, \ell)$ with seven queue-attachments x_1, \ldots, x_7 in $\overline{G}(k-2, \ell)$. By Lemma 3, all seven vertices have to lie between u and v; w.l.o.g. assume that $u \prec x_1 \prec \ldots \prec x_7 \prec v$.

For any edge (u, x_i) or (v, x_i) with $2 \le i \le 6$ belonging to $\overline{G}(k-1, \ell)$, consider an attachment w of this edge. By Lemma 1, we can assume that w is not a stack attachment. Further, if (w, x_i) is a queue-edge, then it forms a 2-rainbow with either (u, v), (u, x_1) , or (v, x_7) ; see Fig. 4a. Hence, we assume that every selected attachment w of (u, x_i) or (v, x_i) with $2 \le i \le 6$ in $\overline{G}(k-1, \ell)$ is a mixed-attachment with stack-edge (w, x_i) . We prove Claims 1–4 for edges (v, x_i) ; for (u, x_i) symmetric arguments work; see Fig. 4.

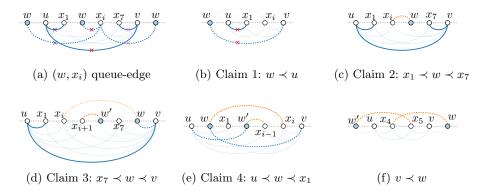


Fig. 4: Illustrations for the proof of Lemma 4.

Claim 1. There is no mixed-attachment w of (v, x_i) with $2 \le i \le 6$ and $w \prec u$ and there is no mixed-attachment w of (u, x_i) with $2 \le i \le 6$ and $v \prec w$.

Proof. Otherwise, the queue-edges (v, w) and (u, x_1) form a 2-rainbow.

Claim 2. There is no mixed-attachment w of (v, x_i) or (u, x_i) with $2 \le i \le 6$ and $x_1 \prec w \prec x_7$.

Proof. Otherwise, there is a smiley face $\langle u, x_1, x_i, w, x_7, v \rangle$ or $\langle u, x_1, w, x_i, x_7, v \rangle$ in $G(k-1,\ell)$, based on whether $x_i \prec w$ or $w \prec x_i$, contradicting Lemma 2. \square

Claim 3. There is no mixed-attachment w of (v, x_i) with $2 \le i \le 6$ and $x_7 \prec w \prec v$ and no mixed-attachment w of (u, x_i) with $2 \le i \le 6$ and $u \prec w \prec x_1$.

Proof. Let to the contrary w' be a mixed-attachment of (v, x_{i+1}) . We have $x_i \prec w' \prec w$, as otherwise the stack-edges (w', x_{i+1}) and (x_i, w) would cross. Then a smiley face $\langle u, x_1, x_{i+1}, w', w, v \rangle$ exists in $G(k-1, \ell)$, contradicting Lemma 2. \square

Claim 4. There is no mixed-attachment w of (v, x_i) with $3 \le i \le 5$ and $u \prec w \prec x_1$ and no mixed-attachment w of (u, x_i) with $3 \le i \le 5$ and $x_7 \prec w \prec v$.

Proof. Let to the contrary w' be a mixed-attachment of (u, x_{i-1}) . We have $u \prec w \prec w' \prec x_i$, as otherwise the stack-edges (w', x_{i-1}) and (x_i, w) would cross. However, by Claims 2 and 3, this leads to a contradiction.

Now consider a mixed-attachment w of (v, x_4) and a mixed-attachment w' of (u, x_5) . By Claims 1–4, we must have $v \prec w$ and $w' \prec u$; see Fig. 4f. However, then the stack-edges (x_4, w) and (x_5, w') cross. This concludes the proof. \Box

Lemmas 1 and 4 imply the following

Corollary 1. Let \mathcal{L} be a mixed layout of $G(k,\ell)$ with $k > 4, \ell > 8$. Then, every queue-edge of $G(k-4,\ell)$ has at least $\ell-8$ mixed-attachments in \mathcal{L} .

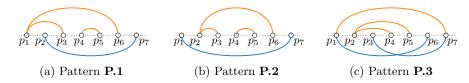


Fig. 5: Illustration of different patterns.

Next we define three patterns **P.1–P.3** and prove that they are forbidden in a mixed layout. Each pattern is denoted by $\langle p_1, \ldots, p_7 \rangle$, as it is defined on a set of seven vertices for which either $p_1 \prec \ldots \prec p_7$ or $p_7 \prec \ldots \prec p_1$ holds in \mathcal{L} ; see Fig. 5. The involved edges in each pattern and their types are as follows.

- **P.1** Stack-edges (p_1, p_3) , (p_1, p_6) and (p_4, p_5) , and a queue-edge (p_2, p_7) .
- **P.2** Stack-edges (p_2, p_3) , (p_2, p_6) and (p_4, p_5) , and a queue-edge (p_1, p_7) .
- **P.3** Stack-edges (p_1, p_7) , (p_2, p_4) and (p_2, p_5) , and queue-edges (p_1, p_6) and (p_3, p_7) .

Lemma 5. Let \mathcal{L} be a mixed layout of $G(k, \ell)$ with $k > 1, \ell > 4$. Then, $G(k-1, \ell)$ does not contain Patterns **P.1–P.3** in \mathcal{L} .

Proof sketch. For a contradiction, let $\langle p_1, \ldots, p_7 \rangle$ be Pattern **P.1** contained in $G(k-1,\ell)$; see Fig. 6. We first argue that at least one of the $\ell > 4$ vertices attached to (p_4,p_5) in $\overline{G}(k,\ell)$ has to be a mixed-attachment. By Lemma 1, at most two of them can be stack-attachments. If more than two of these vertices are queue-attachments, then by Lemma 3, they all appear between p_4 and p_5 in \mathcal{L} , and thus any queue-edge incident to them creates a 2-rainbow with the queue-edge (p_2,p_7) . Hence, there is at least one mixed-attachment x of (p_4,p_5) . Let e and e' be the stack- and queue-edge incident to x, respectively. Then, $p_3 \prec x \prec p_6$, as otherwise e would cross one of the stack-edges (p_1,p_3) and (p_1,p_6) . However, then e' forms a 2-rainbow with the queue-edge (p_2,p_7) ; a contradiction. Similarly we argue for Pattern **P.2**. For Pattern **P.3** see the appendix.

We are now ready to prove the main result of this paper.

Theorem 1. $G(k, \ell)$ does not admit a mixed layout if $k \geq 5, \ell \geq 33$.

Proof sketch. Assume to the contrary that G(5,33) admits a mixed layout \mathcal{L} . By Lemma 1, there is at least one queue-edge (u,v) in G(2,33). W.l.o.g., let $u \prec v$ in \mathcal{L} . By Corollary 1, G(3,33) contains at least 25 mixed-attachments, say x_1, \ldots, x_{25} , of (u,v). For every $i=1,\ldots,25$, one of the following applies: $x_i \prec u$,



Fig. 6: Illustration for the proof of Pattern P.1 in Lemma 5.

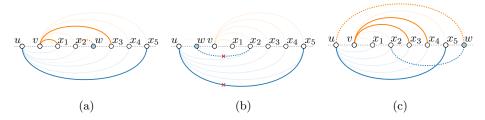


Fig. 7: Illustration for the first case of Theorem 1.

or $u \prec x_i \prec v$, or $v \prec x_i$. For each of the cases, we further distinguish whether the edge (u, x_i) is a stack-edge or a queue-edge. This defines six configurations for x_i . Thus, at least five vertices, say w.l.o.g., x_1, \ldots, x_5 , are attached with the same configuration to (u, v); we assume w.l.o.g. that $x_1 \prec \ldots \prec x_5$. We show a contradiction in the case when $v \prec x_i$ and (u, x_i) is a queue-edge for all $i = 1, \ldots, 5$; the remaining cases are in the appendix.

By Corollary 1, G(4,33) contains at least one mixed-attachment w of (u,x_2) . Thus, either (x_2,w) or (u,w) is a stack-edge. In the former case, the stack-edges (v,x_1) and (v,x_3) enforce $x_1 \prec w \prec x_3$; see Fig. 7a. Hence, $\langle u,v,x_1,x_2,w,x_3,x_5 \rangle$ or $\langle u,v,x_1,w,x_2,x_3,x_5 \rangle$ of G(4,33) form Pattern **P.2** in \mathcal{L} . This contradicts Lemma 5. In the latter case, the stack-edge (v,x_5) enforces either $w \prec v$ or $x_5 \prec w$. We consider three subcases. If $w \prec u$, then the queue-edges (w,x_2) and (u,x_1) form a 2-rainbow. If $u \prec w \prec v$, then the queue-edges (w,x_2) and (u,x_5) form a 2-rainbow; see Fig. 7b. Otherwise, $x_5 \prec w$ holds. It follows that $\langle u,v,x_2,x_3,x_4,x_5,w \rangle$ of G(4,33) form Pattern **P.3** in \mathcal{L} ; see Fig. 7c.

3 Open Problems

In this paper, we proved that 2-trees do not admit mixed 1-stack 1-queue layouts. Since 2-trees admit 2-stack layouts and 3-queue layouts [25], it is natural to ask whether they admit mixed 1-stack 2-queue layouts. We conclude with an algorithmic question, namely, what is the complexity of recognizing graphs that admit mixed 1-stack 1-queue layouts, even for 2-trees? Note that recently de Col et al. [9] showed that testing whether a (not necessarily planar) graph admits a mixed 2-stack 1-queue layout is NP-complete.

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Appendix

In this appendix, we give proofs that were omitted in the main part due to space constraints.

Lemma 5. Let \mathcal{L} be a mixed layout of $G(k, \ell)$ with $k > 1, \ell > 4$. Then, $G(k-1, \ell)$ does not contain Patterns **P.1–P.3** in \mathcal{L} .

Proof. We proved in the main part that $G(k-1,\ell)$ does not contain Pattern **P.1**. We complete the proof of this lemma by showing that $G(k-1,\ell)$ contains neither Pattern **P.2** nor Pattern **P.3**.

As already mentioned, the proof that $G(k-1,\ell)$ does not contain Pattern **P.2** is similar to the corresponding one for Pattern **P.1**. Here, we give the proof only for the sake of completeness. For a contradiction, let $\langle p_1, \ldots, p_7 \rangle$ be Pattern **P.2** contained in $G(k-1,\ell)$; see Fig. 8a. Consider a mixed-attachment x of edge (p_4,p_5) in $\overline{G}(k,\ell)$, whose existence is proven based on Lemmas 1 and 3 as in Pattern **P.1**. Vertex x has to lie between p_3 and p_6 in \mathcal{L} , as otherwise the stackedge incident to x would cross either the stack-edge (p_2,p_6) or the stack-edge (p_2,p_3) . In this case, however, the queue-edge incident to x forms a 2-rainbow with the queue-edge (p_1,p_7) ; a contradiction.

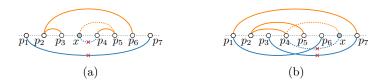


Fig. 8: Illustrations for the proofs of Patterns P.2 and P.3.

For a contradiction, let now $\langle p_1, \ldots, p_7 \rangle$ be Pattern **P.3** contained in G(k-1) $(1,\ell)$; refer to Fig. 8b. Similar to the proof of Pattern **P.1**, we first argue that at least one of the $\ell > 4$ vertices attached to the edge (p_2, p_4) in $G(k, \ell)$ has to be a mixed-attachment. Indeed, by Lemma 1, at most two of these vertices can be stack-attachments. If more than two of these vertices are queue-attachments, then by Lemma 3 they all appear between p_2 and p_4 in \mathcal{L} , which is not possible as any queue-edge incident to them would create a 2-rainbow with the queue-edge (p_1, p_6) . Hence, at least one vertex x attached to (p_2, p_4) is a mixed-attachment. Let e and e' be the stack- and queue-edge incident to x, respectively. Then, x has to lie between p_1 and p_7 in \mathcal{L} , as otherwise e would cross the stack-edge (p_1, p_7) . Also, x cannot lie between p_1 and p_6 , as otherwise e' would form a 2-rainbow with the queue-edge (p_1, p_6) . Hence, x has to lie between p_6 and p_7 in \mathcal{L} . If the edge (p_4, x) is a queue-edge, i.e., $e' = (p_4, x)$, then it forms a 2-rainbow with the queue-edge (p_3, p_7) . Otherwise, the edge (p_4, x) is a stack-edge, i.e., $e = (p_4, x)$, which implies that it crosses the stack-edge (p_2, p_5) . In both cases, we have a contradiction.

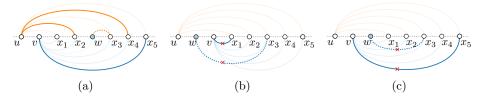


Fig. 9: Illustration for Case 2 of Theorem 1.

Theorem 1. $G(k, \ell)$ does not admit a mixed layout if $k \geq 5, \ell \geq 33$.

Proof. Assume to the contrary that G(5,33) admits a mixed layout \mathcal{L} . Consider the subgraph G(1,33) of G(5,33). By definition, this subgraph is a single edge (a,b). By Lemma 1, in the subgraph G(2,33) of G(5,33), which is obtained by attaching 33 vertices to edge (a,b), there is at least one queue-edge (u,v). W.l.o.g., we assume that $u \prec v$ in \mathcal{L} . Consider now the subgraph G(3,33) of G(5,33). This subgraph contains 33 attachments of edge (u,v). By Corollary 1, at least 25 of them are mixed-attachments. Denote them by x_1, \ldots, x_{25} . For each vertex x_i with $i=1,\ldots,25$, one of the following applies: $x_i \prec u$, or $u \prec x_i \prec v$, or $v \prec x_i$. For each of them, we further distinguish whether the edge (u,x_i) is a stack or a queue-edge. This defines six possible configurations for vertex x_i . By the pigeonhole principle, there exist at least five vertices, say w.l.o.g., x_1, \ldots, x_5 , that are attached with the same configuration to edge (u,v). In the following, we find a contradiction in each of these configurations, assuming w.l.o.g. $x_1 \prec x_2 \prec x_3 \prec x_4 \prec x_5$ in L.

Case 1. For i = 1, ..., 5, $v \prec x_i$ and edge (u, x_i) is a queue-edge: The subgraph G(4,33) of G(5,33) contains 33 attachments to the queue-edge (u, x_2) . By Corollary 1, at least 25 of them are mixed-attachments. Let w be such an attachment. It follows that either (x_2, w) or (u, w) is a stack-edge.

In the former case, the stack-edges (v, x_1) and (v, x_3) enforce $x_1 \prec w \prec x_3$; see Fig. 7a. It follows that $\langle u, v, x_1, w, x_2, x_4, x_5 \rangle$ or $\langle u, v, x_1, w, x_2, x_4, x_5 \rangle$ of G(4, 33) form Pattern **P.2** in \mathcal{L} , depending on whether $x_1 \prec w \prec x_2$ or $x_2 \prec w \prec x_3$, respectively. This contradicts Lemma 5.

In the latter case, the stack-edge (v, x_5) enforces that either $w \prec v$ or $x_5 \prec w$. We consider three subcases. If $w \prec u$, then a 2-rainbow is formed by the queue-edges (w, x_2) and (u, x_1) . If $u \prec w \prec v$, then a 2-rainbow is formed by the queue-edges (w, x_2) and (u, x_5) ; see Fig. 7b. Otherwise, $x_5 \prec w$ holds. It follows that $\langle u, v, x_2, x_3, x_4, x_5, w \rangle$ of G(4, 33) form Pattern **P.3** in \mathcal{L} ; see Fig. 7c. All three cases lead to a contradiction.

Case 2. For $i=1,\ldots,5,\ v\prec x_i$ and edge (u,x_i) is a stack-edge: The subgraph G(4,33) of G(5,33) contains 33 attachments to the queue-edge (v,x_3) . By Corollary 1, at least 25 of them are mixed-attachments. Let w be such an attachment. It follows that either (x_3,w) or (v,w) is a stack-edge.

In the former case, the stack-edges (u, x_2) and (u, x_4) enforce $x_2 \prec w \prec x_4$; see Fig. 9a. It follows that $\langle u, v, x_2, w, x_3, x_4, x_5 \rangle$ or $\langle u, v, x_2, x_3, w, x_4, x_5 \rangle$

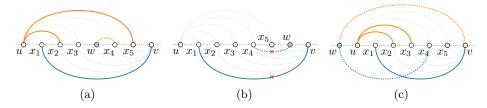


Fig. 10: Illustration for Case 3 of Theorem 1.

of G(4,33) form Pattern **P.1** in \mathcal{L} , depending on whether $x_2 \prec w \prec x_3$ or $x_3 \prec w \prec x_4$, respectively. This contradicts Lemma 5.

In the latter case, the stack-edge (u, x_1) enforces $u \prec w \prec x_1$. We consider two subcases. If $u \prec w \prec v$, then a 2-rainbow is formed by the queue-edges (w, x_3) and (v, x_1) ; see Fig. 9b. Otherwise, $v \prec w \prec x_1$ holds, in which case a 2-rainbow is formed by the queue-edges (v, x_5) and (w, x_3) ; see Fig. 9c. Both cases lead to a contradiction.

Case 3. For $i=1,\ldots,5$, $u \prec x_i \prec v$ and edge (u,x_i) is a stack-edge: As in the previous cases, we first observe that the subgraph G(4,33) of G(5,33) contains 33 attachments to the queue-edge (v,x_4) . By Corollary 1, at least 25 of them are mixed-attachments. Let w be such an attachment. It follows that either (x_4,w) or (v,w) is a stack-edge.

In the former case, the stack-edges (u, x_3) and (u, x_5) enforce $x_3 \prec w \prec x_5$; see Fig. 10a. It follows that $\langle u, x_1, x_2, w, x_4, x_5, v \rangle$ or $\langle u, x_1, x_2, x_4, w, x_5, v \rangle$ of G(4,33) form Pattern **P.1** in \mathcal{L} , depending on whether $x_3 \prec w \prec x_4$ or $x_4 \prec w \prec x_5$, respectively. This contradicts Lemma 5.

In the latter case, the stack-edge (u, x_5) enforces that either $w \prec u$ or $x_5 \prec w$. We consider three subcases. If $v \prec w$, then a 2-rainbow is formed by the queue-edges (w, x_4) and (v, x_5) . If $x_5 \prec w \prec v$, then a 2-rainbow is formed by the queue-edges (w, x_4) and (v, x_1) ; see Fig. 10b. Otherwise, $w \prec u$ holds. It follows that $\langle w, u, x_1, x_2, x_3, x_4, v \rangle$ of G(4, 33) form Pattern **P.3** in \mathcal{L} ; see Fig. 10c. All three cases lead to a contradiction.

Case 4. For i = 1, ..., 5, $x_i \prec u$ and edge (u, x_i) is a stack-edge: This case is symmetric to Case 1.

Case 5. For i = 1, ..., 5, $x_i \prec u$ and edge (u, x_i) is a queue-edge: This case is symmetric to Case 2.

Case 6. For i = 1, ..., 5, $u \prec x_i \prec v$ and edge (u, x_i) is a queue-edge: This case is symmetric to Case 3.

Since Cases 1–6 have led to a contradiction, G(5,33) does not admit any mixed layout, as desired.