Large-scale structures in random graphs

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Abstract

In recent years there has been much progress in graph theory on questions of the following type. What is the threshold for a certain large substructure to appear in a random graph? When does a random graph contain all structures from a given family? And when does it contain them so robustly that even an adversary who is allowed to perturb the graph cannot destroy all of them? I will survey this progress, and highlight the vital role played by some newly developed methods, such as the sparse regularity method, the absorbing method, and the container method. I will also mention many open questions that remain in this area.

1 Introduction

Erdős and Rényi introduced the notion of a random graph in their seminal paper [49]. They thus initiated the study of which type of property typical graphs of a certain density have or do not have, which turned out to be immensely influential in graph theory as well as in other related mathematical areas. The books [26, 63, 79] provide an excellent and extensive overview of the theory of random graphs and its applications.

This survey is concerned with a particular type of properties of random graphs, namely the appearance of given large-scale subgraphs. In the past two decades, the theory of large-scale structures in random graphs G(n, p) underwent swift development and originated powerful new tools. The following three main directions of research can be distinguished in this area.

Firstly and naturally, one may study for which edge probability p the random graph G(n,p) is likely to possess one particular spanning (or large) structure. This structure could for example be a perfect matching, a Hamilton cycle, or a disjoint collection of triangles covering as many vertices of G(n,p) as possible. More generally, for any sequence (H_n) of graphs one can ask when H_n is a subgraph of G(n,p). Questions of this type were pursued since the early days of the theory of random graphs, and some turned out to be extremely challenging.

Secondly and more generally, instead of considering a single sequence (H_n) of subgraphs one may ask for G(n,p) to be *universal* for a a given sequence of families (\mathcal{H}_n) of graphs, that is, to simultaneously contain a copy of each graph in \mathcal{H}_n . A typical example of a question of this type

is for which p the random graph G(n,p) is likely to contain every binary tree on n vertices. Since the number of such trees is huge, it is clear that an answer to this question is not trivially entailed by a result on the appearance of any fixed spanning binary tree in G(n,p). Hence, such universality questions are in general harder than the questions for single subgraph sequences. Universality questions were originally motivated by problems in circuit design, data representation, and parallel computing (see [23] for relevant references, and more history concerning universality). Their study in random graphs is more recent and it is often observed or conjectured that when G(n,p) is likely to contain any fixed graph from \mathcal{H}_n then it is already universal for \mathcal{H}_n .

Finally, one may ask how resiliently G(n, p) possesses certain structures. In other words, if G(n,p) is known to contain a subgraph H, but an adversary is allowed to delete edges from G(n, p) under certain restrictions, when is the adversary likely able to destroy all copies of H. As it turns out the random graph is very robust towards such adversarial edge deletions. Another way of motivating resilience-type questions is from the perspective of extremal graph theory. Two main directions of research in extremal graph theory are the investigation of Turán-type questions, and of Dirac-type questions. Turán's theorem [132] states that K_r is a subgraph of any graph G on n vertices with more edges than the balanced complete (r-1)-partite graph on n vertices contains K_r as a subgraph, that is, graphs G with edge density at least $\frac{r-2}{r-1} + o(1)$ contain K_r . Dirac's theorem [47], on the other hand, asserts that any graph G with minimum degree $\delta(G) \geq \frac{1}{2}v(G)$ contains a Hamilton cycle. Resilience-type questions then ask for the transference of such results to sparse random graphs. For example, when does any subgraph of G(n, p) with sufficiently many edges contain K_r , and when does any subgraph of G(n,p) with sufficiently high minimum degree contain a Hamilton cycle? The former of these two questions proved to be surprisingly deep and both questions and their generalisations inspired much recent work in the area.

In this survey I attempt to give an overview of the progress in these three main directions. Let me stress that there is no material covered here that does not appear elsewhere. Instead, I try to outline the exciting developments in the area, and also give credit to the important new methods that allowed this progress. In some cases I will give simple examples of how these methods can be applied. These necessarily have to be brief, but pointers to further literature will be given.

What is not covered? There are several other important topics which recently received much attention and are closely connected to the developments described in this survey in that progress in these areas influenced

or was influenced by the methods and results provided in the following, but which are, to limit scope, not covered here. These topics include Ramsey theoretic results in random graphs, packing results in random graphs, and embedding results in various types of pseudorandom graphs. I also omit analogous results in random directed graphs and random hypergraphs, and embedding results for induced subgraphs in random graphs.

Organisation. The survey is structured as follows. Section 2 provides basic definitions and the relevant concepts from the theory of random graphs. Section 3 then collects, mainly for comparison, results on the appearance of fixed graphs H in G(n,p). Section 4 reviews results on the appearance of a fixed sequence (H_n) in G(n,p), where the graphs in (H_n) grow with n, while Section 5 considers corresponding universality results. Section 6 surveys progress on resilience results for large subgraphs of G(n,p), and Section 7 discusses an important tool for this type of problem, the sparse blow-up lemma in random graphs.

2 Basic definitions and notation

For easy reference, this section collects the basic definitions we need in this survey. Throughout, we use the natural logarithm $\log x = \log_e x$. The set of the first n natural numbers is denoted by $[n] = \{1, \ldots, n\}$, and $(n)_k = n \cdot (n-1) \cdot \ldots \cdot (n-k)$ is the falling factorial. As is common in the area, ceilings and floors are omitted whenever they are not essential.

For a graph G = (V, E) we denote by v(G) the number of its vertices |V| and by e(G) the number of its edges |E|. The minimum degree of G is $\delta(G)$, while the maximum degree is $\Delta(G)$. The chromatic number of G is denoted by $\chi(G)$. The girth of a graph is the length of its shortest cycle. If H is a (not necessarily induced) subgraph of G we write $H \subseteq G$. An H-copy in G is a (not necessarily induced) copy of H in G. The automorphism group of G is denoted by Aut(G).

For a vertex $v \in V$ we write $N_G(v)$ for the neighbourhood of v in G, and $\deg_G(v) = |N_G(v)|$ for its degree. Similarly, if $U \subseteq V$ then $N_G(v; U)$ is the neighbourhood in G of v in the set U and $\deg_G(v; U) = |N_G(v; U)|$. When the graph G is clear from the context we often omit the subscript G in this notation.

2.1 Graph classes

The graph properties considered in this survey mainly concern the existence of certain subgraphs, and hence are monotone increasing. A monotone increasing graph property is a family \mathcal{P} of graphs such that for any $G \in \mathcal{P}$ we have that a graph G' obtained from G by adding any edge

is also in \mathcal{P} . A monotone increasing property is non-trivial if K_n is in \mathcal{P} but the complement of K_n not, where K_n denotes the complete graph on n vertices.

A balanced r-partite graph is an r-partite graph whose partition classes are as equal as possible; it is complete if all the edges between all partition classes are present. The cycle on n vertices is denoted by C_n , and P_n is the n-vertex path. A Hamilton cycle (or path) of a graph G is a cycle (or path) containing all the vertices of G. A graph is called Hamiltonian if it has a Hamilton cycle. Let H be a fixed graph and G be a graph on n vertices. Then an H-factor in G is a collection of $\lfloor n/v(H) \rfloor$ vertex disjoint copies of H. In particular, when v(H) divides v(G) then an H-factor is a spanning subgraph of G. The d-dimensional cube Q_d is the graph on vertex set $\{0,1\}^d$ with edges uv whenever u and v differ in exactly one coordinate. The $k \times k$ -square grid L_k is the graph on vertex set $[k] \times [k]$, with edges uv whenever u and v differ in exactly one coordinate by exactly one.

The k-th power of a graph H is the graph obtained from H by adding all edges between vertices of distance at most k. The 2-nd power of H is also called the square of H. We also denote the k-th power of H by H^k . In particular, C_n^k is the k-th power of a cycle C_n on n vertices. A graph H is d-degenerate if every subgraph of H contains a vertex of degree at most d. Equivalently, the vertices of H can be ordered in such a way that each vertex v sends at most d edges to vertices preceding v in this order. The bandwidth $\mathrm{bw}(H)$ of a graph H is the smallest integer b such that there is a labelling of V(H) using all the integers [v(H)] for which $|u-v| \leq b$ for each edge $uv \in E(H)$.

Further, the following classes of graphs are considered. Let $\mathcal{H}(n, \Delta)$ be the family of all graphs on n vertices with maximum degree at most Δ , and $\mathcal{H}(n, n, \Delta)$ be the class of all bipartite graphs with partition classes of order n each, and with maximum degree Δ . The class $\mathcal{T}(n, \Delta)$ contains all trees on n vertices with maximum degree Δ . Let me remark that sometimes these graph classes will be used to refer to small linear sized graphs, such as the class $\mathcal{H}(\gamma n, \Delta)$ for some small $\gamma > 0$, where one really should write $\mathcal{H}(|\gamma n|, \Delta)$, but I omit the floors and ceilings for simplicity.

2.2 Random graphs

The binomial random graph G(n, p) is obtained by pairwise independently including each of the possible $\binom{n}{2}$ edges on n vertices with probability p = p(n). The uniform random graph G(n, m), on the other hand,

 $^{^1}$ This model is also often called the Erdős–Rényi model, though this is objected to by part of the community because the model that Erdős and Rényi used in their papers

assigns each graph on vertex set [n] with m edges probability $1/\binom{\binom{n}{2}}{m}$. An event holds asymptotically almost surely (abbreviated a.a.s.) in G(n,p) (or in G(n,m)) if its probability tends to 1 as n tends to infinity. For a monotone increasing graph property \mathcal{P} we say that $\tilde{p} = \tilde{p}(n)$ is a threshold for \mathcal{P} if

$$\mathbb{P}\big(G(n,p)\in\mathcal{P}\big)\to \begin{cases} 0 & \text{if} \quad p/\tilde{p}\to 0\,,\\ 1 & \text{if} \quad p/\tilde{p}\to\infty\,. \end{cases}$$

As is common, in this case \tilde{p} will also be called the threshold, even though it is not unique. Bollobás and Thomason [31] proved that every non-trivial monotone increasing property has a threshold. Moreover if the threshold is of the form $\log^a n/n^b$ with a,b>0 fixed reals, as will be encountered frequently in this survey, then there is a sharp threshold \tilde{p} , that is, for any $\varepsilon>0$

$$\mathbb{P}\big(G(n,p)\in\mathcal{P}\big)\to\begin{cases} 0 & \text{if} \quad p\leq (1-\varepsilon)\tilde{p}\,,\\ 1 & \text{if} \quad p\geq (1+\varepsilon)\tilde{p}\,. \end{cases}$$

As explained in [61] this follows from the celebrated characterisation of sharp thresholds by Friedgut [60].

In this survey many results are considered that concern spanning subgraphs of G(n,p) as n tends to infinity. These results therefore do not concern a single fixed subgraph, but rather a sequence of subgraphs, one for each value of n. Sometimes this fact is implicitly assumed when stating a result, but usually it is stressed by stating that we are given a sequence $H = (H_n)$ of graphs and that G(n,p) contains H_n under certain conditions a.a.s.

2.3 Density parameters

In the results on subgraphs H of G(n,p) that we will discuss, different density parameters are used, which will be defined next.² The first of these parameters is called *maximum* 0-density, and is given by

$$m_0(H) = \max_{H' \subseteq H} \frac{e(H')}{v(H')}.$$

The maximum 0-density is usually simply called maximum density in the literature. Let H^* be a subgraph of H realising the maximum in $m_0(H)$, and let X^* be the random variable counting the number of unlabelled

pioneering the area is the model G(n, m).

²It is not true that these parameters are densities in the sense of being between 0 and 1. Rather, they are variations on the average degree of a graph.

copies of H^* in G(n,p). Then

$$\mathbb{E}(X^*) = ((n)_{v(H^*)} / |\operatorname{Aut}(H^*)|) p^{e(H^*)} \approx n^{v(H^*)} p^{e(H^*)},$$

which tends to infinity if $p \cdot n^{1/m_0(H)} \to \infty$. So, informally, we can say that in expectation the densest subgraph of H appears around $p = n^{-1/m_0(H)}$ in G(n,p). Hence, it is natural to guess that this probability is the threshold for the appearance of H-copies in G(n,p), which is indeed the case (see Theorem 3.1).

The other density parameters are slight variations on this first definition (which can, however, have an important influence on the resulting values of these parameters). These variations have similarly natural motivations as the 0-density. The $maximum\ 1$ -density of a graph H with at least two vertices is

$$m_1(H) = \max_{\substack{H' \subseteq H \\ v(H') > 1}} \frac{e(H')}{v(H') - 1}.$$

This parameter is also called fractional arboricity in [12]. Again, let H^* be a subgraph of H realising the maximum in $m_1(H)$ and let v be a fixed vertex in G(n,p). Then the expected number of H^* -copies in G(n,p) containing v tends to infinity if $p \cdot n^{1/m_1(H)} \to \infty$. The threshold for the property that each vertex of G(n,p) is contained in an H-copy is related to, but is not precisely equal to, $n^{-1/m_1(H)}$ (see also the explanations in Section 3).

The maximum 2-density of a graph H with at least one edge is

$$m_2(H) = \max_{\substack{H' \subseteq H \\ e(H') > 0}} \begin{cases} \frac{e(H') - 1}{v(H') - 2}, & \text{if } v(H') > 2\\ \frac{1}{2}, & \text{if } v(H') = 2. \end{cases}$$

If $p \cdot n^{1/m_2(H^*)} \to \infty$, in expectation a fixed edge of G(n, p) is contained in many H^* -copies, where H^* realises the maximum in $m_2(H)$.

A graph H is called 0-balanced (or 1-balanced, or 2-balanced) if H is a maximiser in $m_0(H)$ (or $m_1(H)$, or $m_2(H)$, respectively). If H is the unique maximiser, then it is called *strictly* 0-balanced (or 1-balanced, or 2-balanced, respectively).

Riordan [120] defined a different density parameter $m_{\rm R}(H)$ for H on at least 3 vertices, which I will call the maximum Riordan-density here and which is given by

$$m_{\rm R}(H) = \max_{\substack{H' \subseteq H \\ v(H') > 2}} \frac{e(H')}{v(H') - 2}.$$

Note that, again, if H is a fixed graph and the maximum in $m_{\mathbb{R}}(H)$ is realised by H^* , then a fixed pair of vertices in G(n,p) is in expectation contained in many H^* -copies if $p \cdot n^{1/m_{\mathbb{R}}(H^*)} \to \infty$. Riordan, however, uses this density in a result about copies of spanning graphs $H = (H_n)$ in G(n,p). For such a graph H, the maximum in $m_{\mathbb{R}}(H)$ may well be realised by some $H^* = (H_n^*)$ with $v(H_n^*) \to \infty$, in which case $m_{\mathbb{R}}(H)$ is asymptotically equal to the maximum 0-density (or maximum 1-density) of H.

3 Small subgraphs

This survey focuses on large subgraphs of G(n, p). Before we turn to the many results in this area, we will briefly review what is known for small, that is, fixed subgraphs H. Some of the relevant results will turn out useful for later comparison.

3.1 The appearance of small subgraphs

There are a number of natural questions that one may ask concerning the existence of a fixed subgraph H in G(n, p):

- 1. What is the threshold for the appearance of an *H*-copy?
- 2. How many *H*-copies are there in G(n, p)?
- 3. How are the *H*-copies distributed in G(n,p)?

The second question is beyond the scope of this survey, though important and strong results were obtained in this direction (see, e.g., [79, Chapter 6] or [63, Chapter 5]). We shall concentrate on the other two, starting with the first. A classical result by Bollobás [24] in the theory of random graphs states that the threshold for the appearance of an H-copy in G(n,p) is determined by its maximum 0-density.

Theorem 3.1 (see, e.g., Theorem 3.4 in [79]) Let H be a graph with (and at least one edge). The threshold for G(n,p) to contain a copy of H is

$$n^{-1/m_0(H)}$$
.

This answers the first question. Note that for 0-balanced graphs this threshold was already established by Erdős and Rényi [50].

So let us turn to the third question, which is phrased rather vaguely. In fact there are two meaningful interpretations which will play a more prominent role in this survey. One the one hand, one could ask: When do we find many vertex disjoint copies of H, or possibly even an H-factor? The latter is a difficult question, and we shall return to it in Section 4.3.

But a related question, considering a property which is clearly necessary for an H-factor, is much easier: When is every vertex of H contained in an H-copy? This question was answered by Ruciński [123] and Spencer [128]. For strictly 1-balanced graphs H the threshold is mainly influenced by the maximum 1-density of H.

Theorem 3.2 (see, e.g., Theorem 3.22 in [79]) Let H be a strictly 1-balanced graph (with at least 2 vertices) and let COV_H be the event that every vertex of G(n,p) is contained in a copy of H. The threshold for COV_H is

 $\frac{(\log n)^{1/e(H)}}{n^{1/m_1(H)}} \, .$

Similar results for non-strictly 1-balanced graphs exist (see [79, Theorem 3.22]). But these are more complicated: they need to take into account all the different ways of rooting the graph H at some vertex and all the different subgraphs H' of H containing this vertex. It is true, however, that the threshold for the event COV_H of Theorem 3.2 is $\Omega(\frac{(\log n)^{1/e(H)}}{n^{(v(H)-1)/e(H)}})$ for every H.

The appearance of the log-factor in the threshold is not surprising. Recall that the expected number of H-copies in G(n,p) containing a fixed vertex v is of order $n^{v(H)-1}p^{e(H)}$. Since we are asking for an H copy at every vertex of G(n,p) it is natural to require that this quantity grows at least like $\log n$ (to allow for concentration), which is precisely the case for $p = \frac{(\log n)^{1/e(H)}}{n^{1/m_1(H)}}$ if H is strictly 1-balanced. On the other hand, one could interpret the third question above as

On the other hand, one could interpret the third question above as asking if G(n, p) has a large subgraph without any H-copies. It is easy to show that this is the case below the 2-density-threshold.

Proposition 3.3 (see, e.g., Proposition 8.9 in [79]) For all 0 < a < 1 and all H with $\Delta(H) \ge 2$ there is a constant c > 0 such that the following holds. If $p \le cn^{1/m_2(H)}$ then G(n,p) a.a.s. has an H-free subgraph G with $e(G) \ge a \cdot e(G_{n,p})$.

So H-copies in G(n,p) are easy to delete once we are below the 2-density threshold. The reason for this is that the likely number of H-copies is comparable to the likely number of edges at this threshold. Above the threshold this changes, which is addressed in the following section.

3.2 The Erdős-Stone theorem in random graphs

What is the maximum number of edges in an H-free subgraph of G(n, p)? This question has inspired much research in the theory of random

graphs. To understand what the answer to this question could reasonably be, let us first turn to dense graphs.

The Erdős–Stone theorem, one of the cornerstones of extremal graph theory, is a Turán-type theorem which states that the crucial property of a fixed graph H for determining the maximum number of edges in an H-free graph is its chromatic number.

Theorem 3.4 (Erdős, Stone [53]) For each fixed graph H and every $\varepsilon > 0$ there is an n_0 such that for all $n \geq n_0$ the following holds. Any n-vertex graph G with at least $(\frac{\chi(H)-2}{\chi(H)-1}+\varepsilon)\binom{n}{2}$ edges contains H as a subgraph.

As a balanced complete $(\chi(H)-1)$ -partite graph has about $\frac{\chi(H)-2}{\chi(H)-1}\binom{n}{2}$ edges and is obviously H-free this is tight up to lower order terms. Similarly, a $(\chi(H)-1)$ -partite subgraph of G(n,p) with (roughly) equal sized random partition classes and all G(n,p)-edges between the partition classes contains $(\frac{\chi(H)-2}{\chi(H)-1}+o(1))e(G(n,p))$ edges. Hence, when we ask for the maximum number of edges in an H-free subgraph of G(n,p) we cannot go below this quantity. Moreover, as explained at the end of the last section, below the 2-density threshold the answer becomes (almost) trivial as all H-copies in G(n,p) can be destroyed by deleting just a tiny fraction of the edges.

The question then is if these two observations already tell the whole story. The following breakthrough result on the transference of the Erdős–Stone theorem to sparse random graphs confirms that this is indeed the case, and was obtained independently by Conlon and Gowers [39] (for 2-balanced H) and Schacht [125] (for general H).

Theorem 3.5 (Schacht [125], Conlon, Gowers [39])

For every fixed graph H and every $\varepsilon > 0$ there are constants 0 < c < C such that the following holds. Let $\mathcal A$ be the property that the maximum number of edges in an H-free subgraph of G(n,p) is at most $(\frac{\chi(H)-2}{\chi(H)-1}+\varepsilon)e(G(n,p))$. Then

$$\mathbb{P}[\mathcal{A}] \to \begin{cases} 0 & \text{if } p \le cn^{-1/m_2(H)}, \\ 1 & \text{if } p \ge Cn^{-1/m_2(H)}. \end{cases}$$

Earlier results in this direction were obtained for special graphs H in [59, 65, 66, 69, 70, 74, 75, 89, 90], and for larger lower bounds on p in [93, 130]. In fact, the results of Conlon and Gowers [39] and of Schacht [125] are much more general statements, allowing the transference of a variety of extremal results on graphs, hypergraphs and sets of integers to sparse

random structures. Both proofs reduce these problems to the analysis of random vertex subsets in certain auxiliary hypergraphs. In the case of Theorem 3.5 the vertices of the auxiliary hypergraph \mathcal{H} are the edges of K_n , and the hyperedges are all e(H)-tuples that form H-copies. Hence, the random graph G(n,p) corresponds to a random subset S of $V(\mathcal{H})$, and an H-free subgraph of G(n,p) to an independent set in $\mathcal{H}[S]$.

Recently, a very general approach has been developed to analyse such independent sets in hypergraphs, the so-called container method developed independently by Balogh, Morris and Samotij [19], and Saxton and Thomasson [124], which has already proved tremendously useful for solving a variety of other problems as well.

The idea, in the language of the Erdős–Stone theorem in random graphs, is as follows. One naive approach to prove that $\mathbb{P}[\mathcal{A}] \to 1$ if $p \geq Cn^{-1/m_2(H)}$ in Theorem 3.5 is to first fix an H-free graph G on vertex set [n], to calculate the probability that G(n,p) contains more than $(\frac{\chi(H)-2}{\chi(H)-1}+\varepsilon)p\binom{n}{2}$ edges of G, and then use a union bound over all choices of G to conclude that a.a.s. G(n,p) does not contain $(\frac{\chi(H)-2}{\chi(H)-1}+\varepsilon)p\binom{n}{2}$ edges of any H-free graph, so that any subgraph of G(n,p) with that many edges cannot be H-free. This, of course, does not work because there are too many choices for G (and we did not use anything about the structure of H-free graphs). The crucial idea of the container method is to show that the set G of G of G of that is, for each $G \in G$ there is $G \in G$ such that $G \subseteq G$ and $G(G) \leq (\frac{\chi(H)-2}{\chi(H)-1}+\frac{1}{2}\varepsilon)\binom{n}{2}$. This then basically allows us to run the union bound argument over G. In reality, things are not quite so simple, and more properties are required of G (see also the excellent explanations in [124, Section 2.2]).

Restricting to the case when $H=K_r$, the analogous structural question to Theorem 3.5 of when the K_r -free subgraph of G(n,p) with the most edges is (r-1)-partite was first considered by Babai, Simonovits and Spencer [14], whose result was improved on by Brightwell, Panagiotou and Steger [35]. Finally, DeMarco and Kahn [46, 45] showed that this is a.a.s. the case when $p \geq C(\log n)^{1/(e(K_r)-1)}/n^{m_2(K_r)}$, which is optimal up to the value of C. For other graphs H a corresponding structural result has not yet been established. Observe that, in general, this is a difficult problem since we do not even know precise structural results in dense graphs for all H.

Question 3.6 For some fixed H different from a complete graph, what is the structure of an H-free subgraph of G(n,p) with the most edges?

This question is already interesting when $H = C_5$, for example, when this subgraph should be bipartite (for p sufficiently large).

4 Large subgraphs

In this section we shall consider the question when the random graph G(n,p) contains a fixed sequence of spanning graphs $H=(H_n)$ as subgraphs. Answers to this question come in various levels of accuracy. For some classes of graphs H we only know non-matching lower and upper bounds on the threshold probability, while for others the threshold has been established. Even stronger hitting time results could so far only rarely be obtained.

We start this section with the most classical subgraphs of G(n,p) to be considered: matchings and Hamilton cycles. In Section 4.2 we present a theorem of Alon and Füredi concerning more general spanning subgraphs and the powerful improvement on this result by Riordan. We also analyse the bounds on the threshold for various classes of graphs that Riordan's result gives. Section 4.3 turns to the deep Johansson–Kahn–Vu theorem which establishes, among others, the threshold for K_r -factors, while Section 4.4 considers the threshold for the containment of spanning bounded degree trees. The question of when a bounded degree graph H appears in G(n,p) is addressed in Section 4.5. In Section 4.6 we discuss a very general conjecture of Kahn and Kalai concerning the form a threshold for the containment of some sequence (H_n) can take in G(n,p). In Section 4.7, finally, we consider the question of algorithmically finding a spanning H-copy in G(n,p).

4.1 Matchings and Hamilton cycles

Two of the most natural questions concerning spanning substructures of random graphs asks for the threshold of G(n,p) to contain a perfect matching or to be Hamiltonian. These questions are as well understood as one can hope for.

Already Erdős and Rényi [51, 52] showed that the threshold for containing a perfect matching is $\log n/n$. Bollobás and Thomason [30] established a hitting time result, which considers G(n,m) as a graph process, where we start from the empty graph on n vertices and randomly add edges one-by-one. The hitting time result then states that a.a.s. precisely the edge in this process which eliminates the last isolated vertex creates a perfect matching (if n is even). In other words, avoiding the most trivial obstacle for containing a perfect matching in fact guarantees a perfect matching (see also [28] for an alternative proof and related results). Luczak and

Ruciński [111] extended these results, showing that the same hitting time result is true for T-factors for any non-trivial tree T.

Turning to the Hamiltonicity problem, Pósa [119] and Korshunov [103, 104] showed that also the threshold for a Hamilton cycle (as well as for a Hamilton path) is $\log n/n$. Improving on this result, Komlós and Szemerédi [102] determined an exact formula for the probability of the existence of a Hamilton cycle. Bollobás [25] established the corresponding hitting time result, stating that as soon as G(n,p) gets minimum degree 2, it also contains a Hamilton cycle. Hence, if $\phi(n)$ is any function tending to infinity, then G(n,p) a.a.s. is Hamiltonian if $p \geq (\log n + \log \log n + \phi(n)/n$ and not Hamiltonian if $p \leq (\log n + \log \log n - \phi(n)/n$.

Algorithmic results for finding Hamilton cycles – a problem which is NP-hard in general – in random graphs above the threshold probability were also obtained. Gurevich and Shelah [71] and Thomason [131] obtained linear expected time algorithms when p is well above the threshold. Improving on polynomial time randomised algorithms by Angluin and Valiant [13] and Shamir [127], Bollobás, Fenner and Frieze [27] gave a deterministic polynomial time algorithm with a success probability that matches the probability that a Hamilton cycle exists given by Komlós and Szemerédi [102].

4.2 The Alon-Füredi theorem and Riordan's theorem

Let us now turn to results concerning more general results on spanning subgraphs of G(n,p). Motivated by a question of Bollobás asking for a non-trivial probability p such that G(n,p) with $n=2^d$ a.a.s. contains a copy of the d-dimensional hypercube Q_d , Alon and Füredi [10] established the following result, providing an upper bound on the threshold for the appearance of a spanning graph with a given maximum degree. Their theorem can be seen as a first general result concerning the appearance of spanning subgraphs in G(n,p), thus stimulating research in the area.

Theorem 4.1 (Alon, Füredi [10]) Let $H = (H_n)$ be a fixed sequence of graphs on n vertices with maximum degree $\Delta(H) \leq \sqrt{\sqrt{n}-1}$. If

$$p \ge \left(\frac{20\Delta(H)^2 \log n}{n}\right)^{1/\Delta(H)}$$

then G(n, p) a.a.s. contains a copy of H.

Their proof uses the following simple strategy, which is based on a multi-round exposure of G(n, p). Apply the Hajnal–Szemerédi theorem [72] to the square H^2 of H to obtain an equitable $(\Delta^2 + 1)$ -colouring of H^2 ,

that is, a partition of $V(H^2) = V(H)$ into $\Delta^2 + 1$ parts $X_1 \dot{\cup} \dots \dot{\cup} X_{\Delta^2 + 1}$ which are as equal in size as possible and form independent sets in H^2 . Observe that this implies that between each pair of these parts H induces a matching. Partition the vertices of G(n,p) into sets $V_1 \dot{\cup} \dots \dot{\cup} V_{\Delta^2 + 1}$ of sizes equal to these parts. Then embed X_i into V_i one by one, revealing the edges between V_i and $\bigcup_{j < i} V_j$, and showing that the partial embedding from the previous round can be extended. This is possible because for any $x \in X_i$ the set $N^-(x)$ of already embedded neighbours is of size at most Δ and disjoint from any $N^-(x')$ with $x \neq x' \in X_i$, and a random bipartite graph with edge probability p^Δ and partition classes of size $n/(\Delta^2 + 1)$ contains a perfect matching. Ideas from this basic strategy were re-used in many of the results on universality and local resilience we will mention later.

The theorem of Alon and Füredi was improved on by Riordan. He proved the following surprisingly powerful result.

Theorem 4.2 (Riordan's theorem [120]) Let $H = (H_n)$ be a fixed sequence of graphs with v(H) = n and e(H) > n/2 and let p = p(n) < 1 satisfy

$$\frac{np^{m_R(H)}}{\Delta(H)^4} \to \infty.$$

Then a.a.s. G(n,p) contains a copy of H.

This result can be found in this form in [118] (where it is in addition verified that this result also remains true for H with fewer edges but $\delta(H) \geq 2$). Observe, that the condition on p in Riordan's theorem implies that $\Delta(H)$ grows slower than $n^{1/4}$. In most of this survey, however, we will consider bounded degree graphs only, for which Theorem 4.2 requires that p grows faster than $n^{-1/m_{\rm R}(H)}$.

Let me mention that in [120] this result is stated for G(n,m) instead of G(n,p) and it is in addition required that $p\binom{n}{2} \to \infty$, $\binom{n}{2} - 2e(H) \to \infty$ and $(1-p)\sqrt{n} \to \infty$. However, the result for G(n,p) follows from a standard argument (e.g. [26, Theorem 2.2]) and the first additional requirement on p follows from the requirement in Theorem 4.2 since $m_{\rm R}(H) \ge \frac{n/2}{n-2} > \frac{1}{2}$ because e(H) > n/2. The second and third additional requirements are satisfied if we take p as small as possible while still satisfying the conditions in Theorem 4.2 because $\Delta(H)$ grows slower than $n^{1/4}$ and $m_{\rm R}(H) \le \Delta(H)$. The conclusion then still remains true for larger p because the property of containing H is monotone increasing.

The heart of the proof of Riordan's theorem is an elegant second moment argument in the G(n, m) model, which shows that the variance of

the number of H-copies is small by bounding from above how much one H-copy in G(n,m) can make another H-copy more likely. Using the same approach in G(n,p) is not possible because if H contains many edges and one conditions on the appearance of a fixed H-copy in G(n,p), then this boosts the number of edges in G(n,p) sufficiently to make other H-copies significantly more likely.

To illustrate the power of Riordan's theorem a few straightforward consequences are collected in the following. The first two of these were already given by Riordan [120], and the third was observed by Kühn and Osthus [108].

Hypercubes. If $n = 2^d$ and

$$p \ge \frac{1}{4} + 6 \frac{\log d}{d}$$

then a.a.s. G(n,p) contains a copy of the d-dimensional cube Q_d , because $m_{\rm R}(Q_d) = \frac{dn}{2(n-2)}$ (that is, Q_d is the maximiser in $m_{\rm R}(Q_d)$). This results is close to best possible since for $p = \frac{1}{4}$ the expected number of Q_d -copies is $(n!/|\operatorname{Aut}(Q_d)|)(\frac{1}{4})^{\frac{1}{2}n\log n} \leq (n!/|\operatorname{Aut}(Q_d)|) \cdot n^{-n}$, which tends to zero as n tends to infinity.

Square grids. If $n = k^2$ and

$$p\cdot n^{1/2}\to\infty$$

then a.a.s. G(n, p) contains a copy of the $k \times k$ -square grid L_k , because $m_{\rm R}(L_k) = 2$ (that is, C_4 is the maximiser in $m_{\rm R}(L_k)$). Again, an easy first moment calculation show that for $p = n^{-1/2}$ the probability that G(n, p) contains L_k tends to 0.

Powers of Hamilton cycles. If $k \geq 3$ and

$$p \cdot n^{1/k} \to \infty$$

then G contains the k-th power of a Hamilton cycle C_n^k , because $m_R(C_n^k) \le k + \frac{(k+1)k^2}{n}$ as shown in [108]. For $p \le ((1-\varepsilon)e/n)^{1/k}$ the probability that G(n,p) contains the k-th power of a Hamilton cycle tends to 0 (using again the first moment).

For k=2 Riordan's theorem does not provide a (close to) optimal result, because $m_{\rm R}(C_n^k)=m_{\rm R}(K_3)=3$. An approximately tight result has been obtained by Kühn and Osthus [108] though, who showed that G(n,p) a.a.s. contains C_n^2 if $p\geq n^{\varepsilon-1/2}$ for any fixed $\varepsilon>0$. This was improved on by Nenadov and Škorić [116] who require $p\geq C\log^4 n/n^{1/2}$. Both the result of Kühn and Osthus and the result of Nenadov and Škorić use an

absorbing-type method. Very recently, using a second moment argument again, Bennett, Dudek, and Frieze [22] announced a proof showing that $p = \sqrt{1/n}$ is the threshold for G(n,p) to contain the square of a Hamilton cycle.

Trees. For trees T on at least 3 vertices we have $m_{\rm R}(T)=2$, where the path on 3 vertices is the maximiser in $m_{\rm R}(T)$. It follows that if $T=(T_n)$ is a fixed sequence of bounded degree trees then G(n,p) a.a.s. contains T if $p \cdot n^{1/2} \to \infty$. This is far from the best known upper bound of $\log^5 n/n$ for the threshold for containing such trees [113], to which we shall return in Section 4.4. Riordan's theorem allows to also consider trees with growing maximum degrees. However, the resulting threshold bounds are again far from the best known bounds (see Section 4.4).

Planar graphs. For a planar graph H' we have $e(H')/(v(H')-2) \leq 3$, and hence any n-vertex planar graph H satisfies $m_{\mathbb{R}}(H) \leq 3$, with equality when H is a triangulation. Hence, if H has bounded degree, then a.a.s. G(n,p) contains H if

$$p \cdot n^{1/3} \to \infty$$
.

As was observed by Bollobás and Frieze [29] for $p=c/n^{1/3}$ with $c=(27e/256)^{1/3}$ the random graph G(n,p) a.a.s. contains no spanning triangulation.

A planar graph H drawn uniformly from all planar graphs on n vertices a.a.s. has maximum degree less than $3\log n$ [112, 48]. It follows that for such graphs H the random graph G(n,p) a.a.s. contains H if $p \cdot \frac{n^{1/3}}{\log^{4/3} n} \to \infty$.

 K_r -factors. For K_r -factors H we have $m_R(H) = m_R(K_r) = \frac{1}{2}r(r-1)/(r-2)$ and hence G(n,p) a.a.s. has a K_r -factor when

$$p\cdot n^{\frac{2}{r}-\frac{2}{r(r-1)}}\to\infty\,.$$

The power in the exponent of n is surprisingly close to the right one, which is $-1/m_1(K_r) = -2/r$ (ignoring log-factors), as given by Theorem 4.3 in the next section.

Bounded degree graphs. Graphs H' with maximum degree $\Delta(H') \leq \Delta$ satisfy $e(H')/(v(H')-2) \leq \frac{1}{2}\Delta + \Delta/(v(H')-2)$. To maximise this quantity we should set $v(H') = \Delta + 1$ (since for smaller v(H') an even better bound on e(H')/(v(H')-2) holds). Hence, for a maximum degree Δ graph H we have $m_{\rm R}(H) \leq \frac{1}{2}(\Delta+1)\Delta/(\Delta-1)$ and thus G(n,p) a.a.s. contains H when

$$p \cdot n^{\frac{2}{\Delta+1} - \frac{2}{\Delta(\Delta+1)}} \to \infty \,.$$

This again is close to the lower bound, which is given by the lower bound for containing a $K_{\Delta+1}$ -factor.

D-degenerate graphs. For *D*-degenerate graphs H' we have $e(H') \leq (v(H') - D)D + \binom{D}{2} \leq v(H')D - 2D$ for $D \geq 3$. It follows that a *D*-degenerate graph H satisfies $m_R(H) \leq D$ for $D \geq 3$. So, if further the maximum degree of H is bounded by a constant (potentially much larger than D) then G(n, p) a.a.s. contains H when

$$p\cdot n^{1/D}\to\infty\,.$$

As mentioned earlier for $p \leq ((1-\varepsilon)e/n)^{1/D}$ the probability that G(n,p) contains the D-th power of a Hamilton cycle tends to 0. Since the D-th power of a Hamilton path is D-degenerate this shows that the bound given by Riordan's theorem is close to best possible. Observe also that this bound is much better than the known bounds in universality results for D-degenerate graphs discussed in Section 5.1.

These examples illustrate that Riordan's theorem often, though not always, gives optimal or close to optimal bounds. As indicated, for K_r -factors and bounded degree trees better bounds have been obtained in recent years, and I shall discuss these in the following sections.

For spanning bounded degree graphs H the gap between lower bounds and the bound given by Riordan's theorem remains, though very recently near-optimal bounds have been obtained for almost spanning H and we shall return to this topic in Section 4.5.

4.3 The Johansson-Kahn-Vu Theorem

It is not too difficult to prove (see, e.g., Theorem 4.9 of [79], or [123]) that the threshold in G(n,p) for an almost spanning H-factor, that is, a collection of vertex disjoint copies of H covering all but at most εn vertices, is $n^{-1/m_1(H)}$. For obtaining a spanning H-factor we need to go above this threshold by at least some (power of a) logarithmic factor in some cases: For strictly 1-balanced H, if p grows slower than $(\log n)^{1/e(H)}/n^{1/m_1(H)}$ then by Theorem 3.2 a.a.s. not every vertex of G(n,p) is covered by a copy of H, hence G(n,p) contains no spanning H-factor.

Ruciński [123] showed that if $np^{\delta^*(H)} - \log n \to \infty$, where $\delta^*(H) = \max\{\delta(H'): H' \subseteq H\}$, then G(n,p) a.a.s. contains an H-factor. This implies that the threshold for a K_r -factor is at most $(\log n/n)^{1/(r-1)}$. This was improved on by Krivelevich [105], who proved that for each r there is a constant C = C(r) such that if $p \geq Cn^{-2r/((r-1)(r+2))}$ then G(n,p) a.a.s. contains a K_r -factor (see [79, Section 4.3] for a short exposition of the interesting proof of this result in the case r=3). Observe that this bound on the threshold is also better than the one implied by Riordan's theorem (Theorem 4.2).

Finally, in a celebrated result, Johansson, Kahn, and Vu [81] proved that for strictly 1-balanced H the threshold for an H-factor does indeed coincide with the H-cover threshold, that is, the threshold for every vertex of G(n, p) to be contained in an H-copy.

Theorem 4.3 (Johansson, Kahn, Vu [81])

For a strictly 1-balanced graph H the threshold for G(n,p) to contain an H-factor is

$$\frac{(\log n)^{1/e(H)}}{n^{1/m_1(H)}} \, .$$

Johansson, Kahn, and Vu prove this theorem more generally for hypergraphs in [81]. When H is a single edge, that is, we are asking for a perfect hypergraph matching, it thus solves the famous Shamir problem. A good exposition of the proof in this case is given in [15].

In their proof Johansson, Kahn and Vu work (for some part of the argument) in G(n,m). The basic idea is to think of G(n,m) as a random graph obtained from K_n by successively deleting random edges until only m edges remain. They then show with the help of a martingale argument and certain entropy results that in each deletion step not too many H-factors get destroyed, implying that the number of H-factors in G(n,m) is close to expectation.

Already Ruciński [123] and Alon and Yuster [12] observed that not for every H the H-factor threshold is the same as the H-cover threshold. Indeed, it was shown in [12, 123] that for graphs H with $\delta(H) < m_1(H)$ the H-factor threshold is at least $n^{-1/m_1(H)}$, while the H-cover threshold is of lower order of magnitude. It is not surprising that the thresholds for these two properties do not always coincide since there may be some vertex $x \in V(H)$ such that among all H-copies in G the vertex x is only mapped to few vertices u of G. Alon and Yuster [12] conjectured, however, that for each graph H with e(H) > 0 the threshold for an H-factor is

$$n^{-(1/m_1(H))+o(1)}$$

Johansson, Kahn and Vu [81] prove this conjecture as well. Further, they conjecture that the obstacle identified in the last paragraph is the only one, that is, that the H-factor threshold coincides with the threshold for the property $LCOV_H$ that in an n-vertex graph G

- 1. each vertex of G is contained in an H-copy, and
- 2. for each $x \in V(H)$ there are at least n/v(H) vertices $u \in V(G)$ such that some H-copy in G maps x to u.

Conjecture 4.4 (Johansson, Kahn, Vu [81]) The threshold for containing an H-factor is the same as that for $LCOV_H$.

A related conjecture appears also already in [123]. Johansson, Kahn, and Vu [81] think it even possible that a hitting time version of conjecture 4.4 is true. Further, they state that the threshold of $LCOV_H$ is as follows (for a proof see the arXiv version of [68, Lemma 2.5]). The local 1-density of H at $x \in V(H)$ is

$$m_1(x, H) = \max_{\substack{H' \subseteq H \\ x \in V(H')}} \frac{e(H')}{v(H') - 1}.$$

We call H vertex-1-balanced if $m_1(x, H) = m_1(H)$ for all $x \in V(H)$. Let s(x, H) denote the minimum number of edges of a maximiser H' in $m_1(x, H)$, and let s(H) be the maximum among all s(x, H). The threshold of $LCOV_H$, which, following [81], we denote by th^[2](n), then satisfies

$$\operatorname{th}^{[2]}(n) = \begin{cases} \frac{(\log n)^{1/s(H)}}{n^{1/m_1(H)}} & \text{if H is vertex-1-balanced}\,, \\ n^{-1/m_1(H)} & \text{otherwise}\,. \end{cases}$$

Gerke and McDowell [68] proved Conjecture 4.4 for graphs which are not vertex-1-balanced. Hence, the only open case now is that of vertex-1-balanced graphs which are not strictly 1-balanced.

Theorem 4.5 (Gerke, McDowell [68]) For a graph H which is not vertex-1-balanced the threshold for an H-factor in G(n,p) is $n^{-1/m_1(H)}$.

The idea of [68] is to identify dense subgraphs H' of H (which do not cover all vertices because H is non-vertex-1-balanced) and first embed a corresponding non-spanning H'-factor into G(n,p). They then use a variant of Theorem 4.3 to complete the embedding. For obtaining this variant they verify that a partite version of the Johansson–Kahn–Vu theorem holds, which is also useful in other applications.

In fact, the method of Gerke and McDowell allows a proof of Conjecture 4.4 also in the case of many H which are vertex-1-balanced and not strictly 1-balanced. Moreover, for all other H (as for example a triangle and a C_4 glued along one edge) the upper bound given by their method is within a constant log-power of the conjectured bound (see the discussions in the concluding remarks of [68]).

4.4 Trees

The appearance of long paths in G(n,p) was another topic considered early on in the theory of random graphs. As explained in Section 4.1 the threshold in G(n,p) for a Hamilton path is $\log n/n$, where the lower bound follows from the fact that for $p < \log n/n$ there are a.a.s. isolated vertices in G(n,p). Many related results were obtained in the sequel. To give an example, in [2, 57] paths of length cn in G(n,p) for 0 < c < 1 are considered. But one very natural question, which turned out to be difficult, is if the threshold result for Hamilton paths extends to other spanning trees with bounded maximum degree. The following conjecture, which claims that this is indeed the case and has prompted much recent work, is attributed to Kahn (see [84]), but also appears in [11].

Conjecture 4.6 For every fixed Δ there is some constant C such that if $T = (T_n)$ is a fixed sequence of trees on n vertices with $\Delta(T) \leq \Delta$ then G(n,p) a.a.s. contains T if $p \geq C \log n/n$.

In the following I will summarise the progress that has been made towards proving this conjecture. Trees of small linear size were considered by Fernandez de la Vega [58], who proved that there are (large) constants C, C' such that for any fixed Δ and any fixed sequence $T = (T_n)$ of trees with $v(T) \leq n/C$ and $\Delta(T) \leq \Delta$, if $p \geq C'\Delta/n$ then G(n, p) a.a.s. contains T. Alon, Krivelevich and Sudakov [11] improved on this and showed that the threshold in G(n, p) for any sequence of almost spanning trees of bounded degree is 1/n.

Theorem 4.7 (Alon, Krivelevich, Sudakov [11]) Given $\Delta \geq 2$ and $0 < \varepsilon < \frac{1}{2}$, let $C = 10^6 \Delta^3 \varepsilon^{-1} \log \Delta \log^2(2/\varepsilon)$. If $p \geq C/n$ then G(n, p) a.a.s. contains all trees T with $\Delta(T) \leq \Delta$ and $v(T) \leq (1 - \varepsilon)n$.

Observe that Theorem 4.7 is a universality result, stating that G(n, p) contains all these trees *simultaneously*. We shall discuss universality results in G(n, p) in more detail in Section 5. Obtaining such a universality result is possible for Alon, Krivelevich, and Sudakov because they do not prove their result directly for G(n, p), but instead for any graph satisfying certain degree and expansion properties. Their proof uses the well-known embedding result for small (linear sized) trees by Friedman and Pippenger [62]. Balogh, Csaba, Pei and Samotij [16] showed that using instead a related tree embedding result of Haxell [73], which works for larger trees, one can improve the constant in Theorem 4.7 to $C = \max\{1000\Delta\log(20\Delta), 30\Delta\varepsilon^{-1}\log(4e\varepsilon^{-1})\}$. This was further improved by

Montgomery [114] to $C = 30\Delta\varepsilon^{-1}\log(4e\varepsilon^{-1})$), which comes close to the $C = \Theta(\Delta\log\varepsilon^{-1})$ believed possible in [11].

Alon, Krivelevich and Sudakov also observed in [11] that for every $\varepsilon > 0$ Theorem 4.7 immediately implies Conjecture 4.6 for trees T with εn leaves (for $p \geq C(\varepsilon, \Delta) \log n/n$), by using a two-round exposure of G(n,p), finding in the first round a copy of T minus $(\varepsilon n/\Delta)$ leaves with distinct parents, and then embedding these leaves in the second round, which is easy because all it requires is to find a certain matching. Hefetz, Krivelevich, and Szabó observe in [77] that a similar strategy can be used for embedding trees T with a linearly sized bare path, that is a path whose inner vertices have degree 2 in T, also for $p \geq C(\varepsilon, \Delta) \log n/n$.

This leaves the case of trees with few leaves (and no long bare path) of Conjecture 4.6. Since each tree has average degree less than 2, however, these trees have many vertices of degree 2, and hence a linear number of (arbitrarily long) constant length bare paths. Krivelevich [106] used this fact and showed that the same strategy as outlined for trees with many leaves in the previous paragraph can be used for trees with many bare paths by replacing the matching argument by a partite version of the Johansson–Kahn–Vu theorem for embedding the bare paths. Krivelevich's strategy leads to the following result.

Theorem 4.8 (Krivelevich [106]) For every $\varepsilon > 0$ and every sequence $T = (T_n)$ of trees with $v(T) \le n$ the random graph G(n, p) a.a.s. contains T if

$$p \ge \frac{40\Delta(T)\varepsilon^{-1}\log n + n^{\varepsilon}}{n}.$$

In this result $\Delta(T)$ is allowed to grow with n (in particular, a different strategy than Theorem 4.7 is used for obtaining an almost spanning embedding).

Further progress on various classes of trees has been obtained by various groups. Hefetz, Krivelevich, and Szabó [77] show that trees with linearly many leaves and trees with linear sized bare paths, and Montgomery [114] that trees with $\alpha n/\log^9 n$ bare paths of length $\log^9 n$ for any $\alpha > 0$, are already a.a.s. contained in G(n, p) for

$$p = (1 + \varepsilon) \log n / n$$
.

Hefetz, Krivelevich, and Szabó [77] also argue that for the same p the random graph G(n,p) a.a.s. contains any typical random tree T, that is, a tree with maximum degree $(1 + o(1)) \log n / \log \log n$ as shown in [115].

Investigating a class of special trees called combs was suggested by Kahn (see [84]). A *comb* is a tree consisting of a path on n/k vertices with

disjoint k-paths beginning at each of its vertices. Observe that, for example for $k = \sqrt{n}$, combs neither have linearly many leaves nor linear sized bare paths. Kahn, Lubetzky, and Wormald [84, 83] established Conjecture 4.6 for combs. This was improved on and generalised by Montgomery [114] who proved the following result. A tooth of length k in a tree is a bare path of length k where one end-vertex is a leaf. Montgomery showed that for any fixed $\alpha > 0$ a tree T with at least $\alpha n/k$ teeth of length k is contained a.a.s. in G(n,p) for $p = (1+\varepsilon) \log n/n$.

Finally, a result for general bounded degree trees has recently been established by Montgomery [113], which comes very close to the conjectured threshold.

Theorem 4.9 (Montgomery [113]) If $T = (T_n)$ is a fixed sequence of trees on n vertices with maximum degree $\Delta = \Delta(n)$ then G(n,p) a.a.s. contains T if $p \ge \Delta \log^5 n/n$.

Montgomery also announced in [113] further work in progress leading to the proof of Conjecture 4.6. For proving Theorem 4.9 Montgomery follows the basic strategy outlined above of first finding an almost spanning subtree of T, leaving some bare paths to be embedded in a second stage (since the case of trees with many leaves is solved already). For embedding these bare paths, however, Montgomery uses an absorbing-type method.

4.5 Bounded degree graphs

Now we turn to the question of when G(n,p) contains given spanning graphs of bounded maximum degree. Let Δ be a constant and $H=(H_n)$ be sequence of graphs with $\Delta(H) \leq \Delta$ and $v(H) \leq n$. Recall that the Theorem of Alon and Füredi (Theorem 4.1) implies that G(n,p) a.a.s. contains H if $p \geq (C(\Delta) \log n/n)^{1/\Delta}$, and Riordan's theorem (Theorem 4.2) implies the same if $p \cdot n^{\frac{2}{\Delta+1} - \frac{2}{\Delta(\Delta+1)}} \to \infty$. This is unlikely to be optimal, though it cannot be far off. The optimum is widely believed to be as follows (see, e.g., [55]).

Conjecture 4.10 Let $H = (H_n)$ be a sequence of graphs with $\Delta(H) \leq \Delta$ and $v(H) \leq n$. Then G(n, p) a.a.s. contains H if

$$p \cdot \frac{n^{2/(\Delta+1)}}{(\log n)^{1/\binom{\Delta+1}{2}}} \to \infty. \tag{4.1}$$

In other words, the conjecture states that G(n,p) contains H from above the threshold for a $K_{\Delta+1}$ -factor. Ferber, Luh and Nguyen [55] prove Conjecture 4.10 for almost spanning H.

Theorem 4.11 (Ferber, Luh, Nguyen [55]) Let $\varepsilon > 0$ and Δ be fixed. Let $H = (H_n)$ be a fixed sequence of graphs with $\Delta(H) \leq \Delta$ and $v(H) \leq (1 - \varepsilon)n$. Then G(n, p) a.a.s. contains H if p satisfies (4.1).

The strategy for the proof of Theorem 4.11 is as follows. Ferber, Luh and Nguyen show that H can be partitioned into a sparse part H', which is sparse enough to be embedded with the help of Riordan's theorem, and a dense part which consists of a collection of induced subgraphs, each of constant size. Given an embedding of H' they then in constantly many rounds extend this embedding successively to embed also the constant size dense bits of H by finding a matching in a suitable auxiliary hypergraph, using a hypergraph Hall-type theorem of Aharoni and Haxell [1] (a similar idea was already used in [38]).

In fact, it is widely believed that even a universality version of Conjecture 4.10 is true (see Conjecture 5.2). Further recent advances were made in this direction, which we shall return to in Section 5.1.

4.6 The Kahn-Kalai conjecture

Let us round off the results presented in the previous sections with a farreaching and appealing conjecture of Kahn and Kalai. We first need some motivation and definitions. Theorem 3.1 states that for fixed graphs H the threshold for the appearance of H in G(n,p) coincides with what Kahn and Kalai [82] call the expectation threshold for H, written $p_{\mathbb{E}}(H,n)$, which is the least p=p(n) such that for each subgraph H' of H the expected number of H' in G(n,p) is at least 1. The expectation threshold can be defined analogously for sequences $H=(H_n)$ of graphs. In particular, for any $(H_n)=H$ we have that $p_{\mathbb{E}}(H,n)$ is the least p=p(n) such that for every subgraph H' of H we have

$$\frac{(n)_{v(H')}}{|\operatorname{Aut}(H')|} p^{e(H')} \ge 1.$$

For example, if H is an F-factor and F' is any subgraph of F then let H' be the vertex disjoint union of $\ell = n/v(F)$ copies of F'. Then the condition above requires that

$$\frac{(n)_{\ell v(F')}}{\ell!|\operatorname{Aut}(F')|}p^{\ell e(F')} \ge 1\,,$$

which can easily be calculated to be equivalent to $p \ge Cn^{-(v(F')-1)/e(F')}$ for some constant C, and hence $p_{\mathbb{E}}(H,n)$ is of the order $n^{-m_1(F)}$ for F-factors. So, by Theorem 4.3, in this case $p_{\mathbb{E}}(H,n)$ is different from the threshold for the appearance of H if F is strictly balanced – but only by

less than a $\log n$ factor. Kahn and Kalai [82] conjectured that this is the case for every H.

Conjecture 4.12 (Kahn, Kalai [82]) There is a universal constant C such that for any sequence $H = (H_n)$ of graphs the threshold for G(n, p) to contain H is at most $Cp_{\mathbb{R}}(H, n) \log n$.

Conjecture 4.6 on trees is a special case of Conjecture 4.12 because $p_{\mathbb{E}}(T,n)$ is of order 1/n for bounded degree trees T. Conjecture 4.4 on H-factors and Conjecture 4.10 on bounded degree graphs, on the other hand, are somewhat stronger than what is implied by Conjecture 4.12 because they specify a smaller log-power.

4.7 Constructive proofs

One question we have only occasionally taken up in the preceding sections is if the results on the various structures that exist in G(n, p) a.a.s. for certain probabilities have constructive proofs, allowing for a deterministic or randomised algorithm which finds the desired structure. This question is important for two reasons:

- 1. Such constructive proofs often lead to polynomial time algorithms, making it possible to find the structures efficiently.
- 2. Constructive proofs often allow the identification of certain pseudorandom properties, that is, properties which G(n,p) a.a.s. enjoys, which are sufficient for the construction to work. In this case universality results may become possible.

In particular, two prominent results we discussed, whose proofs were not constructive but used the second moment method, were Riordan's theorem and the Johansson–Kahn–Vu theorem. As outlined, these were also used as tools in the proof of other results, such as Theorem 4.5, Theorem 4.8, or Theorem 4.11. This motivates the following problem.

Problem 4.13 Give a constructive proof of Riordan's theorem (Theorem 4.2) or the Johansson–Kahn–Vu theorem (Theorem 4.3).

As I shall explain in Sections 5 and 6, many constructive proofs for embedding classes of spanning or almost spanning graphs H in G(n,p) (or in subgraphs of G(n,p)) we know of follow a greedy-type paradigm: They embed H (or a suitable subgraph of H) vertex by vertex (or class of vertices by class of vertices), aiming at guaranteeing that unembedded common H-neighbours of already embedded H-vertices can still be embedded in the future. In this sense they crucially rely on the fact that all common

neighbourhoods in G(n, p) of $\Delta(H)$ vertices (or of D vertices if H is D-degenerate) are large, which fails to be true for $p \leq n^{-1/\Delta(H)}$. Hence, in Sections 5 and 6 probability bounds of this order shall often form a natural barrier not yet overcome in many instances, though they are not believed to be the right bounds.

5 Universality of random graphs

In this section we consider the question of when the random graph is a.a.s. universal for certain classes of graphs. More precisely, a graph G on n vertices is said to be universal for a class \mathcal{H} of graphs, if it contains a copy of every graph $H \in \mathcal{H}$. The crucial difference for G(n,p) to contain some $H \in \mathcal{H}$ a.a.s. and to be a.a.s. universal for \mathcal{H} (if \mathcal{H} is large) is that in the latter case we require a typical graph from G(n,p) to contain all these $H \in \mathcal{H}$ simultaneously.

The graph classes for which universality results have been established, and which we shall consider in this section are bounded degree graphs, bounded degree graphs which further have (smaller) bounded maximum 0-density or bounded degeneracy, and bounded degree trees. Let me stress that none of the results presented in this section is believed to be optimal, indicating that the methods we have at hand for proving universality are still limited. Moreover, there are many other natural graph classes still to be considered. The following is just one example.

Question 5.1 When is G(n,p) a.a.s. universal for the class of all planar graphs with maximum degree Δ ; or more generally for all maximum degree Δ graphs which are F-minor free for some fixed F?

As an aside, n-vertex universal graphs with $O(n \log n)$ edges for n-vertex planar graphs with maximum degree Δ were constructed in [23], and graphs G with v(G) + e(G) = O(n) that are universal for this class of graphs in [36]. For more background on constructions of universal graphs see the survey of Alon [6].

5.1 Universality for bounded degree graphs

Before we turn to results concerning the universality of G(n,p) for the family $\mathcal{H}(n,\Delta)$ of all n-vertex graphs with maximum degree at most Δ , let us first briefly recall some lower bounds. A counting argument shows that any graph G that is universal for $\mathcal{H}(n,\Delta)$ must have edge density at least $\Omega(n^{-2/\Delta})$. This was observed in [9], and follows from the fact that $\sum_{i \leq \Delta n/2} \binom{e(G)}{i} \geq |\mathcal{H}(\Delta,n)|$ and well-known estimates of the number

of Δ -regular graphs (for details see [9]). It is interesting to observe that this lower bound was matched by constructive results: Alon and Capalbo constructed graphs that are universal for $\mathcal{H}(n,\Delta)$ and have n vertices and $C(\Delta)n^{2-2/\Delta}\log^{4/\Delta}n$ edges in [7], and $(1+\varepsilon)n$ vertices and $C_2(\Delta,\varepsilon)n^{2-2/\Delta}$ edges for every $\varepsilon > 0$ in [8] (see also [6]).

For G(n,p) the only better lower bound we know is the following, which is only slightly better and only appeals to one particular graph in $\mathcal{H}(n,\Delta)$ instead of universality. By Theorem 3.2, If p grows slower than $(\log n)^{1/\binom{\Delta+1}{2}}/n^{2/(\Delta+1)}$ then a.a.s. G(n,p) contains no spanning $K_{\Delta+1}$ -factor. If one turns to universality for smaller, but linearly sized graphs H, the known lower bound is not much smaller. Indeed, if $p \leq cn^{-2/(\Delta+1)}$ for some sufficiently small $c = c(\eta) > 0$ then G(n,p) is not universal even for $\mathcal{H}(\eta n, \Delta)$ as it does not contain a vertex disjoint union of $K_{\Delta+1}$ covering ηn vertices because the expected number of $K_{\Delta+1}$ in G(n,p) is at most $n^{\Delta+1}p^{(\Delta+1)\Delta/2} \leq c^{(\Delta+1)\Delta/2}n$.

As mentioned earlier, it is widely believed (see, e.g., [44, 55]) that the lower bound above reflects the truth, that is, when G(n, p) starts containing every fixed sequence (H_n) of graphs from $\mathcal{H}(n, \Delta)$ a.a.s. then it is already universal for $\mathcal{H}(n, \Delta)$ (cf. Conjecture 4.10).

Conjecture 5.2 G(n,p) is a.a.s. universal for $\mathcal{H}(n,\Delta)$ if

$$p \cdot \frac{n^{2/(\Delta+1)}}{(\log n)^{1/(\frac{\Delta+1}{2})}} \to \infty$$
.

At present we are still far from verifying Conjecture 5.2, though this problem attracted considerable attention since the turn of the millennium. Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [9] considered almost spanning graphs and showed that for every $\varepsilon > 0$ and Δ there is C such that for $p \geq C(\log n/n)^{1/\Delta}$ the random graph G(n,p) is a.a.s. universal for $\mathcal{H}((1-\varepsilon)n,\Delta)$. After improvements in [43], Dellamonica, Kohayakawa, Rödl, and Ruciński [44] showed that for this probability G(n,p) is also universal for spanning bounded degree graphs.

Theorem 5.3 (Dellamonica, Kohayakawa, Rödl, Ruciński [44]) For each $\Delta \geq 3$ there is C such that G(n, p) is a.a.s. universal for the family $\mathcal{H}(n, \Delta)$ if

$$p \ge C \left(\frac{\log n}{n}\right)^{1/\Delta}$$
.

Using a simpler argument (but the same basic strategy), Kim and Lee [85] showed that this result also holds for $\Delta = 2$. For proving their

theorem Dellamonica, Kohayakawa, Rödl, and Ruciński present a randomised algorithm that uses a certain set of pseudorandom properties which G(n,p) has a.a.s. and embeds every $H \in \mathcal{H}(n,\Delta)$ a.a.s. in every graph G with these pseudorandom properties. This algorithm is inspired by the various known techniques for proving the blow-up lemma in dense graphs [97, 98, 121, 122], and the underlying idea of using an embedding strategy based on matchings goes back to the proof of the theorem of Alon and Füredi (Theorem 4.1) outlined in Section 4.2.

As mentioned in Section 4.7 the exponent $1/\Delta$ forms a natural barrier to further improvement. So far, this barrier was broken only for almost spanning subgraphs and in the case $\Delta = 2$.

Theorem 5.4 (Conlon, Ferber, Nenadov, Škorić [38])

For every $\varepsilon > 0$ and $\Delta \geq 3$ the random graph G(n,p) is a.a.s. universal for $\mathcal{H}((1-\varepsilon)n,\Delta)$ if

$$p \cdot \frac{n^{1/(\Delta - 1)}}{\log^5 n} \to \infty.$$

For the case of maximum degree $\Delta=2$, Conlon, Ferber, Nenadov, and Škorić [38] also state that similar arguments as those used for showing this theorem show that G(n,p) is a.a.s. universal for $\mathcal{H}\big((1-\varepsilon)n,2\big)$ if $p\geq Cn^{-2/3}$, which is best possible up to the value of C. Moreover, Ferber, Kronenberg, and Luh [54] very recently showed that G(n,p) is a.a.s. universal for $\mathcal{H}(n,2)$ if $p\geq C(\log n/n^2)^{1/3}$, which is again best possible up to the value of C. Their proof combines the Johansson–Kahn–Vu Theorem with arguments from Montgomery's [113] proof of Theorem 4.9.

The strategy of the proof of Theorem 5.4 is as follows. Each graph H under consideration is partitioned into a set of (small) components with at most $\log^4 n$ vertices, a set of induced cycles of length at most $2\log n$, and the graph H' induced on the remaining vertices. They then show that any induced subgraph of G(n,p) on $\frac{1}{2}\varepsilon n$ vertices is universal for $\mathcal{H}(\log^4 n, \Delta)$ and can thus be used for embedding the small components, that H' has a structure suitable for a technical embedding result of Ferber, Nenadov and Peter [56], and that the remaining short cycles can be embedded with the help of the hypergraph matching criterion of Aharoni and Haxell [1].

In [56] Ferber, Nenadov and Peter use the technical result just mentioned for a spanning universality result under additional constraints. More precisely, they consider graphs in $\mathcal{H}(n,\Delta)$ with maximum 0-density at most m_0 , and provide a better bound than Theorem 5.3 for $m_0 < \Delta/4$. Note that for any H we have $m_0(H) \leq \Delta(H)/2$.

Theorem 5.5 (Ferber, Nenadov, Peter [56]) For $\Delta = \Delta(n) > 1$ and $m_0 = m_0(n) > 1$ the random graph G(n, p) is a.a.s. universal

(a) for all $H \in \mathcal{H}(n,\Delta)$ with $m_0(H) \leq m_0$ if

$$p \cdot \frac{\Delta^{12} n^{1/(4m_0)}}{\log^3 n} \to \infty$$
, and

(b) for all $H \in \mathcal{H}(n,\Delta)$ with $m_0(H) \leq m_0$ and girth at least 7 if

$$p \cdot \frac{\Delta^{12} n^{1/(2m_0)}}{\log^3 n} \to \infty.$$

Ferber, Nenadov and Peter prove this result by using a similar embedding strategy (and a similar decomposition of the graphs H) as Dellamonica, Kohayakawa, Rödl, Ruciński [44] and Kim and Lee [85].

A related result is proven in [5], where D-degenerate graphs H in $\mathcal{H}(n,\Delta)$ are considered. It is not difficult to see that the degeneracy D(H)of any graph H satisfies $m_0(H) \leq D(H) \leq 2m_0(H)$. The bound in the first part of the following result is better than that in the first part of Theorem 5.5 if $D(H) < 2m_0(H) - \frac{1}{2}$. The bound in the second part is better than that in Theorem 5.4 if $D(H) < (\Delta(H) - 1)/2$.

Theorem 5.6 (Allen, Böttcher, Hàn, Kohayakawa, Person [5]) For every $\varepsilon > 0$, $\Delta \geq 1$ and $D \geq 1$ there is C such that the random graph G(n,p) a.a.s. is universal

- (a) for all D-degenerate $H \in \mathcal{H}(n,\Delta)$ if $p \geq C(\frac{\log n}{n})^{1/(2D+1)}$, and (b) for all D-degenerate $H \in \mathcal{H}((1-\varepsilon)n,\Delta)$ if $p \geq C(\frac{\log n}{n})^{1/(2D)}$.

This result is a direct consequence of a sparse blow-up lemma for graphs with bounded degeneracy (and maximum degree) established in [5], which we shall return to in Section 7.

Universality for bounded degree trees

Recall that the result of Alon, Krivelevich and Sudakov [11] (Theorem 4.7) states that already for $p = C(\Delta, \varepsilon)/n$ the random graph G(n, p)is a.a.s. universal for the family $\mathcal{T}((1-\varepsilon)n,\Delta)$ of (almost spanning) trees on $(1-\varepsilon)n$ vertices and maximum degree at most Δ .

For spanning trees the situation is less well understood. Hefetz, Krivelevich, and Szabó [77] showed that spanning trees with linearly long bare paths are universally a.a.s. contained in G(n, p) for $p = (1 + \varepsilon) \log n/n$. The first universality result in G(n,p) for the entire class $\mathcal{T}(n,\Delta)$ was obtained by Johannsen, Krivelevich, Samotij [80]. This is a consequence of

the following universality result for graphs with certain natural expansion properties. The proof of this result relies on the embedding result of Haxell [73] for large trees in graphs with suitable expansion properties and a result of Hefetz, Krivelevich, and Szabó [76] on Hamilton paths between any pair of vertices in graphs with certain different expansion properties.

Theorem 5.7 (Johannsen, Krivelevich, Samotij [80])

There is a constant c such that for any n and Δ with $\log n \leq \Delta \leq cn^{1/3}$ every graph G on n vertices with

- (i) $|N_G(X)| \ge 7\Delta n^{2/3}|X|$ for all $X \subseteq V(G)$ with $1 \le |X| < \frac{n^{1/3}}{14\Delta}$, and (ii) $e_G(X,Y) > 0$ for all disjoint $X,Y \subseteq V(G)$ with $|X| = |Y| = \lceil \frac{n^{1/3}}{14\Delta} \rceil$ is universal for $\mathcal{T}(n, \Delta)$.

This directly implies that if $\Delta \geq \log n$ then G(n, p) is a.a.s. universal for $\mathcal{T}(n,\Delta)$ if $p \geq C\Delta \log n/n^{1/3}$, and hence universality for $\mathcal{T}(n,\Delta)$ with constant Δ if $p \ge C \log^2 n/n^{1/3}$.

The result of Ferber, Nenadov, and Peter [56] discussed in the previous section improved on this when Δ grows slower than $n^{1/66}/(\log n)^{1/22}$. Indeed, it follows from the second part of Theorem 5.5 that G(n, p) is a.a.s. universal for $\mathcal{T}(n,\Delta)$ if

$$p \cdot n^{1/2}/(\Delta^{12} \log^3 n) \to \infty$$
.

Further, Montgomery announced in [113] that, using refinements of his method for proving Theorem 4.9, establishing universality of G(n,p) for $\mathcal{T}(n,\Delta)$ with $p=C(\Delta)\log^2 n/n$ is now within reach.

Finally, let us remark that, again, G(n,p) has no chance in giving the sparsest graph that is universal for $\mathcal{T}(n,\Delta)$. Indeed, Bhatt, Chung, Leighton, and Rosenberg [23] constructed n-vertex graphs which are universal for $\mathcal{T}(n,\Delta)$ with constant maximum degree $C(\Delta)$. See the references in [6] for earlier constructions.

Resilience of random graphs

In this section we study the question of how easily an adversary can destroy copies of a graph H in G(n,p). Questions of this type date back (at least³) to [9] where this phenomenon was dubbed fault tolerance (which also appears in [87]), but lately the term resilience has come into vogue, following Sudakov and Vu [129].

³Of course Turán-type problems in random graphs also fall in this category and were studied even earlier (cf. Section 3.2).

Let \mathcal{P} be a monotone increasing graph property and Γ be a graph. The global resilience of Γ with respect to \mathcal{P} is the minimum $\eta \in \mathbb{R}$ such that deleting a suitable set of $\eta e(\Gamma)$ edges from Γ results in a graph not in \mathcal{P} . In other words, whenever an adversary deletes less than a η -fraction of the edges of Γ , the resulting graph will still be in \mathcal{P} . Similarly, in the definition of local resilience the adversary is allowed to destroy a certain fraction of the edges incident to each vertex. Formally, the local resilience of Γ with respect to \mathcal{P} is the minimum $\eta \in \mathbb{R}$ such that deleting a suitable set of edges, while respecting the restriction that for every vertex $v \in V(\Gamma)$ at most $\eta \deg_{\Gamma}(v)$ edges containing v are removed, results in a graph not in \mathcal{P} . For $\Gamma = G(n,p)$ with $p \geq C \log n/n$ for C sufficiently large (where we have degree concentration) this means that for any $\eta' > \eta$ any subgraph G of Γ with minimum degree at least $(1 - \eta')pn$ is in \mathcal{P} .

For the random graph G(n,p) we may then ask what is the local or global resilience of G(n,p) a.a.s. with respect to a property \mathcal{P} for a given p? It turns out that the answer to this question usually is either trivial, that is, basically 0 or 1, or provided by some extremal result in dense graphs (in other words, it is as in G(n,p) with p=1). It is thus not surprising that the local resilience is heavily influenced by the chromatic number of the graphs under study. To the best of my knowledge, at present we do not know of any (subgraph) property which does not follow the pattern just described.

Question 6.1 Let $\pi(\mathcal{H}_n)$ be the local resilience of G(n,1) with respect to containing all graphs from \mathcal{H}_n . Is there any (interesting) family $\mathcal{H} = (\mathcal{H}_n)$ of graphs such that $\pi(\mathcal{H}) = \lim_{n \to \infty} \pi(\mathcal{H}_n)$ exists, and the limit as n tends to infinity of the local resilience of G(n,p) with respect to containing all graphs in \mathcal{H}_n exists but is not in $\{0, 1 - \pi(\mathcal{H})\}$?

Let me remark that resilience and universality are orthogonal properties in the following sense. We might ask for which probabilities G(n, p) has a.a.s. a certain resilience with respect to containing any fixed graph sequence $H = (H_n)$ from a family \mathcal{H} , or with respect to being universal for \mathcal{H} and there is a priori no reason why the answers should turn out the same (though we typically expect them to be). However, in contrast to some results discussed in the previous two sections, at present the methods available for proving resilience generally are constructive and hence allow for universality results. On the other hand, a side effect of this is that many of the probability bounds obtained are far from best-possible.

I will start this section with a global resilience result for small linear sized bounded degree bipartite graphs in Section 6.1, which I also use to outline one approach often used for obtaining resilience results that

relies on the sparse regularity lemma. I then review local resilience results for cycles in Section 6.2, for trees in Section 6.3, for triangle factors in Section 6.4, and for graphs of low bandwidth in Section 6.5.

6.1 Global resilience

Obviously, any graph must have trivial global resilience with respect to the containment of any spanning graph H, since an adversary can delete all copies of H by simply deleting all edges at some vertex. For small linearly sized bipartite graphs H, however, Alon, Capalbo, Kohayakawa, Ruciński [9] and Szemerédi, in a paper initiating research into the area of the resilience of random graphs, proved the following result.

Theorem 6.2 ([9]) For every $\Delta \geq 2$ and $\gamma > 0$ there exist $\eta > 0$ and C such that if $p \geq C(\frac{\log n}{n})^{1/\Delta}$ then G(n,p) a.a.s. has global resilience at least $1 - \gamma$ with respect to universality for the family $\mathcal{H}(\eta n, \eta n, \Delta)$ of all bipartite graphs with partition classes of size $\lfloor \eta n \rfloor$ and maximum degree at most Δ .

Note that this shows that G(n,p) contains many copies of all graphs in $\mathcal{H}(\eta n, \eta n, \Delta)$ everywhere. It is clear that such a result cannot hold for non-bipartite H because, as any other graph, G(n,p) can be made bipartite by deleting half of its edges. The lower bound on p though is unlikely to be optimal.

Problem 6.3 Improve the lower bound on p in Theorem 6.2.

The proof in [9] of Theorem 6.2 uses the sparse regularity lemma, which I will present and explain in more detail in Section 7. The strategy is as follows. First, the sparse regularity lemma is applied to the graph G to obtain a sparse ε -regular partition of V(G). It is then easy to show that some pair of clusters in this partition forms a sparse ε -regular pair (V_1, V_2) with sufficient density. The authors of [9] then develop an embedding result for bounded degree bipartite graphs with partition classes of size $\eta'|V_1|$ and $\eta'|V_2|$ in such a pair.⁴

Most other resilience results (with the exception of the results on cycles in the next section) mentioned in the following use proof strategies which are variations on this basic strategy: They use the sparse regularity lemma to obtain a regular partition, then use a result from dense extremal

⁴This result is only stated for $p \ge C(\log n/n)^{1/2\Delta}$ in [9] though, for example, with the bipartite sparse blow-up lemma inferred in [32] from their techniques and from newer regularity inheritance results, one easily obtains from their proof the probability bound claimed in Theorem 6.2.

graph theory on the so-called reduced graph to obtain a suitable structure of regular pairs in this partition, and then use or develop a suitable embedding lemma in such structures of regular pairs, which allows one (often with substantial extra work) to embed the desired graphs.

6.2 Local resilience for cycles

In the language of local resilience, Dirac's theorem [47] states that K_n has local resilience 1/2 - o(1) with respect to containing a Hamilton cycle. In this section we shall consider sparse analogues of this result in G(n, p).

Clearly, the local resilience of G(n,p) with respect to containing any graph on more than n/2 vertices is at most $\frac{1}{2}-o(1)$, since by deleting the edges of G(n,p) in a random balanced cut we obtain a disconnected graph with components of size at most $\frac{1}{2}n$, and it can easily be shown that each vertex loses at most $(\frac{1}{2}-o(1))pn$ of its edges. Sudakov and Vu [129] then showed a corresponding lower bound. They proved that for every $\gamma>0$ the local resilience of G(n,p) with respect to containing a Hamilton cycle is a.a.s. at least $\frac{1}{2}-\gamma$ if $p>\log^4 n/n$.

Smaller probabilities were first considered by Frieze, Krivelevich [64], who proved that there are C and η such that for $p \geq C \log n/n$ the local resilience of G(n,p) for containing a Hamilton cycle is a.a.s. at least η . Ben-Shimon, Krivelevich, Sudakov [20] then were able to replace η with $\frac{1}{6}(1-\gamma)$, and then in [21] with $\frac{1}{3}(1-\gamma)$. Finally Lee and Sudakov [110] showed that also for this range of p the local resilience is $\frac{1}{2}-o(1)$.

Theorem 6.4 (Lee, Sudakov [110]) For every $\gamma > 0$ there is a constant C such that the local resilience of G(n,p) with respect to containing a Hamilton cycle is a.a.s. at least $\frac{1}{2} - \gamma$ if $p > C \log n/n$.

In [21] probabilities as close as possible to the threshold for Hamiltonicity, that is, $p \geq (\log n + \log \log n + \omega(1))/n$, are investigated, at which point the results need to be of a different form, because vertices of degree 2 may exist in G(n,p). That Hamilton cycles are so well understood is connected to the fact that with the Pósa rotation-extension technique (see, e.g., [119]), which is used in the proof of all the aforementioned results, we have a powerful tool at hand for finding Hamilton cycles.

Even smaller probabilities, where we cannot hope for Hamilton cycles any longer, were considered by Dellamonica, Kohayakawa, Marciniszyn and Steger [42]. They show that a.a.s. the local resilience of G(n,p) with respect to containing a cycle of length at least $(1-\alpha)n$ is $\frac{1}{2}-o(1)$ for any $0<\alpha<\frac{1}{2}$ if $p\cdot n\to\infty$.

Finally, Krivelevich, Lee and Sudakov [107] proved that if $p \cdot n^{1/2} \to \infty$ then the local resilience of G(n, p) with respect to being pancyclic, that

is, having cycles of all lengths between 3 and n, is a.a.s. $\frac{1}{2} - o(1)$. Here the probability required is higher than in the results on Hamilton cycles, which is necessary for ensuring the adversary cannot delete all triangles (see also the remarks in Section 6.4). An even stronger result was proved by Lee and Samotij in [109] who show that for the same probability a.a.s. every Hamiltonian subgraph of G(n,p) containing at least $\left(\frac{1}{2} + o(1)\right)pn$ edges is pancyclic.

6.3 Local resilience for trees

Komlós, Sárközy, and Szemerédi [96] showed that for every $\gamma > 0$ and every Δ every sufficiently large n-vertex graph G with minimum degree at least $(\frac{1}{2} + \gamma)n$ contains a copy of any spanning tree T with maximum degree at most Δ . In [99] they then extended this result to trees with maximum degree at most $cn/\log n$. An analogue of the former result for random graphs in the case that T is almost spanning was obtained by Balogh, Csaba, and Samotij [17]. Recall that $\mathcal{T}(n, \Delta)$ is the family of all n-vertex trees with maximum degree at most Δ .

Theorem 6.5 (Balogh, Csaba, Samotij [17]) For all $\Delta \geq 2$ and $\gamma > 0$ there is a constant C such that for $p \geq C/n$ the local resilience of G(n, p) with respect to being universal for $\mathcal{T}((1-\gamma)n, \Delta)$ a.a.s. is at least $\frac{1}{2} - \gamma$.

The surprising aspect about this theorem is the small probability for which it was proven to hold. Clearly, this bound on p is sharp up to the value of C, since for smaller p the biggest component of G(n,p) gets too small to contain a tree on (1-o(1))n vertices. Moreover, as argued in [17], at this probability we cannot ask for, say, balanced D(n)-ary trees on (1-o(1))n vertices for $D(n) \to \infty$, since we do not have enough vertices of degree D(n). Further, the factor $\frac{1}{2}$ in this result is best possible by the discussion in the second paragraph of the previous section. To prove their result Balogh, Csaba, and Samotij [17] use an approach based on the regularity lemma and an embedding result for trees which is a suitable modification of the tree embedding result by Friedman and Pippenger [62].

The only local resilience result for spanning trees that I am aware of follows from Theorem 6.9 on the resilience of G(n,p) for low-bandwidth graphs, which is presented in Section 6.5. It was proven by Chung [37] that trees with constant maximum degree have bandwidth at most $O(n/\log n)$.

Theorem 6.6 (Allen, Böttcher, Ehrenmüller, Taraz [3])

For all $\Delta \geq 2$ and $\gamma > 0$ there is C such that for $p \geq C(\frac{\log n}{n})^{1/3}$ the local resilience of G(n,p) with respect to being universal for $\mathcal{T}(n,\Delta)$ a.a.s. is at least $\frac{1}{2} - \gamma$.

This probability is not believed to be optimal. Indeed, it is conceivable that this result remains true down to the conjectured universality threshold for $\mathcal{T}(n,\Delta)$.

Conjecture 6.7 The conclusion of Theorem 6.6 is true for $p \ge C \log n/n$.

6.4 Local resilience for triangle factors

Corrádi and Hajnal [41] proved that any graph G with $\delta(G) \geq \frac{2}{3}n$ contains a triangle factor. One could then ask if this result can be transferred to G(n,p) for p sufficiently large, that is, if the local resilience of G(n,p) with respect to containing a triangle factor a.a.s. is $\frac{1}{3} - o(1)$. Huang, Lee, and Sudakov [78] observed that this is not the case even for constant p. Indeed, every vertex v in G(n,p) has a.a.s. a neighbourhood N(v) of roughly size pn, and every $w \in N(v)$ has deg $(w; N(v)) \approx p^2 n$ neighbours in N(v). Therefore, we can delete all triangles containing v by removing at most roughly $p^2n < \gamma pn$ edges at each w if p is small compared to γ and hence obtain a graph without a triangle factor. With a more careful analysis it is possible to show that we can actually choose $O(p^{-2})$ vertices and delete all triangles containing any of these vertices by removing less that γpn edges at each vertex (for the details see [78, Proposition 6.3]).

So the question above should be refined to ask for an almost spanning triangle factor, covering all but $O(p^{-2})$ vertices. Balogh, Lee and Samotij [18] showed that this is indeed true if $p \geq C\left(\frac{\log n}{n}\right)^{1/2}$. Observe that this probability is larger than the threshold $\log^{1/3} n/n^{2/3}$ for a triangle factor as given by Theorem 4.3. If p grows slower than $n^{-1/2}$, however, the $O(p^{-2})$ term becomes trivial.

Theorem 6.8 (Balogh, Lee, Samotij [18]) For every $\gamma > 0$ there are constants C and D such that for $p \geq C\left(\frac{\log n}{n}\right)^{1/2}$ the local resilience of G(n,p) with respect to the containment of an almost spanning triangle factor covering all but at most Dp^{-2} vertices is a.a.s. at least $\frac{1}{3} - \gamma$.

It should be remarked that a corresponding result with Dp^{-2} replaced by εn follows easily from the conjecture of Kohayakawa, Łuczak, and Rödl [90, Conjecture 23], which has long been known for triangles and was proved in full generality in [19, 40, 124]. This argument will be sketched for the purpose of illustrating the sparse regularity lemma in Section 7.1.

For proving their result Balogh, Lee, Samotij [18] develop a sparse analogue of the blow-up lemma for the special case of triangle factors. We shall discuss (more general) blow-up lemmas in Section 7.

Analogous questions concerning H-factors for general H were considered for constant p in [78], but the currently best bounds follow from Theorem 6.9, which we discuss in the next section.

6.5 The bandwidth theorem in random graphs

In [34] it was shown that for every Δ , r and $\gamma > 0$ there is $\beta > 0$ such that any sufficiently large n-vertex graph G with $\delta(G) \geq (\frac{r-1}{r} + \gamma)n$ contains any r-colourable $H \in \mathcal{H}(n,\Delta)$ with bandwidth bw $(H) \leq \beta n$. This proved a conjecture of Bollobás and Komlós and is often referred to as the bandwidth theorem. It is easy to argue that some restriction like the bandwidth restriction in this result is necessary, and also that the $\frac{r-1}{r}$ in the minimum degree is best possible; it is also known that we cannot have $\gamma = 0$ (for details see the discussions in [34]). Further, as shown in [33], the bandwidth condition does not excessively restrict the class of embeddable graphs. Indeed, requiring the bandwidth of a bounded degree n-vertex graph to be o(n) is equivalent to requiring the treewidth to be o(n) or to have no large expanding subgraphs. This implies that bounded degree planar graphs, and more generally bounded degree graphs defined by some (or several) forbidden minor, have bandwidth o(n).

A transference of the bandwidth theorem to G(n,p) for constant p was obtained by Huang, Lee and Sudakov [78]. As discussed in the last section in such a result we cannot hope to cover all the graphs H embedded by the bandwidth theorem. More precisely we have to ask for at least $O(p^{-2})$ vertices of H not to be contained in a triangle. A result for smaller p in the special case of almost spanning bipartite graphs in $\mathcal{H}((1-o(1)n,\Delta))$ with bandwidth at most βn was obtained in [32] for $p \geq C(\log n/n)^{1/\Delta}$. Recently a general sparse analogue of the bandwidth theorem became possible with the help of the sparse blow-up lemma (see Section 7). For a concise statement, let $\mathcal{H}(n,\Delta,r,\beta)$ be the class of all r-colourable n-vertex graphs with maximum degree Δ and bandwidth at most βn .

Theorem 6.9 (Allen, Böttcher, Ehrenmüller, Taraz [3])

For all Δ , D, r and $\gamma > 0$ there are $\beta > 0$ and C such that $(\frac{1}{r} - \gamma)$ is a.a.s. a lower bound on the local resilience of G(n, p) with respect to universality for all $H \in \mathcal{H}(n, \Delta, r, \beta)$ such that either

- (a) at least $C \max\{p^{-2}, p^{-1} \log n\}$ vertices of H are not in triangles, and $p \ge C(\log n/n)^{1/\Delta}$, or
- (b) at least $C \max\{p^{-2}, p^{-1} \log n\}$ vertices of H are in neither triangles nor $C_4 s$, and H is D-degenerate, and $p \geq C(\log n/n)^{1/(2D+1)}$.

Here, the term $p^{-1} \log n$ in the bound on the vertices not in triangles

is only relevant for relatively large probabilities $p > 1/\log n$. It is an artefact of our proof and we do not believe it is necessary. Similarly, the requirement on vertices not being contained in C_4 in (b) can probably be removed, but we need it for our proof.

Observe that this theorem provides two different lower bounds on the probability, where the second one is better if the degeneracy of H is much smaller than its maximum degree (note though that even in this case we require a constant bound on the maximum degree). We do not believe these bounds to be optimal, but the bound in (a) matches the corresponding currently known universality bound in Theorem 5.3 and is thus well justified. Hence, the following problem is hard.

Problem 6.10 Improve the bounds on p in Theorem 6.9.

The exponent of n in p cannot be improved beyond $1/m_2(K_{\Delta+1}) = 2/(\Delta+2)$. Indeed, if p grows slower than $n^{-1/m_2(H)}$ then in G(n,p) the expected number of H-copies containing any fixed vertex is o(pn) and one can show, using a concentration inequality of Kim and Vu [86], that in fact a.a.s. every vertex of G(n,p) lies in at most γpn copies of H (for the details see, e.g., [4, Lemma 3.3]). Hence, in this case an adversary can even easily delete all H-copies without removing more than a 2γ -fraction of the edges at each vertex.

It is possible that $2/(\Delta + 2)$ is indeed the correct exponent. A more precise conjecture is offered in the concluding remarks of [3].

A better probability bound than that in Theorem 6.9 was very recently obtained by Noever and Steger [117] for the special case of almost spanning squares of Hamilton cycles, which is approximately optimal.

Theorem 6.11 (Noever, Steger [117]) For all $\gamma > 0$ and $p \ge n^{\gamma - 1/2}$ the local resilience of G(n,p) with respect to containing the square of a cycle on at least $(1-\gamma)n$ vertices is a.a.s. at least $\frac{1}{2} - \gamma$.

7 The blow-up lemma for sparse graphs

Szemerédi's regularity lemma proved extremely important for much of the progress in extremal graph theory (and other areas) over the past few decades. Together with the blow-up lemma it also allowed for a wealth of results on spanning substructures of dense graphs. For sparse graphs, such as sparse random graphs or their subgraphs, the error terms appearing in the regularity lemma though are too coarse. This inspired the development of sparse analogues of this machinery – which turned out to be a difficult task. In this section these sparse analogues are surveyed and some very

simple example applications are provided to demonstrate how they are used. Section 7.1 introduces the sparse regularity lemma and explains how it is used for obtaining resilience results. Section 7.2 states so-called inheritance lemmas for sparse regular pairs, which are needed to work with the sparse blow-up lemma. Section 7.3 provides the sparse blow-up lemma for random graphs, and Section 7.4 outlines how it is applied.

To a certain degree I assume familiarity of the reader with the dense regularity lemma and blow-up lemma, and refer to the surveys [95, 100, 101] for the relevant background.

7.1 The sparse regularity lemma

In sparse versions of the regularity lemma, all edge densities are taken relative to an ambient density p. In our applications here, where we are interested in subgraphs G of some random graph, we may always take the edge probability of the random graph as the ambient density p. In order to state a sparse regularity lemma we need some definitions.

Let G=(V,E) be a graph, and suppose $p\in(0,1]$ and $\varepsilon>0$ are reals. For disjoint nonempty sets $U,W\subseteq V$ the p-density of the pair (U,W) is defined as $d_{G,p}(U,W)=e_G(U,W)/(p|U||W|)$. The pair (U,W) is (ε,d,p) -regular (or (ε,d,p) -lower-regular) if there is $d'\geq d$ such that $d_{G,p}(U',W')=d'\pm\varepsilon$ (or if $d_{G,p}(U',W')\geq d-\varepsilon$, respectively) for all $U'\subseteq U$ and $W'\subseteq W$ with $|U'|\geq \varepsilon|U|$ and $|W'|\geq \varepsilon|W|$. We say that (U,W) is (ε,p) -regular (or (ε,p) -lower-regular), if it is (ε,d,p) -regular (or (ε,d,p) -lower-regular) for some $d\geq d_{G,p}(U,W)-\varepsilon$.

An ε -equipartition of V is a partition $V = V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_r$ with $|V_0| \leq \varepsilon |V|$ and $|V_1| = \dots = |V_r|$. An (ε, p) -regular partition (or an (ε, p) -lower-regular partition) of G = (V, E) is an ε -equipartition $V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_r$ of $V_0 \dot{\cup} V_1 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_r$ of $V_0 \dot{\cup} V_1 \dot{\cup}$

The sparse regularity lemma by Kohayakawa and Rödl [88, 92] and Scott [126] asserts the existence of (ε,p) -regular partitions for sparse graphs G. In applications of this sparse regularity lemma one often only makes use of sufficiently dense regular pairs in the regular partition, and the reduced graph of the partition captures where these dense pairs are. Formally, an ε -equipartition $V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_r$ of a graph G = (V, E) is an (ε, d, p) -regular partition (or (ε, d, p) -lower-regular partition) with reduced graph R if V(R) = [r] and the pair (V_i, V_j) is (ε, d, p) -regular (or (ε, d, p) -lower-regular) in G whenever $ij \in E(R)$. Observe that, given d > 0, an (ε, p) -regular partition gives rise to an (ε, d, p) -regular partition of G with reduced graph R, where R contains exactly the edges ij such that (V_i, V_j)

is (ε, p) -regular and $d_{G,p}(V_i, V_j) \geq d - \varepsilon$.

It then is a consequence of the sparse regularity lemma that graphs G with sufficiently large minimum degree relative to the ambient density p (and which do not have linear sized subgraphs of density much above p) allow for (ε, d, p) -regular partitions with a reduced graph R of high minimum degree. In this sense R inherits the minimum degree of G. The following lemma, which can be found e.g. in [3], makes this precise.

Lemma 7.1 (sparse regularity lemma, min. degree version)

For each $\varepsilon > 0$, $\alpha \in [0,1]$, and $r_0 \ge 1$ there exists $r_1 \ge 1$ with the following property. For any $d \in [0,1]$, any p > 0, and any n-vertex graph G with $\delta(G) \ge \alpha \cdot pn$ such that for any disjoint $X,Y \subseteq V(G)$ with $|X|,|Y| \ge \varepsilon \frac{n}{r_1}$ we have $e(X,Y) \le (1+\frac{\varepsilon^2}{10^3})p|X||Y|$, there is an (ε,d,p) -regular partition of V(G) with reduced graph R with $\delta(R) \ge (\alpha - d - \varepsilon)|V(R)|$ and $r_0 \le |V(R)| \le r_1$.

The crucial point is that the reduced graph R in this lemma is a *dense* graph, which means that we can apply extremal graph theory results for dense graphs to R. It should be noted that analogous lemmas can easily be formulated where other properties are inherited by the reduced graph, such as the (relative) density of G.

The regularity lemma then becomes useful in conjunction with suitable embedding lemmas. These come in different flavours. Embedding constant sized graphs H in systems of regular pairs in G(n,p) is allowed by the so-called counting lemma, which even allows to give good estimates on the number of H-copies. In a major breakthrough such counting lemmas were recently established for the correct p (that is, the threshold was established) in [19, 40, 124], verifying a conjecture of Kohayakawa, Łuczak, and Rödl [90, Conjecture 23]. An embedding lemma for H of small linear size, on the other hand, was provided in [94] for $p \geq C(\log n/n)^{-1/\Delta}$. This range of p is not believed to be best possible, but again matches the natural barrier. Finally, the blow-up lemma, which is stated in Section 7.3, handles spanning graphs (for the same edge probability p).

To illustrate how the sparse regularity lemma and the embedding lemmas interact, let us briefly sketch how to show that for $p \geq C(\log n/n)^{1/2}$ a.a.s. a subgraph G of G(n,p) with $\delta(G) \geq (\frac{2}{3} + \gamma)pn$ has a triangle factor covering at least $(1-\gamma)n$ vertices for every $\gamma > 0$ and C sufficiently large. Indeed, if we apply the minimum degree version of the sparse regularity lemma (Lemma 7.1) to G, with $\varepsilon \ll d$ sufficiently small and $r_0 = 3$, we obtain a reduced graph R with $\delta(R) \geq (\frac{2}{3} + \frac{\gamma}{2})v(R)$, which thus contains a (spanning) triangle factor by the theorem of Corrádi and Hajnal [41]. One triangle in this triangle factor corresponds to three (ε, d, p) -regular

pairs in G, in which, according to the sparse counting lemma, we find one triangle. After removing the three vertices of this triangle, what remains of the three pairs is still (ε',d,p) -regular for ε' almost as big as ε . Hence, we can apply the counting lemma again to find another triangle. In fact, we can repeat this process until, say, a $\frac{1}{2}\gamma$ -fraction of the original three pairs is left. Repeating this for each triangle in the triangle factor of R, we obtain an almost spanning triangle factor in G covering all but at most the εn vertices of the exceptional set V_0 and a $\frac{1}{2}\gamma$ -fraction of $V \setminus V_0$.

7.2 Regularity inheritance in G(n, p)

In the dense setting, when embedding graphs H in systems of regular pairs one often proceeds in rounds, and for later rounds crucially relies on the following fact (and a two-sided version thereof). Assume (X,Y) is a regular pair into which we want to embed an edge x'y' of H. Assume further that some neighbour z' of x' was embedded in previous rounds in a pair Z. Then the setup of the blow-up lemma will be such that (Z,X) is also a regular pair, and we will have chosen the image z of z' carefully enough so that z is "typical" in the pair (Z,X) in the sense that N(z;X) will be of size $d|X| \gg \varepsilon |X|$ for a suitable constant d. It then easily follows from the definition of ε -regularity that the pair (N(z;X),Y) is still a regular pair (with reduced regularity parameter), that is (N(z;X),Y) inherits regularity from (X,Y). This then makes it easy to embed the edge x'y' in (X,Y) such that x' is embedded into N(z;X).

Trying to use a similar approach in sparse graphs we encounter the following problem: If (X,Y) and (Y,Z) are (ε,d,p) -regular pairs then a "typical" vertex $z \in Z$ has a neighbourhood of size about dp|X| in X, which is much smaller than $\varepsilon|X|$ if p goes to 0. Hence, it is not clear any more that (N(z;X),Y) inherits regularity from (X,Y) – in fact, this is false in general. Fortunately, however, if we consider regular pairs (X,Y) and (Y,Z) in a subgraph G of G(n,p), then it is true for most $z \in Z$ that $(N_G(z;X),Y)$ inherits regularity from (X,Y). This phenomenon was observed in [67, 91, 94]. Based on the techniques developed in these papers, the following regularity inheritance lemmas are shown in [5].

Lemma 7.2 (One-sided regularity inheritance [5]) For each ε' , d > 0 there are $\varepsilon_0 > 0$ and C such that for all $0 < \varepsilon < \varepsilon_0$ and $0 , a.a.s. <math>\Gamma = G(n,p)$ has the following property. Let $G \subseteq \Gamma$ be a graph and X,Y be disjoint subsets of $V(\Gamma)$. If (X,Y) is (ε,d,p) -lower-regular in G and

$$|X| \ge C \max(p^{-2}, p^{-1} \log n)$$
 and $|Y| \ge Cp^{-1} \log n$,

then the pair $(N_{\Gamma}(z;X),Y)$ is not (ε',d,p) -lower-regular in G for at most $Cp^{-1}\log n$ vertices $z \in V(\Gamma)$.

Observe that this lemma consider neighbourhoods in $\Gamma = G(n,p)$, rather than directly in G. More specifically, Lemma 7.2 establishes lower-regularity of $(N_{\Gamma}(z;X),Y)$. However, since for most vertices $z \in Z$ the order of magnitude of $\deg_G(z;X)$ and $\deg_{\Gamma}(z;X)$ differs by a factor of at most 2d, the pair $(N_G(z;X),Y)$ then easily inherits regularity from $(N_{\Gamma}(z;X),Y)$.

Lemma 7.2 is complemented by the following two-sided version, which guarantees lower-regularity of the pair $(N_{\Gamma}(z;X), N_{\Gamma}(z;Y))$. This plays an important role when we want to embed triangles.

Lemma 7.3 (Two-sided regularity inheritance [5]) For each ε' , d > 0 there are $\varepsilon_0 > 0$ and C such that for all $0 < \varepsilon < \varepsilon_0$ and $0 , a.a.s. <math>\Gamma = G(n,p)$ has the following property. Let $G \subseteq \Gamma$ be a graph and X, Y be disjoint subsets of $V(\Gamma)$. If (X,Y) is (ε,d,p) -lower-regular in G and

$$|X|, |Y| \ge C \max (p^{-2}, p^{-1} \log n),$$

then the pair $(N_{\Gamma}(z;X), N_{\Gamma}(z;Y))$ is not (ε', d, p) -lower-regular in G for at most $C \max(p^{-2}, p^{-1} \log n)$ vertices $z \in V(\Gamma)$.

These two lemmas are similar to [94, Proposition 15], and in fact equivalent when $p = \Theta\left((\log n/n)^{1/\Delta}\right)$, but not for larger p, when the bounds on |X| and |Y| and the number of vertices z are different, which is sometimes useful in applications.

They are moreover proved for lower-regular pairs rather than for regular pairs (which leads to a less strong assumption, but also to a weaker conclusion). In fact, it would be interesting to obtain analogous lemmas for sparse regular pairs.

Problem 7.4 Prove analogues of Lemmas 7.2 and 7.3 for (ε, d, p) -regular pairs in subgraphs of G(n, p).

Lemmas 7.2 and 7.3 state that most vertices in Z satisfy regularity inheritance properties. In the sparse blow-up lemma, however, we will require this property from all vertices in a cluster. More precisely, let X, Y and Z be vertex sets in $G \subseteq \Gamma$, where X and Y are disjoint and X and Z are disjoint, but we do allow Y = Z. We say that (Z, X, Y) has one-sided (ε, d, p) -inheritance if for each $z \in Z$ the pair $(N_{\Gamma}(z, X), Y)$ is (ε, d, p) -lower-regular. If in addition X and Z are disjoint, then we say that (Z, X, Y) has two-sided (ε, d, p) -inheritance if for each $z \in Z$ the pair

 $(N_{\Gamma}(z, X), N_{\Gamma}(z, Y))$ is (ε, d, p) -lower-regular. When applying the sparse blow-up lemma, our approach will be to simply remove the few vertices from each cluster whose neighbourhoods in certain other clusters do not inherit lower-regularity (and deal with them separately).

7.3 The random graphs blow-up lemma

The purpose of this section is to state a slightly simplified version of the blow-up lemma for random graphs proven in [5]. The setup in this blow-up lemma is as follows. We are given two graphs G and H on the same number of vertices, where G is a subgraph of the random graph $\Gamma = G(n, p)$. The graphs G and H are endowed with partitions $\mathcal{V} = \{V_i\}_{i \in [r]}$ and $\mathcal{X} = \{X_i\}_{i \in [r]}$ of their respective vertex sets, of which we require certain properties. Firstly, the partitions \mathcal{V} and \mathcal{X} need to be size-compatible, that is, $|V_i| = |X_i|$ for all $i \in [r]$. Secondly, (G, \mathcal{V}) needs to be κ -balanced, that is, there exists m such that $m \leq |V_i| \leq \kappa m$ for all $i, j \in [r]$.

Further, we will have two reduced graphs R and $R' \subseteq R$ on r vertices, where R represents the regular pairs of (G, \mathcal{V}) . In fact, we work with lower-regularity instead of regularity, because that is what the inheritance lemmas discussed in the last section provide. Hence, we say that (G, \mathcal{V}) is an (ε, d, p) -lower-regular R-partition if for each edge $ij \in R$ the pair (V_i, V_j) is (ε, d, p) -lower-regular. We require that H has edges only along lower-regular pairs of this partition. Formally, (H, \mathcal{X}) is an R-partition if each part of \mathcal{X} is empty, and whenever there are edges of H between X_i and X_j , the pair ij is an edge of R.

As in the dense blow-up lemma, we cannot hope to embed a spanning graph solely in systems of regular pairs, as these may contain isolated vertices. Therefore, we will require certain pairs to be super-regular, that is to additionally satisfy a minimum degree condition. Where these super-regular pairs are is captured by the second reduced graph R'. A pair (X,Y) in $G \subseteq \Gamma$ is called (ε,d,p) -super-regular (in G) if it is (ε,d,p) -lower-regular and for every $x \in X$ and $y \in Y$ we have

$$\deg_G(x;Y) > (d-\varepsilon) \max\{p|Y|, \deg_{\Gamma}(x;Y)/2\},$$

$$\deg_G(y;X) > (d-\varepsilon) \max\{p|X|, \deg_{\Gamma}(y;X)/2\}.$$

The second term in these maxima is technically necessary to treat vertices x of exceptionally high Γ -degree into Y, but can be ignored for most

⁵Observe that this differs from an (ε, d, p) -lower-regular partition with reduced graph R in that we do not require the partition to be an ε -equipartition. In fact, in the partitions referred to in the blow-up lemma the exceptional set is omitted.

purposes (see also the discussion in [5]). The partition (G, \mathcal{V}) is (ε, d, p) super-regular on R' if for every $ij \in E(R')$ the pair (V_i, V_j) is (ε, d, p) super-regular.

But even requiring super-regularity is not enough, as for super-regular pairs (X,Y), (Y,Z), (X,Z) in G(n,p) there may be vertices $z \in Z$ with no edge in $(N_G(z;X), N_G(z;Y))$, which prevents us for example from embedding a triangle factor in (Z,X,Y). However, as argued in the previous section, lower-regularity does not get inherited on neighbourhoods for only a few vertices in (Z,X,Y). Hence, omitting these we can circumvent this problem. In the blow-up lemma we will thus require regularity inheritance along R'. Formally, (G,\mathcal{V}) has one-sided inheritance on R' if (V_i,V_j,V_k) has one-sided (ε,d,p) -inheritance for every $ij,jk\in E(R')$, where we do allow i=k. Similarly, (G,\mathcal{V}) has two-sided inheritance on R' if (V_i,V_j,V_k) has two-sided (ε,d,p) -inheritance for every $ij,jk,ik\in R'$.

It remains to describe which of the edges of H are required to go along the super-regular pairs captured by R'. It turns out that we only need to restrict a small linear fraction of the vertices of each X_i to having their neighbours and second neighbours along R'. We collect these special vertices in a so-called buffer. Formally, a family $\tilde{X} = \{\tilde{X}_i\}_{i \in [r]}$ of subsets $\tilde{X}_i \subseteq X_i$ is an (α, R') -buffer for H if for each $i \in [r]$ we have $|\tilde{X}_i| \ge \alpha |X_i|$ and for each $x \in \tilde{X}_i$ and each $xy, yz \in E(H)$ with $y \in X_j$ and $z \in X_k$ we have $ij \in R'$ and $jk \in R'$. The buffer sets can be chosen by the user of the blow-up lemma, which asserts that for any graphs H and G with the setup as just described we can embed H into G (if p is sufficiently large).

Lemma 7.5 (Blow-up lemma for $G_{n,p}$ [5]) For all $\Delta \geq 2$, $\Delta_{R'} \geq 1$, $\kappa \geq 1$, and $\alpha, d > 0$ there exists $\varepsilon > 0$ such that for all r_1 there is a C such that for $p \geq C(\log n/n)^{1/\Delta}$ the random graph $\Gamma = G(n,p)$ a.a.s. satisfies the following. Let R be a graph on $r \leq r_1$ vertices and let $R' \subseteq R$ be a spanning subgraph with $\Delta(R') \leq \Delta_{R'}$. Let H and $G \subseteq \Gamma$ be graphs with κ -balanced size-compatible vertex partitions $\mathcal{X} = \{X_i\}_{i \in [r]}$ and $\mathcal{V} = \{V_i\}_{i \in [r]}$, respectively, which have parts of size at least $m \geq n/(\kappa r_1)$. Let $\tilde{\mathcal{X}} = \{\tilde{X}_i\}_{i \in [r]}$ be a family of subsets of V(H) and suppose that

- (i) $\Delta(H) \leq \Delta$, (H, \mathcal{X}) is an R-partition, and $\tilde{\mathcal{X}}$ an (α, R') -buffer for H,
- (ii) (G, \mathcal{V}) is an (ε, d, p) -lower-regular R-partition, which is (ε, d, p) super-regular on R', and has one- and two-sided inheritance on R'.

Then there is an embedding of H into G.

A number of remarks are in place. Firstly, one of the advantages in this formulation of the blow-up lemma, compared to that of the dense blow-up lemma, is that the required regularity constant ε does not depend

on the number of clusters r of the reduced graph R, but only on the maximum degree of R'. This makes it possible to apply this blow-up lemma to the whole reduced graph of a regular partition given by the sparse regularity lemma, instead of the repeated applications of the blow-up lemma to small parts of the reduced graph together with a technique for "glueing" the different so-obtained subgraphs together that were the norm when applying the dense blow-up lemma. Since for p=1 we recover the dense setting, this technique can now also be used for dense graphs G.

Secondly, the version of this blow-up lemma given in [5] is stronger in the following senses. One difference is that in [5] we only require two-sided inheritance on triangles of R' in which we want to embed some triangle of H containing a vertex in the buffer. In particular, we do not need two-sided inheritance at all if H has no triangles. This is useful in some applications, as explained in [5]. The other difference is that in [5] so-called image restrictions are allowed, that is, for some vertices x of H we are allowed to specify a relatively small set of vertices in G into which x is to be embedded. These image restrictions have somewhat more complex requirements than in the dense case, hence we omit them here, but the basic philosophy is that the requirements are those needed to guarantee compatibility with super-regularity and regularity inheritance in the remainder of the partition of G (and they are generalisations of the image restrictions in the dense case).

But why do we need image restrictions? In the dense case such image restrictions were usually used for "glueing" different blow-up lemma applications together, which is now no longer needed, as described above. However, as we will describe in the next section, when we want to apply the blow-up lemma to a partition obtained from the sparse regularity lemma, we will need to exclude a number of vertices from each cluster to guarantee super-regularity and regularity inheritance. In the dense case, usually all of these vertices can be redistributed to other clusters without destroying these properties, but in the sparse case this is not necessarily possible. Hence, if we want to obtain a spanning embedding result we will first need to embed certain H-vertices on these exceptional vertices of G by hand, which lead to image restrictions, before we can apply the sparse blow-up lemma to embed the remainder of H (see, e.g., [3] for more details).

Finally, again, the lower bound on p is unlikely to be best possible.

Problem 7.6 Improve the exponent of n in the lower bound on p in Lemma 7.5.

Even in a version of this lemma for only small linear sized graphs H

this would for example directly lead to an improvement on the known bounds on so-called size Ramsey numbers (cf. [94]), among many others.

Let me remark that in [5] additionally a version of the blow-up lemma for D-degenerate graphs H with maximum degree Δ is given. In this version the exponent in the power of n in the bound on p depends only on D (but the constant C still depends on Δ). Often we can choose this exponent to be 2D+1, and in some cases even smaller, but the details depend on the choice of a suitable buffer and are more involved. In applications image restrictions are often needed in addition, which complicate the statement of a corresponding blow-up lemma even further. It is this version of the sparse blow-up lemma which is used to prove Theorem 6.9(b).

7.4 Applying the blow-up lemma

As an example application of the sparse blow-up lemma presented in the last section, let us briefly sketch how it can be used to show that for every $\gamma > 0$, if C is sufficiently large and $p \geq C \left(\log n/n\right)^{1/4}$, a.a.s. any subgraph G of $\Gamma = G(n,p)$ with $\delta(G) \geq (\frac{1}{2} + \gamma)pn$ contains a copy of the $k \times k$ square grid $H = L_k$ with $k = (1 - \gamma)\sqrt{n}$.

We start by preparing G for the sparse blow-up lemma. To this end, we first apply the minimum degree version of the sparse regularity lemma (Lemma 7.1) to G and obtain an (ε, d, p) -regular partition $V_0 \dot{\cup} V_1 \dot{\cup} \dots \dot{\cup} V_r$ with reduced graph R of minimum degree bigger than $\frac{1}{2}v(R)$. Hence, R has a Hamilton cycle C, which contains a perfect matching if v(R) is even (otherwise first add one cluster to the exceptional set V_0). This matching is our second reduced graph R'. We assume without loss of generality that the Hamilton cycle is $1, 2, \dots, r$, and that the matching $R' \subseteq C$ is $\{1, 2\}, \{3, 4\}, \dots, \{r-1, r\}$.

We then have to transform the regular-partition of G into a superregular partition with regularity inheritance. Hence, we remove from each cluster all those vertices violating super-regularity on R', which are at most $\varepsilon |V_i|$ vertices per cluster V_i , and all those vertices violating (onesided) regularity inheritance on R', which by Lemmas 7.2 are at most $Cp^{-1} \log n$ vertices per cluster V_i .

Next we prepare H. For embedding the grid H we want to use the linear cycle structure of the Hamilton cycle C in the reduced graph R. Therefore, let us first show that we can cut H into roughly equal pieces along a linear structure. Indeed, any diagonal of H has at most \sqrt{n} vertices, hence by choosing appropriate diagonals as cuts (that is, we "cut" along the diagonal) we can partition H into $\frac{r}{2}$ sets $Y_1, \ldots, Y_{r/2}$ of size $(2n/r) \pm \sqrt{n}$. A C-partition X_1, \ldots, X_r of H is then obtained by letting X_{i-1} and X_i be the two colour-classes of $H[Y_{i/2}]$ for every even $i \in [r]$. Observe that

most edges of H then go along the matching edges of R'. Since vertices of the buffer \tilde{X}_i for $i \in [r]$ should have their first and second neighbourhood along R', we simply choose αn vertices in X_i as \tilde{X}_i which are on diagonals of distance at least 3 to any of the cut diagonals.

It is easy to check that we have $|V_i| \ge |X_i|$ for each i, so we can add isolated vertices to each part X_i of H to ensure size-compatibility, and then we can apply the sparse blow-up lemma, Lemma 7.5, to embed H into the remainder of G.

In fact, for embedding the almost spanning H into G we could have chosen a much simpler setup: We could have added αn isolated vertices to each X_i , and used these for the buffer \tilde{X}_i . Then we could have set R' to be the empty graph. We chose to describe the more complicated setting here though, because it is this setting which is necessary for generalising this approach to obtain a spanning H-copy in G.

The idea of how to generalise the above proof is roughly as follows. We would like to "redistribute" the vertices v of G we deleted to different clusters of G where they do not violate the required properties. Because of the minimum degree condition on G this can easily be shown to be possible for vertices v which satisfy regularity inheritance for any pair of clusters (X_i, X_i) with $ij \in R$, and which further have roughly the expected Γ -degree in each X_i with $i \in [r]$. The latter condition is necessary because of the second term in the maximum in the definition of sparse super-regular pairs, but it can be shown that a.a.s. all but at most r. $Cp^{-1}\log n$ vertices satisfy this condition. The remaining $r\cdot Cp^{-1}\log n$ $r^3 \cdot Cp^{-1} \log n$ vertices cannot be redistributed, and need to be dealt with "by hand". The sparse blow-up lemma with image restrictions can then be used to complete the embedding. The details are more complicated, because the redistribution process is iterative. In particular, we need to ensure that during the redistribution no new violations of other vertices are created. Moreover, we also have to adapt the sizes of the clusters of G to match the actual sizes of the X_i . The details are omitted (see [3] for more explanations).

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References

[1] R. Aharoni and P. Haxell, *Hall's theorem for hypergraphs*, J. Graph Theory **35** (2000), no. 2, 83–88.

- [2] M. Ajtai, J. Komlós, and E. Szemerédi, *The longest path in a random graph*, Combinatorica 1 (1981), no. 1, 1–12.
- [3] P. Allen, J. Böttcher, J. Ehrenmüller, and A. Taraz, *The bandwidth theorem in sparse graphs*, arXiv:1612.00661.
- [4] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa, and R. Morris, Chromatic thresholds in sparse random graphs, Random Structures Algorithms, accepted, arXiv:1508.03875.
- [5] P. Allen, J. Böttcher, H. Hàn, Y. Kohayakawa, and Y. Person, *Blow-up lemmas for sparse graphs*, arXiv:1612.00622.
- [6] N. Alon, Universality, tolerance, chaos and order, An irregular mind, Bolyai Soc. Math. Stud., vol. 21, János Bolyai Math. Soc., Budapest, 2010, pp. 21–37.
- [7] N. Alon and M. Capalbo, Sparse universal graphs for bounded-degree graphs, Random Structures Algorithms 31 (2007), no. 2, 123–133.
- [8] ______, Optimal universal graphs with deterministic embedding, Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, ACM, New York, 2008, pp. 373–378.
- [9] N. Alon, M. Capalbo, Y. Kohayakawa, V. Rödl, A. Ruciński, and E. Szemerédi, *Universality and tolerance (extended abstract)*, 41st Annual Symposium on Foundations of Computer Science (Redondo Beach, CA, 2000), IEEE Comput. Soc. Press, Los Alamitos, CA, 2000, pp. 14–21.
- [10] N. Alon and Z. Füredi, Spanning subgraphs of random graphs, Graphs Combin. 8 (1992), no. 1, 91–94.
- [11] N. Alon, M. Krivelevich, and B. Sudakov, *Embedding nearly-spanning bounded degree trees*, Combinatorica **27** (2007), no. 6, 629–644.
- [12] N. Alon and R. Yuster, *Threshold functions for H-factors*, Combin. Probab. Comput. **2** (1993), no. 2, 137–144.
- [13] D. Angluin and L. G. Valiant, Fast probabilistic algorithms for Hamiltonian circuits and matchings, J. Comput. System Sci. 18 (1979), no. 2, 155–193.
- [14] L. Babai, M. Simonovits, and J. Spencer, Extremal subgraphs of random graphs, J. Graph Theory 14 (1990), no. 5, 599–622.

[15] D. Bal and A. Frieze, The Johansson-Kahn-Vu solution of the Shamir problem, https://www.math.cmu.edu/~af1p/Teaching/ATIRS/Papers/FRH/Shamir.pdf.

- [16] J. Balogh, B. Csaba, M. Pei, and W. Samotij, Large bounded degree trees in expanding graphs, Electron. J. Combin. 17 (2010), no. 1, Research Paper 6, 9.
- [17] J. Balogh, B. Csaba, and W. Samotij, *Local resilience of almost spanning trees in random graphs*, Random Structures Algorithms **38** (2011), no. 1-2, 121–139.
- [18] J. Balogh, C. Lee, and W. Samotij, Corrádi and Hajnal's theorem for sparse random graphs, Combin. Probab. Comput. 21 (2012), no. 1-2, 23–55.
- [19] J. Balogh, R. Morris, and W. Samotij, Independent sets in hypergraphs, J. Amer. Math. Soc. 28 (2015), no. 3, 669–709.
- [20] S. Ben-Shimon, M. Krivelevich, and B. Sudakov, *Local resilience* and *Hamiltonicity maker-breaker games in random regular graphs*, Combin. Probab. Comput. **20** (2011), no. 2, 173–211.
- [21] ______, On the resilience of Hamiltonicity and optimal packing of Hamilton cycles in random graphs, SIAM J. Discrete Math. 25 (2011), no. 3, 1176–1193.
- [22] P. Bennett, A. Dudek, and A. Frieze, Square of a Hamilton cycle in a random graph, arXiv:1611.06570.
- [23] S. N. Bhatt, F. R. K. Chung, F. T. Leighton, and A. L. Rosenberg, Universal graphs for bounded-degree trees and planar graphs, SIAM J. Discrete Math. 2 (1989), no. 2, 145–155.
- [24] B. Bollobás, *Threshold functions for small subgraphs*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 2, 197–206.
- [25] ______, The evolution of sparse graphs, Graph theory and combinatorics (Cambridge, 1983), Academic Press, London, 1984, pp. 35–57.
- [26] ______, Random graphs, second ed., Cambridge Studies in Advanced Mathematics, vol. 73, Cambridge University Press, Cambridge, 2001.
- [27] B. Bollobás, T. I. Fenner, and A. M. Frieze, An algorithm for finding Hamilton paths and cycles in random graphs, Combinatorica 7 (1987), no. 4, 327–341.

- [28] B. Bollobás and A. M. Frieze, On matchings and Hamiltonian cycles in random graphs, Ann. Discrete Math. 28 (1985), 23–46.
- [29] _____, Spanning maximal planar subgraphs of random graphs, Random Structures Algorithms 2 (1991), no. 2, 225–231.
- [30] B. Bollobás and A. Thomason, Random graphs of small order, Random graphs '83 (Poznań, 1983), North-Holland Math. Stud., vol. 118, North-Holland, Amsterdam, 1985, pp. 47–97.
- [31] ______, Threshold functions, Combinatorica 7 (1987), no. 1, 35–38.
- [32] J. Böttcher, Y. Kohayakawa, and A. Taraz, Almost spanning subgraphs of random graphs after adversarial edge removal, Combin. Probab. Comput. **22** (2013), no. 5, 639–683.
- [33] J. Böttcher, K. P. Pruessmann, A. Taraz, and A. Würfl, *Bandwidth*, expansion, treewidth, separators and universality for bounded-degree graphs, European J. Combin. **31** (2010), no. 5, 1217–1227.
- [34] J. Böttcher, M. Schacht, and A. Taraz, Proof of the bandwidth conjecture of Bollobás and Komlós, Math. Ann. 343 (2009), no. 1, 175–205.
- [35] G. Brightwell, K. Panagiotou, and A. Steger, *Extremal subgraphs of random graphs*, Random Structures Algorithms **41** (2012), no. 2, 147–178.
- [36] M. R. Capalbo, A small universal graph for bounded-degree planar graphs, Proceedings of the Tenth Annual ACM-SIAM Symposium on Discrete Algorithms (Baltimore, MD, 1999), ACM, New York, 1999, pp. 156–160.
- [37] F. R. K. Chung, *Labelings of graphs*, Selected topics in graph theory, 3, Academic Press, San Diego, CA, 1988, pp. 151–168.
- [38] D. Conlon, A. Ferber, R. Nenadov, and N. Škorić, *Almost-spanning universality in random graphs*, Random Structures Algorithms, accepted, arXiv:1503.05612.
- [39] D. Conlon and W. T. Gowers, Combinatorial theorems in sparse random sets, Ann. of Math. (2) **184** (2016), no. 2, 367–454.
- [40] D. Conlon, W. T. Gowers, W. Samotij, and M. Schacht, On the KLR conjecture in random graphs, Israel J. Math. 203 (2014), no. 1, 535– 580.

[41] K. Corrádi and A. Hajnal, On the maximal number of independent circuits in a graph, Acta Math. Acad. Sci. Hungar. 14 (1963), 423–439.

- [42] D. Dellamonica, Y. Kohayakawa, M. Marciniszyn, and A. Steger, On the resilience of long cycles in random graphs, Electron. J. Combin. 15 (2008), 26 pp., R32.
- [43] D. Dellamonica, Y. Kohayakawa, V. Rödl, and A. Ruciński, *Universality of random graphs*, SIAM J. Discrete Math. 26 (2012), no. 1, 353–374.
- [44] ______, An improved upper bound on the density of universal random graphs, Random Structures Algorithms **46** (2015), no. 2, 274–299.
- [45] B. DeMarco and J. Kahn, Turán's Theorem for random graphs, arXiv:1501.01340.
- [46] ______, Mantel's theorem for random graphs, Random Structures Algorithms 47 (2015), no. 1, 59–72.
- [47] G. A. Dirac, Some theorems on abstract graphs, Proc. London Math. Soc. (3) 2 (1952), 69–81.
- [48] M. Drmota, O. Giménez, M. Noy, K. Panagiotou, and A. Steger, The maximum degree of random planar graphs, Proc. Lond. Math. Soc. (3) 109 (2014), no. 4, 892–920.
- [49] P. Erdős and A. Rényi, On random graphs. I, Publ. Math. Debrecen 6 (1959), 290–297.
- [50] _____, On the evolution of random graphs, Magyar Tud. Akad. Mat. Kutató Int. Közl. 5 (1960), 17–61.
- [51] ______, On random matrices, Magyar Tud. Akad. Mat. Kutató Int. Közl. 8 (1964), 455–461 (1964).
- [52] ______, On the existence of a factor of degree one of a connected random graph, Acta Math. Acad. Sci. Hungar. 17 (1966), 359–368.
- [53] P. Erdős and A. H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946), 1087–1091.
- [54] A. Ferber, G. Kronenberg, and K. Luh, Optimal threshold for a random graph to be 2-universal, arXiv:1612.06026.

- [55] A. Ferber, K. Luh, and O. Nguyen, *Embedding large graphs into a random graph*, arXiv:1606.05923.
- [56] A. Ferber, R. Nenadov, and U. Peter, *Universality of random graphs and rainbow embedding*, Random Structures Algorithms **48** (2016), no. 3, 546–564.
- [57] W. Fernandez de la Vega, Long paths in random graphs, Studia Sci. Math. Hungar. 14 (1979), no. 4, 335–340.
- [58] ______, *Trees in sparse random graphs*, J. Combin. Theory Ser. B **45** (1988), no. 1, 77–85.
- [59] P. Frankl and V. Rödl, Large triangle-free subgraphs in graphs without K_4 , Graphs Combin. 2 (1986), no. 2, 135–144.
- [60] E. Friedgut, Sharp thresholds of graph properties, and the k-sat problem, J. Amer. Math. Soc. 12 (1999), no. 4, 1017–1054, With an appendix by Jean Bourgain.
- [61] ______, Hunting for sharp thresholds, Random Structures Algorithms **26** (2005), no. 1-2, 37–51.
- [62] J. Friedman and N. Pippenger, Expanding graphs contain all small trees, Combinatorica 7 (1987), no. 1, 71–76.
- [63] A. Frieze and M. Karoński, *Introduction to random graphs*, Cambridge University Press, 2015.
- [64] A. Frieze and M. Krivelevich, On two Hamilton cycle problems in random graphs, Israel J. Math. 166 (2008), 221–234.
- [65] Z. Füredi, Random Ramsey graphs for the four-cycle, Discrete Math. 126 (1994), no. 1-3, 407–410.
- [66] S. Gerke, Random graphs with constraints, 2005, Habilitationsschrift, Institut für Informatik, TU München.
- [67] S. Gerke, Y. Kohayakawa, V. Rödl, and A. Steger, Small subsets inherit sparse ϵ -regularity, J. Combin. Theory Ser. B **97** (2007), no. 1, 34–56.
- [68] S. Gerke and A. McDowell, *Nonvertex-balanced factors in random graphs*, J. Graph Theory **78** (2015), no. 4, 269–286, (arXiv:1304.3000).

[69] S. Gerke, H. J. Prömel, T. Schickinger, A. Steger, and A. Taraz, K_4 -free subgraphs of random graphs revisited, Combinatorica **27** (2007), no. 3, 329–365.

- [70] S. Gerke, T. Schickinger, and A. Steger, K_5 -free subgraphs of random graphs, Random Structures Algorithms 24 (2004), no. 2, 194–232.
- [71] Y. Gurevich and S. Shelah, Expected computation time for Hamiltonian path problem, SIAM J. Comput. 16 (1987), no. 3, 486–502.
- [72] A. Hajnal and E. Szemerédi, Proof of a conjecture of P. Erdős, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), North-Holland, Amsterdam, 1970, pp. 601–623.
- [73] P. E. Haxell, Tree embeddings, J. Graph Theory 36 (2001), no. 3, 121–130.
- [74] P. E. Haxell, Y. Kohayakawa, and T. Luczak, Turán's extremal problem in random graphs: forbidding even cycles, J. Combin. Theory Ser. B 64 (1995), no. 2, 273–287.
- [75] ______, Turán's extremal problem in random graphs: forbidding odd cycles, Combinatorica **16** (1996), no. 1, 107–122.
- [76] D. Hefetz, M. Krivelevich, and T. Szabó, Hamilton cycles in highly connected and expanding graphs, Combinatorica 29 (2009), no. 5, 547–568.
- [77] ______, Sharp threshold for the appearance of certain spanning trees in random graphs, Random Structures Algorithms 41 (2012), no. 4, 391–412.
- [78] H. Huang, C. Lee, and B. Sudakov, *Bandwidth theorem for random graphs*, J. Combin. Theory Ser. B **102** (2012), no. 1, 14–37.
- [79] S. Janson, T. Łuczak, and A. Ruciński, Random graphs, Wiley-Interscience, New York, 2000.
- [80] D. Johannsen, M. Krivelevich, and W. Samotij, *Expanders are universal for the class of all spanning trees*, Combin. Probab. Comput. **22** (2013), no. 2, 253–281.
- [81] A. Johansson, J. Kahn, and V. Vu, Factors in random graphs, Random Structures Algorithms 33 (2008), no. 1, 1–28.
- [82] J. Kahn and G. Kalai, *Thresholds and expectation thresholds*, Combin. Probab. Comput. **16** (2007), no. 3, 495–502.

- [83] J. Kahn, E. Lubetzky, and N. Wormald, Cycle factors and renewal theory, Comm. Pure Appl. Math., accepted, arXiv:1401.2707.
- [84] ______, The threshold for combs in random graphs, Random Structures Algorithms 48 (2016), no. 4, 794–802.
- [85] J. H. Kim and S. J. Lee, Universality of random graphs for graphs of maximum degree two, SIAM J. Discrete Math. 28 (2014), no. 3, 1467–1478.
- [86] J. H. Kim and V. H. Vu, Concentration of multivariate polynomials and its applications, Combinatorica 20 (2000), no. 3, 417–434.
- [87] ______, Sandwiching random graphs: universality between random graph models, Adv. Math. 188 (2004), no. 2, 444–469.
- [88] Y. Kohayakawa, Szemerédi's regularity lemma for sparse graphs, Foundations of computational mathematics, Springer, 1997, pp. 216–230.
- [89] Y. Kohayakawa, B. Kreuter, and A. Steger, An extremal problem for random graphs and the number of graphs with large even-girth, Combinatorica 18 (1998), no. 1, 101–120.
- [90] Y. Kohayakawa, T. Łuczak, and V. Rödl, On K⁴-free subgraphs of random graphs, Combinatorica 17 (1997), no. 2, 173–213.
- [91] Y. Kohayakawa and V. Rödl, Regular pairs in sparse random graphs. I, Random Structures Algorithms 22 (2003), no. 4, 359–434.
- [92] _____, Szemerédi's regularity lemma and quasi-randomness, Recent advances in algorithms and combinatorics, Springer, 2003, pp. 289–351.
- [93] Y. Kohayakawa, V. Rödl, and M. Schacht, The Turán theorem for random graphs, Combin. Probab. Comput. 13 (2004), no. 1, 61–91.
- [94] Y. Kohayakawa, V. Rödl, M. Schacht, and E. Szemerédi, Sparse partition universal graphs for graphs of bounded degree, Adv. Math. **226** (2011), no. 6, 5041–5065.
- [95] J. Komlós, *The blow-up lemma*, Combin. Probab. Comput. **8** (1999), no. 1-2, 161–176, Recent trends in combinatorics (Mátraháza, 1995).
- [96] J. Komlós, G. N. Sárközy, and E. Szemerédi, Proof of a packing conjecture of Bollobás, Combin. Probab. Comput. 4 (1995), no. 3, 241–255.

- [97] ______, Blow-up lemma, Combinatorica 17 (1997), no. 1, 109–123.
- [98] ______, An algorithmic version of the blow-up lemma, Random Structures Algorithms 12 (1998), no. 3, 297–312.
- [99] ______, Spanning trees in dense graphs, Combin. Probab. Comput. 10 (2001), no. 5, 397–416.
- [100] J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi, The regularity lemma and its applications in graph theory, Theoretical aspects of computer science (Tehran, 2000), Lecture Notes in Comput. Sci., vol. 2292, Springer, Berlin, 2002, pp. 84–112.
- [101] J. Komlós and M. Simonovits, Szemerédi's regularity lemma and its applications in graph theory, Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.
- [102] J. Komlós and E. Szemerédi, Limit distribution for the existence of Hamiltonian cycles in a random graph, Discrete Math. 43 (1983), no. 1, 55–63.
- [103] A. Korshunov, Solution of a problem of Erdős and Rényi on Hamiltonian cycles in nonoriented graphs., Sov. Math., Dokl. 17 (1976), 760–764.
- [104] ______, Solution of a problem of P. Erdős and A. Rényi on Hamiltonian cycles in undirected graphs, Metody Diskretn. Anal. **31** (1977), 17–56.
- [105] M. Krivelevich, *Triangle factors in random graphs*, Combin. Probab. Comput. **6** (1997), no. 3, 337–347.
- [106] _____, Embedding spanning trees in random graphs, SIAM J. Discrete Math. **24** (2010), no. 4, 1495–1500.
- [107] M. Krivelevich, C. Lee, and B. Sudakov, Resilient pancyclicity of random and pseudorandom graphs, SIAM J. Discrete Math. 24 (2010), no. 1, 1–16.
- [108] D. Kühn and D. Osthus, On Pósa's conjecture for random graphs, SIAM J. Discrete Math. 26 (2012), no. 3, 1440–1457.
- [109] C. Lee and W. Samotij, *Pancyclic subgraphs of random graphs*, J. Graph Theory **71** (2012), no. 2, 142–158.

- [110] C. Lee and B. Sudakov, *Dirac's theorem for random graphs*, Random Structures Algorithms **41** (2012), no. 3, 293–305.
- [111] T. Łuczak and A. Ruciński, Tree-matchings in graph processes, SIAM J. Discrete Math. 4 (1991), no. 1, 107–120.
- [112] C. McDiarmid and B. Reed, On the maximum degree of a random planar graph, Combin. Probab. Comput. 17 (2008), no. 4, 591–601.
- [113] R. Montgomery, Embedding bounded degree spanning trees in random graphs, arXiv:1405.6559.
- [114] _____, Sharp threshold for embedding combs and other spanning trees in random graphs, arXiv:1405.6560.
- [115] J. W. Moon, On the maximum degree in a random tree, Michigan Math. J. 15 (1968), 429–432.
- [116] R. Nenadov and N. Škorić, Powers of cycles in random graphs and hypergraphs, arXiv:1601.04034.
- [117] A. Noever and A. Steger, Local resilience for squares of almost spanning cycles in sparse random graphs, arXiv:1606.02958.
- [118] O. Parczyk and Y. Person, Spanning structures and universality in sparse hypergraphs, Random Structures Algorithms, accepted, arXiv:1504.02243.
- [119] L. Pósa, Hamiltonian circuits in random graphs, Discrete Math. 14 (1976), no. 4, 359–364.
- [120] O. Riordan, Spanning subgraphs of random graphs, Combin. Probab. Comput. 9 (2000), no. 2, 125–148.
- [121] V. Rödl and A. Ruciński, Perfect matchings in ε -regular graphs and the blow-up lemma, Combinatorica 19 (1999), no. 3, 437–452.
- [122] V. Rödl, A. Ruciński, and A. Taraz, Hypergraph packing and graph embedding, Combin. Probab. Comput. 8 (1999), no. 4, 363–376, Random graphs and combinatorial structures (Oberwolfach, 1997).
- [123] A. Ruciński, Matching and covering the vertices of a random graph by copies of a given graph, Discrete Math. 105 (1992), no. 1-3, 185– 197.
- [124] D. Saxton and A. Thomason, *Hypergraph containers*, Invent. Math. **201** (2015), no. 3, 925–992.

[125] M. Schacht, Extremal results for random discrete structures, Ann. of Math. (2) **184** (2016), no. 2, 333–365.

- [126] A. Scott, Szemerédi's regularity lemma for matrices and sparse graphs, Combin. Probab. Comput. **20** (2011), no. 3, 455–466.
- [127] E. Shamir, How many random edges make a graph Hamiltonian?, Combinatorica 3 (1983), no. 1, 123–131.
- [128] J. Spencer, Threshold functions for extension statements, J. Combin. Theory Ser. A **53** (1990), no. 2, 286–305.
- [129] B. Sudakov and V. H. Vu, *Local resilience of graphs*, Random Structures Algorithms **33** (2008), no. 4, 409–433.
- [130] T. Szabó and V. H. Vu, *Turán's theorem in sparse random graphs*, Random Structures Algorithms **23** (2003), no. 3, 225–234.
- [131] A. Thomason, A simple linear expected time algorithm for finding a Hamilton path, Discrete Math. **75** (1989), no. 1-3, 373–379, Graph theory and combinatorics (Cambridge, 1988).
- [132] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, Mat. Fiz. Lapok 48 (1941), 436–452.

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