5 Zeroth-Order Variational Principles

In the previous chapter we addressed principles based on the conservation of vector quantities, specifically translational and angular momentum. In this chapter we will address principles rooted in the variation of scalar quantities like work and energy. We will begin with d'Alembert's Principle of Virtual Work, which is an extension of Bernoulli's static principle to dynamics. The principle is based on the notion of virtual displacement. We will refer to d'Alembert's Principle as a zeroth-order variational principle to denote that it is based on the variation of the zeroth-order derivative of displacement. This is in contrast to higher-order variations related to velocity and acceleration, which will be discussed in subsequent chapters.

5.1 Virtual Displacements

Virtual displacements refer to all displacements of a system that satisfy the scleronomic constraints of the system. Scleronomic constraints refer to constraints that are not explicitly dependent on time, as opposed to rheonomic constraints, which are explicitly dependent on time. In the case of virtual displacements, time is frozen or stationary.

5.2 D'Alembert's Principle of Virtual Work

PRINCIPLE 5.1 *The virtual work of a system is stationary. That is,*

$$\delta W = 0. \tag{5.1}$$

Additionally, the constraints of the system perform no virtual work:

$$\delta W_c = 0. \tag{5.2}$$

This is known as ***d'Alembert's Principle****.*

It is noted that while d'Alembert's Principle can be seen as providing an alternate statement of Newton's second law, for interacting bodies, a law of action and reaction (Newton's third law) is still needed. Therefore, when we use d'Alembert's Principle to derive the equations of motion for systems of particles/bodies, we will invoke the law of action and reaction.

5.2.1 A Single Particle

D'Alembert's Principle for a single point mass with a discrete set of n_f external forces, $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n_f}\}$, acting on it is expressed as

$$\begin{gathered} \delta W = \sum_{i=1}^{n_f} \boldsymbol{f}_i \cdot \delta \boldsymbol{r} - Mg\hat{\boldsymbol{e}}_3 \cdot \delta \boldsymbol{r} - M\boldsymbol{a} \cdot \delta \boldsymbol{r} = 0, \\ \forall \delta \boldsymbol{r} \in \mathbb{R}^3, \end{gathered} \tag{5.3}$$

where $\delta \boldsymbol{r}$ represents the displacement variations. During these variations time is stationary. That is, $\delta t = 0$. More concisely, we can express

$$\begin{gathered} \delta W = \left(\sum_{i=1}^{n_f} \boldsymbol{f}_i - Mg\hat{\boldsymbol{e}}_3 - M\boldsymbol{a} \right) \cdot \delta \boldsymbol{r} = 0, \\ \forall \delta \boldsymbol{r} \in \mathbb{R}^3, \end{gathered} \tag{5.4}$$

which implies

$$\sum_{i=1}^{n_f} \boldsymbol{f}_i - Mg\hat{\boldsymbol{e}}_3 - M\boldsymbol{a} = \boldsymbol{0}. \tag{5.5}$$

5.2.2 A Single Rigid Body

D'Alembert's Principle for a single rigid body with a discrete set of n_f external forces, $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n_f}\}$, and n_φ external moments, $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_{n_\varphi}\}$, acting on it is expressed as

$$\begin{gathered} \delta W = \sum_{i=1}^{n_f} \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_{P_i} + \sum_{i=1}^{n_\varphi} \boldsymbol{\varphi}_j \cdot \delta \boldsymbol{\theta} - Mg\hat{\boldsymbol{e}}_3 \cdot \delta \boldsymbol{r}_G - M\boldsymbol{a}_G \cdot \delta \boldsymbol{r}_G - (\boldsymbol{I}^G \boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{I}^G \boldsymbol{\omega}) \cdot \delta \boldsymbol{\theta} = 0, \\ \forall \delta \boldsymbol{r}_G \in \mathbb{R}^3, \quad \text{and} \quad \forall \delta \boldsymbol{\theta} \in \mathbb{R}^3, \end{gathered} \tag{5.6}$$

where the $\delta \boldsymbol{r}$ and $\delta \boldsymbol{\theta}$ terms represent all displacement variations consistent with the rigid-body constraint. This is depicted in Figure 5.1. During these variations, time is stationary. That is, $\delta t = 0$. D'Alembert's Principle states that the virtual work associated with all internal forces and moments consistent with the rigid body constraint is *zero* ($W_c = 0$). We further note that

$$\delta \boldsymbol{r}_{P_i} = \delta \boldsymbol{r}_G + \delta \boldsymbol{\theta} \times \boldsymbol{d}_{\overrightarrow{GP_i}}. \tag{5.7}$$

Therefore,

$$\sum_{i=1}^{n_f} \boldsymbol{f}_i \cdot \delta \boldsymbol{r}_{P_i} = \sum_{i=1}^{n_f} \boldsymbol{f}_i \cdot (\delta \boldsymbol{r}_G + \delta \boldsymbol{\theta} \times \boldsymbol{d}_{\overrightarrow{GP_i}}) = \left(\sum_{i=1}^{n_f} \boldsymbol{f}_i \right) \cdot \delta \boldsymbol{r}_G + \sum_{i=1}^{n_f} \boldsymbol{f}_i \cdot (\delta \boldsymbol{\theta} \times \boldsymbol{d}_{\overrightarrow{GP_i}}). \tag{5.8}$$

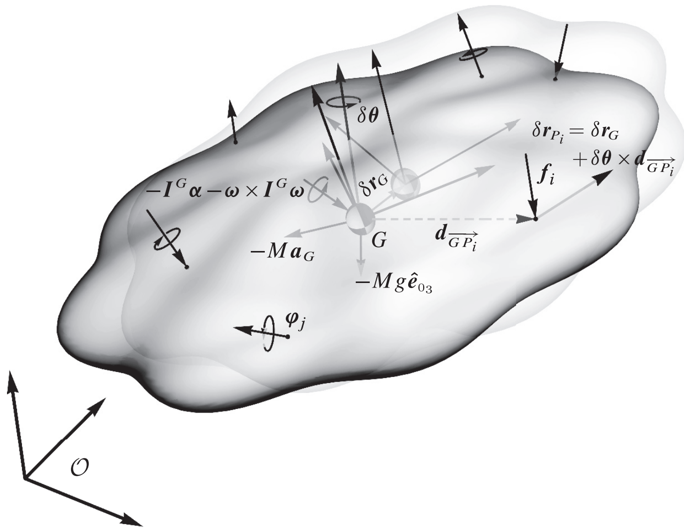

Figure 5.1 A virtual displacement of a rigid body consisting of a translational, $\delta\boldsymbol{r}_G$, and rotational, $\delta\boldsymbol{\theta}$, displacement. We are concerned with all such displacement variations consistent with the rigid-body constraint. During these variations, time is stationary.

Since $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{c}\times\boldsymbol{a})\cdot\boldsymbol{b}$,

$$\sum_{i=1}^{n_f}\boldsymbol{f}_i\cdot(\delta\boldsymbol{\theta}\times\boldsymbol{d}_{\overrightarrow{GP_i}}) = \sum_{i=1}^{n_f}(\boldsymbol{d}_{\overrightarrow{GP_i}}\times\boldsymbol{f}_i)\cdot\delta\boldsymbol{\theta} = \left(\sum_{i=1}^{n_f}\boldsymbol{d}_{\overrightarrow{GP_i}}\times\boldsymbol{f}_i\right)\cdot\delta\boldsymbol{\theta}. \tag{5.9}$$

So,

$$\sum_{i=1}^{n_f}\boldsymbol{f}_i\cdot\delta\boldsymbol{r}_{P_i} = \left(\sum_{i=1}^{n_f}\boldsymbol{f}_i\right)\cdot\delta\boldsymbol{r}_G + \left(\sum_{i=1}^{n_f}\boldsymbol{d}_{\overrightarrow{GP_i}}\times\boldsymbol{f}_i\right)\cdot\delta\boldsymbol{\theta}. \tag{5.10}$$

Substituting (5.10) into (5.6), we get

$$\begin{aligned}\delta W = &\left(\sum_{i=1}^{n_f}\boldsymbol{f}_i - Mg\hat{\boldsymbol{e}}_3 - M\boldsymbol{a}_G\right)\cdot\delta\boldsymbol{r}_G\\ &+\left(\sum_{i=1}^{n_f}\boldsymbol{d}_{\overrightarrow{GP_i}}\times\boldsymbol{f}_i + \sum_{i=1}^{n_\varphi}\boldsymbol{\varphi}_j - \boldsymbol{I}^G\boldsymbol{\alpha} - \boldsymbol{\omega}\times\boldsymbol{I}^G\boldsymbol{\omega}\right)\cdot\delta\boldsymbol{\theta} = 0,\end{aligned} \tag{5.11}$$

or

$$\begin{pmatrix}\sum_{i=1}^{n_f}\boldsymbol{f}_i - Mg\hat{\boldsymbol{e}}_3 - M\boldsymbol{a}_G\\ \sum_{i=1}^{n_f}\boldsymbol{d}_{\overrightarrow{GP_i}}\times\boldsymbol{f}_i + \sum_{i=1}^{n_\varphi}\boldsymbol{\varphi}_j - \boldsymbol{I}^G\boldsymbol{\alpha} - \boldsymbol{\omega}\times\boldsymbol{I}^G\boldsymbol{\omega}\end{pmatrix}\cdot\begin{pmatrix}\delta\boldsymbol{r}_G\\ \delta\boldsymbol{\theta}\end{pmatrix} = 0, \tag{5.12}$$

$$\forall\delta\boldsymbol{r}_G\in\mathbb{R}^3,\quad\text{and}\quad\forall\delta\boldsymbol{\theta}\in\mathbb{R}^3,$$

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which implies

$$\sum_{i=1}^{n_f} \boldsymbol{f}_i - Mg\hat{\boldsymbol{e}}_3 - M\boldsymbol{a}_G = \boldsymbol{0} \tag{5.13}$$

$$\sum_{i=1}^{n_f} \boldsymbol{d}_{\overrightarrow{GP_i}} \times \boldsymbol{f}_i + \sum_{i=1}^{n_\varphi} \boldsymbol{\varphi}_j - \boldsymbol{I}^G\boldsymbol{\alpha} - \boldsymbol{\omega} \times \boldsymbol{I}^G\boldsymbol{\omega} = \boldsymbol{0}. \tag{5.14}$$

5.2.3 A System of Particles

We now apply d'Alembert's Principle to a system of n_p particles with a discrete set of forces acting on them. The virtual work associated with a given particle i is given by

$$\delta W_i = \left(\sum_{j=0}^{n_p} \boldsymbol{f}_i^j\right) \cdot \delta\boldsymbol{r}_i - M_i(\boldsymbol{a}_i + g\hat{\boldsymbol{e}}_3) \cdot \delta\boldsymbol{r}_i = 0, \quad \forall \delta\boldsymbol{r}_i \in \mathbb{R}^3, \tag{5.15}$$

for $i = 1, \ldots, n_p$. The term $\boldsymbol{f}_i^j$ is the force that particle j exerts on particle i, where $\boldsymbol{f}_i^0$ is the force exerted by ground (inertial reference frame) on the ith particle. Equation (5.15) is true for any particle i in the system, under all independent variations, $\delta\boldsymbol{r}_i$.

If we sum (5.15) over all particles, we obtain

$$\delta W = \sum_{i=1}^{n_p} \delta W_i = \sum_{i=1}^{n_p} \left(\sum_{j=0}^{n_p} \boldsymbol{f}_i^j\right) \cdot \delta\boldsymbol{r}_i - \sum_{i=1}^{n_p} M_i(\boldsymbol{a}_i + g\hat{\boldsymbol{e}}_3) \cdot \delta\boldsymbol{r}_i = 0. \tag{5.16}$$

The first summation is associated with the virtual work performed by the interparticle reaction forces. It will be useful to rearrange the summation so as to pair up the equal and opposite reaction forces, $\boldsymbol{f}_i^j$ and $\boldsymbol{f}_j^i$. Doing this, we have

$$\sum_{i=1}^{n_p} \left(\sum_{j=0}^{n_p} \boldsymbol{f}_i^j\right) \cdot \delta\boldsymbol{r}_i = \sum_{i=0}^{n_p-1} \sum_{j=i+1}^{n_p} (\boldsymbol{f}_i^j \cdot \delta\boldsymbol{r}_i + \boldsymbol{f}_j^i \cdot \delta\boldsymbol{r}_j) + \sum_{i=1}^{n_p} \boldsymbol{f}_i^i \cdot \delta\boldsymbol{r}_i - \sum_{j=1}^{n_p} \boldsymbol{f}_0^j \cdot \delta\boldsymbol{r}_0. \tag{5.17}$$

Since $\boldsymbol{f}_i^i = \boldsymbol{0}$ and $\delta\boldsymbol{r}_0 = \boldsymbol{0}$, we have

$$\sum_{i=1}^{n_p} \left(\sum_{j=0}^{n_p} \boldsymbol{f}_i^j\right) \cdot \delta\boldsymbol{r}_i = \sum_{i=0}^{n_p-1} \sum_{j=i+1}^{n_p} (\boldsymbol{f}_i^j \cdot \delta\boldsymbol{r}_i + \boldsymbol{f}_j^i \cdot \delta\boldsymbol{r}_j). \tag{5.18}$$

We note that $\boldsymbol{f}_j^i = -\boldsymbol{f}_i^j$. Thus,

$$\sum_{i=1}^{n_p} \left(\sum_{j=0}^{n_p} \boldsymbol{f}_i^j\right) \cdot \delta\boldsymbol{r}_i = \sum_{i=0}^{n_p-1} \sum_{j=i+1}^{n_p} \boldsymbol{f}_i^j \cdot (\delta\boldsymbol{r}_i - \delta\boldsymbol{r}_j). \tag{5.19}$$

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This reflects the virtual work done by interparticle reaction forces, summed over all particles.

If we consider that the particles are subject to a set of holonomic constraints, the positions, $\boldsymbol{r}_i$, are not independent. In this case we will assume that the positions can be expressed in terms of a set of n_q independent generalized coordinates, $\boldsymbol{q}$. If we now consider only the variations, $\delta\boldsymbol{r}_i$, that are consistent with the kinematic constraints, then (5.19) reflects a projection of the interparticle forces on the direction of the interparticle motion. D'Alembert's Principle states that the virtual work associated with all forces orthogonal to the interparticle motion (reaction forces) is *zero* ($W_c = 0$) and only the generalized force acting in the direction of interparticle motion produces virtual work. Thus,

$$\sum_{i=0}^{n_p-1} \sum_{j=i+1}^{n_p} \boldsymbol{f}_i^j \cdot (\delta\boldsymbol{r}_i - \delta\boldsymbol{r}_j) = \sum_{i=1}^{n_q} \tau_i \cdot \delta q_i, \tag{5.20}$$

and (5.16) can be expressed as

$$\sum_{i=1}^{n_q} \tau_i \cdot \delta q_i - \sum_{i=0}^{n_p} M_i(\boldsymbol{a}_i + g\hat{\boldsymbol{e}}_3) \cdot \delta\boldsymbol{r}_i = 0 \tag{5.21}$$

for all variations, $\delta\boldsymbol{r}_i$, that are consistent with the kinematic constraints.

Expressing the variations in terms of variations in the generalized coordinates, we have

$$\delta\boldsymbol{r}_i = \boldsymbol{\Gamma}_i \delta\boldsymbol{q}. \tag{5.22}$$

So, (5.21) can be expressed as

$$\begin{gathered}\boldsymbol{\tau} \cdot \delta\boldsymbol{q} - \sum_{i=1}^{n_p} M_i(\boldsymbol{a}_i + g\hat{\boldsymbol{e}}_3) \cdot (\boldsymbol{\Gamma}_i \delta\boldsymbol{q}) = 0, \\ \forall \delta\boldsymbol{q} \in \mathbb{R}^n,\end{gathered} \tag{5.23}$$

or

$$\begin{gathered}\left[\boldsymbol{\tau} - \sum_{i=1}^{n_p} \boldsymbol{\Gamma}_i^T (M_i\boldsymbol{a}_i + M_i g\hat{\boldsymbol{e}}_3)\right] \cdot \delta\boldsymbol{q} = 0 \\ \forall \delta\boldsymbol{q} \in \mathbb{R}^n,\end{gathered} \tag{5.24}$$

which implies

$$\boldsymbol{\tau} = \sum_{i=1}^{n_p} \boldsymbol{\Gamma}_i^T (M_i\boldsymbol{a}_i + M_i g\hat{\boldsymbol{e}}_3). \tag{5.25}$$

Noting that

$$\boldsymbol{a}_i = \boldsymbol{\Gamma}_i \ddot{\boldsymbol{q}}, \tag{5.26}$$

we have

$$\boldsymbol{\tau} = \left(\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i\right) \ddot{\boldsymbol{q}} + g \sum_{i=1}^{n_q} M_i \boldsymbol{\Gamma}_i^T \hat{\boldsymbol{e}}_3. \tag{5.27}$$

Defining the system mass matrix, $\boldsymbol{M}(\boldsymbol{q})$, and the vector of generalized gravity forces, $\boldsymbol{g}(\boldsymbol{q})$,

$$\boldsymbol{M}(\boldsymbol{q}) \triangleq \sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i \tag{5.28}$$

$$\boldsymbol{g}(\boldsymbol{q}) \triangleq g \sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \hat{\boldsymbol{e}}_3, \tag{5.29}$$

we have

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}). \tag{5.30}$$

5.2.4 A System of Rigid Bodies

We now apply d'Alembert's Principle to a system of n_b rigid bodies forming a serial chain, with a discrete set of forces acting on them. The virtual work associated with a given body i is given by

$$\begin{aligned} \delta W_i = \boldsymbol{f}_i^{i-1} \cdot \delta \boldsymbol{r}_i^{i-1} + \boldsymbol{f}_i^{i+1} \cdot \delta \boldsymbol{r}_i^{i+1} - M_i(\boldsymbol{a}_{G_i} + g\hat{\boldsymbol{e}}_{0_3}) \cdot \delta \boldsymbol{r}_{G_i} + \boldsymbol{\varphi}_i^{i-1} \cdot \delta \boldsymbol{\theta}_i \\ + \boldsymbol{\varphi}_i^{i+1} \cdot \delta \boldsymbol{\theta}_i - (\boldsymbol{I}_i^{G_i} \boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i^{G_i} \boldsymbol{\omega}_i) \cdot \delta \boldsymbol{\theta}_i = 0, \\ \forall \delta \boldsymbol{r}_{G_i} \in \mathbb{R}^3, \quad \text{and} \quad \forall \delta \boldsymbol{\theta}_i \in \mathbb{R}^3, \end{aligned} \tag{5.31}$$

for $i = 1, \ldots, n_b$. The term $\boldsymbol{f}_i^{i-1}$ is the force that body $i-1$ exerts on body i. Likewise, $\boldsymbol{\varphi}_i^{i-1}$ is the moment that body $i-1$ exerts on body i. The term $\boldsymbol{r}_i^{i-1}$ is the point on body i to which body $i-1$ attaches. Equation (5.31) is true for any rigid body i in the system, under all independent variations, $\delta \boldsymbol{r}_{G_i}$ and $\delta \boldsymbol{\theta}_i$ (we note that $\delta \boldsymbol{r}_i^{i-1}$ and $\delta \boldsymbol{r}_i^{i+1}$ are functions of $\delta \boldsymbol{r}_{G_i}$ and $\delta \boldsymbol{\theta}_i$ due to the rigid-body constraint). If we sum (5.31) over all rigid bodies, we obtain

$$\begin{aligned} \delta W = \sum_{i=1}^{n_b} \delta W_i = \sum_{i=1}^{n_b} (\boldsymbol{f}_i^{i-1} \cdot \delta \boldsymbol{r}_i^{i-1} + \boldsymbol{f}_i^{i+1} \cdot \delta \boldsymbol{r}_i^{i+1}) \\ - \sum_{i=1}^{n_b} M_i(\boldsymbol{a}_{G_i} + g\hat{\boldsymbol{e}}_{0_3}) \cdot \delta \boldsymbol{r}_{G_i} + \sum_{i=1}^{n_b} (\boldsymbol{\varphi}_i^{i-1} \cdot \delta \boldsymbol{\theta}_i + \boldsymbol{\varphi}_i^{i+1} \cdot \delta \boldsymbol{\theta}_i) \\ - \sum_{i=1}^{n_b} (\boldsymbol{I}_i^{G_i} \boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i^{G_i} \boldsymbol{\omega}_i) \cdot \delta \boldsymbol{\theta}_i = 0. \end{aligned} \tag{5.32}$$

The first summation is associated with the virtual work performed by the interlink reaction forces. It will be useful to rearrange the summation so as to pair up the equal

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and opposite reaction forces, $\boldsymbol{f}_i^{i-1}$ and $\boldsymbol{f}_{i-1}^{i}$, acting through each joint i. Doing this, we have

$$\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{i-1}\cdot\delta\boldsymbol{r}_i^{i-1}+\boldsymbol{f}_i^{i+1}\cdot\delta\boldsymbol{r}_i^{i+1})=\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{i-1}\cdot\delta\boldsymbol{r}_i^{i-1}+\boldsymbol{f}_{i-1}^{i}\cdot\delta\boldsymbol{r}_{i-1}^{i})+\boldsymbol{f}_{n_b}^{n_b+1}\cdot\delta\boldsymbol{r}_{n_b}^{n_b+1}-\boldsymbol{f}_0^1\cdot\delta\boldsymbol{r}_0^1. \tag{5.33}$$

Since $\boldsymbol{f}_{n_b}^{n_b+1}=\boldsymbol{0}$ and $\delta\boldsymbol{r}_0^1=\boldsymbol{0}$, we have

$$\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{i-1}\cdot\delta\boldsymbol{r}_i^{i-1}+\boldsymbol{f}_i^{i+1}\cdot\delta\boldsymbol{r}_i^{i+1})=\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{i-1}\cdot\delta\boldsymbol{r}_i^{i-1}+\boldsymbol{f}_{i-1}^{i}\cdot\delta\boldsymbol{r}_{i-1}^{i}). \tag{5.34}$$

We note that $\boldsymbol{f}_{i-1}^{i}=-\boldsymbol{f}_i^{i-1}$. Thus,

$$\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{i-1}\cdot\delta\boldsymbol{r}_i^{i-1}+\boldsymbol{f}_i^{i+1}\cdot\delta\boldsymbol{r}_i^{i+1})=\sum_{i=1}^{n_b}\boldsymbol{f}_i^{i-1}\cdot(\delta\boldsymbol{r}_i^{i-1}-\delta\boldsymbol{r}_{i-1}^{i}). \tag{5.35}$$

This reflects the virtual work done by reaction forces at each joint, summed over all joints.

As with the reaction forces, it will be useful to rearrange the summation associated with the virtual work performed by the interlink reaction moments so as to pair up the equal and opposite reaction moments, $\boldsymbol{\varphi}_i^{i-1}$ and $\boldsymbol{\varphi}_{i-1}^{i}$, acting about each joint. Using the same procedure as with the reaction forces, we have

$$\begin{aligned}\sum_{i=1}^{n_b}(\boldsymbol{\varphi}_i^{i-1}\cdot\delta\boldsymbol{\theta}_i+\boldsymbol{\varphi}_i^{i+1}\cdot\delta\boldsymbol{\theta}_i)&=\sum_{i=1}^{n_b}(\boldsymbol{\varphi}_i^{i-1}\cdot\delta\boldsymbol{\theta}_i+\boldsymbol{\varphi}_{i-1}^{i}\cdot\delta\boldsymbol{\theta}_{i-1})\\&=\sum_{i=1}^{n_b}\boldsymbol{\varphi}_i^{i-1}\cdot(\delta\boldsymbol{\theta}_i-\delta\boldsymbol{\theta}_{i-1}).\end{aligned} \tag{5.36}$$

This reflects the virtual work done by reaction moments at each joint, summed over all joints. So, the total virtual work performed by the interlink reaction forces and moments can be expressed compactly as

$$\begin{aligned}&\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{i-1}\cdot\delta\boldsymbol{r}_i^{i-1}+\boldsymbol{f}_i^{i+1}\cdot\delta\boldsymbol{r}_i^{i+1})+\sum_{i=1}^{n_b}(\boldsymbol{\varphi}_i^{i-1}\cdot\delta\boldsymbol{\theta}_i+\boldsymbol{\varphi}_i^{i+1}\cdot\delta\boldsymbol{\theta}_i)\\&\qquad=\sum_{i=1}^{n_b}\begin{pmatrix}\boldsymbol{f}_i^{i-1}\\\boldsymbol{\varphi}_i^{i-1}\end{pmatrix}\cdot\begin{pmatrix}\delta\boldsymbol{r}_i^{i-1}-\delta\boldsymbol{r}_{i-1}^{i}\\\delta\boldsymbol{\theta}_i-\delta\boldsymbol{\theta}_{i-1}\end{pmatrix}.\end{aligned} \tag{5.37}$$

If we now consider only the variations $\delta\boldsymbol{r}_i^{i-1}$, $\delta\boldsymbol{r}_{i-1}^{i}$, $\delta\boldsymbol{\theta}_i$, and $\delta\boldsymbol{\theta}_{i-1}$ that are consistent with the kinematic constraints, then (5.37) reflects a projection of the interlink forces and moments on the direction of the joint motion. D'Alembert's Principle states that the virtual work associated with all forces and moments orthogonal to the joint motion (reaction forces/moments) is *zero* ($W_c=0$) and only the generalized force acting in the

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direction of joint motion produces virtual work. Thus,

$$\sum_{i=1}^{n_b} \begin{pmatrix} \boldsymbol{f}_i^{i-1} \\ \boldsymbol{\varphi}_i^{i-1} \end{pmatrix} \cdot \begin{pmatrix} \delta \boldsymbol{r}_i^{i-1} - \delta \boldsymbol{r}_{i-1}^{i} \\ \delta \boldsymbol{\theta}_i - \delta \boldsymbol{\theta}_{i-1} \end{pmatrix} = \sum_{i=1}^{n_q} \tau_i \cdot \delta q_i, \tag{5.38}$$

and (5.32) can be expressed as

$$\sum_{i=1}^{n_q} \tau_i \cdot \delta q_i - \sum_{i=1}^{n_b} M_i(\boldsymbol{a}_{G_i} + g\hat{\boldsymbol{e}}_{0_3}) \cdot \delta \boldsymbol{r}_{G_i} - \sum_{i=1}^{n_b} (\boldsymbol{I}_i^{G_i} \boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i^{G_i} \boldsymbol{\omega}_i) \cdot \delta \boldsymbol{\theta}_i = 0 \tag{5.39}$$

for all variations, $\delta \boldsymbol{r}_{G_i}$ and $\delta \boldsymbol{\theta}_i$, that are consistent with the kinematic constraints.

Using d'Alembert's Principle, we can address a system of n_b rigid bodies forming a branching chain in a similar manner. With a serial chain, the parent/child structure is implicit to the numbering scheme. Every link i has a single parent link, $i-1$, at its proximal end and a single child link, $i+1$, at its distal end (except for the nth link). The ith joint is at the proximal end of the ith link. With a branching chain the numbering of links is more arbitrary, without a parent/child structure implicit to the numbering scheme. We can explicitly capture the parent/child structure, however, by defining three additional parameters, λ_i, c_i, μ_{ij}. The term λ_i is the parent link number of the ith link, c_i is the number of child links for the ith link, and $\mu_{i1} \cdots \mu_{ic_i}$ are the child link numbers of the ith link. Given these parameters, the virtual work associated with a given body i is given by

$$\begin{aligned} \delta W_i = \boldsymbol{f}_i^{\lambda_i} \cdot \delta \boldsymbol{r}_i^{\lambda_i} + \sum_{j=1}^{c_i} \boldsymbol{f}_i^{\mu_{ij}} \cdot \delta \boldsymbol{r}_i^{\mu_{ij}} - M_i(\boldsymbol{a}_{G_i} + g\hat{\boldsymbol{e}}_{0_3}) \cdot \delta \boldsymbol{r}_{G_i} + \boldsymbol{\varphi}_i^{\lambda_i} \cdot \delta \boldsymbol{\theta}_i \\ + \sum_{j=1}^{c_i} \boldsymbol{\varphi}_i^{\mu_{ij}} \cdot \delta \boldsymbol{\theta}_i - (\boldsymbol{I}_i^{G_i} \boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i^{G_i} \boldsymbol{\omega}_i) \cdot \delta \boldsymbol{\theta}_i = 0, \\ \forall \delta \boldsymbol{r}_{G_i} \in \mathbb{R}^3, \quad \text{and} \quad \forall \delta \boldsymbol{\theta}_i \in \mathbb{R}^3, \end{aligned} \tag{5.40}$$

for $i = 1, \ldots, n_b$. The term $\boldsymbol{r}_i^{\lambda_i}$ is the point on body i to which body (parent) λ_i attaches, and likewise $\boldsymbol{r}_i^{\mu_{i1}} \cdots \boldsymbol{r}_i^{\mu_{ic_i}}$ are the points on body i to which bodies (children) $\mu_{i1} \cdots \mu_{ic_i}$ attach. We note that $\delta \boldsymbol{r}_i^{\lambda_i}$ and $\delta \boldsymbol{r}_i^{\mu_{i1}} \cdots \delta \boldsymbol{r}_i^{\mu_{ic_i}}$ are functions of $\delta \boldsymbol{r}_{G_i}$ and $\delta \boldsymbol{\theta}_i$ due to the rigid-body constraint. If we sum (5.40) over all rigid bodies, we obtain

$$\begin{aligned} \delta W = \sum_{i=1}^{n_b} \delta W_i = \sum_{i=1}^{n_b} \left(\boldsymbol{f}_i^{\lambda_i} \cdot \delta \boldsymbol{r}_i^{\lambda_i} + \sum_{j=1}^{c_i} \boldsymbol{f}_i^{\mu_{ij}} \cdot \delta \boldsymbol{r}_i^{\mu_{ij}} \right) \\ - \sum_{i=1}^{n_b} M_i(\boldsymbol{a}_{G_i} + g\hat{\boldsymbol{e}}_{0_3}) \cdot \delta \boldsymbol{r}_{G_i} + \sum_{i=1}^{n_b} \left(\boldsymbol{\varphi}_i^{\lambda_i} \cdot \delta \boldsymbol{\theta}_i + \sum_{j=1}^{c_i} \boldsymbol{\varphi}_i^{\mu_{ij}} \cdot \delta \boldsymbol{\theta}_i \right) \\ - \sum_{i=1}^{n_b} (\boldsymbol{I}_i^{G_i} \boldsymbol{\alpha}_i + \boldsymbol{\omega}_i \times \boldsymbol{I}_i^{G_i} \boldsymbol{\omega}_i) \cdot \delta \boldsymbol{\theta}_i = 0. \end{aligned} \tag{5.41}$$

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The term associated with the virtual work performed by the interlink reaction forces is

$$\sum_{i=1}^{n_b}\left(\boldsymbol{f}_i^{\lambda_i}\cdot\delta\boldsymbol{r}_i^{\lambda_i}+\sum_{j=1}^{c_i}\boldsymbol{f}_i^{\mu_{ij}}\cdot\delta\boldsymbol{r}_i^{\mu_{ij}}\right). \tag{5.42}$$

It will be useful to rearrange the summation so as to pair up the equal and opposite reaction forces, $\boldsymbol{f}_i^{\lambda_i}$ and $\boldsymbol{f}_{\lambda_i}^{i}$, acting through each joint i. We can rewrite the summation of (5.42) based on considering the virtual work at each joint. The sum over all joints then gives us

$$\sum_{i=1}^{n_b}(\boldsymbol{f}_i^{\lambda_i}\cdot\delta\boldsymbol{r}_i^{\lambda_i}+\boldsymbol{f}_{\lambda_i}^{i}\cdot\delta\boldsymbol{r}_{\lambda_i}^{i})=\sum_{i=1}^{n_b}\boldsymbol{f}_i^{\lambda_i}\cdot(\delta\boldsymbol{r}_i^{\lambda_i}-\delta\boldsymbol{r}_{\lambda_i}^{i}). \tag{5.43}$$

The term associated with the virtual work performed by the interlink reaction moments is

$$\sum_{i=1}^{n_b}\left(\boldsymbol{\varphi}_i^{\lambda_i}\cdot\delta\boldsymbol{\theta}_i+\sum_{j=1}^{c_i}\boldsymbol{\varphi}_i^{\mu_{ij}}\cdot\delta\boldsymbol{\theta}_i\right). \tag{5.44}$$

In a similar manner as before, we can rewrite this summation based on considering the virtual work at each joint. The sum over all joints then gives us

$$\sum_{i=1}^{n_b}(\boldsymbol{\varphi}_i^{\lambda_i}\cdot\delta\boldsymbol{\theta}_i+\boldsymbol{\varphi}_{\lambda_i}^{i}\cdot\delta\boldsymbol{\theta}_{\lambda_i})=\sum_{i=1}^{n_b}\boldsymbol{\varphi}_i^{\lambda_i}\cdot(\delta\boldsymbol{\theta}_i-\delta\boldsymbol{\theta}_{\lambda_i}). \tag{5.45}$$

So, the total virtual work performed by the interlink reaction forces and moments can be expressed compactly as

$$\begin{aligned}\sum_{i=1}^{n_b}\left(\boldsymbol{f}_i^{\lambda_i}\cdot\delta\boldsymbol{r}_i^{\lambda_i}+\sum_{j=1}^{c_i}\boldsymbol{f}_i^{\mu_{ij}}\cdot\delta\boldsymbol{r}_i^{\mu_{ij}}\right)+\sum_{i=1}^{n_b}\left(\boldsymbol{\varphi}_i^{\lambda_i}\cdot\delta\boldsymbol{\theta}_i+\sum_{j=1}^{c_i}\boldsymbol{\varphi}_i^{\mu_{ij}}\cdot\delta\boldsymbol{\theta}_i\right)\\ =\sum_{i=1}^{n_b}\begin{pmatrix}\boldsymbol{f}_i^{\lambda_i}\\ \boldsymbol{\varphi}_i^{\lambda_i}\end{pmatrix}\cdot\begin{pmatrix}\delta\boldsymbol{r}_i^{\lambda_i}-\delta\boldsymbol{r}_{\lambda_i}^{i}\\ \delta\boldsymbol{\theta}_i-\delta\boldsymbol{\theta}_{\lambda_i}\end{pmatrix}.\end{aligned} \tag{5.46}$$

If we now consider only the variations $\delta\boldsymbol{r}_i^{\lambda_i}$, $\delta\boldsymbol{r}_{\lambda_i}^{i}$, $\delta\boldsymbol{\theta}_i$, and $\delta\boldsymbol{\theta}_{\lambda_i}$ that are consistent with the kinematic constraints, then (5.46) reflects a projection of the interlink forces and moments on the direction of the joint motion. D'Alembert's Principle states that the virtual work associated with all forces and moments orthogonal to the joint motion (reaction forces/moments) is *zero* ($W_c = 0$) and only the generalized force acting in the direction of joint motion produces virtual work. Thus,

$$\sum_{i=1}^{n_b}\begin{pmatrix}\boldsymbol{f}_i^{\lambda_i}\\ \boldsymbol{\varphi}_i^{\lambda_i}\end{pmatrix}\cdot\begin{pmatrix}\delta\boldsymbol{r}_i^{\lambda_i}-\delta\boldsymbol{r}_{\lambda_i}^{i}\\ \delta\boldsymbol{\theta}_i-\delta\boldsymbol{\theta}_{\lambda_i}\end{pmatrix}=\sum_{i=1}^{n_q}\tau_i\cdot\delta q_i, \tag{5.47}$$

and (5.41) can be expressed as

$$\sum_{i=1}^{n_q}\tau_i\cdot\delta q_i-\sum_{i=1}^{n_b}M_i(\boldsymbol{a}_{G_i}+g\hat{\boldsymbol{e}}_{0_3})\cdot\delta\boldsymbol{r}_{G_i}-\sum_{i=1}^{n_b}(\boldsymbol{I}_i^{G_i}\boldsymbol{\alpha}_i+\boldsymbol{\omega}_i\times\boldsymbol{I}_i^{G_i}\boldsymbol{\omega}_i)\cdot\delta\boldsymbol{\theta}_i=0, \tag{5.48}$$

for all variations, $\delta \boldsymbol{r}_{G_i}$ and $\delta \boldsymbol{\theta}_i$, that are consistent with the kinematic constraints. Expressing the variations in terms of variations in the generalized coordinates, we have

$$ {}^i\delta \boldsymbol{r}_{G_i} = {}^i\boldsymbol{\Gamma}_{G_i}\delta \boldsymbol{q} \quad \text{and} \quad {}^i\delta\boldsymbol{\theta}_i = {}^i\boldsymbol{\Pi}_i\delta\boldsymbol{q}, \tag{5.49} $$

where the terms are expressed in the local link frame i for convenience. So, (5.48) can be expressed as

$$ \begin{gathered} \boldsymbol{\tau}\cdot\delta\boldsymbol{q} - \sum_{i=1}^{n_b} M_i({}^i\boldsymbol{a}_{G_i} + g\,{}^i\hat{\boldsymbol{e}}_{0_3})\cdot({}^i\boldsymbol{\Gamma}_{G_i}\delta\boldsymbol{q}) - \sum_{i=1}^{n_b}({}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\alpha}_i + {}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\omega}_i)\cdot({}^i\boldsymbol{\Pi}_i\delta\boldsymbol{q}) = 0, \\ \forall \delta\boldsymbol{q} \in \mathbb{R}^n, \end{gathered} \tag{5.50} $$

or

$$ \begin{gathered} \boldsymbol{\tau}\cdot\delta\boldsymbol{q} - \sum_{i=1}^{n_b}\left[M_i\,{}^i\boldsymbol{\Gamma}_{G_i}^T\,{}^i\boldsymbol{a}_{G_i} + M_i g\,{}^i\boldsymbol{\Gamma}_{G_i}^T\,{}^i\hat{\boldsymbol{e}}_{0_3} + {}^i\boldsymbol{\Pi}_i^T({}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\alpha}_i + {}^i\boldsymbol{\omega}_i\times{}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\omega}_i)\right]\cdot\delta\boldsymbol{q} = 0 \\ \forall\delta\boldsymbol{q}\in\mathbb{R}^n. \end{gathered} \tag{5.51} $$

In matrix form we have

$$ \begin{gathered} \left[\boldsymbol{\tau} - \sum_{i=1}^{n_b}\begin{pmatrix}{}^i\boldsymbol{\Gamma}_{G_i}^T & {}^i\boldsymbol{\Pi}_i^T\end{pmatrix}\begin{pmatrix} M_i\,{}^i\boldsymbol{a}_{G_i} + M_i g\,{}^i\hat{\boldsymbol{e}}_{0_3} \\ {}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\alpha}_i + {}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\omega}_i\end{pmatrix}\right]\cdot\delta\boldsymbol{q} = 0, \\ \forall\delta\boldsymbol{q}\in\mathbb{R}^n, \end{gathered} \tag{5.52} $$

which implies

$$ \boldsymbol{\tau} = \sum_{i=1}^{n_b}\begin{pmatrix}{}^i\boldsymbol{\Gamma}_{G_i}^T & {}^i\boldsymbol{\Pi}_i^T\end{pmatrix}\begin{pmatrix} M_i\,{}^i\boldsymbol{a}_{G_i} + M_i g\,{}^i\hat{\boldsymbol{e}}_{0_3} \\ {}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\alpha}_i + {}^i\boldsymbol{\Omega}_i\,{}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\omega}_i\end{pmatrix}. \tag{5.53} $$

Noting that

$$ {}^i\boldsymbol{\omega}_i = {}^i\boldsymbol{\Pi}_i\dot{\boldsymbol{q}}, \qquad {}^i\boldsymbol{a}_{G_i} = {}^i\boldsymbol{\Gamma}_{G_i}\ddot{\boldsymbol{q}} + ({}^i\dot{\boldsymbol{\Gamma}}_{G_i} + {}^i\boldsymbol{\Omega}_i\,{}^i\boldsymbol{\Gamma}_{G_i})\dot{\boldsymbol{q}}, \qquad \text{and} \qquad {}^i\boldsymbol{\alpha}_i = {}^i\boldsymbol{\Pi}_i\ddot{\boldsymbol{q}} + {}^i\dot{\boldsymbol{\Pi}}_i\dot{\boldsymbol{q}}, \tag{5.54} $$

we have

$$ \begin{aligned} \boldsymbol{\tau} = {}& \left[\sum_{i=1}^{n_b}(M_i\,{}^i\boldsymbol{\Gamma}_{G_i}^T\,{}^i\boldsymbol{\Gamma}_{G_i} + {}^i\boldsymbol{\Pi}_i^T\,{}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\Pi}_i)\right]\ddot{\boldsymbol{q}} \\ & + \left[\sum_{i=1}^{n_b}\left(M_i\,{}^i\boldsymbol{\Gamma}_{G_i}^T({}^i\dot{\boldsymbol{\Gamma}}_{G_i} + {}^i\boldsymbol{\Omega}_i\,{}^i\boldsymbol{\Gamma}_{G_i}) + {}^i\boldsymbol{\Pi}_i^T({}^i\boldsymbol{I}_i^{G_i}\,{}^i\dot{\boldsymbol{\Pi}}_i + {}^i\boldsymbol{\Omega}_i\,{}^i\boldsymbol{I}_i^{G_i}\,{}^i\boldsymbol{\Pi}_i)\right)\right]\dot{\boldsymbol{q}} + g\sum_{i=1}^{n_b} M_i\,{}^i\boldsymbol{\Gamma}_{G_i}^T\,{}^i\hat{\boldsymbol{e}}_{0_3}. \end{aligned} \tag{5.55} $$

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Algorithm 3 Second-order method for integrating the equations of motion

1: $\boldsymbol{q}_0 = \boldsymbol{q}_o$ {initialization}
2: $\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}_o$ {initialization}
3: **for** $i = 0$ to $n_s - 1$ **do**
4: $\quad \ddot{\boldsymbol{q}}_i = \boldsymbol{M}_i^{-1}(\boldsymbol{\tau}_i - \boldsymbol{b}_i - \boldsymbol{g}_i)$
5: $\quad \dot{\boldsymbol{q}}_{i+1} = \dot{\boldsymbol{q}}_i + \ddot{\boldsymbol{q}}_i \Delta t$
6: $\quad \boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \dot{\boldsymbol{q}}_i \Delta t + \frac{1}{2}\ddot{\boldsymbol{q}}_i \Delta t^2$
7: **end for**

Defining the symmetric positive definite mass matrix, $\boldsymbol{M}(\boldsymbol{q})$, the centrifugal and Coriolis force vector, $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, and the gravity force vector, $\boldsymbol{g}(\boldsymbol{q})$,

$$\boldsymbol{M}(\boldsymbol{q}) \triangleq \sum_{i=1}^{n_b} (M_i {}^i\boldsymbol{\Gamma}_{G_i}^T {}^i\boldsymbol{\Gamma}_{G_i} + {}^i\boldsymbol{\Pi}_i^T {}^i\boldsymbol{I}_i^{G_i} {}^i\boldsymbol{\Pi}_i), \tag{5.56}$$

$$\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \triangleq \left[\sum_{i=1}^{n_b} \left(M_i {}^i\boldsymbol{\Gamma}_{G_i}^T ({}^i\dot{\boldsymbol{\Gamma}}_{G_i} + {}^i\boldsymbol{\Omega}_i {}^i\boldsymbol{\Gamma}_{G_i}) + {}^i\boldsymbol{\Pi}_i^T ({}^i\boldsymbol{I}_i^{G_i} {}^i\dot{\boldsymbol{\Pi}}_i + {}^i\boldsymbol{\Omega}_i {}^i\boldsymbol{I}_i^{G_i} {}^i\boldsymbol{\Pi}_i) \right) \right] \dot{\boldsymbol{q}}, \tag{5.57}$$

$$\boldsymbol{g}(\boldsymbol{q}) \triangleq g \sum_{i=1}^{n_b} M_i {}^i\boldsymbol{\Gamma}_{G_i}^T {}^i\hat{\boldsymbol{e}}_{0_3}, \tag{5.58}$$

we have

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \tag{5.59}$$

Numerical Integration

We can numerically integrate (5.59) using a second-order method. Solving for the generalized accelerations, we have

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g}). \tag{5.60}$$

The second-order method for integrating this system can be summarized as shown in Algorithm 3.

Example: We consider a serial chain robot and parameterize the system using four generalized coordinates as shown in Figure 5.2. The centers of mass of each link can be

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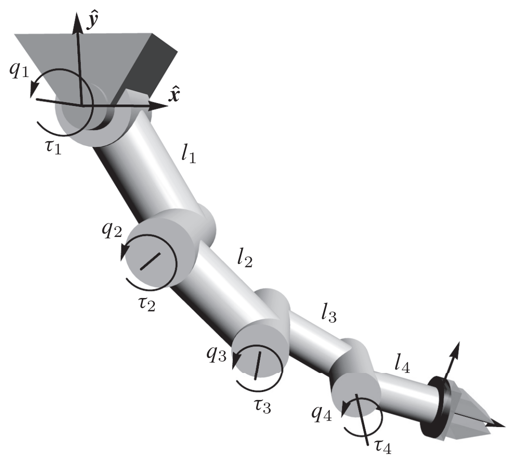

Figure 5.2 A 4 degree-of-freedom serial chain robot parameterized using four generalized coordinates.

computed in a straightforward fashion:

$$\boldsymbol{r}_{G_1} = \begin{pmatrix} \frac{l_1}{2}\cos(q_1) \\ \frac{l_1}{2}\sin(q_1) \end{pmatrix}, \tag{5.61}$$

$$\boldsymbol{r}_{G_2} = \begin{pmatrix} l_1\cos(q_1) + \frac{l_2}{2}\cos(q_1+q_2) \\ l_1\sin(q_1) + \frac{l_2}{2}\sin(q_1+q_2) \end{pmatrix}, \tag{5.62}$$

$$\boldsymbol{r}_{G_3} = \begin{pmatrix} l_1\cos(q_1) + l_2\cos(q_1+q_2) + \frac{l_3}{2}\cos(q_1+q_2+q_3) \\ l_1\sin(q_1) + l_2\sin(q_1+q_2) + \frac{l_3}{2}\sin(q_1+q_2+q_3) \end{pmatrix}, \tag{5.63}$$

$$\boldsymbol{r}_{G_4} = \left(\begin{matrix} l_1\cos(q_1) + l_2\cos(q_1+q_2) + l_3\cos(q_1+q_2+q_3) \\ l_1\sin(q_1) + l_2\sin(q_1+q_2) + l_3\sin(q_1+q_2+q_3) \end{matrix} \right. \tag{5.64}$$

$$\left. \dots \begin{matrix} +\frac{l_4}{2}\cos(q_1+q_2+q_3+q_4) \\ +\frac{l_4}{2}\sin(q_1+q_2+q_3+q_4) \end{matrix} \right). \tag{5.65}$$

Similarly, the angular velocities of each link can be computed easily:

$$\boldsymbol{\omega}_1 = \dot{q}_1, \tag{5.66}$$

$$\boldsymbol{\omega}_2 = \dot{q}_1 + \dot{q}_2, \tag{5.67}$$

$$\boldsymbol{\omega}_3 = \dot{q}_1 + \dot{q}_2 + \dot{q}_3, \tag{5.68}$$

$$\boldsymbol{\omega}_4 = \dot{q}_1 + \dot{q}_2 + \dot{q}_3 + \dot{q}_4. \tag{5.69}$$

The translational Jacobians are computed as

$$\boldsymbol{\Gamma}_{G_i} = \frac{\partial \boldsymbol{r}_{G_i}}{\partial \boldsymbol{q}}, \qquad \text{for } i = 1, \dots, 3, \tag{5.70}$$

and the angular velocity Jacobians as

$$\boldsymbol{\Pi}_i = \frac{\partial \boldsymbol{\omega}_i}{\partial \dot{\boldsymbol{q}}}, \qquad \text{for } i = 1, \dots, 3. \tag{5.71}$$

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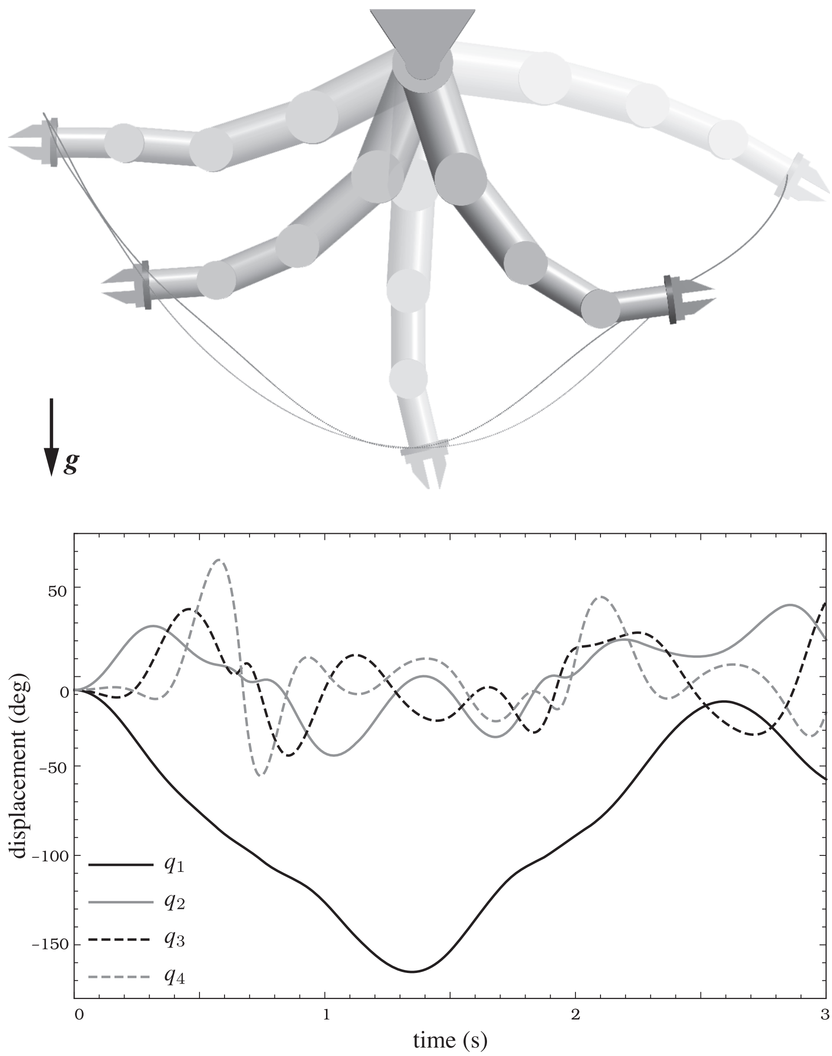

Figure 5.3 (Top) Animation frames from the simulation of a serial chain robot. (Bottom) Time history of the robot generalized coordinates (q_1, q_2, q_3, q_4).

The dynamical terms are

$$\boldsymbol{M}(\boldsymbol{q}) = \sum_{i=1}^{4} (M_i \boldsymbol{\Gamma}_{G_i}^T \boldsymbol{\Gamma}_{G_i} + \boldsymbol{\Pi}_i^T \boldsymbol{I}_i^{G_i} \boldsymbol{\Pi}_i), \tag{5.72}$$

$$\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{i=1}^{4} M_i \boldsymbol{\Gamma}_{G_i}^T \dot{\boldsymbol{\Gamma}}_{G_i} \dot{\boldsymbol{q}}, \tag{5.73}$$

$$\boldsymbol{g}(\boldsymbol{q}) = g \sum_{i=1}^{4} M_i \boldsymbol{\Gamma}_{G_i}^T \hat{\boldsymbol{e}}_2, \tag{5.74}$$

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where we take the rotational inertias to be

$$I_i^{G_i} = \frac{M_i r_i^2}{4} + \frac{M_i l_i^2}{12}. \tag{5.75}$$

The simulation results are displayed in Figure 5.3. The following values were used for the constants:

$$\begin{aligned} &r_1 = 0.15, \quad r_2 = 0.125, \quad r_3 = 0.1, \quad r_4 = 0.0875, \\ &l_1 = 0.7, \quad l_2 = 0.6, \quad l_3 = 0.5, \quad l_4 = 0.4, \\ &M_1 = 2.5, \quad M_2 = 1.5, \quad M_3 = 1.0, \quad M_4 = 0.75. \end{aligned} \tag{5.76}$$

The initial conditions used were $q_i = -\pi/24$ and $\dot{q}_i = 0$.

5.2.5 Auxiliary Constraints

Holonomic Constraints

The standard taxonomy of multibody systems by kinematic topology consists of branching or tree-like structures and graph or closed-loop structures. In the case of branching structures a set of independent generalized coordinates is chosen. The kinematic constraints between the bodies are implied by the choice of these generalized coordinates, as we saw in Section 5.2.4. Since these generalized coordinates are independent (unconstrained), we will refer to branching structures as unconstrained with respect to configuration space.

Analysis of graph or closed-loop structures typically involves breaking the loop(s) and deriving the dynamics of the resulting branching structures. The last step involves the imposition of a set of holonomic constraint equations, $\boldsymbol{\phi}(\boldsymbol{q}) = \mathbf{0}$, to enforce the loop closures. A closed-loop topology is depicted in Figure 5.4 (left), with the nodes representing the bodies and the edges representing the joints. Closed-loop structures represent a subset of the larger class of holonomically constrained multibody systems. Such systems involve holonomic constraints in the form of general algebraic dependencies between generalized coordinates, as depicted in Figure 5.4 (right). Again, a set of holonomic constraint equations is imposed in conjunction with the unconstrained equations of motion. Since the systems of Figure 5.4 involve explicit, or auxilliary, constraints between the generalized coordinates (in addition to the implicit constraints between the bodies suggested by the choice of generalized coordinates), we will refer to these structures as constrained with respect to configuration space. In Section 3.4 we considered the number of degrees of freedom of holonomically constrained systems. The number of degrees of freedom, p, is given by

$$p = n - m, \tag{5.77}$$

where n is the number of generalized coordinates and m is the number of independent constraint equations.

The constrained systems described thus far exclusively involve holonomic constraints. Nonholonomic systems, which involve nonintegrable constraints on the generalized velocities, will be addressed in the subsequent chapters. They can be handled with

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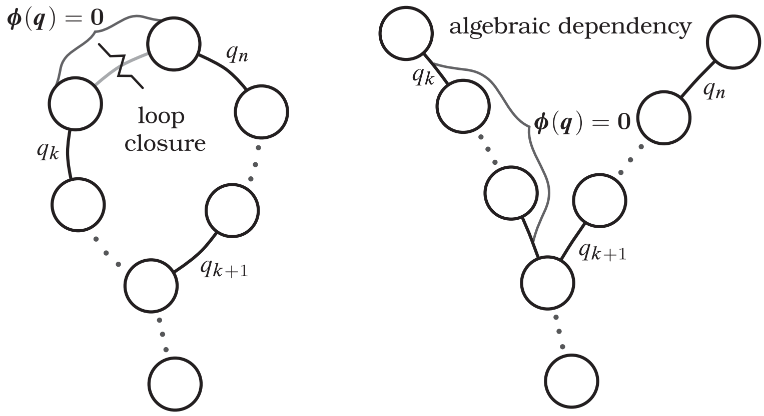

Figure 5.4 Constrained structures involving closed loops (Left) and general algebraic dependencies (Right). The dynamics of both systems can be derived from the unconstrained system dynamics through the imposition of a set of holonomic constraint equations, $\boldsymbol{\phi}(\boldsymbol{q}) = \mathbf{0}$.

higher-order variational principles (Flannery 2005) and methods like Jourdain's Principle, Kane's method, Gauss's Principle, or the Gibbs-Appell method. Jourdain's Principle and Gauss's Principle deal with the constraints explicitly, while Kane's method and the Gibbs-Appell method deal with the costraints implicitly by defining a set of independent quasi-velocities (generalized speeds) and/or quasi-accelerations.

We now consider the general case of auxiliary holonomic constraints. Given the multibody system

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}), \tag{5.78}$$

subject to the holonomic constraints

$$\boldsymbol{\phi}(\boldsymbol{q}) = \mathbf{0}, \tag{5.79}$$

we begin by first expressing the zeroth-order variational equation associated with d'Alembert's Principle:

$$\boldsymbol{\tau}_C \cdot \delta\boldsymbol{q} + (\boldsymbol{\tau} - \boldsymbol{M}\boldsymbol{q} - \boldsymbol{b} - \boldsymbol{g}) \cdot \delta\boldsymbol{q} = 0, \tag{5.80}$$

where $\boldsymbol{\tau}_C$ is the vector of generalized constraint forces. The virtual displacements, $\delta\boldsymbol{q}$, refer to all displacement variations which satisfy the constraints, while time is fixed. With $\delta t = 0$ the variation of the constraint equation yields

$$\delta\boldsymbol{\phi} = \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{q}}\delta\boldsymbol{q} = \boldsymbol{\Phi}\delta\boldsymbol{q} = \mathbf{0}, \tag{5.81}$$

which implies that $\delta\boldsymbol{q} \in \ker(\boldsymbol{\Phi})$, where $\boldsymbol{\Phi} = \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{q}}$ is the constraint matrix. Under this condition, (5.80) can be restricted to constraint-consistent virtual displacements:

$$\begin{gathered}\boldsymbol{\tau}_C \cdot \delta\boldsymbol{q} + (\boldsymbol{\tau} - \boldsymbol{M}\boldsymbol{q} - \boldsymbol{b} - \boldsymbol{g}) \cdot \delta\boldsymbol{q} = 0 \\ \forall\delta\boldsymbol{q} \in \ker(\boldsymbol{\Phi}).\end{gathered} \tag{5.82}$$

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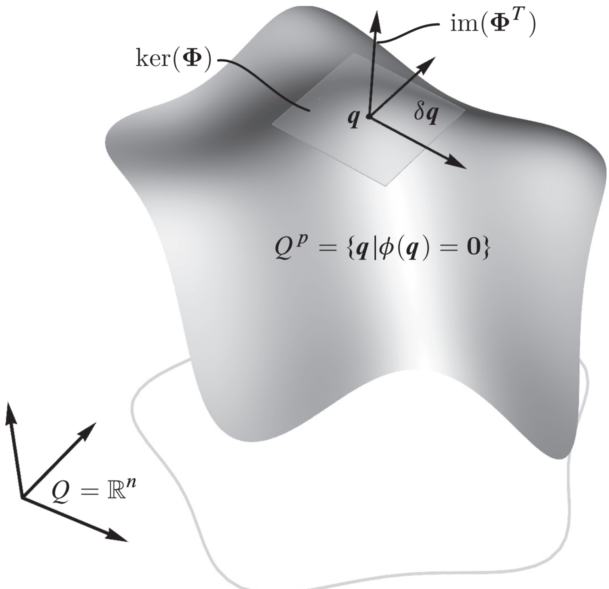

Figure 5.5 The configuration space constrained-motion manifold, Q^p. All constraint-consistent virtual variations, $\delta \boldsymbol{q}$, lie in the tangent space of Q^p and are orthogonal to the constraint forces.

We have

$$\begin{gathered}\boldsymbol{\tau}_C \perp \delta \boldsymbol{q} \\ \forall \delta \boldsymbol{q} \in \ker(\boldsymbol{\Phi}).\end{gathered} \tag{5.83}$$

The $\ker(\boldsymbol{\Phi})$ represents the tangent space of the constrained-motion manifold, Q^p, at a point, $\boldsymbol{q}$, in configuration space. The constraint-consistent virtual displacements, $\delta \boldsymbol{q}$, lie in this tangent space, and the generalized constraint forces, $\boldsymbol{\tau}_C$, are orthogonal to it. This is illustrated in Figure 5.5. Based on this, the following is implied:

$$\boldsymbol{\tau}_C \in \ker(\boldsymbol{\Phi})^{\perp} = \mathrm{im}(\boldsymbol{\Phi}^T). \tag{5.84}$$

Thus, the generalized constraint force, $\boldsymbol{\tau}_C$, can be represented as a linear combination of the columns of $\boldsymbol{\Phi}^T$. That is, $\boldsymbol{\tau}_C = \boldsymbol{\Phi}^T \boldsymbol{\lambda}$, where $\boldsymbol{\lambda}$ is a vector of unknown Lagrange multipliers. The term $\boldsymbol{\tau}_C \cdot \delta \boldsymbol{q}$ vanishes from (5.82) and we have the orthogonality relation

$$\begin{gathered}(\boldsymbol{M}\boldsymbol{q} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\tau}) \cdot \delta \boldsymbol{q} = 0 \\ \forall \delta \boldsymbol{q} \in \ker(\boldsymbol{\Phi}).\end{gathered} \tag{5.85}$$

The constrained multibody equations of motion are then

$$\boldsymbol{\tau} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\Phi}^T \boldsymbol{\lambda}, \tag{5.86}$$

subject to

$$\boldsymbol{\phi}(\boldsymbol{q}) = \boldsymbol{0} \qquad \Rightarrow \qquad \dot{\boldsymbol{\phi}} = \boldsymbol{0}, \ddot{\boldsymbol{\phi}} = \boldsymbol{0}. \tag{5.87}$$

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Solution of the Constrained Dynamics Problem

We can arrive at an explicit solution of the constrained dynamics problem,

$$\boldsymbol{M}\ddot{\boldsymbol{q}} = \boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g} + \boldsymbol{\Phi}^T \boldsymbol{\lambda} \tag{5.88}$$

$$\boldsymbol{\Phi}\ddot{\boldsymbol{q}} = -\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}}. \tag{5.89}$$

It is useful to define the $m_C \times m_C$ constraint space mass matrix, which reflects the system inertia projected at the constraint:

$$\mathbf{H} \triangleq (\boldsymbol{\Phi}\boldsymbol{M}^{-1}\boldsymbol{\Phi}^T)^{-1}. \tag{5.90}$$

We now express the mass-weighted (right) inverse of $\boldsymbol{\Phi}$:

$$\bar{\boldsymbol{\Phi}} = \boldsymbol{M}^{-1}\boldsymbol{\Phi}^T(\boldsymbol{\Phi}\boldsymbol{M}^{-1}\boldsymbol{\Phi}^T)^{-1}, \tag{5.91}$$

where $\boldsymbol{\Phi}\bar{\boldsymbol{\Phi}} = \mathbf{1}$ and, equivalently, $\bar{\boldsymbol{\Phi}}^T\boldsymbol{\Phi}^T = \mathbf{1}$. Defining the $n \times n$ constraint null space matrix, $\boldsymbol{\Theta} \triangleq \mathbf{1} - \bar{\boldsymbol{\Phi}}\boldsymbol{\Phi}$, we can solve the system. This yields

$$\ddot{\boldsymbol{q}} = -\bar{\boldsymbol{\Phi}}\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}} + \boldsymbol{\Theta}\boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g}) \tag{5.92}$$

$$\boldsymbol{\lambda} = -\bar{\boldsymbol{\Phi}}^T(\boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g}) - \mathbf{H}\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}}. \tag{5.93}$$

It is noted that $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}$ satisfy the condition $\boldsymbol{\Phi}\boldsymbol{\Theta} = \mathbf{0}$ and, equivalently, $\boldsymbol{\Theta}^T\boldsymbol{\Phi}^T = \mathbf{0}$. Furthermore, if we form the projection matrix $\boldsymbol{P}^T = \boldsymbol{P}$ which projects any vector in $\mathbb{R}^n$ onto the null space of $\boldsymbol{\Phi}$, we have

$$\boldsymbol{P}^T = \boldsymbol{W}\boldsymbol{W}^T = \mathbf{1} - \boldsymbol{\Phi}^T\boldsymbol{\Phi}^{+T} = \mathbf{1} - \boldsymbol{\Phi}^T(\boldsymbol{\Phi}\boldsymbol{\Phi}^T)^{-1}\boldsymbol{\Phi}, \tag{5.94}$$

where $\boldsymbol{W}$ spans the null space of $\boldsymbol{\Phi}$, and $\boldsymbol{\Phi}^+ = \boldsymbol{\Phi}^T(\boldsymbol{\Phi}\boldsymbol{\Phi}^T)^{-1}$ is the pseudoinverse (right inverse) of $\boldsymbol{\Phi}$. The expression for $\boldsymbol{P}^T$ in (5.94) has a similar form as the expression

$$\boldsymbol{\Theta}^T = \mathbf{1} - \boldsymbol{\Phi}^T\bar{\boldsymbol{\Phi}}^T = \mathbf{1} - \boldsymbol{\Phi}^T(\boldsymbol{\Phi}\boldsymbol{M}^{-1}\boldsymbol{\Phi}^T)^{-1}\boldsymbol{\Phi}\boldsymbol{M}^{-1}. \tag{5.95}$$

Consequently $\boldsymbol{P}^T = \boldsymbol{W}\boldsymbol{W}^T$ can be regarded as a *kinematic* constraint null space projection matrix and $\boldsymbol{\Theta}^T$ can be regarded as a *mass-weighted* constraint null space projection matrix. The physical and geometric meaning of $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}$ will be discussed further in Section 8.1.3.

Numerical Integration

In the previous section we solved the constrained dynamical system for the generalized accelerations and the Lagrange multipliers. In practice the integration of the forward dynamics would also require constraint stabilization to mitigate drift in the constraints. Baumgarte stabilization (Baumgarte 1972) involves replacing our original acceleration constraint equation with a linear combination of acceleration, velocity, and position constraint terms:

$$\ddot{\boldsymbol{\phi}} + \beta\dot{\boldsymbol{\phi}} + \alpha\boldsymbol{\phi} = \mathbf{0} \tag{5.96}$$

or

$$\boldsymbol{\Phi}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}} + \beta\boldsymbol{\Phi}\dot{\boldsymbol{q}} + \alpha\boldsymbol{\phi} = \mathbf{0}, \tag{5.97}$$

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Algorithm 4 Second-order method for integrating the zeroth-order holonomically constrained equations of motion

$\boldsymbol{q}_0 = \boldsymbol{q}_o$ {initialization}

$\dot{\boldsymbol{q}}_0 = \dot{\boldsymbol{q}}_o$ {initialization}

for $i = 0$ to $n_s - 1$ **do**

  $\ddot{\boldsymbol{q}}_i = -\bar{\boldsymbol{\Phi}}_i \dot{\boldsymbol{\Phi}}_i \dot{\boldsymbol{q}}_i + \boldsymbol{\Theta}_i \boldsymbol{M}_i^{-1}(\boldsymbol{\tau}_i - \boldsymbol{b}_i - \boldsymbol{g}_i) - \bar{\boldsymbol{\Phi}}_i(\alpha \boldsymbol{\phi}_i + \beta \boldsymbol{\Phi}_i \dot{\boldsymbol{q}}_i)$

  $\boldsymbol{\lambda}_i = -\bar{\boldsymbol{\Phi}}_i^T(\boldsymbol{\tau}_i - \boldsymbol{b}_i - \boldsymbol{g}_i) - \mathbf{H}_i \dot{\boldsymbol{\Phi}}_i \dot{\boldsymbol{q}}_i - \mathbf{H}_i(\alpha \boldsymbol{\phi}_i + \beta \boldsymbol{\Phi}_i \dot{\boldsymbol{q}}_i)$

  $\dot{\boldsymbol{q}}_{i+1} = \dot{\boldsymbol{q}}_i + \ddot{\boldsymbol{q}}_i \Delta t$

  $\boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \dot{\boldsymbol{q}}_i \Delta t + \frac{1}{2}\ddot{\boldsymbol{q}}_i \Delta t^2$

end for

where α and β are constant parameters chosen to drive the first- and zeroth-order derivatives of the holonomic constraint equations to *zero*, thereby compensating for position and velocity drift in the constraints. The constraint stabilized equations of motion are then

$$\begin{pmatrix} \boldsymbol{M} & -\boldsymbol{\Phi}^T \\ -\boldsymbol{\Phi} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g} \\ \dot{\boldsymbol{\Phi}} \dot{\boldsymbol{q}} + \beta \boldsymbol{\Phi} \dot{\boldsymbol{q}} + \alpha \boldsymbol{\phi} \end{pmatrix}, \tag{5.98}$$

and the solution of this system is

$$\ddot{\boldsymbol{q}} = -\bar{\boldsymbol{\Phi}} \dot{\boldsymbol{\Phi}} \dot{\boldsymbol{q}} + \boldsymbol{\Theta} \boldsymbol{M}^{-1}(\boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g}) - \bar{\boldsymbol{\Phi}}(\alpha \boldsymbol{\phi} + \beta \boldsymbol{\Phi} \dot{\boldsymbol{q}}) \tag{5.99}$$
$$\boldsymbol{\lambda} = -\bar{\boldsymbol{\Phi}}^T(\boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g}) - \mathbf{H} \dot{\boldsymbol{\Phi}} \dot{\boldsymbol{q}} - \mathbf{H}(\alpha \boldsymbol{\phi} + \beta \boldsymbol{\Phi} \dot{\boldsymbol{q}}). \tag{5.100}$$

The term $-\mathbf{H}(\alpha \boldsymbol{\phi} + \beta \boldsymbol{\Phi} \dot{\boldsymbol{q}})$ in the expression for $\boldsymbol{\lambda}$ can be physically interpreted as a corrective constraint force term used to compensate for any drift in the constraints. This is analogous to a proportional-derivative (PD) control law in a feedback system.

A second-order method for integrating this system can be summarized as shown in Algorithm 4.

Example: A Stewart platform parallel mechanism can be described by a set of 24 generalized coordinates, as shown in Figure 5.6.

The constraint equations associated with the loop closures are given by

$$\boldsymbol{r}_{l_i} = \boldsymbol{r}_{p_i} \qquad \text{for } i = 1, \cdots, 6, \tag{5.101}$$

where $\boldsymbol{r}_{l_i}$ is the terminal point of the ith strut subsystem which connects to $\boldsymbol{r}_{p_i}$, the ith position on the platform subsystem. In vector form we have

$$\boldsymbol{\phi} = \begin{pmatrix} \boldsymbol{r}_{l_1} - \boldsymbol{r}_{p_1} \\ \vdots \\ \boldsymbol{r}_{l_6} - \boldsymbol{r}_{p_6} \end{pmatrix} = \boldsymbol{0}. \tag{5.102}$$

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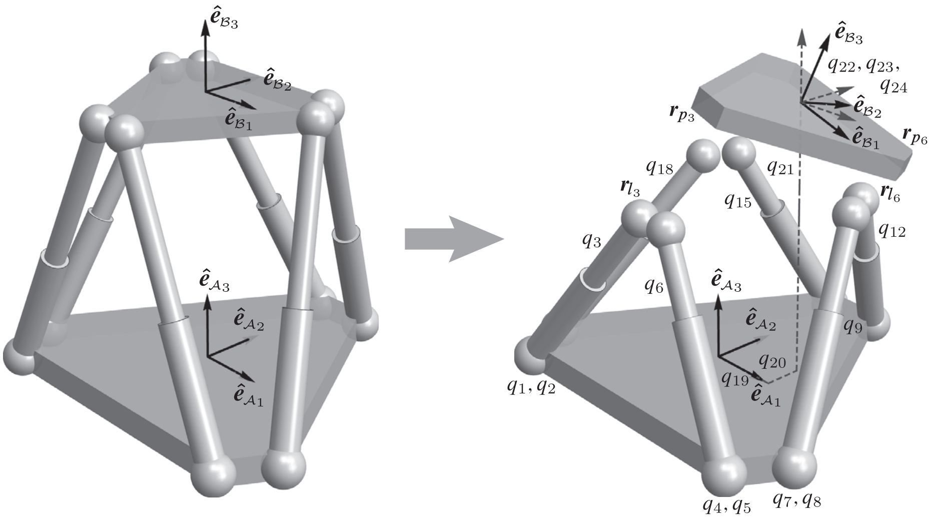

Figure 5.6 (Left) Stewart platform actuated by six prismatic struts (remaining joints are passive). (Right) The closed-loop mechanism is cut at various locations to create serial chains described by the set of generalized coordinates, $q_1, \ldots, q_{24}$.

Taking the derivative yields

$$\dot{\boldsymbol{\phi}} = \begin{pmatrix} \dot{\boldsymbol{r}}_{l_1} - \dot{\boldsymbol{r}}_{p_1} \\ \vdots \\ \dot{\boldsymbol{r}}_{l_6} - \dot{\boldsymbol{r}}_{p_6} \end{pmatrix} = \boldsymbol{\Phi}\dot{\boldsymbol{q}} = \boldsymbol{0}, \tag{5.103}$$

where

$$\dot{\boldsymbol{r}}_{l_i} = \boldsymbol{\Gamma}_{l_i} \begin{pmatrix} \dot{q}_{3i-2} \\ \dot{q}_{3i-1} \\ \dot{q}_{3i} \end{pmatrix} \quad \text{and,} \quad \dot{\boldsymbol{r}}_{p_i} = \boldsymbol{\Gamma}_{p_i} \begin{pmatrix} \dot{q}_{19} \\ \vdots \\ \dot{q}_{24} \end{pmatrix}, \tag{5.104}$$

$$\text{for } i = 1, \cdots, 6.$$

The terms, $\boldsymbol{\Gamma}_{l_i}$ and $\boldsymbol{\Gamma}_{p_i}$, are the corresponding Jacobians of $\boldsymbol{r}_{l_i}$ and $\boldsymbol{r}_{p_i}$, respectively. So,

$$\boldsymbol{\Phi}\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{\Gamma}_{l_1} & \ldots & \boldsymbol{0} & -\boldsymbol{\Gamma}_{p_1} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \ldots & \boldsymbol{\Gamma}_{l_6} & -\boldsymbol{\Gamma}_{p_6} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_{24} \end{pmatrix} = \boldsymbol{0}, \tag{5.105}$$

where

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Gamma}_{l_1} & \ldots & \boldsymbol{0} & -\boldsymbol{\Gamma}_{p_1} \\ \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \ldots & \boldsymbol{\Gamma}_{l_6} & -\boldsymbol{\Gamma}_{p_6} \end{pmatrix}. \tag{5.106}$$

The constraint forces, $\boldsymbol{\lambda}$, are shown in Figure 5.7.

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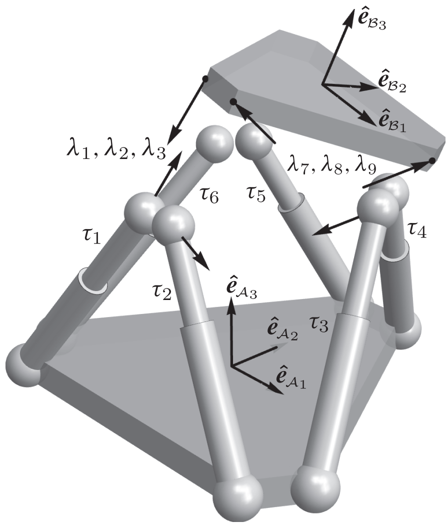

Figure 5.7 Constraint forces associated with loop closures. Lagrange multipliers, $\boldsymbol{\lambda}$, represent constraint forces at various locations between the serial chains to enforce the constraints associated with the loop closures.

The unconstrained equations of motion for the six strut serial subsystems are

$$\begin{pmatrix} 0 \\ 0 \\ \tau_1 \end{pmatrix} = \boldsymbol{M}_{l_1} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{pmatrix} + \boldsymbol{b}_{l_1}(q_1, \ldots, \dot{q}_3) + \boldsymbol{g}_{l_1}(q_1, \ldots, q_3), \tag{5.107}$$

$$\vdots$$

$$\begin{pmatrix} 0 \\ 0 \\ \tau_6 \end{pmatrix} = \boldsymbol{M}_{l_6} \begin{pmatrix} \ddot{q}_{16} \\ \ddot{q}_{17} \\ \ddot{q}_{18} \end{pmatrix} + \boldsymbol{b}_{l_6}(q_{16}, \ldots, \dot{q}_{18}) + \boldsymbol{g}_{l_6}(q_{16}, \ldots, q_{18}), \tag{5.108}$$

and for the platform subsystem we have

$$\mathbf{0} = \boldsymbol{M}_p \begin{pmatrix} \ddot{q}_{19} \\ \vdots \\ \ddot{q}_{24} \end{pmatrix} + \boldsymbol{b}_p(q_{19}, \ldots, \dot{q}_{24}) + \boldsymbol{g}_p(q_{19}, \ldots, q_{24}). \tag{5.109}$$

The entire unconstrained system is described by

$$\begin{pmatrix} \mathbf{0} \\ \tau_1 \\ \vdots \\ \mathbf{0} \\ \tau_6 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{l_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \boldsymbol{M}_{l_6} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{M}_p \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_{24} \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_{l_1} \\ \vdots \\ \boldsymbol{b}_{l_6} \\ \boldsymbol{b}_p \end{pmatrix} + \begin{pmatrix} \boldsymbol{g}_{l_1} \\ \vdots \\ \boldsymbol{g}_{l_6} \\ \boldsymbol{g}_p \end{pmatrix}, \tag{5.110}$$

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and the constrained system is

$$\begin{pmatrix} \mathbf{0} \\ \tau_1 \\ \vdots \\ \mathbf{0} \\ \tau_6 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{l_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \boldsymbol{M}_{l_6} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{M}_p \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_{24} \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_{l_1} \\ \vdots \\ \boldsymbol{b}_{l_6} \\ \boldsymbol{b}_p \end{pmatrix} + \begin{pmatrix} \boldsymbol{g}_{l_1} \\ \vdots \\ \boldsymbol{g}_{l_6} \\ \boldsymbol{g}_p \end{pmatrix}$$
$$+ \begin{pmatrix} \boldsymbol{\Gamma}_{l_1}^T & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \boldsymbol{\Gamma}_{l_6}^T \\ -\boldsymbol{\Gamma}_{p_1}^T & \cdots & -\boldsymbol{\Gamma}_{p_6}^T \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{18} \end{pmatrix}, \tag{5.111}$$

where the dimensions of the terms are,

$$\boldsymbol{M} \in \mathbb{R}^{n \times n}, \qquad \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{\tau} \in \mathbb{R}^n, \qquad \boldsymbol{\Phi} \in \mathbb{R}^{m \times n}, \qquad \boldsymbol{\lambda} \in \mathbb{R}^m, \tag{5.112}$$

and $n = 24$ and $m = 18$. The constrained system has $p = n - m = 6$ degrees of freedom.

Noting that the generalized constraint force $\boldsymbol{\tau}_C$ is given by

$$\boldsymbol{\tau}_C = \boldsymbol{\Phi}^T \boldsymbol{\lambda}, \tag{5.113}$$

the virtual work of the constraint forces is given by

$$\boldsymbol{\tau}_C \cdot \delta \boldsymbol{q} = \boldsymbol{\tau}_C^T \delta \boldsymbol{q} = \boldsymbol{\lambda}^T \boldsymbol{\Phi} \delta \boldsymbol{q}. \tag{5.114}$$

So,

$$\begin{gathered} \boldsymbol{\tau}_C \cdot \delta \boldsymbol{q} = \mathbf{0} \\ \forall \delta \boldsymbol{q} \in \delta Q^p, \end{gathered} \tag{5.115}$$

where $\delta Q^p = \ker(\boldsymbol{\Phi})$. Furthermore, we note that

$$\boldsymbol{\Phi} \delta \boldsymbol{q} = \begin{pmatrix} \delta \boldsymbol{r}_{l_1} - \delta \boldsymbol{r}_{p_1} \\ \vdots \\ \delta \boldsymbol{r}_{l_6} - \delta \boldsymbol{r}_{p_6} \end{pmatrix}. \tag{5.116}$$

So,

$$\boldsymbol{\tau}_C \cdot \delta \boldsymbol{q} = \boldsymbol{\lambda}^T \boldsymbol{\Phi} \delta \boldsymbol{q} = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{18} \end{pmatrix}^T \begin{pmatrix} \delta \boldsymbol{r}_{l_1} - \delta \boldsymbol{r}_{p_1} \\ \vdots \\ \delta \boldsymbol{r}_{l_6} - \delta \boldsymbol{r}_{p_6} \end{pmatrix}. \tag{5.117}$$

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Thus, the virtual work of the constraint forces can also be expressed by

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{18} \end{pmatrix} \cdot \begin{pmatrix} \delta \boldsymbol{r}_{l_1} - \delta \boldsymbol{r}_{p_1} \\ \vdots \\ \delta \boldsymbol{r}_{l_6} - \delta \boldsymbol{r}_{p_6} \end{pmatrix} = \mathbf{0}, \tag{5.118}$$

$$\forall \delta \boldsymbol{r}_{l_i}, \delta \boldsymbol{r}_{p_i} | \delta \boldsymbol{r}_{l_i} = \delta \boldsymbol{r}_{p_i}, \quad i = 1, \ldots, 6.$$

Example: A parallel mechanism is depicted in Figure 5.8. The constraint equations describe the loop closures and are given by

$$\boldsymbol{\phi}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{r}_{l_1} - \boldsymbol{r}_{p_1} \\ \boldsymbol{r}_{l_2} - \boldsymbol{r}_{p_2} \\ \boldsymbol{r}_{l_3} - \boldsymbol{r}_{p_3} \end{pmatrix}, \tag{5.119}$$

where $\boldsymbol{r}_{l_i}$ is the terminal point of the ith elbow chain subsystem which connects to $\boldsymbol{r}_{p_i}$, the ith position on the platform subsystem. Taking the derivative yields

$$\dot{\boldsymbol{\phi}} = \begin{pmatrix} \dot{\boldsymbol{r}}_{l_1} - \dot{\boldsymbol{r}}_{p_1} \\ \dot{\boldsymbol{r}}_{l_2} - \dot{\boldsymbol{r}}_{p_2} \\ \dot{\boldsymbol{r}}_{l_3} - \dot{\boldsymbol{r}}_{p_3} \end{pmatrix} = \boldsymbol{\Phi} \dot{\boldsymbol{q}} = \mathbf{0}, \tag{5.120}$$

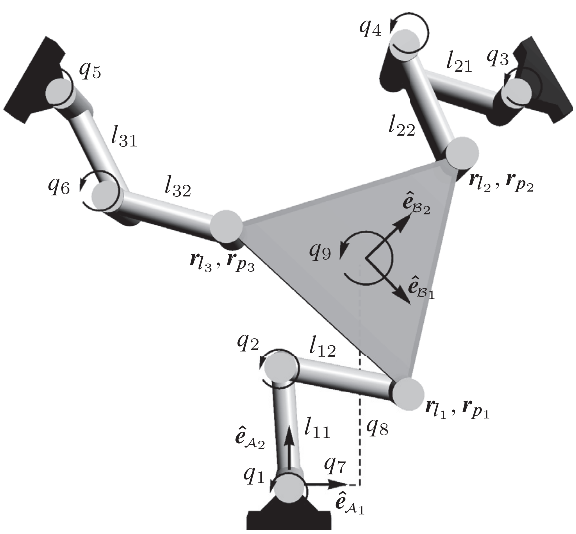

Figure 5.8 Parallel mechanism consisting of serial chains with loop closures. The three elbow joints are actuated, while the remaining joints are passive.

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where

$$\dot{\boldsymbol{r}}_{l_i} = \boldsymbol{\Gamma}_{l_i} \begin{pmatrix} \dot{q}_{2i-1} \\ \dot{q}_{2i} \end{pmatrix} \quad \text{and,} \quad \dot{\boldsymbol{r}}_{p_i} = \boldsymbol{\Gamma}_{p_i} \begin{pmatrix} \dot{q}_7 \\ \dot{q}_8 \\ \dot{q}_9 \end{pmatrix}, \qquad \text{for } i = 1, \ldots, 3. \tag{5.121}$$

The terms, $\boldsymbol{\Gamma}_{l_i}$ and $\boldsymbol{\Gamma}_{p_i}$ are the corresponding Jacobians of $\boldsymbol{r}_{l_i}$ and $\boldsymbol{r}_{p_i}$, respectively. So,

$$\boldsymbol{\Phi}\dot{\boldsymbol{q}} = \begin{pmatrix} \boldsymbol{\Gamma}_{l_1} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\Gamma}_{p_1} \\ \mathbf{0} & \boldsymbol{\Gamma}_{l_2} & \mathbf{0} & -\boldsymbol{\Gamma}_{p_2} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_{l_3} & -\boldsymbol{\Gamma}_{p_3} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_9 \end{pmatrix} = \mathbf{0}, \tag{5.122}$$

where

$$\boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Gamma}_{l_1} & \mathbf{0} & \mathbf{0} & -\boldsymbol{\Gamma}_{p_1} \\ \mathbf{0} & \boldsymbol{\Gamma}_{l_2} & \mathbf{0} & -\boldsymbol{\Gamma}_{p_2} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_{l_3} & -\boldsymbol{\Gamma}_{p_3} \end{pmatrix}. \tag{5.123}$$

For the three elbow link chains we compute

$$\boldsymbol{\Gamma}_{l_i} = \begin{pmatrix} -l_{i1}\sin(q_{2i-1}) - l_{i2}\sin(q_{2i-1}+q_{2i}) & -l_{i2}\sin(q_{2i-1}+q_{2i}) \\ l_{i1}\cos(q_{2i-1}) + l_{i2}\cos(q_{2i-1}+q_{2i}) & l_{i2}\cos(q_{2i-1}+q_{2i}) \end{pmatrix}, \qquad \text{for } i = 1, \ldots, 3, \tag{5.124}$$

and for the platform we compute

$$\boldsymbol{\Gamma}_{p_1} = \begin{pmatrix} -1 & 0 & -1.08\sin(0.47 - q_9) \\ 0 & -1 & -1.08\cos(0.47 - q_9) \end{pmatrix}, \tag{5.125}$$

$$\boldsymbol{\Gamma}_{p_2} = \begin{pmatrix} -1 & 0 & 1.08\sin(1.62 + q_9) \\ 0 & -1 & -1.08\cos(1.62 + q_9) \end{pmatrix}, \tag{5.126}$$

$$\boldsymbol{\Gamma}_{p_3} = \begin{pmatrix} -1 & 0 & 1.08\sin(3.72 + q_9) \\ 0 & -1 & -1.08\cos(3.72 + q_9) \end{pmatrix}, \tag{5.127}$$

$$\text{for } i = 1, \ldots, 3.$$

For the six elbow link centers of mass (one proximal and one distal link for each chain) we compute

$$\boldsymbol{\Gamma}_{G_{l_{i1}}} = \begin{pmatrix} -\frac{l_{i1}}{2}\sin(q_{2i-1}) & 0 \\ \frac{l_{i1}}{2}\cos(q_{2i-1}) & 0 \end{pmatrix}, \qquad \text{for } i = 1, \ldots, 3, \tag{5.128}$$

and

$$\boldsymbol{\Gamma}_{G_{l_{i2}}} = \begin{pmatrix} -l_{i1}\sin(q_{2i-1}) - \frac{l_{i2}}{2}\sin(q_{2i-1}+q_{2i}) & -\frac{l_{i2}}{2}\sin(q_{2i-1}+q_{2i}) \\ l_{i1}\cos(q_{2i-1}) + \frac{l_{i2}}{2}\cos(q_{2i-1}+q_{2i}) & \frac{l_{i2}}{2}\cos(q_{2i-1}+q_{2i}) \end{pmatrix}, \qquad \text{for } i = 1, \ldots, 3, \tag{5.129}$$

The terms, $\boldsymbol{\Gamma}_{G_{l_{ij}}}$ and l_{ij}, are the center of mass Jacobian and link length, respectively, of the jth link of the ith elbow chain subsystem. The unconstrained equations of motion for the three elbow chain subsystems are

$$\begin{pmatrix} 0 \\ \tau_i \end{pmatrix} = \boldsymbol{M}_{l_i} \begin{pmatrix} \ddot{q}_{2i-1} \\ \ddot{q}_{2i} \end{pmatrix} + \boldsymbol{b}_{l_i}(q_{2i-1}, \ldots, \dot{q}_{2i}) + \boldsymbol{g}_{l_1}(q_{2i-1}, q_{2i}), \quad \text{for } i = 1, \ldots, 3, \tag{5.130}$$

where, taking the rotational inertia of the individual links to be *zero*, we have

$$\boldsymbol{M}_{l_i} = \sum_{j=1}^{2} M_{l_{ij}} \boldsymbol{\Gamma}^T_{G_{l_{ij}}} \boldsymbol{\Gamma}_{G_{l_{ij}}}, \tag{5.131}$$

$$\boldsymbol{b}_{l_i} = \left(\sum_{j=1}^{2} M_{l_{ij}} \boldsymbol{\Gamma}^T_{G_{l_{ij}}} \dot{\boldsymbol{\Gamma}}_{G_{l_{ij}}} \right) \dot{\boldsymbol{q}}, \tag{5.132}$$

$$\boldsymbol{g}_{l_i} = g \sum_{j=1}^{2} M_{l_{ij}} \boldsymbol{\Gamma}^T_{G_{l_{ij}}} \hat{\boldsymbol{e}}_2, \quad \text{for } i = 1, \ldots, 3. \tag{5.133}$$

The term, $M_{l_{ij}}$, is the mass of the jth link of the ith elbow chain subsystem. The terms are computed as

$$\boldsymbol{M}_{l_i} = \begin{pmatrix} \frac{1}{4}[l_{i2}^2 M_{l_{i2}} + l_{i1}^2(M_{l_{i1}} + 4M_{l_{i2}}) + 4 l_{i1} l_{i2} M_{l_{i2}} \cos(q_{2i})] & \frac{1}{4} l_{i2} M_{l_{i2}} [l_{i2} + 2 l_{i1} \cos(q_{2i})] \\ \frac{1}{4} l_{i2} M_{l_{i2}} [l_{i2} + 2 l_{i1} \cos(q_{2i})] & \frac{1}{4} l_{i2}^2 M_{l_{i2}} \end{pmatrix}, \tag{5.134}$$

$$\boldsymbol{b}_{l_i} = \begin{pmatrix} -\frac{1}{2} l_{i1} l_{i2} M_{l_{i2}} \sin(q_{2i}) \dot{q}_{2i} (2\dot{q}_{2i-1} + \dot{q}_{2i}) \\ \frac{1}{2} l_{i1} l_{i2} M_{l_{i2}} \sin(q_{2i}) \dot{q}_{2i-1}^2 \end{pmatrix}, \tag{5.135}$$

$$\boldsymbol{g}_{l_i} = \begin{pmatrix} \frac{1}{2} g [l_{i1}(M_{l_{i1}} + 2M_{l_{i2}}) \cos(q_{2i-1}) + l_{i2} M_{l_{i2}} \cos(q_{2i-1} + q_{2i})] \\ \frac{1}{2} g l_{i2} M_{l_{i2}} \cos(q_{2i-1} + q_{2i}) \end{pmatrix}, \quad \text{for } i = 1, \ldots, 3. \tag{5.136}$$

For the platform subsystem we have

$$\boldsymbol{0} = \boldsymbol{M}_p \begin{pmatrix} \ddot{q}_7 \\ \ddot{q}_8 \\ \ddot{q}_9 \end{pmatrix} + \boldsymbol{b}_p(q_7, \ldots, \dot{q}_9) + \boldsymbol{g}_p(q_7, q_8, q_9), \tag{5.137}$$

where

$$\boldsymbol{M}_p = M_p \boldsymbol{\Gamma}^T_{G_p} \boldsymbol{\Gamma}_{G_p} + I_p^{G_p} \boldsymbol{\Pi}_p^T \boldsymbol{\Pi}_p, \tag{5.138}$$

$$\boldsymbol{b}_p = \left(M_p \boldsymbol{\Gamma}^T_{G_p} \dot{\boldsymbol{\Gamma}}_{G_p} + I_p^{G_p} \boldsymbol{\Pi}_p^T \dot{\boldsymbol{\Pi}}_p \right) \dot{\boldsymbol{q}}, \tag{5.139}$$

$$\boldsymbol{g}_p = g M_p \boldsymbol{\Gamma}^T_{G_p} \hat{\boldsymbol{e}}_2. \tag{5.140}$$

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The Jacobians are simply $\boldsymbol{\Gamma}_{G_p} = \mathbf{1}_{2\times 2}$ and $\boldsymbol{\Pi}_p = 1$. The dynamical terms are then computed as

$$\boldsymbol{M}_p = \begin{pmatrix} M_p & 0 & 0 \\ 0 & M_p & 0 \\ 0 & 0 & I_p^{G_p} \end{pmatrix}, \tag{5.141}$$

$$\boldsymbol{b}_p = \mathbf{0}, \tag{5.142}$$

$$\boldsymbol{g}_p = \begin{pmatrix} 0 \\ M_p g \\ 0 \end{pmatrix}. \tag{5.143}$$

The entire unconstrained system is composed as

$$\begin{pmatrix} 0 \\ \tau_1 \\ 0 \\ \tau_2 \\ 0 \\ \tau_3 \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}_{l_2} & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}_{l_3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M}_p \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_9 \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_{l_1} \\ \boldsymbol{b}_{l_2} \\ \boldsymbol{b}_{l_3} \\ \boldsymbol{b}_p \end{pmatrix} + \begin{pmatrix} \boldsymbol{g}_{l_1} \\ \boldsymbol{g}_{l_2} \\ \boldsymbol{g}_{l_3} \\ \boldsymbol{g}_p \end{pmatrix}, \tag{5.144}$$

and the constrained system is

$$\begin{aligned} \begin{pmatrix} 0 \\ \tau_1 \\ 0 \\ \tau_2 \\ 0 \\ \tau_3 \\ \mathbf{0} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{M}_{l_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}_{l_2} & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & \boldsymbol{M}_{l_3} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{M}_p \end{pmatrix} \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_9 \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_{l_1} \\ \boldsymbol{b}_{l_2} \\ \boldsymbol{b}_{l_3} \\ \boldsymbol{b}_p \end{pmatrix} + \begin{pmatrix} \boldsymbol{g}_{l_1} \\ \boldsymbol{g}_{l_2} \\ \boldsymbol{g}_{l_3} \\ \boldsymbol{g}_p \end{pmatrix} \\ &+ \begin{pmatrix} \boldsymbol{\Gamma}_{l_1}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}_{l_2}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Gamma}_{l_3}^T \\ -\boldsymbol{\Gamma}_{p_1}^T & -\boldsymbol{\Gamma}_{p_2}^T & -\boldsymbol{\Gamma}_{p_3}^T \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_6 \end{pmatrix}, \end{aligned} \tag{5.145}$$

where the dimensions of the terms are

$$\boldsymbol{M} \in \mathbb{R}^{n\times n}, \qquad \boldsymbol{b}, \boldsymbol{g}, \boldsymbol{\tau} \in \mathbb{R}^n, \qquad \boldsymbol{\Phi} \in \mathbb{R}^{m\times n}, \qquad \boldsymbol{\lambda} \in \mathbb{R}^m, \tag{5.146}$$

and $n = 9$ and $m = 6$. The constrained system has $p = n - m = 3$ degrees of freedom.

Results from simulation of the system under gravity are shown in Figures 5.9 and 5.10. All geometric and inertial constants were chosen to be 1:

$$\begin{gathered} l_{ij} = 1, \; M_{ij} = 1, \; M_p = 1, \; I_p^{G_p} = 1, \\ \text{for } i = 1, \ldots, 3, \quad \text{and, } j = 1, 2. \end{gathered} \tag{5.147}$$

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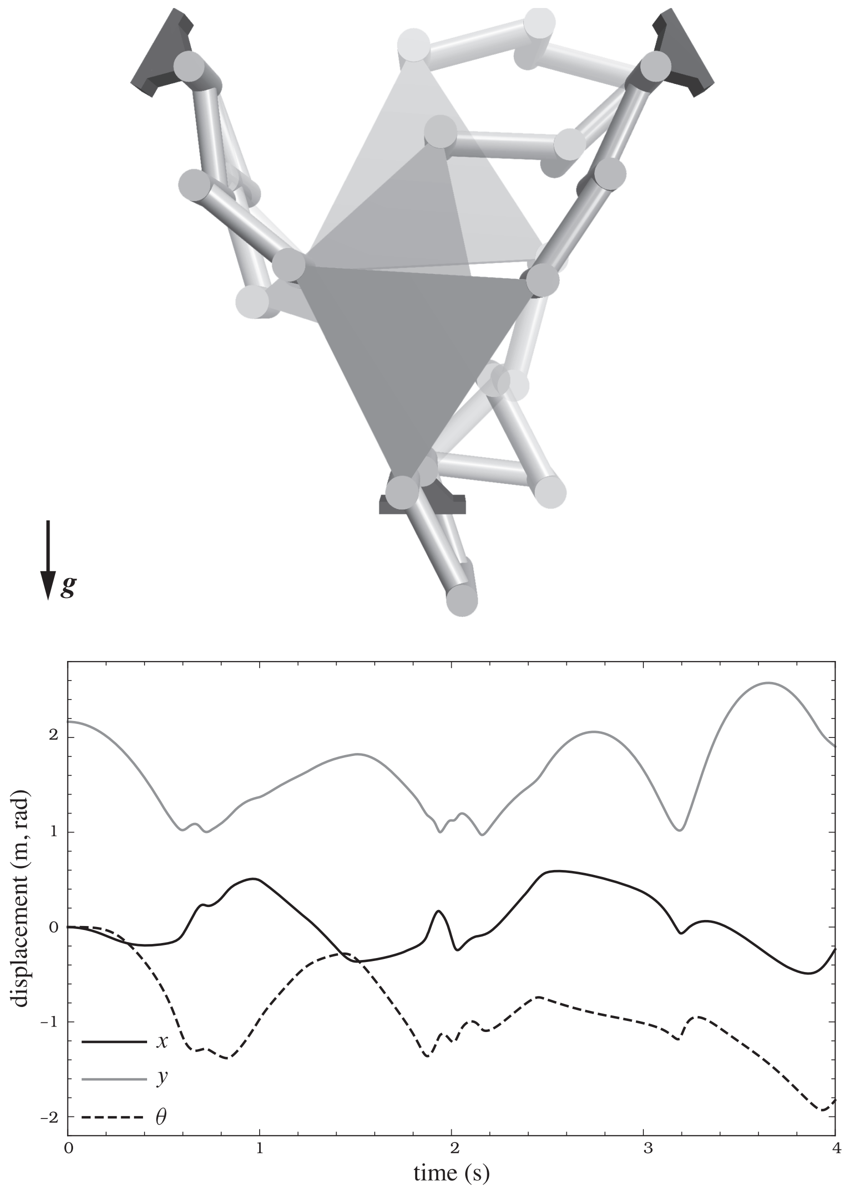

Figure 5.9 (Top) Animation frames from the simulation of the parallel mechanism falling under gravity. (Bottom) Time history of the motion of the platform (x, y, θ).

The initial conditions used were

$$\boldsymbol{q}_o = \begin{pmatrix} \pi/4 & \pi/6 & 11\pi/12 & \pi/6 & -5\pi/12 & \pi/6 & 0 & 2.165 & 0 \end{pmatrix}^T \tag{5.148}$$

$$\dot{\boldsymbol{q}}_o = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T. \tag{5.149}$$

It can be verified that these satisfy the constraints.

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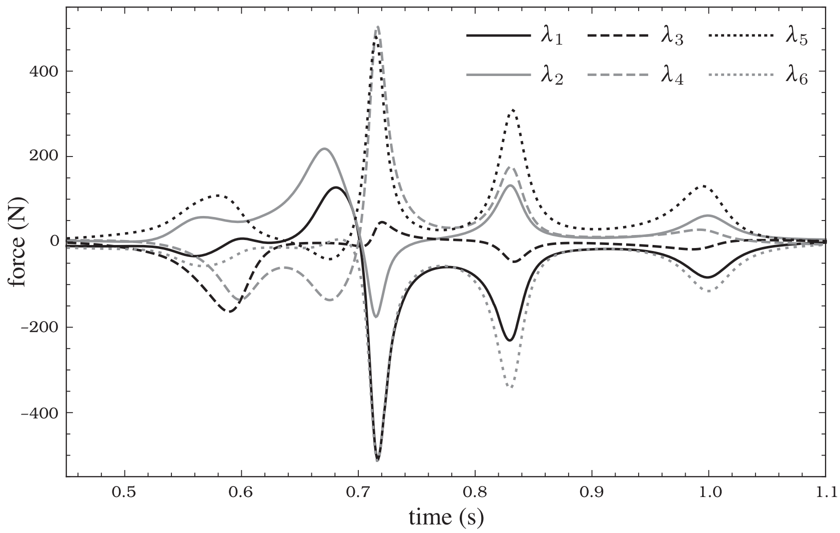

Figure 5.10 Lagrange multipliers (constraint forces) for the parallel mechanism falling under gravity.

5.2.6 Generalized Constrained Equation of Motion

Given the explicit solution of the constrained dynamics problem from Section 5.2.5, we wish to express an alternate form of the constrained dynamical equations of motion. We begin by recalling (5.93):

$$\boldsymbol{\lambda} = -\bar{\boldsymbol{\Phi}}^T(\boldsymbol{\tau} - \boldsymbol{b} - \boldsymbol{g}) - \mathbf{H}\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}}. \tag{5.150}$$

Substituting (5.150) into (5.86) yields

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} = -\boldsymbol{\Phi}^T\mathbf{H}\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}} + (\mathbf{1} - \boldsymbol{\Phi}^T\bar{\boldsymbol{\Phi}}^T)\boldsymbol{\tau} + \boldsymbol{\Phi}^T\bar{\boldsymbol{\Phi}}^T(\boldsymbol{b} + \boldsymbol{g}). \tag{5.151}$$

We now define the $m_C \times 1$ vector of centrifugal and Coriolis forces projected at the constraint,

$$\boldsymbol{\alpha} \triangleq \bar{\boldsymbol{\Phi}}^T\boldsymbol{b} - \mathbf{H}\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}}, \tag{5.152}$$

and the $m_C \times 1$ vector of gravity forces projected at the constraint,

$$\boldsymbol{\rho} \triangleq \bar{\boldsymbol{\Phi}}^T\boldsymbol{g}. \tag{5.153}$$

Thus, we have the concise expression, which we will refer to as the *generalized constrained equation of motion* (De Sapio, Khatib, and Delp 2006):

$$\boldsymbol{\Theta}^T\boldsymbol{\tau} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\Phi}^T(\boldsymbol{\alpha} + \boldsymbol{\rho}). \tag{5.154}$$

An alternative means of deriving this equation involves directly mapping the configuration space equation (5.86) into the constraint null space using $\boldsymbol{\Theta}^T$,

$$\boldsymbol{\Theta}^T\boldsymbol{\tau} = \boldsymbol{\Theta}^T\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{\Theta}^T\boldsymbol{b} + \boldsymbol{\Theta}^T\boldsymbol{g} - \boldsymbol{\Theta}^T\boldsymbol{\Phi}^T\boldsymbol{\lambda}. \tag{5.155}$$

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Noting that $\boldsymbol{\Theta}^T \boldsymbol{\Phi}^T = \mathbf{0}$ and manipulating, we have

$$\begin{aligned}\boldsymbol{\Theta}^T \boldsymbol{\tau} &= \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\Phi}^T \bar{\boldsymbol{\Phi}}^T \boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{\Phi}^T \bar{\boldsymbol{\Phi}}^T \boldsymbol{b} - \boldsymbol{\Phi}^T \bar{\boldsymbol{\Phi}}^T \boldsymbol{g} \\ &= \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\Phi}^T \mathbf{H}\boldsymbol{\Phi}\ddot{\boldsymbol{q}} - \boldsymbol{\Phi}^T(\boldsymbol{p} + \boldsymbol{\alpha}) - \boldsymbol{\Phi}^T \mathbf{H}\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}}.\end{aligned} \tag{5.156}$$

Substituting in our constraint condition, $\dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}} = -\boldsymbol{\Phi}\ddot{\boldsymbol{q}}$, yields

$$\boldsymbol{\Theta}^T \boldsymbol{\tau} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\Phi}^T(\boldsymbol{p} + \boldsymbol{\alpha}). \tag{5.157}$$

5.3 Hamilton’s Principle of Least Action

An alternate way of arriving at the the equations of motion is through a least action principle. The fundamental theme associated with least action principles is that the evolution of a dynamical system can be revealed by examining the stationary condition for an appropriately defined action integral.

While least action can refer to a general family of variational principles, perhaps the most significant among these is Hamilton’s *Principle of Least Action*. This principle states that the path, $\boldsymbol{q}(t)$, of a system in configuration space over an interval, $[t_o, t_f]$, is such that the action is *stationary* under all path variations that vanish at the endpoints, $\boldsymbol{q}(t_o)$ and $\boldsymbol{q}(t_f)$. It is noted that this does not strictly imply a minimization of the action, as the name of the principle suggests, but rather an extremization of the action.

PRINCIPLE 5.2 *For scleronomic systems (no explicit time dependence) the path of a system in configuration space during an interval, $[t_o, t_f]$, is such that the action*

$$I = \int_{t_o}^{t_f} \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})dt \tag{5.158}$$

is stationary under all path variations. The scalar term, $\mathcal{L}$, is the Lagrangian, defined as

$$\mathcal{L} \triangleq T - V, \tag{5.159}$$

where T and V are the system kinetic and potential energies, respectively. Furthermore

$$\begin{gathered}\delta I = 0 \\ \forall \delta | \delta \boldsymbol{q}(t_o) = \delta \boldsymbol{q}(t_f) = \mathbf{0}.\end{gathered} \tag{5.160}$$

*This is known as **Hamilton’s Principle of Least Action**.*

For forced systems the Principle of Least Action is modified such that the variation in the action is given by

$$\delta I = \delta \int_{t_o}^{t_f} \mathcal{L}\, dt + \int_{t_o}^{t_f} \boldsymbol{\tau} \cdot \delta \boldsymbol{q}\, dt. \tag{5.161}$$

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5.3.1 Euler-Lagrange Equations

It is straightforward to apply calculus of variations to this problem (Goldstein, Poole, and Safko 2002). We can express the first term in (5.161) as

$$\delta \int_{t_o}^{t_f} \mathcal{L}\,dt = \int_{t_o}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \dot{\boldsymbol{q}} \right) dt = \int_{t_o}^{t_f} \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q}\,dt + \int_{t_o}^{t_f} \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d(\delta \boldsymbol{q})}{dt}\,dt. \tag{5.162}$$

Noting that

$$\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d(\delta \boldsymbol{q})}{dt} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \right) - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q}, \tag{5.163}$$

we have

$$\begin{aligned} \int_{t_o}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d(\delta \boldsymbol{q})}{dt} \right) dt &= \int_{t_o}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \right) dt \\ &\quad + \int_{t_o}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \cdot \delta \boldsymbol{q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \right) dt. \end{aligned} \tag{5.164}$$

Since the variations, $\delta \boldsymbol{q}$, vanish at the endpoints,

$$\int_{t_o}^{t_f} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \right) dt = \left. \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot \delta \boldsymbol{q} \right|_{t_o}^{t_f} = 0. \tag{5.165}$$

Thus,

$$\delta \int_{t_o}^{t_f} \mathcal{L}\,dt = \int_{t_o}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \right) \cdot \delta \boldsymbol{q}\,dt \tag{5.166}$$

and

$$\delta I = \int_{t_o}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} + \boldsymbol{\tau} \right) \cdot \delta \boldsymbol{q}\,dt. \tag{5.167}$$

The condition

$$\begin{gathered} \delta I = 0 \\ \forall \delta \boldsymbol{q}, \end{gathered} \tag{5.168}$$

implies the following *Euler-Lagrange* equations:

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}. \tag{5.169}$$

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We can express the Euler-Lagrange equations as

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}}. \tag{5.170}$$

5.3.2 A Single Particle

For a single point mass with a discrete set of n_f external forces, $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n_f}\}$, acting on it, the Euler-Lagrange equations are

$$\sum_{i=1}^{n_f} \boldsymbol{f}_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{r}}} - \frac{\partial T}{\partial \boldsymbol{r}} + \frac{\partial V}{\partial \boldsymbol{r}}. \tag{5.171}$$

The kinetic energy, T, is given by

$$T = \frac{1}{2}M\boldsymbol{v}^T\boldsymbol{v} = \frac{1}{2}M\dot{\boldsymbol{r}}^T\dot{\boldsymbol{r}}. \tag{5.172}$$

So,

$$\frac{\partial T}{\partial \dot{\boldsymbol{r}}} = M\dot{\boldsymbol{r}} \tag{5.173}$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{r}}} = M\ddot{\boldsymbol{r}} = M\boldsymbol{a}. \tag{5.174}$$

We note that

$$\frac{\partial T}{\partial \boldsymbol{r}} = 0. \tag{5.175}$$

The potential energy, V, is given by

$$V = M\left\langle \boldsymbol{r}, g\hat{\boldsymbol{e}}_3 \right\rangle = Mg\boldsymbol{r}^T\hat{\boldsymbol{e}}_3. \tag{5.176}$$

So,

$$\frac{\partial V}{\partial \boldsymbol{r}} = Mg\hat{\boldsymbol{e}}_3. \tag{5.177}$$

Thus,

$$\sum_{i=1}^{n_f} \boldsymbol{f}_i = M\boldsymbol{a} + Mg\hat{\boldsymbol{e}}_3. \tag{5.178}$$

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5.3.3 A Single Rigid Body

For a single rigid body with a discrete set of n_f external forces, $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_{n_f}\}$, and n_φ external moments, $\{\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_{n_\varphi}\}$, acting on it, the Euler-Lagrange equations are

$$\begin{pmatrix} \sum_{i=1}^{n_f} \boldsymbol{f}_i \\ \sum_{i=1}^{n_f} \boldsymbol{d}_{\overrightarrow{GP_i}} \times \boldsymbol{f}_i + \sum_{i=1}^{n_\varphi} \boldsymbol{\varphi}_j \end{pmatrix} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{x}}} - \frac{\partial T}{\partial \mathbf{x}} + \frac{\partial V}{\partial \mathbf{x}}, \tag{5.179}$$

where

$$\mathbf{x} = \begin{pmatrix} \boldsymbol{r}_G \\ \boldsymbol{\theta} \end{pmatrix} \quad \text{and} \quad \dot{\mathbf{x}} = \begin{pmatrix} \boldsymbol{v}_G \\ \boldsymbol{\omega} \end{pmatrix}. \tag{5.180}$$

The kinetic energy, T, is given by

$$T = \frac{1}{2}(M\boldsymbol{v}_G^T\boldsymbol{v}_G + \boldsymbol{\omega}^T\boldsymbol{I}^G\boldsymbol{\omega}). \tag{5.181}$$

So,

$$\frac{\partial T}{\partial \dot{\mathbf{x}}} = \begin{pmatrix} M\boldsymbol{v}_G \\ \boldsymbol{I}^G\boldsymbol{\omega} \end{pmatrix} \tag{5.182}$$

and

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{x}}} = \begin{pmatrix} M\boldsymbol{a}_G \\ \boldsymbol{I}^G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{I}^G\boldsymbol{\omega} \end{pmatrix}. \tag{5.183}$$

The potential energy, V, is given by

$$V = M\,\langle \boldsymbol{r}, g\hat{\boldsymbol{e}}_3 \rangle = Mg\boldsymbol{r}^T\hat{\boldsymbol{e}}_3. \tag{5.184}$$

So,

$$\frac{\partial V}{\partial \mathbf{x}} = \begin{pmatrix} Mg\hat{\boldsymbol{e}}_3 \\ \mathbf{0} \end{pmatrix}. \tag{5.185}$$

Thus,

$$\begin{pmatrix} \sum_{i=1}^{n_f} \boldsymbol{f}_i \\ \sum_{i=1}^{n_f} \boldsymbol{d}_{\overrightarrow{GP_i}} \times \boldsymbol{f}_i + \sum_{i=1}^{n_\varphi} \boldsymbol{\varphi}_j \end{pmatrix} = \begin{pmatrix} M\boldsymbol{a}_G + Mg\hat{\boldsymbol{e}}_3 \\ \boldsymbol{I}^G\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{I}^G\boldsymbol{\omega} \end{pmatrix}. \tag{5.186}$$

The Kinetic Energy Ellipsoid

Rotational kinetic energy is given by

$$T = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{I}^G\boldsymbol{\omega}. \tag{5.187}$$

We will assume this expression is represented in the base frame for convenience. Since $\boldsymbol{I}^G$ is symmetric positive definite, it has positive eigenvalues and an orthogonal

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eigenbasis (principal axes), $\mathcal{E}$. Therefore,

$$T = \frac{1}{2} {}^{\mathcal{O}}\boldsymbol{\omega}^{T\,\mathcal{O}}\boldsymbol{I}^{G\,\mathcal{O}}\boldsymbol{\omega} = \frac{1}{2} {}^{\mathcal{E}}\boldsymbol{\omega}^{T\,\mathcal{O}}_{\;\mathcal{E}}\boldsymbol{Q}^{T\,\mathcal{O}}\boldsymbol{I}^{G\,\mathcal{O}}_{\;\mathcal{E}}\boldsymbol{Q}^{\mathcal{E}}\boldsymbol{\omega}, \tag{5.188}$$

where the columns of ${}^{\mathcal{O}}_{\mathcal{E}}\boldsymbol{Q}$ are the eigenvectors, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, of ${}^{\mathcal{O}}\boldsymbol{I}^G$,

$${}^{\mathcal{O}}_{\mathcal{E}}\boldsymbol{Q} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}. \tag{5.189}$$

We then have

$${}^{\mathcal{O}}_{\mathcal{E}}\boldsymbol{Q}^{T\,\mathcal{O}}\boldsymbol{I}^{G\,\mathcal{O}}_{\;\mathcal{E}}\boldsymbol{Q} = {}^{\mathcal{E}}\boldsymbol{I}^G, \tag{5.190}$$

where ${}^{\mathcal{E}}\boldsymbol{I}^G$ is a diagonal matrix of eigenvalues, $\{\lambda_1, \lambda_2, \lambda_3\}$, of ${}^{\mathcal{O}}\boldsymbol{I}^G$:

$${}^{\mathcal{E}}\boldsymbol{I}^G = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{5.191}$$

In terms of the eigenbasis,

$$T = \frac{1}{2} {}^{\mathcal{E}}\boldsymbol{\omega}^{T\,\mathcal{E}}\boldsymbol{I}^{G\,\mathcal{E}}\boldsymbol{\omega}. \tag{5.192}$$

For constant values of the kinetic energy T, this represents an ellipsoid expressed with respect to the principal axes. In scalar form,

$$T = \frac{1}{2}\left(\lambda_1 {}^{\mathcal{E}}\omega_1^2 + \lambda_2 {}^{\mathcal{E}}\omega_2^2 + \lambda_3 {}^{\mathcal{E}}\omega_3^2\right). \tag{5.193}$$

The semiaxes of the ellipsoid in frame $\mathcal{E}$ are then

$$a_i = \sqrt{2T/\lambda_i}. \tag{5.194}$$

Example: We consider the cone addressed in Section 4.1.2. The inertia tensor about the center of mass is

$$\boldsymbol{I}^G = \begin{pmatrix} \frac{3}{80}M(h^2 + 4R^2) & 0 & 0 \\ 0 & \frac{3}{80}M(h^2 + 4R^2) & 0 \\ 0 & 0 & \frac{3}{10}R^2 \end{pmatrix}. \tag{5.195}$$

Specifying $M = 1$, $h = 1$, and $R = .75$, the eigenvalues are $\lambda_1 = 0.684$, $\lambda_2 = 0.684$, and $\lambda_3 = 0.169$. The ellipsoid associated with a constant kinetic energy, $T = 1$, has semiaxes $a_1 = 1.71$, $a_2 = 1.71$, and $a_3 = 3.443$, aligned with the cone axes depicted in Figure 4.3.

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5.3.4 A System of Particles

For a system of particles with generalized forces, $\boldsymbol{\tau}$, acting on the system the Euler-Lagrange equations are

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}}. \tag{5.196}$$

The kinetic energy, T, is given by

$$T = \frac{1}{2}\sum_{i=1}^{n_p} M_i \boldsymbol{v}_i^T \boldsymbol{v}_i = \frac{1}{2}\dot{\boldsymbol{q}}^T \left(\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i\right)\dot{\boldsymbol{q}}. \tag{5.197}$$

We note that

$$\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = \left(\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i\right)\dot{\boldsymbol{q}}. \tag{5.198}$$

So,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = \left(\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i\right)\ddot{\boldsymbol{q}}. \tag{5.199}$$

The potential energy, V, is given by

$$V = \sum_{i=1}^{n_p} M_i \langle \boldsymbol{r}_i, g\hat{\boldsymbol{e}}_3\rangle = g\sum_{i=1}^{n_p} M_i \boldsymbol{r}_i^T \hat{\boldsymbol{e}}_3. \tag{5.200}$$

So

$$\frac{\partial V}{\partial \boldsymbol{q}} = g\sum_{i=1}^{n_p} M_i \left(\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{q}}\right)^T \hat{\boldsymbol{e}}_3 = g\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \hat{\boldsymbol{e}}_3 \tag{5.201}$$

and

$$\boldsymbol{\tau} = \sum_{i=1}^{n_p} \boldsymbol{\Gamma}_i^T (M_i \boldsymbol{a}_i + M_i g \hat{\boldsymbol{e}}_3). \tag{5.202}$$

Thus,

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = \left(\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i\right)\ddot{\boldsymbol{q}} + g\sum_{i=1}^{n_q} M_i \boldsymbol{\Gamma}_i^T \hat{\boldsymbol{e}}_3. \tag{5.203}$$

For a system of particles, we had previously defined

$$\boldsymbol{M}(\boldsymbol{q}) \triangleq \sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \boldsymbol{\Gamma}_i \tag{5.204}$$

$$\boldsymbol{g}(\boldsymbol{q}) \triangleq g\sum_{i=1}^{n_p} M_i \boldsymbol{\Gamma}_i^T \hat{\boldsymbol{e}}_3. \tag{5.205}$$

So,

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}). \tag{5.206}$$

5.3.5 A System of Rigid Bodies

For a system of rigid bodies with generalized forces, $\boldsymbol{\tau}$, acting on the system, the Euler-Lagrange equations are

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}}. \tag{5.207}$$

The kinetic energy, T, is given by

$$\begin{aligned} T &= \frac{1}{2}\sum_{i=1}^{n_b}\left(M_i {}^i\boldsymbol{v}_{G_i}^T {}^i\boldsymbol{v}_{G_i} + {}^i\boldsymbol{\omega}_i^T {}^i\boldsymbol{I}_i^{G_i} {}^i\boldsymbol{\omega}_i\right) \\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T\left[\sum_{i=1}^{n_b}\left(M_i {}^i\boldsymbol{\Gamma}_{G_i}^T {}^i\boldsymbol{\Gamma}_{G_i} + {}^i\boldsymbol{\Pi}_i^T {}^i\boldsymbol{I}_i^{G_i} {}^i\boldsymbol{\Pi}_i\right)\right]\dot{\boldsymbol{q}}. \end{aligned} \tag{5.208}$$

We note that

$$\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = \left[\sum_{i=1}^{n_b}\left(M_i {}^i\boldsymbol{\Gamma}_{G_i}^T {}^i\boldsymbol{\Gamma}_{G_i} + {}^i\boldsymbol{\Pi}_i^T {}^i\boldsymbol{I}_i^{G_i} {}^i\boldsymbol{\Pi}_i\right)\right]\dot{\boldsymbol{q}}. \tag{5.209}$$

For a system of rigid bodies, we had previously defined

$$\boldsymbol{M}(\boldsymbol{q}) \triangleq \sum_{i=1}^{n_b}\left(M_i {}^i\boldsymbol{\Gamma}_{G_i}^T {}^i\boldsymbol{\Gamma}_{G_i} + {}^i\boldsymbol{\Pi}_i^T {}^i\boldsymbol{I}_i^{G_i} {}^i\boldsymbol{\Pi}_i\right). \tag{5.210}$$

So,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}\dot{\boldsymbol{q}} \tag{5.211}$$

and

$$\frac{\partial T}{\partial \boldsymbol{q}} = \frac{1}{2}\begin{pmatrix} \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_1}\dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_n}\dot{\boldsymbol{q}} \end{pmatrix}. \tag{5.212}$$

The potential energy, V, is given by

$$V = \sum_{i=1}^{n_b} M_i \left\langle {}^i\boldsymbol{r}_{G_i}, g {}^i\hat{\boldsymbol{e}}_{0_3}\right\rangle = g\sum_{i=1}^{n_b} M_i {}^i\boldsymbol{r}_{G_i}^T {}^i\hat{\boldsymbol{e}}_{0_3}. \tag{5.213}$$

So,

$$\frac{\partial V}{\partial \boldsymbol{q}} = g\sum_{i=1}^{n_b} M_i {}^i\boldsymbol{\Gamma}_{G_i}^T {}^i\hat{\boldsymbol{e}}_{0_3}. \tag{5.214}$$

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For a system of rigid bodies, we had previously defined

$$\boldsymbol{g}(\boldsymbol{q}) \triangleq g \sum_{i=1}^{n_b} M_i {}^i\boldsymbol{\Gamma}_{G_i}^{T\,i}\hat{\boldsymbol{e}}_{0_3}. \tag{5.215}$$

Thus,

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}\dot{\boldsymbol{q}} - \frac{1}{2}\begin{pmatrix} \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_1}\dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_n}\dot{\boldsymbol{q}} \end{pmatrix} + \boldsymbol{g}(\boldsymbol{q}). \tag{5.216}$$

Defining $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in an alternate, but consistent, manner as previously defined:

$$\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \triangleq \dot{\boldsymbol{M}}\dot{\boldsymbol{q}} - \frac{1}{2}\begin{pmatrix} \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_1}\dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_n}\dot{\boldsymbol{q}} \end{pmatrix} : \tag{5.217}$$

we have

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}). \tag{5.218}$$

Example: A gimballed gyroscope is depicted in Figure 5.11. The generalized coordinates of the 2-axis gimbal are q_1 and q_2, while the spin angle of the gyroscope is q_3. A

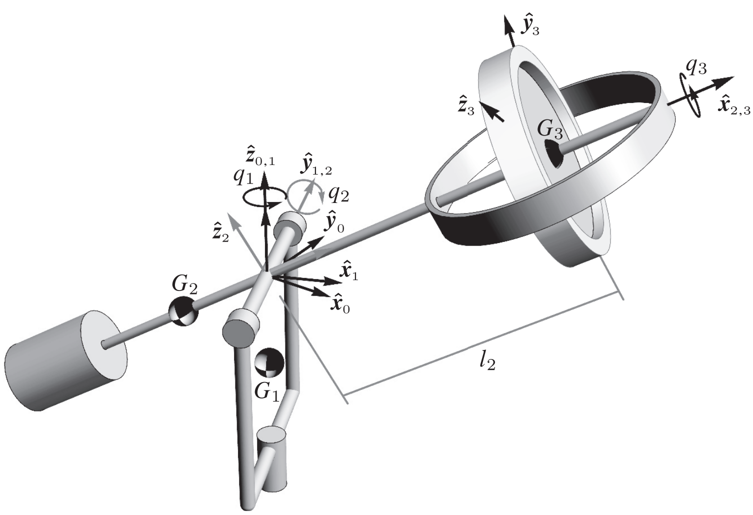

Figure 5.11 A gyroscope supported by a two-axis gimballed frame. The generalized coordinates, q_1 and q_2, parameterize the gimbal frame, and q_3 is the rotor angle. The rotor diameter is r.

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zyx Euler sequence will be used to represent the orientation of the system. So,

$$ {}^0_1\boldsymbol{Q} = \boldsymbol{Q}_z(q_1), \tag{5.219}$$

$$ {}^0_2\boldsymbol{Q} = \boldsymbol{Q}_z(q_1)\boldsymbol{Q}_y(q_2), \tag{5.220}$$

$$ {}^0_3\boldsymbol{Q} = \boldsymbol{Q}_z(q_1)\boldsymbol{Q}_y(q_2)\boldsymbol{Q}_x(q_3). \tag{5.221}$$

The positions of the proximal ends and centers of mass of the links are given by

$$ {}^0\boldsymbol{r}_1 = \boldsymbol{0}, \tag{5.222}$$

$$ {}^0\boldsymbol{r}_{G_1} = {}^0\boldsymbol{r}_1 - l_{G_1}\hat{\boldsymbol{e}}_3, \tag{5.223}$$

$$ {}^0\boldsymbol{r}_2 = {}^0\boldsymbol{r}_1 + \boldsymbol{0}, \tag{5.224}$$

$$ {}^0\boldsymbol{r}_{G_2} = {}^0\boldsymbol{r}_2 - l_{G_2}\,{}^0_2\boldsymbol{Q}\hat{\boldsymbol{e}}_1, \tag{5.225}$$

$$ {}^0\boldsymbol{r}_3 = {}^0\boldsymbol{r}_3 + l_3\,{}^0_2\boldsymbol{Q}\hat{\boldsymbol{e}}_1, \tag{5.226}$$

$$ {}^0\boldsymbol{r}_{G_3} = {}^0\boldsymbol{r}_3. \tag{5.227}$$

We can compute the angular velocity in the local frame by noting that

$$ {}^1\boldsymbol{\Omega}_1 = {}^0_1\boldsymbol{Q}^T\,{}^0_1\dot{\boldsymbol{Q}}, \tag{5.228}$$

$$ {}^2\boldsymbol{\Omega}_2 = {}^0_2\boldsymbol{Q}^T\,{}^0_2\dot{\boldsymbol{Q}}, \tag{5.229}$$

$$ {}^3\boldsymbol{\Omega}_3 = {}^0_3\boldsymbol{Q}^T\,{}^0_3\dot{\boldsymbol{Q}}. \tag{5.230}$$

Carrying this operation out, we determine

$$ {}^1\boldsymbol{\omega}_1 = \begin{pmatrix} 0 \\ 0 \\ \dot{q}_1 \end{pmatrix} \tag{5.231}$$

and

$$ {}^2\boldsymbol{\omega}_2 = \begin{pmatrix} -\sin(q_2)\dot{q}_1 \\ \dot{q}_2 \\ \cos(q_2)\dot{q}_1 \end{pmatrix} \tag{5.232}$$

and

$$ {}^3\boldsymbol{\omega}_3 = \begin{pmatrix} \dot{q}_3 - \sin(q_2)\dot{q}_1 \\ \cos(q_2)\sin(q_3)\dot{q}_1 + \cos(q_3)\dot{q}_2 \\ \cos(q_2)\cos(q_3)\dot{q}_1 - \sin(q_3)\dot{q}_2 \end{pmatrix} \tag{5.233}$$

Alternately, we could have arrived at the same results by propagating the rotation rates forward, where

$$ {}^1\boldsymbol{\omega}_1 = \dot{q}_1\hat{\boldsymbol{e}}_3, \tag{5.234}$$

$$ {}^2\boldsymbol{\omega}_2 = \boldsymbol{Q}_y^T(q_2)\,{}^1\boldsymbol{\omega}_1 + \dot{q}_2\hat{\boldsymbol{e}}_2, \tag{5.235}$$

$$ {}^3\boldsymbol{\omega}_3 = \boldsymbol{Q}_x^T(q_3)\,{}^2\boldsymbol{\omega}_2 + \dot{q}_3\hat{\boldsymbol{e}}_1. \tag{5.236}$$

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The translational velocity Jacobians are given by

$$ {}^0\boldsymbol{\Gamma}_{G_1} = \frac{\partial {}^0\boldsymbol{r}_{G_1}}{\partial \boldsymbol{q}} = \boldsymbol{0}, \tag{5.237} $$

$$ {}^0\boldsymbol{\Gamma}_{G_2} = \frac{\partial {}^0\boldsymbol{r}_{G_2}}{\partial \boldsymbol{q}} = \begin{pmatrix} l_{G_2}\sin(q_1)\cos(q_2) & l_{G_2}\cos(q_1)\sin(q_2) & 0 \\ -l_{G_2}\cos(q_1)\cos(q_2) & l_{G_2}\sin(q_1)\sin(q_2) & 0 \\ 0 & l_{G_2}\cos(q_2) & 0 \end{pmatrix}, \tag{5.238} $$

$$ {}^0\boldsymbol{\Gamma}_{G_3} = \frac{\partial {}^0\boldsymbol{r}_{G_3}}{\partial \boldsymbol{q}} = \begin{pmatrix} -l_3\sin(q_1)\cos(q_2) & -l_3\cos(q_1)\sin(q_2) & 0 \\ l_3\cos(q_1)\cos(q_2) & -l_3\sin(q_1)\sin(q_2) & 0 \\ 0 & -l_3\cos(q_2) & 0 \end{pmatrix}, \tag{5.239} $$

and the angular velocity Jacobians are given by

$$ {}^1\boldsymbol{\Pi}_1 = \frac{\partial {}^1\boldsymbol{\omega}_1}{\partial \dot{\boldsymbol{q}}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{5.240} $$

$$ {}^2\boldsymbol{\Pi}_2 = \frac{\partial {}^2\boldsymbol{\omega}_2}{\partial \dot{\boldsymbol{q}}} = \begin{pmatrix} -\sin(q_2) & 0 & 0 \\ 0 & 1 & 0 \\ \cos(q_2) & 0 & 0 \end{pmatrix}, \tag{5.241} $$

$$ {}^3\boldsymbol{\Pi}_3 = \frac{\partial {}^3\boldsymbol{\omega}_3}{\partial \dot{\boldsymbol{q}}} = \begin{pmatrix} -\sin(q_2) & 0 & 1 \\ \cos(q_2)\sin(q_3) & \cos(q_3) & 0 \\ \cos(q_2)\cos(q_3) & -\sin(q_3) & 0 \end{pmatrix}. \tag{5.242} $$

Our dynamical terms are then

$$ \boldsymbol{M}(\boldsymbol{q}) = \sum_{i=1}^{3} \left(M_i {}^0\boldsymbol{\Gamma}_{G_i}^T {}^0\boldsymbol{\Gamma}_{G_i} + {}^i\boldsymbol{\Pi}_i^T {}^i\boldsymbol{I}_i^{Gi} {}^i\boldsymbol{\Pi}_i \right), \tag{5.243} $$

$$ \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}\dot{\boldsymbol{q}} - \frac{1}{2} \begin{pmatrix} \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_1} \dot{\boldsymbol{q}} \\ \vdots \\ \dot{\boldsymbol{q}}^T \frac{\partial \boldsymbol{M}}{\partial q_3} \dot{\boldsymbol{q}} \end{pmatrix}, \tag{5.244} $$

$$ \boldsymbol{g}(\boldsymbol{q}) = g \sum_{i=1}^{3} M_i {}^0\boldsymbol{\Gamma}_{G_i}^T \hat{\boldsymbol{e}}_3. \tag{5.245} $$

The inertia tensors will be taken as

$$ {}^1\boldsymbol{I}_1^{G_1} = \begin{pmatrix} \frac{1}{3}M_1 & 0 & 0 \\ 0 & \frac{1}{3}M_1 & 0 \\ 0 & 0 & \frac{1}{2}M_1 \end{pmatrix}, \tag{5.246} $$

$$ {}^2\boldsymbol{I}_2^{G_2} = \begin{pmatrix} 0.125M_2 & 0 & 0 \\ 0 & 3.0625M_2 & 0 \\ 0 & 0 & 3.083M_2 \end{pmatrix}, \tag{5.247} $$

$$ {}^3\boldsymbol{I}_3^{G_3} = \begin{pmatrix} \frac{1}{2}M_3r^2 & 0 & 0 \\ 0 & \frac{1}{12}h^2M_3 + \frac{1}{4}M_3r^2 & 0 \\ 0 & 0 & \frac{1}{12}h^2M_3 + \frac{1}{4}M_3r^2 \end{pmatrix}. \tag{5.248} $$

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We then have

$$M(q) = \begin{pmatrix} M_{11} & 0 & -\frac{1}{2}M_3r^2\sin(q_2) \\ 0 & M_{22} & -\frac{1}{2}M_3r^2\sin(q_2) \\ -\frac{1}{2}M_3r^2\sin(q_2) & 0 & \frac{1}{2}M_3r^2 \end{pmatrix},$$

$$\begin{aligned} M_{11} &= 0.5M_1 + 1.604M_2 + 0.5l_{G_2}^2M_2 + 0.042h^2M_3 + 0.5l_3^2M_3 + 0.375M_3r^2 \\ &\quad + [(1.479 + 0.5l_{G_2}^2)M_2 + M_3(0.042h^2 + 0.5l_3^2 - 0.125r^2)]\cos(2q_2), \\ M_{22} &= (3.0625 + l_{G_2}^2)M_2 + M_3(0.083h^2 + l_3^2 + 0.25r^2), \end{aligned} \tag{5.249}$$

$$b(q, \dot{q}) = \begin{pmatrix} b_1 \\ b_2 \\ -\frac{1}{2}M_3r^2\cos(q_2)\dot{q}_1\dot{q}_2 \end{pmatrix},$$

$$\begin{aligned} b_1 &= -2\cos(q_2)\dot{q}_2([(2.958 + l_{G_2}^2)M_2 \\ &\quad + M_3(0.083h^2 + l_3^2 - 0.25r^2)]\sin(q_2)\dot{q}_1 + 0.25M_3r^2\dot{q}_3), \\ b_2 &= \cos(q_2)\dot{q}_1([(2.958 + l_{G_2}^2)M_2 \\ &\quad + M_3(0.083h^2 + l_3^2 - 0.25r^2)]\sin(q_2)\dot{q}_1 + 0.5M_3r^2\dot{q}_3), \end{aligned} \tag{5.250}$$

$$g(q) = \begin{pmatrix} 0 \\ g(l_{G_2}M_2 - l_3M_3)\cos(q_2) \\ 0 \end{pmatrix}. \tag{5.251}$$

The simulation results are displayed in Figure 5.12. The values of the constants were $M_1 = M_2 = M_3 = 1$, $l_{G_1} = l_{G_2} = 1$, $l_3 = 3$, $r = 1$, and $h = 1$. The initial conditions used were

$$q_o = \begin{pmatrix} 0 & -\pi/6 & 0 \end{pmatrix}^T \tag{5.252}$$

$$\dot{q}_o = \begin{pmatrix} 0 & 0 & 2\pi/8 \end{pmatrix}^T. \tag{5.253}$$

The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{3}\left(M_i{}^0\Gamma_{G_i}^T{}^0\Gamma_{G_i} + {}^i\Pi_i^T{}^iI_i^{Gi}\Pi_i\right)\right]\dot{q} = \frac{1}{2}\dot{q}^TM(\dot{q}), \\ &= \frac{1}{2}[0.5M_1 + 1.604M_2 + 0.5l_{G_2}^2M_2 + 0.042h^2M_3 + 0.5l_3^2M_3 + 0.375M_3r^2 \\ &\quad + (1.479M_2 + 0.5l_{G_2}^2M_2 + 0.042h^2M_3 + 0.5l_3^2M_3 - 0.125M_3r^2)\cos(2q_2)]\dot{q}_1^2 \\ &\quad + (3.063M_2 + l_{G_2}^2M_2 + 0.083h^2M_3 + l_3^2M_3 + 0.25M_3r^2)\dot{q}_2^2 \\ &\quad - M_3r^2\sin(q_2)\dot{q}_1\dot{q}_3 + 0.5M_3r^2\dot{q}_3^2. \end{aligned} \tag{5.254}$$

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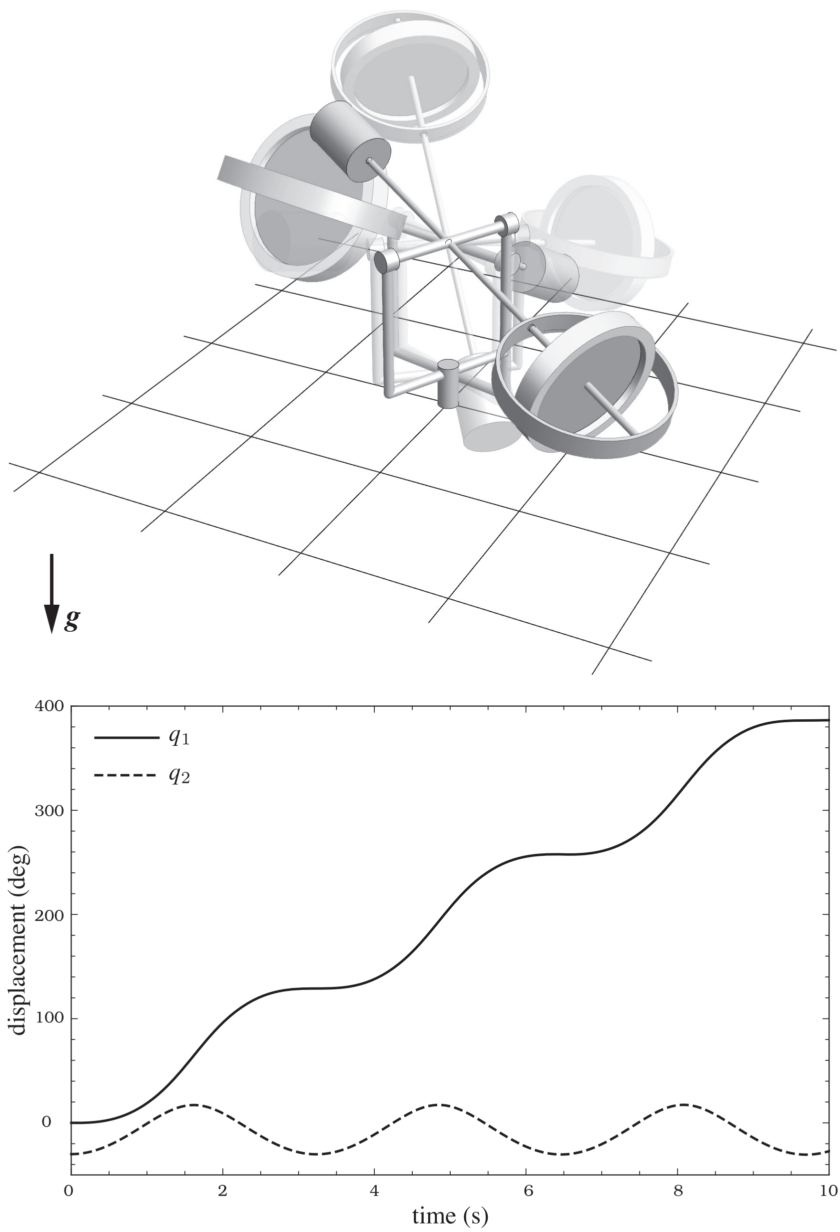

Figure 5.12 (Top) Animation frames from the simulation of the gimballed gyroscope. (Bottom) Time history of the gimbal angles, q_1 and q_2. The rotor of the gyroscope is spun up to $2\pi/8$ rad/s. Precession of the gyroscope can be observed.

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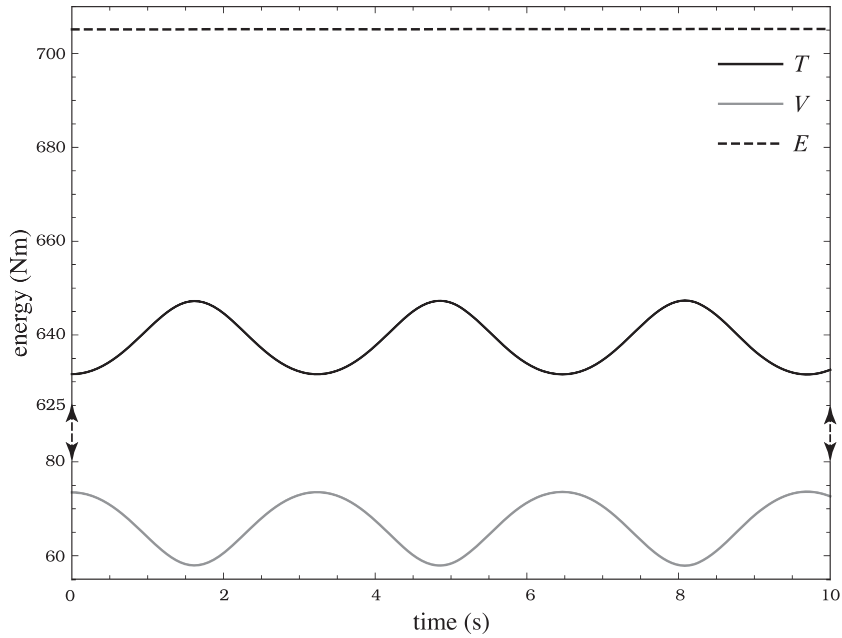

Figure 5.13 Fluctuations of kinetic and potential energy and conservation of total energy.

The potential energy (relative to $z = -2.5$) of the system is

$$V = g\sum_{i=1}^{n_b} M_i {}^0\boldsymbol{r}_{G_i}^T \hat{\boldsymbol{e}}_3 = g[(2.5 - l_{G_1})M_1 + 2.5M_2 + 2.5M_3 + (l_{G_2}M_2 - l_3M_3)\sin(q_2)], \tag{5.255}$$

and the total energy is

$$E = T + V. \tag{5.256}$$

The Lagrangian is

$$\mathcal{L} = T - V. \tag{5.257}$$

Figure 5.13 shows the fluctuations of kinetic and potential energy and the conservation of total energy for the gyroscope.

5.3.6 Constrained Least Action

Least action can be applied to multibody systems with auxiliary holonomic constraint equations. We introduce a set of m holonomic (and scleronomic) constraint equations, $\boldsymbol{\phi}(\boldsymbol{q}) = \boldsymbol{0}$. The zeroth-order variation of the constraint equations is $\delta\boldsymbol{\phi} = \boldsymbol{\Phi}\delta\boldsymbol{q} = \boldsymbol{0}$,

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where the matrix $\boldsymbol{\Phi}(\boldsymbol{q}) = \partial\boldsymbol{\phi}/\partial\boldsymbol{q} \in \mathbb{R}^{m_C \times n}$ is the constraint Jacobian. In this case, the Principle of Least Action can be stated as

$$\begin{gathered}\delta I = 0, \\ \forall \delta | \delta \boldsymbol{q}(t_o) = \delta \boldsymbol{q}(t_f) = \boldsymbol{0}, \quad \text{and} \quad \boldsymbol{\Phi}\delta\boldsymbol{q} = \boldsymbol{0}.\end{gathered} \tag{5.258}$$

Thus, least action seeks the path, $\boldsymbol{q}(t)$, in configuration space that results in a stationary value of action, I, under all path variations, $\delta\boldsymbol{q}$, that vanish at the endpoints and *satisfy the constraints*.

We recall the Euler-Lagrange equations for unconstrained systems:

$$\delta I = \int_{t_o}^{t_f} \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} + \boldsymbol{\tau} \right) \cdot \delta\boldsymbol{q}\, dt. \tag{5.259}$$

This makes use of the condition that the path variations vanish at the endpoints. The condition

$$\begin{gathered}\delta I = 0 \\ \forall \delta\boldsymbol{q} | \boldsymbol{\Phi}\delta\boldsymbol{q} = \boldsymbol{0}\end{gathered} \tag{5.260}$$

applied to (5.259) implies the following orthogonality relation at *any instant*:

$$\begin{gathered}\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \boldsymbol{\tau} \right) \cdot \delta\boldsymbol{q} = 0 \\ \forall \delta\boldsymbol{q} \in \ker(\boldsymbol{\Phi}).\end{gathered} \tag{5.261}$$

Thus,

$$\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \boldsymbol{\tau} \right) \in \ker(\boldsymbol{\Phi})^{\perp} = \operatorname{im}(\boldsymbol{\Phi}^T). \tag{5.262}$$

This implies the familiar constrained Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{\tau} + \boldsymbol{\Phi}^T\boldsymbol{\lambda}. \tag{5.263}$$

Identical equations could have been obtained by embedding the constraints directly in the Lagrangian. In this case the Lagrangian in (5.259) would be replaced by the augmented Lagrangian with the constraints adjoined,

$$\mathcal{L}_{\text{aug}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\lambda}) \triangleq \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{\lambda}^T\boldsymbol{\phi}(\boldsymbol{q}), \tag{5.264}$$

and the stationary value of I would be sought for all variations that vanish at the endpoints. We note that

$$\delta\mathcal{L}_{\text{aug}} = \delta\mathcal{L} + \boldsymbol{\lambda}^T\delta\boldsymbol{\phi} = \delta\mathcal{L} + \boldsymbol{\lambda}^T\boldsymbol{\Phi}\delta\boldsymbol{q} \tag{5.265}$$

and

$$\frac{d}{dt}\frac{\partial \mathcal{L}_{\text{aug}}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}_{\text{aug}}}{\partial \boldsymbol{q}} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \boldsymbol{\Phi}^T\boldsymbol{\lambda} = \boldsymbol{\tau}. \tag{5.266}$$

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In standard matrix form we have

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{\Phi}^T \boldsymbol{\lambda}, \tag{5.267}$$

subject to $\boldsymbol{\phi}(\boldsymbol{q}) = \mathbf{0}$. For the stationary value of I to correspond to a minimum requires that the first-order variation of I be greater than or equal to *zero*. This condition is shown to be satisfied (Vujanovic and Atanackovic 2004) for sufficiently small time intervals, $[t_o, t_f]$, if the following is satisfied:

$$\int_{t_o}^{t_f} \delta\dot{\boldsymbol{q}}^T \boldsymbol{M} \delta\dot{\boldsymbol{q}}\, dt \geq 0, \tag{5.268}$$

which corresponds to $\boldsymbol{M}$ being positive definite over the actual path. For classical Lagrangian systems this condition is met.

Since (5.267) forms a set of second-order differential equations, it is appropriate to complement it with the second derivative of the constraint equations,

$$\ddot{\boldsymbol{\phi}} = \boldsymbol{\Phi}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{\Phi}}\dot{\boldsymbol{q}} = \mathbf{0}. \tag{5.269}$$

5.4 Canonical Hamiltonian Formulation

5.4.1 Unconstrained Case

We can express a set of so-called canonical equations of motion by defining an additional set of states, the *generalized momenta*,

$$\boldsymbol{p}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \triangleq \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}. \tag{5.270}$$

In principle, we can invert this expression to represent the generalized velocities in terms of the generalized coordinates and the generalized momenta. With (5.270) we can express (5.169) as

$$\boldsymbol{\tau} = \dot{\boldsymbol{p}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}. \tag{5.271}$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \dot{\boldsymbol{p}} - \boldsymbol{\tau}. \tag{5.272}$$

Given that $\mathcal{L} = \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, the total differential of the Lagrangian is

$$d\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \cdot d\boldsymbol{q} + \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} \cdot d\dot{\boldsymbol{q}}. \tag{5.273}$$

Substituting (5.270) and (5.272) into (5.273), we have

$$d\mathcal{L} = (\dot{\boldsymbol{p}} - \boldsymbol{\tau}) \cdot d\boldsymbol{q} + \boldsymbol{p} \cdot d\dot{\boldsymbol{q}}. \tag{5.274}$$

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Defining the *Hamiltonian*, $\mathcal{H}$, as

$$\mathcal{H} \triangleq \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - \mathcal{L}, \tag{5.275}$$

we note that the total differential of the Hamiltonian is

$$d\mathcal{H} = d(\boldsymbol{p} \cdot \dot{\boldsymbol{q}} - \mathcal{L}) = d(\boldsymbol{p} \cdot \dot{\boldsymbol{q}}) - d\mathcal{L} = (\boldsymbol{\tau} - \dot{\boldsymbol{p}}) \cdot d\boldsymbol{q} + \dot{\boldsymbol{q}} \cdot d\boldsymbol{p}. \tag{5.276}$$

Given that we can express $\mathcal{H} = \mathcal{H}(\boldsymbol{q}, \boldsymbol{p})$ by replacing the generalized velocities in (5.275) with expressions in terms of the generalized momenta, we have

$$d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \cdot d\boldsymbol{q} + \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \cdot d\boldsymbol{p}. \tag{5.277}$$

Comparing this with (5.276), we note that

$$\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} = \boldsymbol{\tau} - \dot{\boldsymbol{p}} \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}. \tag{5.278}$$

The canonical equations are thus the $2n$ first-order differential equations

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \tag{5.279}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{\tau} - \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}. \tag{5.280}$$

where the state vector is $(\, \boldsymbol{q} \;\, \boldsymbol{p} \,)^T$.

It is noted that, in practice, computing the Hamiltonian from the Lagrangian will result in an expression for the Hamiltonian in terms of the generalized coordinates, velocities, and momenta rather than just the generalized coordinates and momenta (canonical states). The procedure for expressing the Hamiltonian exclusively in terms of the generalized coordinates and momenta as follows:

1. Express the generalized momenta, $\boldsymbol{p}$, as a function of the generalized coordinates, $\boldsymbol{q}$, and the generalized velocities, $\dot{\boldsymbol{q}}$, using
$$\boldsymbol{p}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}.$$
2. Invert this expression to represent the generalized velocities as a function of the generalized coordinates and the generalized momenta. That is, determine $\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{p})$.
3. Compute the Hamiltonian, $\mathcal{H}$, using the Lagrangian
$$\mathcal{H}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{p}) = \boldsymbol{p} \cdot \dot{\boldsymbol{q}} - \mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}).$$
4. Express the Hamiltonian as a function of the generalized coordinates and the generalized momenta by replacing the generalized velocities with expressions in terms of the generalized momenta using $\dot{\boldsymbol{q}} = \boldsymbol{f}(\boldsymbol{q}, \boldsymbol{p})$.

Numerical Integration

A first-order method for integrating Hamilton's equations can be summarized as shown in Algorithm 5.

Algorithm 5 First-order method for integrating Hamilton's equations

1: $\boldsymbol{q}_0 = \boldsymbol{q}_o$ {initialization}
2: $\boldsymbol{p}_0 = \boldsymbol{p}_o$ {initialization}
3: **for** $i = 0$ to $n_s - 1$ **do**
4: $\quad \dot{\boldsymbol{q}}_i = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}|_i$
5: $\quad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}|_i$
6: $\quad \boldsymbol{p}_{i+1} = \boldsymbol{p}_i + \dot{\boldsymbol{p}}_i \Delta t$
7: $\quad \boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \dot{\boldsymbol{q}}_i \Delta t$
8: **end for**

5.4.2 Auxiliary Constraints

As in the unconstrained case, we can express a set of canonical equations of motion. Beginning with the constrained Euler-Lagrange equations,

$$\boldsymbol{\tau} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}} - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} - \boldsymbol{\Phi}^T \boldsymbol{\lambda}, \tag{5.281}$$

and following the same procedure as for the unconstrained case, we have the $2n$ first-order differential equations

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \tag{5.282}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{\tau} + \boldsymbol{\Phi}^T \boldsymbol{\lambda} - \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}, \tag{5.283}$$

complemented by the m constraint equations $\boldsymbol{\Phi}\dot{\boldsymbol{q}} = \boldsymbol{0}$. This yields a set of $2n + m$ first-order differential equations. In practice the integration of the forward dynamics would also require constraint stabilization to mitigate drift in the constraints. Unlike second-order systems, where drift occurs at the position and velocity levels, we only need to be concerned with drift at the position level (Naudet et al. 2003). We can replace our original differential constraint equation with

$$\dot{\boldsymbol{\phi}} + \alpha \boldsymbol{\phi} = \boldsymbol{0}, \tag{5.284}$$

or

$$\boldsymbol{\Phi}\dot{\boldsymbol{q}} + \alpha \boldsymbol{\phi} = \boldsymbol{0}. \tag{5.285}$$

Thus, the constraint stabilized canonical equations of motion in compact form are

$$\begin{pmatrix} \boldsymbol{1} & 0 & 0 \\ 0 & \boldsymbol{1} & -\boldsymbol{\Phi}^T \\ \boldsymbol{\Phi} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \\ \boldsymbol{\tau} - \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \\ -\alpha \boldsymbol{\phi} \end{pmatrix}. \tag{5.286}$$

Dirac generalized the handling of constraints in Hamiltonian dynamics (Dirac 1958). If we consider the constraints to be a function of the generalized momenta as well as the

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generalized coordinates, we have

$$\boldsymbol{\phi}(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{0}. \tag{5.287}$$

We can define the augmented Hamiltonian as

$$\mathcal{H}_{\text{aug}}(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{\lambda}) \triangleq \mathcal{H}(\boldsymbol{q}, \boldsymbol{p}) - \boldsymbol{\lambda}^T \boldsymbol{\phi}(\boldsymbol{q}, \boldsymbol{p}). \tag{5.288}$$

The constrained canonical equations are then

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}_{\text{aug}}}{\partial \boldsymbol{p}} \tag{5.289}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{\tau} - \frac{\partial \mathcal{H}_{\text{aug}}}{\partial \boldsymbol{q}} \tag{5.290}$$

or

$$\dot{\boldsymbol{q}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} - \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{p}} \boldsymbol{\lambda} \tag{5.291}$$

$$\dot{\boldsymbol{p}} = \boldsymbol{\tau} - \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} + \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}} \boldsymbol{\lambda}. \tag{5.292}$$

The stabilized differential constraint equation is

$$\dot{\boldsymbol{\phi}} + \alpha \boldsymbol{\phi} = \boldsymbol{0}, \tag{5.293}$$

or

$$\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}} \dot{\boldsymbol{q}} + \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{p}} \dot{\boldsymbol{p}} + \alpha \boldsymbol{\phi} = \boldsymbol{0}. \tag{5.294}$$

The generalized constraint stabilized canonical equations of motion in compact form are

$$\begin{pmatrix} \mathbf{1} & 0 & \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{p}} \\ 0 & \mathbf{1} & -\frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}} \\ \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{q}} & \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{p}} & 0 \end{pmatrix} \begin{pmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \\ \boldsymbol{\tau} - \frac{\partial \mathcal{H}}{\partial \boldsymbol{q}} \\ -\alpha \boldsymbol{\phi} \end{pmatrix}. \tag{5.295}$$

5.5 Elimination of Multipliers

The Lagrange multipliers can be eliminated from (5.86) by first expressing the zeroth-order variational equation:

$$\boldsymbol{\tau}_C \cdot \delta \boldsymbol{q} + (\boldsymbol{\tau} - \boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{b} - \boldsymbol{g}) \cdot \delta \boldsymbol{q} = 0. \tag{5.296}$$

By restricting the variations to constraint-consistent virtual displacements, we have

$$\begin{gathered} \boldsymbol{\tau}_C \cdot \delta \boldsymbol{q} + (\boldsymbol{\tau} - \boldsymbol{M}\ddot{\boldsymbol{q}} - \boldsymbol{b} - \boldsymbol{g}) \cdot \delta \boldsymbol{q} = 0 \\ \forall \delta \boldsymbol{q} \in \ker(\boldsymbol{\Phi}). \end{gathered} \tag{5.297}$$

Recalling (5.83) we note that the generalized constraint forces produce no virtual work under virtual displacements that are consistent with the constraints. Thus, the term

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$\boldsymbol{\tau}_C \cdot \delta\boldsymbol{q}$ vanishes from (5.297), and we have the orthogonality relation

$$(\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{b} + \boldsymbol{g} - \boldsymbol{\tau}) \cdot \delta\boldsymbol{q} = 0 \qquad \forall \delta\boldsymbol{q} \in \ker(\boldsymbol{\Phi}). \tag{5.298}$$

We now define a matrix, $\boldsymbol{W} \in \mathbb{R}^{n\times p}$, whose columns span the null space of $\boldsymbol{\Phi}$. This implies that $\mathrm{im}(\boldsymbol{W}) = \ker(\boldsymbol{\Phi})$. Thus, $\boldsymbol{\Phi W} = \boldsymbol{0}$ and $\boldsymbol{W}^T\boldsymbol{\Phi}^T = \boldsymbol{0}$. In this manner, $\boldsymbol{W}$ orthogonally complements $\boldsymbol{\Phi}$. That is,

$$\mathrm{im}(\boldsymbol{W}) = \ker(\boldsymbol{\Phi}) = \mathrm{im}(\boldsymbol{\Phi}^T)^\perp. \tag{5.299}$$

Geometrically, $\mathrm{im}(\boldsymbol{W})$ represents the tangent space of the constrained-motion manifold, Q^p (see Figure 5.5). These geometric properties are discussed in further detail in Blajer (1997) and Jungnickel (1994). While not required for the subsequent analysis, we specify that the columns of $\boldsymbol{W}$ be mutually orthogonal and thus form an orthogonal basis, $\mathcal{C}$, for the null space of $\boldsymbol{\Phi}$. The constraint-consistent virtual displacements, $\delta\boldsymbol{q} \in \ker(\boldsymbol{\Phi})$, can then be expressed in terms of the virtual displacements of a minimal set of p independent coordinates, $\boldsymbol{q}_p$:

$$\delta\boldsymbol{q} = \boldsymbol{W}\delta\boldsymbol{q}_p. \tag{5.300}$$

Using this relationship, we can express (5.298) over all possible variations of a minimal set of coordinates:

$$\begin{gathered}(\boldsymbol{W}^T\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{W}^T\boldsymbol{b} + \boldsymbol{W}^T\boldsymbol{g} - \boldsymbol{W}^T\boldsymbol{\tau}) \cdot \delta\boldsymbol{q}_p = 0,\\ \forall \delta\boldsymbol{q}_p \in \mathbb{R}^p,\\ \Downarrow\\ \boldsymbol{W}^T\boldsymbol{\tau} = \boldsymbol{W}^T\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{W}^T\boldsymbol{b} + \boldsymbol{W}^T\boldsymbol{g}.\end{gathered} \tag{5.301}$$

Noting that $\dot{\boldsymbol{q}} = \boldsymbol{W}\dot{\boldsymbol{q}}_p$ and $\ddot{\boldsymbol{q}} = \boldsymbol{W}\ddot{\boldsymbol{q}}_p + \dot{\boldsymbol{W}}\dot{\boldsymbol{q}}_p$, we can express (5.301) as

$$\boldsymbol{\tau}_p = \boldsymbol{M}_p(\boldsymbol{q})\ddot{\boldsymbol{q}}_p + \boldsymbol{b}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}_p) + \boldsymbol{g}_p(\boldsymbol{q}), \tag{5.302}$$

where

$$\boldsymbol{M}_p(\boldsymbol{q}) = \boldsymbol{W}^T\boldsymbol{M}\boldsymbol{W}, \tag{5.303}$$

$$\boldsymbol{b}_p(\boldsymbol{q}, \dot{\boldsymbol{q}}_p) = \boldsymbol{W}^T\boldsymbol{b} + \boldsymbol{W}^T\boldsymbol{M}\dot{\boldsymbol{W}}\dot{\boldsymbol{q}}_p, \tag{5.304}$$

$$\boldsymbol{g}_p(\boldsymbol{q}) = \boldsymbol{W}^T\boldsymbol{g}, \tag{5.305}$$

$$\boldsymbol{\tau}_p = \boldsymbol{W}^T\boldsymbol{\tau}. \tag{5.306}$$

The approach outlined here is consistent with the projection method of Blajer (1997). This approach was also used by Russakow et al. for application to serial-to-parallel chain manipulators (Russakow, Khatib, and Rock 1995). We note that (5.302) includes a mix of our initial set of n generalized coordinates, $\boldsymbol{q}$, as well as the minimal set of p independent coordinates, $\boldsymbol{q}_p$. Since the constraints are holonomic, we would expect there to be a mapping, in principle, which could be derived from the constraints that would yield $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}_p)$. In this case $\boldsymbol{W}$ could be computed explicitly from the mapping rather than computing the null space of $\boldsymbol{\Phi}$; that is, $\boldsymbol{W} = \partial\boldsymbol{q}/\partial\boldsymbol{q}_p$. Additionally, the terms

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in (5.302) could be expressed as functions of $\boldsymbol{q}_p$ rather than $\boldsymbol{q}$. Since $\boldsymbol{q}_p$ are independent coordinates, the constraints would be implicitly addressed and the resulting system would be unconstrained with respect to configuration space. However, finding the mapping $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{q}_p)$ would be difficult in general. In such cases a null space method or a coordinate partitioning method (Wehage and Haug 1982) would need to be used to compute $\boldsymbol{W}$.

Additionally, the generalized coordinates, $\boldsymbol{q}_p$, and the generalized forces, $\boldsymbol{\tau}_p$, do not necessarily have a natural and physically intuitive meaning, making it difficult to standardize their use in a numerical algorithm. This is in contrast to the coordinates, $\boldsymbol{q}$, which are chosen specifically to describe the system in the most natural and physically intuitive manner. It is usually desirable to select $\boldsymbol{q}$ in a manner that preserves the physical meaning of the generalized forces as torques about individual joints. Often when using a minimal set of coordinates, this is not the case, since a single generalized coordinate may influence multiple joint displacements. Therefore, from an algorithmic perspective, it is often preferable to deal with a nonminimal but standardized set of generalized coordinates (like joint angles) that are amenable to numerical formulation and to compute the dynamical terms corresponding to that kinematic parametrization.

5.6 Exercises

1. Consider the serial chain robot from Section 4.2, Exercise 3 (shown in Figure 5.14). Recall that link lengths are l_1 and l_2. The link radii are r_1 and r_2, the link masses are M_1 and M_2, and the link inertia tensors are

$$ {}^{i}\boldsymbol{I}_{i}^{G_i} = \begin{pmatrix} \frac{1}{2}M_i r_i^2 & 0 & 0 \\ 0 & \frac{1}{12}l_i^2 M_i + \frac{1}{4}M_i r_i^2 & 0 \\ 0 & 0 & \frac{1}{12}l_i^2 M_i + \frac{1}{4}M_i r_i^2 \end{pmatrix} $$

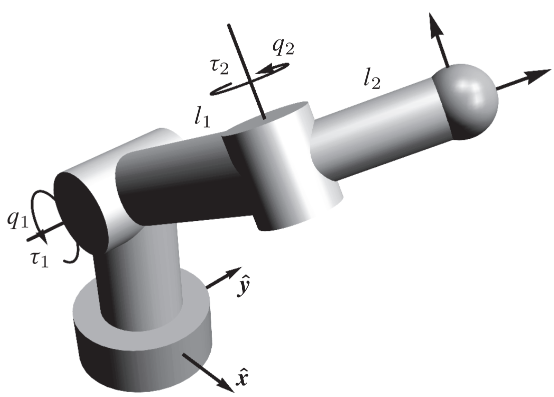

Figure 5.14 A 2 degree-of-freedom serial chain robot parameterized by the generalized coordinates, q_1 and q_2, with generalized forces, τ_1 and τ_2 (Exercises 1, 5, and 7). The link lengths are l_1 and l_2. The centers of mass are at the geometric centers of the links. The link radii are r_1 and r_2, the link masses are M_1 and M_2, and the link principal inertia components are ${}^{i}I_{i_{11}}^{G_i} = \frac{1}{2}M_i r_i^2$ and ${}^{i}I_{i_{22}}^{G_i} = {}^{i}I_{i_{33}}^{G_i} = \frac{1}{12}l_i^2 M_i + \frac{1}{4}M_i r_i^2$ for links $i = 1, 2$.

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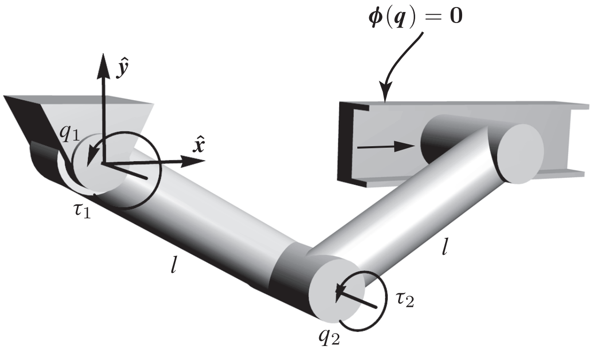

Figure 5.15 A two-link planar slider-crank mechanism (Exercises 2, 6, and 8). The unconstrained system is parameterized by the generalized coordinates, q_1 and q_2, with generalized forces, τ_1 and τ_2. The link lengths are each l and the link masses are each M. The centers of mass are at the geometric centers of the links. The link rotational inertias are taken to be *zero*. Link 2 is constrained from translating in the $\hat{\boldsymbol{y}}$ direction.

for links $i = 1, 2$.

(a) Compute the mass matrix, $\boldsymbol{M}(\boldsymbol{q})$.
(b) Compute the vector of centrifugal and Coriolis forces, $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})$.
(c) Compute the vector of gravity forces, $\boldsymbol{g}(\boldsymbol{q})$.

2. Consider the two-link planar slider-crank mechanism shown in Figure 5.15. The link lengths are each l and the link masses are each M. The link rotational inertias are taken to be *zero*.
   (a) How many degrees of freedom does the constrained system have?
   (b) Compute the mass matrix, $\boldsymbol{M}(\boldsymbol{q})$, for the unconstrained system.
   (c) Compute the vector of centrifugal and Coriolis forces, $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, for the unconstrained system.
   (d) Compute the vector of gravity forces, $\boldsymbol{g}(\boldsymbol{q})$, for the unconstrained system.
   (e) Express the constraint equations, $\boldsymbol{\phi}(\boldsymbol{q})$, for the loop closure.
   (f) Compute the constraint matrix, $\boldsymbol{\Phi}(\boldsymbol{q})$.
3. Consider the planar four-bar linkage shown in Figure 5.16. The link lengths are each l and the link masses are each M. The link rotational inertias are taken to be *zero*.
   (a) How many degrees of freedom does the constrained system have?
   (b) Compute the mass matrix, $\boldsymbol{M}(\boldsymbol{q})$, for the unconstrained system.
   (c) Compute the vector of centrifugal and Coriolis forces, $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, for the unconstrained system.
   (d) Compute the vector of gravity forces, $\boldsymbol{g}(\boldsymbol{q})$, for the unconstrained system.
   (e) Express the constraint equations, $\boldsymbol{\phi}(\boldsymbol{q})$ for the loop closure.
   (f) Compute the constraint matrix, $\boldsymbol{\Phi}(\boldsymbol{q})$.
4. Consider the three-link planar slider-crank mechanism shown in Figure 5.17. The unconstrained system is identical to the four-bar linkage of Exercise 3.
   (a) How many degrees of freedom does the constrained system have?

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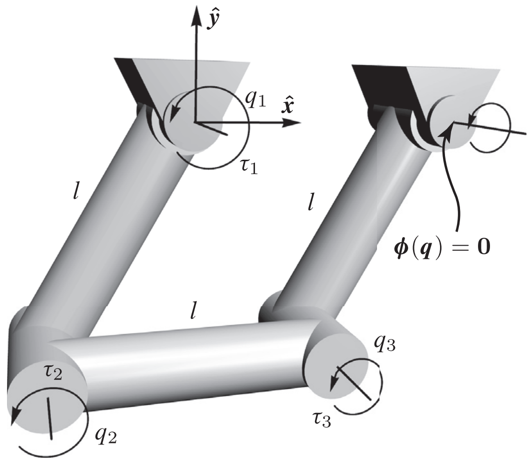

Figure 5.16 A planar four-bar linkage (Exercise 3). The unconstrained system is parameterized by the generalized coordinates, q_1, q_2, and q_3, with generalized forces, τ_1, τ_2, and τ_3. The link lengths are each l and the link masses are each M. The centers of mass are at the geometric centers of the links. The link rotational inertias are taken to be *zero*. The link 3 endpoint is constrained from translating.

(b) If not already completed, compute the mass matrix, $\boldsymbol{M}(\boldsymbol{q})$, the vector of centrifugal and Coriolis forces, $\boldsymbol{b}(\boldsymbol{q}, \dot{\boldsymbol{q}})$, and the gravity forces, $\boldsymbol{g}(\boldsymbol{q})$, for the unconstrained system.

(c) Express the constraint equations, $\boldsymbol{\phi}(\boldsymbol{q})$ for the loop closure.

(d) Compute the constraint matrix, $\boldsymbol{\Phi}(\boldsymbol{q})$.

5. Consider the serial chain robot from Exercise 1.
   (a) Compute the kinetic energy, T, of the system.
   (b) Compute the potential energy, V, of the system.

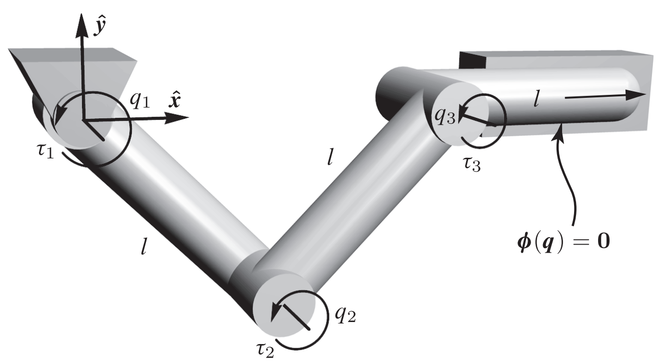

Figure 5.17 A three-link planar slider-crank mechanism (Exercise 4). The unconstrained system is parameterized by the generalized coordinates, q_1, q_2, and q_3, with generalized forces, τ_1, τ_2, and τ_3. The link lengths are each l and the link masses are each M. The centers of mass are at the geometric centers of the links. The link rotational inertias are taken to be *zero*. Link 3 is constrained from rotating as well as translating in the $\hat{\boldsymbol{y}}$ direction.

   (c) Compute the Lagrangian, $\mathcal{L}$, of the system.
   (d) Generate the equations of motion, directly from the Lagrangian, using the Euler-Lagrange equations.
6. Consider the two-link planar slider-crank mechanism from Exercise 2.
   (a) Compute the kinetic energy, T, of the system.
   (b) Compute the potential energy, V, of the system.
   (c) Compute the Lagrangian, $\mathcal{L}$, of the system.
   (d) Generate the equations of motion, directly from the Lagrangian, using the Euler-Lagrange equations.
   (e) Express the constrained equations of motion.
7. Consider the serial chain robot from Exercise 1.
   (a) Express the generalized momenta, $\boldsymbol{p}$, as a function of the generalized coordinates, $\boldsymbol{q}$, and the generalized velocities, $\dot{\boldsymbol{q}}$. Invert this expression to represent the generalized velocities as a function of the generalized coordinates and the generalized momenta.
   (b) Compute the Hamiltonian, $\mathcal{H}$, using the Lagrangian. Express the Hamiltonian as a function of the generalized coordinates and the generalized momenta by replacing the generalized velocities with expressions in terms of the generalized momenta.
   (c) Generate the equations of motion, directly from the Hamiltonian, using Hamilton's canonical equations.
8. Consider the two-link planar slider-crank mechanism from Exercise 2.
   (a) Express the generalized momenta, $\boldsymbol{p}$, as a function of the generalized coordinates, $\boldsymbol{q}$, and the generalized velocities, $\dot{\boldsymbol{q}}$. Invert this expression to represent the generalized velocities as a function of the generalized coordinates and the generalized momenta.
   (b) Compute the Hamiltonian, $\mathcal{H}$, using the Lagrangian. Express the Hamiltonian as a function of the generalized coordinates and the generalized momenta by replacing the generalized velocities with expressions in terms of the generalized momenta.
   (c) Generate the equations of motion, directly from the Hamiltonian, using Hamilton's canonical equations.
   (d) Express the constrained equations of motion.