USING ALMOST-EVERYWHERE THEOREMS FROM ANALYSIS TO STUDY RANDOMNESS

KENSHI MIYABE, ANDRÉ NIES, AND JING ZHANG

ABSTRACT. We study algorithmic randomness notions via effective versions of almost-everywhere theorems from analysis and ergodic theory. The effectivization is in terms of objects described by a computably enumerable set, such as lower semicomputable functions. The corresponding randomness notions are slightly stronger than Martin-Löf (ML) randomness.

We establish several equivalences. Given a ML-random real z, the additional randomness strengths needed for the following are equivalent.
(1) all effectively closed classes containing z have density 1 at z.
(2) all nondecreasing functions with uniformly left-c.e. increments are differentiable at z.
(3) z is a Lebesgue point of each lower semicomputable integrable function.

We also consider convergence of left-c.e. martingales, and convergence in the sense of Birkhoff's pointwise ergodic theorem. Lastly, we study randomness notions related to density of Π^0_n and Σ^1_1 classes at a real.

CONTENTS

Miyabe was supported by JSPS. Nies was supported by the Marsden fund of New Zealand. Zhang was supported by a University of Auckland summer scholarship.

1. Introduction

Several theorems in analysis and ergodic theory express that all functions in a certain class are well-behaved at almost every point. For instance, Lebesgue published the following theorem in 1904. It is often covered in textbooks on analysis, e.g. [8, Ch. 20].

Theorem 1.1 ([26]). *Let $f\colon [0,1] \to \mathbb{R}$ be a nondecreasing function. Then f is differentiable almost-everywhere.*

Another example of such a result is Birkhoff's ergodic theorem; see e.g. [23, Thm. 2.3] for a textbook reference.

Theorem 1.2 ([5]). *Let T be a measure preserving operator on a probability space X. Let f be an integrable function on X. Then for almost every point $z \in X$, the average of $f(z), f(T(z)), \ldots, f(T^{n-1}(z))$ converges as $n \to \infty$. If the operator is ergodic then this limit is the integral of f.*

The theorems involve a null set of exceptions which usually depends on given objects, such as T and f in Theorem 1.2. By an effective version of such a theorem, we mean the following. If the given objects are algorithmic in some sense, then the resulting null set is also algorithmic. (A slightly stronger effective version of such a theorem would also ask that the null set be obtained uniformly from a presentation of the given objects, without the assumption that it is algorithmic; this is usually the case for the examples we consider.)

Brattka, Miller and Nies [7], in their Thm. 4.1 combined with Remark 4.7, show the following effective version of Lebesgue's theorem. The given object is a computable function.

Theorem 1.3 ([7]). *Suppose a nondecreasing function $f\colon [0,1] \to \mathbb{R}$ is computable. There exists a computable martingale that succeeds on the binary presentation of each real z such that $f'(z)$ fails to exist.*

We explain the terms used in this theorem.

(a) The computability of a function is taken in the usual sense of computable analysis [40]. As shown in the last section of the longer arXiv version of [7], the weaker hypothesis is sufficient that $f(q)$ be a computable real uniformly in a rational q.
(b) In randomness theory, a *martingale* is a function $M\colon 2^{<\omega} \to \mathbb{R}^+_0$ such that $2M(\sigma) = M(\sigma 0) + M(\sigma 1)$. A martingale M succeeds on a bit sequence Z if the value of M on initial segments of Z is unbounded. The success set is a null set which is effective in case M is computable.

A real on which no computable martingale succeeds is called computably random, a notion introduced by Schnorr [38]; for a recent reference see [33, Ch. 7] or [12]. The theorem above shows that $f'(z)$ exists for each computably random real z and each nondecreasing computable function f. Brattka et al. also show that conversely, if a real z is not computably random, then some computable monotonic function f fails to be differentiable at z. In this way, this effective form of Lebesgue's Theorem 1.1 is matched to computable randomness. This is an instance of a more general principle: effective versions of "almost-everywhere" theorems often correspond to well-studied algorithmic randomness notions.

Pathak, Rojas and Simpson [35, Theorem 3.15] matched a particular effective form of the Lebesgue differentiation theorem to Schnorr randomness (the direction where a function is turned into a test was independently proven in [18, Thm. 5.1]). We will discuss this in more detail in Subsection 5.1.

V'yugin [39], Gács et al. [19], Bienvenu et al. [2], Franklin et al. [16], and Franklin and Towsner [17] all studied effective versions of Birkhoff's theorem. For instance, in the notation above, if an ergodic operator T is computable, and the integrable function f is lower semicomputable as defined below, then the corresponding notion is Martin-Löf randomness by [2, 16].

Matching such theorems to algorithmic randomness notions has been useful in two ways:

(a) to determine the strength of the theorem, and
(b) to understand the randomness notion.

For an example of (a), Demuth [11] (see [7] for a proof in modern language) showed that Jordan's extension of Lebesgue's result to functions of bounded variation corresponds to Martin-Löf randomness. This notion is stronger than computable randomness; so in a sense this extension is harder to obtain. For an example of (b), Brattka et al. [7] used their results to show that computable randomness of a real does not depend on the choice of base in its digit expansion, even though martingales (which can also be defined with respect to bases other than 2) bet on such an expansion.

The main purpose of this paper is to examine effective versions of almost-everywhere theorems that do not correspond to known randomness notions. This apparently occurred for the first time when Bienvenu et al. showed in [4, Cor. 5.10] that the randomness notion corresponding to the Denjoy-Young-Saks theorem implies computable randomness, but is incomparable with Martin-Löf randomness.

We base our study on Lebesgue's theorems mentioned earlier, and on the following two results. The first, Lebesgue's density theorem [26], asserts that for almost every point z in a measurable class $\mathcal{C} \subseteq [0,1]$, the class is "thick" around z in the sense that the relative measure of $\mathcal{C}$ converges to 1 as one "zooms in" on z. The second, Doob's martingale convergence theorem [14], says that a martingale converges on almost every point.

The main given object will only be effective in the weak sense of computable enumerations. We consider the Lebesgue density theorem for effectively closed sets of reals (the complement is an open set that can be computably enumerated as a union of rational open intervals). We consider Doob's convergence theorem for martingales that uniformly assign left-c.e. reals to strings.

A group or researchers working at the University of Wisconsin at Madison, consisting of Andrews, Cai, Diamondstone, Lempp, and Miller, showed in 2012 that for a real z the following two conditions are equivalent, thereby connecting the two theorems.

(1) z is Martin-Löf random and every effectively closed class containing z has density 1 at z

(2) every left-c.e. martingale converges along the binary expansion of z.

In this paper we provide two further conditions on a real z that are equivalent to the ones above. They are are also linked to well-known classical results of the "almost-everywhere" type where the main given object is in some sense computably enumerable. The conditions are:

(3) every interval-c.e. function f is differentiable at z

(4) z is Martin-Löf random and a Lebesgue point of each integrable lower semicomputable function $g\colon [0,1] \to \mathbb{R} \cup \{\infty\}$.

By default, functions will have domain $[0,1]$. In (3), the relevant classical result is Lebesgue's theorem on monotonic functions discussed above. To say that a monotonic function f is interval-c.e. means that $f(0) = 0$ and $f(q) - f(p)$ is left-c.e. uniformly in rationals $p < q$. In (4), the classical result is Lebesgue's differentiation theorem, which extends the density theorem. A function g is lower semicomputable if $\{x\colon g(x) > q\}$ is Σ^0_1 uniformly in a rational q.

The new randomness notion identifying the strength of each of the conditions (1)–(4) will be called *density randomness*.

The analytic notion of density has already been very useful for resolving open problems on the complexity of sets of numbers, asked for instance in [30]. It was applied in [9] to show that K-triviality coincides with ML-noncuppability. It was further used to solve the so-called covering problem that every K-trivial is Turing below an incomplete ML-random oracle, and in fact below a single such oracle that also is Δ^0_2. See the survey [1] for more detail and references.

Sections 2-6 of the paper are based on the almost-everywhere theorems that serve as an analytic background for our algorithmic investigations: Lebesgue density theorem, Doob martingale convergence, differentiability of monotonic functions [26], Lebesgue differentiation theorem [27], and Birkhoff's theorem [5]. In a final section we will study density for classes that have descriptional complexity higher than Π^0_1.

This work is a mix of survey and research paper. Section 2 introduces the notion of density of a class at a point in detail, and contains basic results on effective aspects of density, some of them new. Section 3 contains a proof of the unpublished 2012 result of the Madison group (with permission). Section 4 elaborates on a conference paper of Nies [34]. The remainder of the paper consists of new results.

2. Lebesgue density theorem

This section presents background material and some initial results. We discuss the theorem that leads to the definition of two central notions for this paper, density-one points and density randomness. We also look at these notions in the setting of Cantor space. M. Khan and J. S. Miller (see [22]) have shown that among the ML-random reals, this choice of a setting does not make a difference. We show that lowness for density randomness is the same as lowness for ML-randomness, or equivalently, K-triviality.

2.1. Density in the setting of reals.

The definitions below follow [4]. Let λ denote Lebesgue measure.

Definition 2.1. We define the lower Lebesgue density of a set $\mathcal{C} \subseteq \mathbb{R}$ at a point z to be the quantity

$$\underline{\varrho}(\mathcal{C}|z) := \liminf_{\gamma,\delta\to 0^+} \frac{\lambda([z-\gamma, z+\delta]\cap\mathcal{C})}{\gamma+\delta}.$$

Note that $0 \leq \underline{\varrho}(\mathcal{C}|z) \leq 1$.

Theorem 2.2 (Lebesgue [26])**.** *Let $\mathcal{C} \subseteq \mathbb{R}$ be a measurable set. Then $\underline{\varrho}(\mathcal{C}|z) = 1$ for almost every $z \in \mathcal{C}$.*

When $\mathcal{C}$ is open, then the lower Lebesgue density is clearly 1. Thus, the simplest non-trivial case is when $\mathcal{C}$ is closed. We use this case to motivate our central definition.

Definition 2.3. We say that a real $z \in [0,1]$ is a *density-one point* if $\underline{\varrho}(\mathcal{C}|z) = 1$ for every effectively closed class $\mathcal{C}$ containing z. We say that z is *density random* if z is a density-one point and Martin-Löf random.

As noted e.g. in [4], being a density-one point by itself is not a reasonable randomness notion: for instance, every 1-generic real is a density-one point, but fails the law of large numbers.

By the Lebesgue density theorem and the fact that there are only countably many effectively closed classes, almost every real z is density random. Recall that a real is weakly-2-random if it does not lie in any Π^0_2 null class. In fact, any such real is density random: for any effectively closed $\mathcal{C}$ and rational $q < 1$, the null class $\{z \in \mathcal{C} \colon \underline{\varrho}(\mathcal{C} \mid z) \leq q\}$ is Π^0_2.

We say that z is a *positive density point* if $\underline{\varrho}(\mathcal{C}|z) > 0$ for every effectively closed class $\mathcal{C}$ containing z. The difference between positive and full density is typical for our algorithmic setting. In classical analysis, null sets are usually negligible, so everything is settled by Lebesgue's theorem. In effective analysis, a result of Day and Miller [10] separates the two cases: for a ML-random real z, to be a full density-one point is a stronger randomness condition than to be a positive density point.

Bienvenu et al. [4] have shown that a ML-random real z is a positive density point if and only if z is Turing incomplete. In contrast, for density-one points, no characterisation in terms of computational complexity among the ML-random reals is known at present.

2.2. **Density in the setting of Cantor space.** We let $2^{\mathbb{N}}$ denote the usual product probability space of infinite bit sequences. For $Z \in 2^{\mathbb{N}}$ we let $Z \upharpoonright_n$ (or $Z \upharpoonright n$ in subscripts) denote the first n bits of Z. Variables σ, τ, η range over strings in $2^{<\omega}$. We denote by $\sigma \preceq \tau$ that σ is an initial segment of τ; $\sigma \prec \tau$ denotes that σ is a proper initial segment of τ; $\sigma \prec Z$ that σ is an initial segment of the infinite bit sequence Z.

For each σ we let $[\sigma]$ denote the clopen set of extensions of σ. For $\mathcal{C} \subseteq 2^{\mathbb{N}}$ we let $\lambda_\sigma(\mathcal{C}) = 2^{|\sigma|}\lambda(\mathcal{C} \cap [\sigma])$ denote the local measure of $\mathcal{C}$ inside $[\sigma]$.

Consider a measurable set $\mathcal{C} \subseteq 2^{\mathbb{N}}$ and $Z \in 2^{\mathbb{N}}$. The *lower density* of $Z \in 2^{\mathbb{N}}$ in $\mathcal{C}$ is defined to be

$$\underline{\varrho}_2(\mathcal{C}|Z) = \liminf_{n\to\infty} \lambda_{Z\upharpoonright n}(\mathcal{C})$$

We say that a real $z \in [0,1]$ is a *dyadic density-one* point if its dyadic expansion is a density one point in Cantor space. We will use the following result.

Theorem 2.4 (Khan and Miller [22])**.** *Let z be a ML-random dyadic density-one point. Then z is a full density-one point.*

Thus, by the usual identification of irrational real numbers in $[0,1]$ with elements in Cantor space, we can equivalently define density randomness for a real as in Definition 2.3, or for the corresponding bit sequence in Cantor space using lower dyadic density.

2.3. **Lowness for Density randomness.** We say that a Turing oracle A is *low for density randomness* if whenever $Z \in 2^{\mathbb{N}}$ is density random, Z is already density random relative to A. Here, z is density random relative to A if z is ML-random relative to A, and $\underline{\varrho}(\mathcal{C}|z) = 1$ for every A-effectively closed class $\mathcal{C}$ containing z. We will show that this is equivalent to lowness for ML-randomness.

By $\mathsf{W2R}$ we denote the class of weakly-2-random sets, i.e. sets that do not lie in any Π^0_2-null class of sets. Low($\mathsf{W2R}$, MLR) denotes the class of oracles A such that $\mathsf{W2R} \subseteq \mathsf{MLR}^A$. Downey, Nies, Weber and Yu [13] have shown that Low($\mathsf{W2R}$, MLR)=Low(MLR)

Lemma 2.5 (Day and Miller [9])**.** *Suppose Z is Martin-Löf random, A is low for ML-randomness, and $\mathcal{P}$ is a $\Pi^{0,A}_1$ class containing Z. Then there exists a Π^0_1 class $\mathcal{Q} \subseteq \mathcal{P}$ such that $A \in \mathcal{Q}$.*

Theorem 2.6. *$A \in 2^{\mathbb{N}}$ is low for ML-randomness $\Leftrightarrow$*
A is low for density randomness.

Proof. $\Leftarrow$: Let DenseR denote the class of density random sets. Since $\mathsf{W2R} \subseteq \mathsf{DenseR} \subseteq \mathsf{MLR}$ and by the result in [13], we have

$$\text{Low}(\mathsf{DenseR}) \subseteq \text{Low}(\mathsf{W2R}, \mathsf{MLR})\text{=Low}(\mathsf{MLR}).$$

$\Rightarrow$: Suppose that A is not low for density randomness, i.e., there exists a set Z that is density random but not density random relative to A. If Z is not even Martin-Löf random relative to A, then A is not low for ML-randomness. Otherwise, Z is Martin-Löf random relative to A but not density random relative to A. Hence there exists a $\Pi^{0,A}_1$ class $\mathcal{P}$ containing

Z such that $\underline{\varrho}_2(\mathcal{P}|Z) < 1$. By Lemma 2.5, there is a Π^0_1 class $\mathcal{Q} \subseteq \mathcal{P}$ such that $Z \in \mathcal{Q}$. Then $\underline{\varrho}_2(\mathcal{Q}|Z) \le \underline{\varrho}_2(\mathcal{P}|Z) < 1$, so Z is not density random, contradiction. ☐

2.4. **Upper density.** The upper density of $\mathcal{C} \subseteq 2^{\mathbb{N}}$ at Z is:

$$\overline{\varrho}_2(\mathcal{C}|Z) = \limsup_{n\to\infty} \lambda_{Z\upharpoonright n}(\mathcal{C})$$

Bienvenu et al. [3, Prop. 5.4] have shown that for any effectively closed set $\mathcal{P}$ and ML-random $Z \in \mathcal{P}$, we have $\overline{\varrho}_2(\mathcal{P} \mid Z) = 1$. Actually ML-randomness of Z was too strong an assumption. The weaker notion of partial computable randomness, defined in terms of partial computable martingales, already suffices. See [33, Ch. 7] for background on this notion.

Proposition 2.7. *Let $\mathcal{P} \subseteq 2^{\mathbb{N}}$ be effectively closed. Let $Z \in \mathcal{P}$.*

(i) *If Z is partial computably random, then $\overline{\varrho}_2(\mathcal{P} \mid Z) = 1$.*
(ii) *Suppose that, in addition, $\lambda\mathcal{P}$ is computable. If Z is Kurtz random, then $\overline{\varrho}_2(\mathcal{P} \mid Z) = 1$.*

Proof. Suppose that there is a rational $q < 1$ and an $n^* \in \mathbb{N}$ such that $\lambda_\eta(\mathcal{P}) < q$ for each $\eta \prec Z$ with $|\eta| \ge n^*$.
(i). We define a partial computable martingale M that succeeds on Z. Let $M(\eta) = 1$ for all strings η with $|\eta| \le n^*$. Now suppose that $M(\eta)$ has been defined, but M is as yet undefined on any extensions of η. Search for $t = t_\eta > |\eta|$ such that $p := |F|2^{-(t-|\eta|)} \le q$, where

$$F = \{\tau \succ \eta \colon\ |\tau| = t \wedge [\tau] \cap \mathcal{P}_t \neq \emptyset\}.$$

If t_η and F are found, bet all the capital existing at η along the strings in F. That is, for $\tau \succeq \eta$, $|\tau| \le t$, let

$$M(\tau) = M(\eta) \cdot |\{\sigma \in F \colon\ \sigma \succeq \tau\}|/p.$$

Then $M(\sigma) = M(\eta)/p \ge M(\eta)/q$ for each $\sigma \in F$. Now continue the procedure with all such strings $\sigma \succ \eta$ of length t.

For each $\eta \prec Z$ of length at least n^*, we have $\lambda_\eta(\mathcal{P}) < q$, so a t_η as above will be found. Since $Z \in \mathcal{P}$, M never decreases along Z. Then, since $q < 1$, M succeeds on Z.
(ii). Under the extra hypothesis on $\mathcal{P}$, we can make M total, and also bound from below its growth at an infinite computable set of positions along Z. This will show that Z is not Kurtz random (see Downey and Hirschfeldt [12, Theorem 7.2.13]).

Note that $\lambda_\eta(\mathcal{P})$ is a computable real uniformly in η. Pick rationals $q' < q < 1$ and an $n^* \in \mathbb{N}$ such that $\lambda_\eta(\mathcal{P}) < q'$ for each $\eta \prec Z$ with $|\eta| \ge n^*$. In the same situation as above, search for $t_\eta > |\eta|$ such that we see $\lambda_\eta(\mathcal{P}) > q'$ at stage t_η, or F is found. One of the cases must occur. If the former case is seen first, we let $M(\tau) = M(\eta)$ for all $\tau \succ \eta$, $\tau \le t_\eta$. Otherwise, we proceed as above.

For the lower bound on the growth, define a computable function by

$$g(n) = \max\{t_\eta \colon\ n^* \le |\eta| \le n\},$$

for $n \ge n^*$, and $g(n) = 0$ otherwise. Let $r(k) = g^{(2k)}(n^*)$. Then $M(Z \upharpoonright r(k)) \ge q^{-k}$ for each k. ☐

It is not known at present whether the partiality of M in (i) is necessary.

Question 2.8. *Is there a Π^0_1 class $\mathcal{P}$ and a computably random $Z \in \mathcal{P}$ such that $\overline{\varrho}_2(\mathcal{P} \mid Z) < 1$?*

In Subsection 5.1 we will continue the study of Π^0_1 classes of computable measure. We show that such a class has density one at every Schnorr random member.

3. Martingale convergence theorem

For background on martingales in probability theory, see for instance Durrett [14, Ch. 4]. The martingale convergence theorem goes back to work of Doob. Recall that for a random variable Y one defines $Y^+ = \max(Y, 0)$.

Theorem 3.1. *Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be a martingale with $\sup_n EX_n^+ < \infty$. Then $X(w) := \lim_n X_n(w)$ exists almost surely, and $E|X| < \infty$.*

The standard proof (see e.g. [14, Ch. 4, (2.10)]) uses Doob's upcrossing inequality. In randomness theory, researchers have so far only used the very restricted form of the powerful notion of a martingale defined in the introduction: The probability space is Cantor space with the usual product measure. The filtration $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ is defined by letting $\mathcal{F}_n$ be the set of events that only depend on the first n bits. If $\langle X_n \rangle$ is adapted to $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$, then X_n has constant value on each $[\sigma]$ for $|\sigma| = n$. Let $M(\sigma)$ be this value. The martingale condition $E(X_{n+1} \mid \mathcal{F}_n) = X_n$ now turns into $\forall\sigma\, M(\sigma 0) + M(\sigma 1) = 2M(\sigma)$. One also requires that the values be non-negative (so that one can reasonably define that a martingale succeeds along a bit sequence).

Note that $EX_0 = M(\langle\rangle) < \infty$. Thus, Theorem 3.1 turns into the following.

Theorem 3.2. *Let $M\colon 2^{<\omega} \to \mathbb{R}^+_0$ be a martingale in the restricted sense above. Then for almost every $Z \in 2^{\mathbb{N}}$, $X(Z) = \lim_n M(Z\restriction_n)$ exists and is finite. Furthermore, $EX < \infty$.*

If $\lim_n M(Z\restriction_n)$ exists and is finite, we say that M converges *along* Z.

We can now analyze the theorem in the effective setting, according to the main plan of the paper. Firstly we discuss the effective form of Theorem 3.2 in terms of computable martingales. It is not hard to show that a computable martingale converges along any computably random bit sequence Z (see [12, Theorem 7.1.3]). In other words, boundedness of all computable martingales along a bit sequence Z already implies their convergence. For the converse see the proof of [18, Thm. 4.2], where success of a computable martingale is turned into oscillation of another. Thus, this effective form of Theorem 3.2 is matched to computable randomness.

Next, we weaken the effectiveness to a notion based on computable enumerability. A martingale $L\colon 2^{<\omega} \to \mathbb{R}^+_0$ is called *left-c.e.* if $L(\sigma)$ is a left-c.e. real uniformly in σ. Note that Z is Martin-Löf-random iff every such martingale is bounded along Z (see e.g. [33, Prop. 7.2.6]). Unlike the case of computable martingales, convergence requires a stronger form of algorithmic randomness than boundedness. For instance, let $\mathcal{U} = [0, \Omega)$ where Ω is a left-c.e. Martin-Löf-ranom real, and let $L(\sigma) = \lambda_\sigma(\mathcal{U})$; then the left-c.e. martingale L is bounded by 1 but diverges on Ω because Ω is Borel normal.

The following theorem matches left-c.e. martingale convergence to density randomness. It is due to unpublished 2012 work of the "Madison Group" consisting of Andrews, Cai, Diamondstone, Lempp and Joseph S. Miller. Recall that by Theorem 2.4, a ML-random z is a full density-one point iff z is a dyadic density-one point.

Theorem 3.3 (Madison group)**.** *The following are equivalent for a ML-random real $z \in [0,1]$ with binary expansion $0.Z$.*

(i) *z is a dyadic density-one point.*

(ii) *Every left-c.e. martingale converges along Z.*

The writeup of the proof below, due to Nies, is based on discussions with Miller, and Miller's slides for his talks at a Semester dedicated to computability, complexity and randomness at Buenos Aires in 2013 [29]. Nies supplied the technical details of the verifications.

Proof. The easier implication (ii) $\to$ (i) was proved in [3, Corollary 5.5]. Simply note that for a Π^0_1 class $\mathcal{P}$, the function $M(\sigma) = 1 - \lambda_\sigma(\mathcal{P})$ is a left-c.e. martingale. Convergence of M along Z means that $\rho(\mathcal{P} \mid Z)$ exists. Prop. 2.7 implies that the upper density $\overline{\rho}(\mathcal{P} \mid Z)$ equals 1. Therefore $\rho(\mathcal{P} \mid Z) = 1$.

(i) $\to$ (ii). We can work within Cantor space because the dyadic density of a class $\mathcal{P} \subseteq [0,1]$ at z is the same as the density of $\mathcal{P}$ at Z when $\mathcal{P}$ is viewed as a subclass of Cantor space. We use the technical concept of a "Madison test". Such a test is intended to capture the oscillation of a left-c.e. martingale along a bit sequence. We will now introduce and motivate this concept. We define the weight of a set $X \subseteq 2^{\mathbb{N}}$ by

$$\mathsf{wt}(X) = \sum_{\sigma \in X} 2^{-|\sigma|}.$$

Let $\sigma^{\prec} = \{\tau \in 2^{<\omega} \colon\ \sigma \prec \tau\}$ denote the set of proper extensions of a string σ.

Definition 3.4. A *Madison test* is a computable sequence $\langle U_s \rangle_{s\in\mathbb{N}}$ of computable subsets of $2^{<\omega}$ such that $U_0 = \emptyset$, there is a constant c such that for each stage s we have $\mathsf{wt}(U_s) \le c$, and for all strings σ, τ,

(a) $\tau \in U_s - U_{s+1} \to \exists \sigma \prec \tau\, [\sigma \in U_{s+1} - U_s]$

(b) $\mathsf{wt}(\sigma^{\prec} \cap U_s) > 2^{-|\sigma|} \to \sigma \in U_s$.

Note that by (a), $U(\sigma) := \lim_s U_s(\sigma)$ exists for each σ; in fact, $U_s(\sigma)$ changes at most $2^{|\sigma|}$ times.

We say that Z *fails* the test $\langle U_s \rangle_{s\in\mathbb{N}}$ if $Z \upharpoonright_n \in U$ for infinitely many n; otherwise Z *passes* $\langle U_s \rangle_{s\in\mathbb{N}}$.

We show that $\mathsf{wt}(U_s) \le \mathsf{wt}(U_{s+1})$, so that $\mathsf{wt}(U) = \sup_s \mathsf{wt}(U_s) < \infty$ is a left-c.e. real. Suppose that σ is minimal under the prefix relation such that $\sigma \in U_{s+1} - U_s$. By (b) and since $\sigma \notin U_s$, we have $\mathsf{wt}(\sigma^{\prec} \cap U_s) \le 2^{-|\sigma|}$. So enumerating σ adds $2^{-|\sigma|}$ to the weight, while the weight of all the strings above σ that are removed from U_s is at most $2^{-|\sigma|}$.

The implication (i)$\to$(ii) is proved in two steps.

Step 1. Lemma 3.5 shows that if $Z \in 2^{\mathbb{N}}$ is a ML-random dyadic density-one point, then Z passes all Madison tests.

Step 2. Lemma 3.8 shows that if Z passes all Madison tests, then every left-c.e. martingale converges along Z.

Lemma 3.5. *Let Z be a ML-random dyadic density-one point. Then Z passes each Madison test.*

Proof. Suppose that a ML-random bit sequence Z fails a Madison test $\langle U_s \rangle_{s\in\mathbb{N}}$. We will build a ML-test $\langle \mathcal{S}^k \rangle_{k\in\mathbb{N}}$ such that $\forall \sigma \in U\, [\lambda_\sigma(\mathcal{S}^k) \geq 2^{-k}]$, and therefore

$$\underline{\varrho}(2^{\mathbb{N}} - \mathcal{S}^k \mid Z) \leq 1 - 2^{-k}.$$

Since Z is ML-random we have $Z \notin \mathcal{S}^k$ for some k. So Z is not a dyadic density-one point, as witnessed by the Π^0_1 class $2^{\mathbb{N}} - \mathcal{S}^k$.

To define $\langle \mathcal{S}^k \rangle_{k\in\mathbb{N}}$ we construct, for each $k, t \in \omega$ and each string $\sigma \in U_t$, clopen sets $\mathcal{A}^k_{\sigma,t} \subseteq [\sigma]$ given by strong indices for finite sets of strings computed from k, σ, t, such that $\lambda(\mathcal{A}^k_{\sigma,t}) = 2^{-|\sigma|-k}$ for each $\sigma \in U_t$. We will let $\mathcal{S}^k$ be the union of these sets over all σ and t. The clopen sets for k and a final string $\sigma \in U$ will be disjoint from the Π^0_1 class $\mathcal{S}^k$. Condition (b) on Madison tests ensures that during the construction, a string σ can inherit the clopen sets belonging to its extensions τ, without risking that the Π^0_1 class becomes empty above σ.

Construction of clopen sets $\mathcal{A}^k_{\sigma,t} \subseteq [\sigma]$ for $\sigma \in U_t$.
At stage 0 no sets need to be defined because $U_0 = \emptyset$. At stage $t+1$, suppose that $\sigma \in U_{t+1} - U_t$. For each $\tau \succ \sigma$ such that $\tau \in U_t - U_{t+1}$, put $\mathcal{A}^k_{\tau,t}$ into an auxiliary clopen set $\widetilde{\mathcal{A}}^k_{\sigma,t+1}$. Since $\sigma \notin U_t$, by condition (b) on Madison tests, we have $\mathsf{wt}(\sigma^{\prec} \cap U_t) \leq 2^{-|\sigma|}$. Inductively we have $\lambda(\mathcal{A}^k_{\tau,t}) = 2^{-|\tau|-k}$ for each τ, and hence

$$\lambda(\widetilde{\mathcal{A}}^k_{\sigma,t+1}) \leq 2^{-|\sigma|-k}.$$

Now, to obtain $\mathcal{A}^k_{\sigma,t+1}$ we simply add mass from $[\sigma]$ to $\widetilde{\mathcal{A}}^k_{\sigma,t+1}$ in order to ensure equality as required.

Let

$$\mathcal{S}^k_t = \bigcup_{\sigma \in U_t} \mathcal{A}^k_{\sigma,t}.$$

Then $\mathcal{S}^k_t \subseteq \mathcal{S}^k_{t+1}$ by condition (a) on Madison tests. Clearly

$$\lambda \mathcal{S}^k_t \leq 2^{-k} \mathsf{wt}(U_t) \leq 2^{-k}.$$

So $\mathcal{S}^k = \bigcup_t \mathcal{S}^k_t$ determines a ML-test. Since Z is ML-random, we have $Z \notin \mathcal{S}^k$ for some k. If $\sigma \in U$ then by construction $\lambda \mathcal{A}^k_{\sigma,s} = 2^{-|\sigma|-k}$ for almost all s. Thus $\lambda_\sigma(\mathcal{S}^k) \geq 2^{-k}$ as required. $\square$

We now take the second step of the argument. We begin with a remark on Madison tests.

Remark 3.6. Consider a computable rational-valued martingale B; that is, $B(\sigma)$ is a rational uniformly computed (as a single output) from σ. Suppose that c, d are rationals, $0 < c < d$, $B(\langle\rangle) < c$, and B oscillates between values less than c and greater than d along a bit sequence Z. An *upcrossing* (for these values) is a pair of strings $\sigma \prec \tau$ such that $B(\sigma) < c$, $B(\tau) > d$, and $B(\eta) \leq d$ for each η such that $\sigma \preceq \eta \prec \tau$.

Dubins' inequality from probability theory limits the amount of oscillation a martingale can have; see, for instance, [14, Exercise 2.14 on pg. 238]. (Note that this inequality implies a version of the better-known Doob upcrossing inequality by taking the sum over all k.) In the restricted setting of martingales on $2^{<\omega}$, Dubins' inequality shows that for each k

$$\lambda\{X\colon \text{ there are } k \text{ upcrossings along } X\} \leq (c/d)^k. \tag{1}$$

See [3, Lemma 5.8] for a proof of this fact using notation close to the one of the present paper.

Suppose now that $2c < d$. We define a Madison test that Z fails. Strings never leave the computable approximation of the test, so (a) holds.

We put the empty string $\langle\rangle$ into U_1. If $\sigma \in U_{s-1}$, put into U_s all strings η such that $B(\tau) > d$ and $B(\eta) < c$ for some $\tau \succ \sigma$ chosen prefix minimal, and $\eta \succ \tau$ chosen prefix minimal. Let $U = \bigcup U_s$ (which is in fact computable). For each σ, by the inequality (1) localised to $[\sigma]$, we have $\mathsf{wt}(\sigma^{\prec} \cap U) \leq 2^{-|\sigma|} \sum_{k \geq 1} (c/d)^k < 2^{-|\sigma|}$, so (b) is satisfied vacuously.

As noted in [3, Section 5], if $B = \sup B_s$ is a left-c.e. martingale where $\langle B_s \rangle_{s\in\mathbb{N}}$ is a uniformly computable sequence of martingales, an upcrossing apparent at stage s can later disappear because $B(\sigma)$ increases. We will see in the proof of Lemma 3.8 that in this case, the full power of the conditions (a) and (b) is needed to obtain a Madison test from the oscillatory behaviour of B.

We use Remark 3.6 for an intermediate fact, which is not as obvious as one might expect.

Lemma 3.7. *Suppose that Z passes each Madison test. Then Z is computably random.*

Proof. Suppose Z is not computably random. Then some computable rational-valued martingale M with the savings property succeeds on Z (see [33, Ex. 7.1.14 and Prop. 7.3.8] or [12]). The proof of [18, Thm. 4.2] turns success of such a martingale into oscillation of another computable rational-valued martingale B. Slightly adapting the (arbitrary) bounds for the oscillation given there, we may assume that B is as in Remark 3.6 for $c = 2, d = 5$: if M succeeds along Z, then there are infinitely many upcrossings $\tau \prec \eta \prec Z$, $B(\tau) < 2$ and $B(\eta) > 5$. Therefore Z fails the Madison test constructed in Remark 3.6. $\square$

We are now ready for the main part of the second step.

Lemma 3.8. *Suppose that Z passes each Madison test. Then every left-c.e. martingale L converges along Z. In particular, Z is ML-random.*

Proof. Let L be a left-c.e. martingale. Then $L(\sigma) = \sup_s L_s(\sigma)$ where $\langle L_s \rangle$ is a uniformly computable sequence of martingales, and $L_0 = 0$ and $L_s(\sigma) \leq L_{s+1}(\sigma)$ for each σ and s. Since Z is computably random, $\lim_n L_s(Z\upharpoonright_n)$ exists for each s. If L diverges along Z, then $\lim_n L(Z\upharpoonright_n) = \infty$ or there is a positive $\varepsilon < L(\langle\rangle)$ such that

$$\limsup_n L(Z\upharpoonright_n) - \liminf_n L(Z\upharpoonright_n) > \varepsilon.$$

Based on this fact we define a Madison test that Z fails. Along with the U_s we define a uniformly computable labelling function $\gamma_s\colon U_s \to \{0, \ldots, s\}$. If $\lim_n L(Z\restriction_n) = \infty$ set $\varepsilon = 1$. The construction is as follows.

> *Let $U_0 = \emptyset$. For $s > 0$ we put the empty string $\langle\rangle$ into U_s and let $\gamma_s(\langle\rangle) = 0$. If already $\sigma \in U_s$ with $\gamma_s(\sigma) = t$, then we also put into U_s all strings $\tau \succ \sigma$ that are minimal under the prefix ordering with $L_s(\tau) - L_t(\tau) > \varepsilon$. Let $\gamma_s(\tau)$ be the least r with $L_r(\tau) - L_t(\tau) > \varepsilon$.*

Note that $\gamma_s(\tau)$ records the greatest stage $r \le s$ at which τ entered U_r. Intuitively, this construction attempts to find upcrossings between values (arbitrarily close to) $\liminf_n L(Z\restriction_n)$ and $\limsup_n L(Z\restriction_n)$. Clearly

$$\lim_n L_t(Z\restriction_n) \le \liminf_n L(Z\restriction_n).$$

So, if a string $\tau \prec Z$ as above is sufficiently long, then we have an upcrossing of the required kind.

We verify that $\langle U_s\rangle_{s\in\mathbb{N}}$ is a Madison test. For condition (a), suppose that $\tau \in U_s - U_{s+1}$. Let $\sigma_0 \prec \sigma_1 \prec \ldots \prec \sigma_n = \tau$ be the prefixes of τ in U_s. We can choose a least $i < n$ such that σ_{i+1} is no longer the minimal extension of σ_i at stage $s+1$. Thus there is η with $\sigma_i \prec \eta \prec \sigma_{i+1}$ and $L_{s+1}(\eta) - L_{\gamma_s(\sigma_i)}(\eta) > \varepsilon$. Then $\eta \in U_{s+1}$ and $\eta \prec \tau$, as required.

We verify condition (b). We fix s, and for $t \le s$ write

$$M_t(\eta) = L_s(\eta) - L_t(\eta).$$

Thus M_t is the increase of L from t to s. Note that M_t is a martingale.

Claim 3.9. *For each $\eta \in U_s$, where $\gamma_s(\eta) = r$, we have*

$$2^{-|\eta|} M_r(\eta) \ge \varepsilon \cdot \mathsf{wt}(U_s \cap \eta^{\prec}).$$

In particular, if $\eta = \langle\rangle$ then $r = 0$; we obtain that $\mathsf{wt}(U_s)$ is bounded by a constant $c = L(\langle\rangle)\varepsilon^{-1} + 1$ (the "+1" is for the empty string in U_s), as required.

For $\sigma \in U_s$ and $k \in \mathbb{N}$, let $U_s^\sigma(k)$ be the set of strings properly extending σ and at a distance to σ of at most k, that is, the set of strings τ such that there is $\sigma = \sigma_0 \prec \ldots \prec \sigma_m = \tau$ on U_s with $m \le k$ and σ_{i+1} a child (i.e., immediate successor) of σ_i for each $i < m$. To establish the claim, we show by induction on k that

$$2^{-|\eta|} M_r(\eta) \ge \varepsilon \cdot \mathsf{wt}(U_s^\eta(k)).$$

If $k = 0$ then $U_s^\eta(k)$ is empty so the right hand side equals 0. Now suppose that $k > 0$. Let F be the set of of children of η on U_s. For $\tau \in F$ write $r_\tau = \gamma_s(\tau)$. Then $s \ge r_\tau > r$ by the definition of the function γ_s. By the inductive hypothesis, we have for each $\tau \in F$

$$\begin{aligned} 2^{-|\tau|} M_r(\tau) &= 2^{-|\tau|}[(L_{r_\tau}(\tau) - L_r(\tau)) + M_{r_\tau}(\tau)] \\ &\ge 2^{-|\tau|} \cdot \varepsilon + \varepsilon \cdot \mathsf{wt}(U_s^\tau(k-1)). \end{aligned} \tag{2}$$

Then, taking the sum over all $\tau \in F$,

$$2^{-|\eta|} M_r(\eta) \ge \sum_{\tau \in F} 2^{-|\tau|} M_r(\tau) \ge \varepsilon \cdot \mathsf{wt}(U_s^\eta(k)).$$

The first inequality holds by a general fact about for martingales attributed to Kolmogorov (see [33, 7.1.8]), and uses that F is an antichain. For the second inequality we have used (2) and that $U_s^\eta(k) = F \cup \bigcup_{\tau \in F} U_s^\tau(k-1)$. This completes the induction and shows the claim.

Now, to obtain (b), suppose that $\mathsf{wt}(U_s \cap \sigma^\prec) > 2^{-|\sigma|}$. We show that $\sigma \in U_s$. Assume otherwise. Let $\eta \prec \sigma$ be in U_s with $|\eta|$ maximal, and let $r = \gamma_s(\eta)$. Let now F be the set of prefix minimal extensions of σ in U_s, and $r_\tau = \gamma_s(\tau)$. Then $L_{r_\tau}(\tau) - L_r(\tau) > \varepsilon$ for $\tau \in F$. Since $\tau \in U_s$, we can apply Claim 3.9 to τ. We now argue similar to the above, but with σ instead of η, and using in the last line that $U_s \cap \sigma^\prec = F \cup \bigcup_{\tau \in F}(U_s \cap \tau^\prec)$:

$$\begin{aligned}
2^{-|\sigma|}M_r(\sigma) &\geq \sum_{\tau \in F} 2^{-|\tau|}M_r(\tau) \\
&= \sum_{\tau \in F} 2^{-|\tau|}[L_{r_\tau}(\tau) - L_r(\tau) + M_{r_\tau}(\tau)] \\
&\geq \sum_{\tau \in F} 2^{-|\tau|}[\varepsilon + \varepsilon \cdot \mathsf{wt}(U_s \cap \tau^\prec)] \\
&\geq \varepsilon \cdot \mathsf{wt}(U_s \cap \sigma^\prec).
\end{aligned}$$

Since $\mathsf{wt}(U_s \cap \sigma^\prec) > 2^{-|\sigma|}$, this implies that $M_r(\sigma) > \varepsilon$. Hence some η' with $\eta \prec \eta' \prec \sigma$ is in U_s, contrary to the maximality of η.

This concludes the verification that $\langle U_s \rangle_{s \in \mathbb{N}}$ is a Madison test. As mentioned, for each r there are infinitely many n with $L(Z\upharpoonright_n) - L_r(Z\upharpoonright_n) > \varepsilon$. This shows that Z fails this test: suppose inductively that we have $\sigma \prec Z$ such that there is a least r with $\sigma \in U_t$ for all $t \geq r$ (so that $\gamma_t(\sigma) = r$ for all such t). Choose $n > |\sigma|$ for this r. Then from some stage on $\tau = Z\upharpoonright_n$ is a viable extension of σ, so τ, or some prefix of it that is longer than σ, is in U. □

This concludes our proof of Thm. 3.3. □

4. Differentiability of non-decreasing functions

We consider an effective version, in the sense of computable enumerability, of Lebesgue's theorem 1.1 that non-decreasing functions are almost everywhere differentiable. Freer, Kjos-Hanssen, Nies and Stephan [18] studied a class of non-decreasing functions they called *interval-c.e.* They showed (with J. Rute) that the continuous interval-c.e. functions are precisely the variation functions of computable functions.

Definition 4.1. A non-decreasing function $f\colon [0,1] \to \mathbb{R}$ is *interval-c.e.* if $f(0) = 0$, and $f(y) - f(x)$ is a left-c.e. real, uniformly in all rationals $x < y$.

We match an effective version of Lebesgue's theorem, stated in terms of interval-c.e. functions, to density randomness. This result is due to Nies in the conference paper [34]. We give a more detailed proof here.

Theorem 4.2 ([34]). *$z \in [0,1]$ is density random $\Leftrightarrow$ $f'(z)$ exists for each interval-c.e. function $f\colon [0,1] \to \mathbb{R}$.*

$\Leftarrow$: If z is not density random then by Theorem 3.3 a left-c.e. martingale M diverges along the binary expansion of z. Let μ_M be the measure on

$[0,1]$ corresponding to M, which is given by $\mu[\sigma] = 2^{-|\sigma|}M(\sigma)$, and let $\mathsf{cdf}_M(x) = \mu_M[0,x)$. Then cdf_M is interval-c.e. and $(\mathsf{cdf}_M)'(z)$ fails to exist.

The rest of this section is devoted to proving the implication $\Rightarrow$. This combines purely analytical arguments with effectiveness considerations.

4.1. **Slopes and martingales.** First we need notation and a few definitions, mostly taken from [7] or [4]. For a function $f\colon [0,1] \to \mathbb{R}$, the *slope* at a pair a, b of distinct reals in its domain is

$$S_f(a,b) = \frac{f(a) - f(b)}{a - b}.$$

For a nontrivial interval A with endpoints a, b, we also write $S_f(A)$ instead of $S_f(a,b)$.

We let σ, τ range over (binary) strings. For such a string σ, by $[\sigma]$ we denote the closed basic dyadic interval $[0.\sigma, 0.\sigma + 2^{-|\sigma|}]$. The corresponding open basic dyadic interval is denoted (σ).

Derivatives. If z is in an open neighborhood of the domain of f, the *upper* and *lower derivatives* of f at z are

$$\overline{D}f(z) = \limsup_{h\to 0} S_f(z, z+h) \quad \text{and} \quad \underline{D}f(z) = \liminf_{h\to 0} S_f(z, z+h),$$

where h ranges over reals. The derivative $f'(z)$ exists if and only if these values coincide and are finite.

We will also consider the upper and lower *pseudo*-derivatives defined by:

$$\widetilde{D}f(x) = \limsup_{h\to 0^+} \{S_f(a,b) \mid a \le x \le b \ \wedge\ 0 < b - a \le h\},$$
$$\underset{\sim}{D}f(x) = \liminf_{h\to 0^+} \{S_f(a,b) \mid a \le x \le b \ \wedge\ 0 < b - a \le h\},$$

where a, b range over rationals in $[0,1]$. We use them because in our arguments it is often convenient to consider rational intervals containing x, rather than intervals that have x as an endpoint.

Remark 4.3. Brattka et al. [7, after Fact 2.4] verified that

$$\underline{D}f(z) \le \underset{\sim}{D}f(z) \le \widetilde{D}f(z) \le \overline{D}f(z)$$

for any real $z \in [0,1]$. To show $\widetilde{D}f(z) \le \overline{D}f(z)$, given any real z and rationals $a \le z \le b$ with $a < b$, we have

$$S_f(a,b) = \tfrac{b-z}{b-a} S_f(b,z) + \tfrac{z-a}{b-a} S_f(z,a) \le \overline{D}f(z).$$

The inequality $\underline{D}f(z) \le \underset{\sim}{D}f(z)$ can be shown in a similar way.

If f is nondecreasing one can in fact verify equality, so the lower and upper pseudo-derivatives of f coincide with the usual lower and upper derivatives.

We will use the subscript 2 to indicate that all the limit operations are restricted to the case of basic dyadic intervals containing z. Thus,

$$\widetilde{D}_2 f(x) \quad = \quad \limsup_{|A|\to 0} \{S_f(A) \mid x \in A \ \wedge\ A \text{ is a basic dyadic interval}\},$$
$$\underset{\sim}{D}_2 f(x) \quad = \quad \liminf_{|A|\to 0} \{S_f(A) \mid x \in A \ \wedge\ A \text{ is a basic dyadic interval}\}.$$

4.2. **Porosity and upper derivatives.** We say that a set $\mathcal{C} \subseteq \mathbb{R}$ is *porous at* z via the constant $\varepsilon > 0$ if there exist arbitrarily small $\beta > 0$ such that $(z-\beta, z+\beta)$ contains an open interval of length $\varepsilon\beta$ that is disjoint from $\mathcal{C}$. We say that $\mathcal{C}$ is *porous at* z if it is porous at z via some $\varepsilon > 0$. This notion originated in the work of Denjoy. See for instance [6, 5.8.124] (but note the typo in the definition there).

Definition 4.4 ([4]). We call z a *porosity point* if some effectively closed class to which it belongs is porous at z. Otherwise, z is a *non-porosity point*.

Clearly, if $\mathcal{C}$ is porous at z then $\underline{\varrho}(\mathcal{C}|z) < 1$, so z is not a density-one point. The converse fails: every Turing incomplete Martin-Löf random real is a non-porosity point by [4]. By [10] there is such a real such that $\underline{\varrho}(\mathcal{C}|z) < 1$ for some Π^0_1 class $\mathcal{C}$. We also note that it is unknown whether a Turing complete Martin-Löf random real can be a non-porosity point. If not, then the sets of positive density and non-porosity ML-random reals coincide.

We show that if the dyadic and full upper/lower derivatives at z are different, then some closed set is porous at z. This extends the idea in the proof of Theorem 2.4 due to Khan and Miller. We begin with the easier case of the upper derivative. The other case will be supplied in Subsection 4.4.

Proposition 4.5. *Let $f\colon [0,1] \to \mathbb{R}$ be interval-c.e. If z is a non-porosity point, then* $\widetilde{D}_2 f(z) = \widetilde{D} f(z)$

Proof. Suppose that $\widetilde{D}_2 f(z) < p < \widetilde{D} f(z)$ for a rational p. Choose $k \in \mathbb{N}$ such that $p(1+2^{-k+1}) < \widetilde{D} f(z)$.

Let $\sigma^* \prec Z$ be any string such that $\forall \sigma\, [\sigma^* \preceq \sigma \prec Z \Rightarrow S_f([\sigma]) \le p]$. It is sufficient to establish the following.

Claim 4.6. *The closed set*

$$\mathcal{C} = [\sigma^*] - \bigcup \{(\sigma) \mid S_f([\sigma]) > p\},$$

which contains z, is porous at z.

If f is interval-c.e., the function $\sigma \to S_f([\sigma])$ is a left-c.e. martingale. In particular, $\mathcal{C}$ is effectively closed, and porous at z.

The proof of the claim is purely analytical, and only uses that f is nondecreasing. We show that there exist arbitrarily large n such that some basic dyadic interval $[a, \tilde{a}]$ of length 2^{-n-k} is disjoint from $\mathcal{C}$, and contained in $[z - 2^{-n+2}, z + 2^{-n+2}]$. In particular, we can choose 2^{-k-2} as a porosity constant.

By choice of k there is an interval $I \ni z$ of arbitrarily short positive length such that $p(1+2^{-k+1}) < S_f(I)$. Let n be such that $2^{-n+1} > |I| \ge 2^{-n}$. Let a_0 be greatest of the form $\ell 2^{-n-k}$, $\ell \in \mathbb{Z}$, such that $a_0 < \min I$. Let $a_v = a_0 + v 2^{-n-k}$. Let r be least such that $a_r \ge \max I$.

Since f is nondecreasing and $a_r - a_0 \le |I| + 2^{-n-k+1} \le (1+2^{-k+1})|I|$, we have

$$S_f(I) \le S_f(a_0, a_r)(1+2^{-k+1}),$$

and therefore $S_f(a_0, a_r) > p$. Since $S_f(a_0, a_r)$ is the average of the slopes $S_f(a_u, a_{u+1})$ for $u < r$, there is a $u < r$ such that

$$S_f(a_u, a_{u+1}) > p.$$

Since $(a_u, a_{u+1}) = (\sigma)$ for some string σ, this gives the required 'hole' in $\mathcal{C}$ which is near $z \in I$ and large on the scale of I: in the definition of porosity at the beginning of this subsection, let $\beta = 2^{-n+2}$ and note that we have $[a_u, a_{u+1}] \subseteq [z - 2^{-n+2}, z + 2^{-n+2}]$ because $z \in I$ and $|I| < 2^{-n+1}$. $\square$

4.3. **Basic dyadic intervals shifted by** $1/3$**.** We will use a basic 'geometric' fact observed, for instance, by Morayne and Solecki [32]. For $m \in \mathbb{N}$ let $\mathcal{D}_m$ be the collection of intervals of the form

$$[k2^{-m}, (k+1)2^{-m}]$$

where $k \in \mathbb{Z}$. Let $\widehat{\mathcal{D}}_m$ be the set of intervals $(1/3) + I$ where $I \in \mathcal{D}_m$.

Lemma 4.7. *Let $m \geq 1$. If $I \in \mathcal{D}_m$ and $J \in \widehat{\mathcal{D}}_m$, then the distance between an endpoint of I and an endpoint of J is at least $1/(3 \cdot 2^m)$.*

To see this, assume that $|k2^{-m} - (p2^{-m} + 1/3)| < 1/(3 \cdot 2^m)$. This yields $|3k - 3p - 2^m|/(3 \cdot 2^m) < 1/(3 \cdot 2^m)$, and hence $3|2^m$, a contradiction.

In order to apply Lemma 4.7, we may need values of nondecreasing functions $f\colon [0,1] \to \mathbb{R}$ at endpoints of any such intervals, which may lie outside $[0,1]$. So we think of f as extended to $[-1,2]$ via $f(x) = f(0)$ for $-1 \leq x < 0$ and $f(y) = f(1)$ for $1 < y \leq 2$. Being interval-c.e. is preserved by this because it suffices to determine the values of the function at rationals.

4.4. **Porosity and lower derivatives.** We complete the proof of the implication "$\Rightarrow$" in Theorem 4.2. We may assume that $z > 1/2$. Note that $z - 1/3$ is a ML-random density-one point, hence a dyadic density-one point. In particular, both z and $z - 1/3$ are non-porosity points. Also, Theorem 3.3, all left c.e. martingales converge on the binary expansions of the reals z and $z - 1/3$.

Let $M = M_f$ be the left-c.e. martingale given by $\sigma \to S_f([\sigma])$. Then M converges on z (recall that we write $M(z)$ for the limit). Thus $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) = M(z)$.

Let $\widehat{f}(x) = f(x + 1/3)$, and let $\hat{M} = M_{\widehat{f}}$. Then $\hat{M}$ converges on $z - 1/3$.

Claim 4.8. $M(z) = \hat{M}(z - 1/3)$.

If $M(z) < \hat{M}(z-1/3)$ then $\widetilde{D}_2 f(z) < \widetilde{D} f(z)$. However, z is a non-porosity point, so this contradicts Proposition 4.5. If $\hat{M}(z - 1/3) < M(z)$ we argue similarly using that $z - 1/3$ is a non-porosity point. This establishes the claim.

We have already shown that $\underset{\sim}{D}_2 f(z) = \widetilde{D}_2 f(z) = \widetilde{D} f(z)$, so to complete the proof of "$\Rightarrow$" in Theorem 4.2, it suffices to show that $\underset{\sim}{D} f(z) = \underset{\sim}{D}_2 f(z)$. Then, since f is nondecreasing, $f'(z)$ exists by Remark 4.3.

Assume for a contradiction that if $\underset{\sim}{D} f(z) < \underset{\sim}{D}_2 f(z)$. We will show that one of z, $z - 1/3$ is a porosity point. First we define porosity in Cantor space.

Definition 4.9. For a closed set $\mathcal{C} \subseteq 2^{\mathbb{N}}$, we say that $\mathcal{C}$ is porous at $Y \in \mathcal{C}$ if there is $r \in \mathbb{N}$ as follows: there exists arbitrarily large m such that

$$\mathcal{C} \cap [(Y\restriction_m)\hat{\ }\tau] = \emptyset \text{ for some } \tau \text{ of length } r.$$

Clearly this implies that $\mathcal{C}$ viewed as a subclass of $[0,1]$ is porous at $0.Y$ (now "holes" on both sides of $0.Y$ are allowed). We will actually define Π^0_1 classes $\mathcal{E}$ and $\widehat{\mathcal{E}}$ in Cantor space such that $\mathcal{E}$ is porous at the binary expansion of z, or $\widehat{\mathcal{E}}$ is porous at the binary expansion of $z-1/3$.

We employ a method similar to the one in Subsection 4.2, but now take into account both dyadic intervals, and dyadic intervals shifted by 1/3 of the same length. Recall that $\underset{\sim}{D}_2 f(z) = M(z)$.

We can choose rationals p, q such that

$$\underset{\sim}{D} f(z) < p < q < M(z) = \hat{M}(z-1/3).$$

Let $k \in \mathbb{N}$ be such that $p < q(1-2^{-k+1})$. Let u, v be rationals such that

$$q < u < M(z) < v \text{ and } v - u \leq 2^{-k-3}(u-q).$$

Recalling the notation in Subsection 4.3, let $n^* \in \mathbb{N}$ be such that for each $n \geq n^*$ and any interval $A \in \mathcal{D}_n \cup \widehat{\mathcal{D}}_n$ containing z, we have $S_f(A) \geq u$. Let

$$\begin{aligned} \mathcal{E} &= \{X \in 2^{\mathbb{N}} \colon \forall n \geq n^* M(X\restriction_n) \leq v\} \\ \widehat{\mathcal{E}} &= \{W \in 2^{\mathbb{N}} \colon \forall n \geq n^* \hat{M}(W\restriction_n) \leq v\} \end{aligned}$$

Since f is interval-c.e., M and $\hat{M}$ are left-c.e. martingales, so these classes are effectively closed.

Let Z be the bit sequence such that $z = 0.Z$. By the choice of n^* we have $Z \in \mathcal{E}$. Let Y be the bit sequence such that $0.Y = z - 1/3$. We have $Y \in \widehat{\mathcal{E}}$.

Consider an interval $I \ni z$ of positive length $\leq 2^{-n^*-3}$ such that $S_f(I) \leq p$. Let n be such that $2^{-n+1} > |I| \geq 2^{-n}$. Let a_0 be least of the form $w2^{-n-k}$ where $w \in \mathbb{Z}$, such that $a_0 \geq \min(I)$. Similarly, let b_0 be least of the form $w2^{-n-k} + 1/3$ such that $b_0 \geq \min(I)$. Let

$$a_i = a_0 + i2^{-n-k} \text{ and } b_j = b_0 + j2^{-n-k}.$$

Let r, s be greatest such that $a_r \leq \max(I)$ and $b_s \leq \max(I)$.

Since f is nondecreasing and

$$a_r - a_0 \geq |I| - 2^{-n-k+1} \geq (1-2^{-k+1})|I|,$$

we have $S_f(I) \geq S_f(a_0, a_r)(1-2^{-k+1})$, and therefore $S_f(a_0, a_r) < q$. Then there is an $i < r$ such that $S_f(a_i, a_{i+1}) < q$. Similarly, there is $j < s$ such that $S_f(b_j, b_{j+1}) < q$.

Claim 4.10. *One of the following is true.*

(i) *z, a_i, a_{i+1} are all contained in a single interval from $\mathcal{D}_{n-3}$.*

(ii) *z, b_j, b_{j+1} are all contained in a single interval from $\widehat{\mathcal{D}}_{n-3}$.*

For suppose that (i) fails. Then there is an endpoint of an interval $A \in \mathcal{D}_{n-3}$ (that is, a number of the form $w2^{-n+3}$ with $w \in \mathbb{Z}$) between $\min(z, a_i)$ and $\max(z, a_{i+1})$. Note that $\min(z, a_i)$ and $\max(z, a_{i+1})$ are in I. By Fact 4.7 and since $|I| < 2^{-n+1}$, there can be no endpoint of an interval $\hat{A} \in \widehat{\mathcal{D}}_{n-3}$ in I. Then, since $b_j, b_{j+1} \in I$, (ii) holds. This establishes the claim.

Suppose I is an interval as above and $2^{-n+1} > |I| \geq 2^{-n}$, where $n \geq n^*+3$. Let $\eta = Z\restriction_{n-3}$ and $\hat{\eta} = Y\restriction_{n-3}$.

If (i) holds for this I then there is a string α of length $k+3$ (where $[\eta\alpha] = [a_i, a_{i+1}]$) such that $M(\eta\alpha) < q$. So by the choice of $q < u < v$ and since $M(\eta) \geq u$ there is β of length $k+3$ such that $M(\eta\beta) > v$. (The decrease along $\eta\alpha$ of the martingale M must be balanced by an increase along some $\eta\beta$.) This yields a "hole" in $\mathcal{E}$, large and near Z on the scale of I, as required for the porosity of $\mathcal{E}$ at Z; in the notation of the Definition 4.9 above, $\mathcal{E}$ is porous at Z via $m = |\eta|$ and $r = k + 3$.

Similarly, if (ii) holds for this I, then there is a string α of length $k + 3$ (where $[\hat{\eta}\alpha] = [b_j, b_{j+1}]$) such that $M(\hat{\eta}\alpha) < q$. So by the choice of $q < u < v$ and since $\hat{M}(\hat{\eta}) \geq u$, there is a string β of length $k+3$ such that $\hat{M}(\hat{\eta}\beta) > v$. This yields a hole in $\widehat{\mathcal{E}}$, large and near Y on the scale of I, as required for the porosity of $\widehat{\mathcal{E}}$ at Y.

Thus, if case (i) applies for arbitrarily short intervals I, then $\mathcal{E}$ is porous at Z, whence z is a porosity point. Otherwise (ii) applies for intervals below a certain length. Then $\widehat{\mathcal{E}}$ is porous at Y, whence $z - 1/3$ is a porosity point. Both cases are contradictory. This concludes the proof of Theorem 4.2.

Nies [34] also uses porosity for an effective version of Lebesgue's theorem 1.1 in the setting of polynomial time computable functions and martingales. The proof can be easily adapted to the original setting of computable functions and martingales, thereby providing a simpler proof of the main result in Brattka et al., Theorem 1.3.

5. Lebesgue differentiation theorem

This section is centred around an effective version, in the c.e. setting, of another result obtained by Lebesgue in 1904 [25].

Definition 5.1. Given an integrable non-negative function g on $[0, 1]$, a point z in the domain of g is called a *weak Lebesgue point* of g if

$$\lim_{Q \to z} \frac{1}{\lambda(Q)} \int_Q g$$

exists, where Q ranges over open intervals containing z with length $\lambda(Q)$ tending to 0; z is called a *Lebesgue point* of g if this value equals $g(z)$.

We note that also a variant of this definition can be found in the literature, where Q is centred at z. This is in fact equivalent to the definition given here; see for instance [37, Thm. 7.10]

Theorem 5.2 (Lebesgue [25]). *Suppose g is an integrable function on $[0, 1]$. Then almost every $z \in [0, 1]$ is a Lebesgue point of g.*

Equivalently, the function $f(z) = \int_{[0,z]} g d\lambda$ is differentiable at almost every z, and $f'(z) = g(z)$.

Several years later, Lebesgue [27] extended this result to higher dimensions; the variable Q now ranges over open cubes containing z.

5.1. Effective Lebesgue Differentiation Theorem via L_1-computability.

Pathak, Rojas and Simpson [35, Theorem 3.15] studied an effective version of Lebesgue's theorem, where the given function is L_1-computable, as defined in [36] (or see [35, Def. 2.6]). They showed that

z is Schnorr random $\Leftrightarrow$

z is a weak Lebesgue point of each L_1-computable function.

The implication "$\Rightarrow$" was independently obtained in [18, Thm. 5.1]. Using this result, we observe that if a Π^0_1 class has computable measure, it has density 1 at every Schnorr random member.

Proposition 5.3. *Let $\mathcal{P} \subseteq [0,1]$ be an effectively closed set such that $\lambda\mathcal{P}$ is computable. Let $z \in \mathcal{P}$ be a Schnorr random real. Then $\underline{\varrho}(\mathcal{P} \mid z) = 1$.*

Proof. Let $P = \bigcap_s P_s$ for a computable sequence $\langle P_s \rangle$ of finite unions of closed intervals. There is a computable function g such that $\lambda(P_{g(n)} - P) \leq 2^{-n}$. Hence the characteristic function 1_P is L_1-computable. Now by [35, Theorem 3.15] or [18, Thm. 5.1], the density of $\mathcal{P}$ at z exists, that is $\underline{\varrho}(P \mid z) = \overline{\varrho}(P \mid z)$.

The binary expansion Z of the real z is Kurtz random, so by Proposition 2.7(ii) we have $\overline{\varrho}_2(P \mid Z) = 1$. Therefore $\underline{\varrho}(P \mid z) = 1$. ☐

5.2. **Dyadic Lebesgue points and integral tests.** Recall that an open basic dyadic interval in $[0,1]$ has the form $(i2^{-n}, (i+1)2^{-n})$ where $i < 2^n$. If a string σ of length n is the binary expansion of i, we also write (σ) for this interval. We say that z is a (weak) *dyadic Lebesgue point* if the limit in Definition 5.1 exists when Q is restricted to open basic dyadic intervals,

As usual let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For a function $f : [0,1] \to \overline{\mathbb{R}}$ and $z \in [0,1]$, let

$$E(f, \sigma) = \frac{\int_{(\sigma)} f \, d\lambda}{2^{-n}}.$$

Then, z is a dyadic Lebesgue point iff $\lim_n E(f, Z \upharpoonright_n) = f(z)$ where $z = 0.Z$.

Recall from the introduction that a function $g : [0,1] \to \mathbb{R} \cup \{\infty\}$ is lower semi-computable if $f^{-1}(\{z \colon z > q\})$ is effectively open, uniformly in a rational q. (This is an effective version of lower semicontinuity.) It is well-known that such functions can be used to characterise Martin-Löf randomness; see for instance Li and Vitányi [28, Subsection 4.5.6].

Definition 5.4. An *integral test* is a non-negative lower semi-computable function $g : [0,1] \to \overline{\mathbb{R}}$ such that $\int g \, d\lambda < \infty$.

Theorem 5.5 (Levin)**.** *A real z is Martin-Löf random if and only if $g(z) < \infty$ for each integral test g.*

Note that if f is an integral test, the function $\sigma \mapsto E(f, \sigma)$ is a left-c.e. martingale. Since f is integrable, $f^{-1}(\{\infty\})$ is a null set.

In Definition 5.1 of [weak] Lebesgue points, we allow functions g that can take the value ∞. For z to be a (weak) Lebesgue point, the limit as the intervals approach z is required to be finite. First we show that for an integral test g, the dyadic versions of the weak and strong conditions in Def. 5.1 coincide at a ML-random real z.

Lemma 5.6. *Let g be an integral test, and let z be a Martin-Löf random real. If z is a dyadic* weak *Lebesgue point of g, then z is in fact a dyadic Lebesgue point of g.*

Proof. Let $\langle g_s \rangle_{s \in \mathbb{N}}$ be an increasing computable sequence of step functions with dyadic points of discontinuity and rational values such that $\sup g_s(z) = g(z)$ for each dyadic irrational (see Miyabe [31, Lemmas 4.6, 4.8], a variant of [39, Prop. 2]). Then, there is a non-decreasing computable function $u \colon \mathbb{N} \to \mathbb{N}$ such that for each σ with $|\sigma| \geq u(s)$

$$E(g_s, \sigma) = E(g_s, \sigma 0) = E(g_s, \sigma 1).$$

Unless z is a dyadic rational, we have $g_t(z) = \lim_n E(g_t, Z\restriction_n)$, where, as usual, $0.Z$ is the binary expansion of z.

By hypothesis, $\lim_n E(g, Z\restriction_n) =: r$ exists. Clearly $g(z) \le r$, because for each t

$$g_t(z) = \lim_n E(g_t, Z\restriction_n) \le \lim_n E(g, Z\restriction_n). \tag{3}$$

Suppose for a contradiction that $g(z) < r$, and let q be a rational number such that $g(z) < q < r$. We build an integral test h such that $h(z) = \infty$, which contradicts our assumption that z is ML-random. To do so, we define a uniformly c.e. sequence of sets $S_n \subseteq 2^{<\omega} \times \omega$. Let $S_0 = \{(\langle\rangle, 0)\}$. Suppose now that $n \ge 1$ and S_{n-1} has been defined. Uniformly in $(\sigma, s) \in S_{n-1}$, let $B \subseteq 2^{<\omega}$ be a c.e. antichain of strings of length $\ge u(s)$ such that

$$[B]^{\prec} = [\{\tau \succ \sigma \colon\ |\tau| \ge u(s) \wedge \exists t\, E(g_t, \tau) > q\}]^{\prec}.$$

For each $\tau \in B$ let $t > s$ be the least corresponding stage and put $\langle \tau, t\rangle$ into S_n.

Let $\mathbf{1}_A$ denote the characteristic function of a set A. For each $(\tau, t) \in S_n$, let

$$h_\tau = (q - E(g_s, \tau))\mathbf{1}_{[\tau]}$$

where $(\sigma, s) \in S_{n-1}$ and $\sigma \prec \tau$. We define h by

$$h = \sum_n \sum_{(\tau,t)\in S_n} h_\tau.$$

We aim to show that h is an integral test and $h(z) = \infty$. So z is not ML-random contrary to our assumption.

To see that h is an integral test, note that h is lower semicomputable. So it suffices to show that, for every N,

$$\sum_{n=0}^{N} \sum_{(\tau,t)\in S_n} \int h_\tau \, d\lambda \le \int g \, d\lambda < \infty.$$

If $(\tau, t) \in S_n$, for $n > 0$, let $(\sigma_{\tau,t}, s_{\tau,t}) \in S_{n-1}$ be the corresponding element for which (τ, t) is enumerated into S_{n+1}.

Notice that

$$\int h_\tau \, d\lambda \le (E(g_t, \tau) - E(g_{s_{\tau,t}}, \tau))2^{-|\tau|} = \int_{[\tau]} (g_t - g_{s_{\tau,t}}) d\lambda,$$

whence

$$\sum_{(\tau,t)\in S_n} \int h_\tau d\lambda \le \sum_{(\tau,t)\in S_n} \int_{[\tau]} (g - g_{s_{\tau,t}}) d\lambda \le \sum_{(\sigma,s)\in S_{n-1}} \int_{[\sigma]} (g - g_s) d\lambda.$$

Then, in case $N \geq 2$,

$$\begin{aligned}\sum_{n=N-1}^{N} \sum_{(\tau,t)\in S_n} \int h_\tau d\lambda &\leq \sum_{(\tau,t)\in S_{N-1}} \int_{[\tau]} (g-g_t)d\lambda + \sum_{(\tau,t)\in S_{N-1}} \int_{[\tau]} (g_t - g_{s_{\tau,t}})d\lambda \\ &\leq \sum_{(\tau,t)\in S_{N-1}} \int_{[\sigma]} (g-g_{s_{\tau,t}})d\lambda \\ &\leq \sum_{(\tau,t)\in S_{N-2}} \int_{[\sigma]} (g-g_t)d\lambda.\end{aligned}$$

By iterating this argument for sums starting at $N-2, N-3, \ldots, 2$, we have

$$\sum_{n=0}^{N} \sum_{(\tau,t)\in S_n} \int h_\tau d\lambda \leq \sum_{(\tau,t)\in S_0} \int_{[\tau]} (g-g_s)d\lambda = \int g \, d\lambda < \infty.$$

Finally, since $\lim_n E(g, Z\restriction_n) = r > q$, for each n there exists $(\tau_n, t_n) \in S_n$ such that $\tau_n \prec z$. Then

$$h(z) = \sum_n (q - E(g_s, \tau_n)) \geq \sum_n (q - g(z)) = \infty.$$

$\square$

Remark 5.7. The proofs of Lemma 5.6 and of Theorem 3.3 are related. In the notation of Lemma 5.6, we have a left-c.e. martingale $L(\sigma) = E(g, \sigma)$ and uniformly computable martingales $L_s(\sigma) = E(g_s, \sigma)$ so that $L(\sigma) = \sup_s L_s(\sigma)$. By definition of the g_s as dyadic step functions, we have a computable function u on $\mathbb{N}$ such that $L_s(\tau) = L_s(\tau\restriction_{u(s)})$ whenever $|\tau| \geq u(s)$. Let us say that a left-c.e. martingale L of this kind is *stationary in approximation*. The obvious inequality

$$\sup_s L(Z\restriction_{u(s)}) \leq \liminf_n L(Z\restriction_n)$$

corresponds to $f(z) \leq r$ before (3). If Z is density random then $\lim_n L(Z\restriction_n)$ exists, and equals $\sup_s L(Z\restriction_{u(s)})$ by an argument similar to the one in the proof of Lemma 5.6.

5.3. **Effective Lebesgue Differentiation Theorem via lower semicomputability.** We show that density randomness is the same as being a Lebesgue point of each integral test. We use as a basic fact: if g is a non-negative integrable function, then $\sigma \to E(g, \sigma)$ is a martingale. By definition, z is a weak dyadic Lebesgue point of g iff this martingale converges along Z.

Theorem 5.8. *The following are equivalent for $z \in [0, 1]$:*

(i) *z is density random.*
(ii) *z is a dyadic Lebesgue point of each integral test.*
(iii) *z is a Lebesgue point of each integral test.*

We could equivalently formulate (ii) and (iii) in terms of integrable lower semicomputable functions, rather than the seemingly more restricted integral tests. For, any lower semicontinuous function on a compact domain is bounded below. So any integrable lower semicomputable function on $[0,1]$ becomes an integral test after adding a constant.

Proof. (ii) $\Rightarrow$ (i). By definition $g(z)$ is finite for each integral test g, whence z is ML-random.

Let $\mathcal{C}$ be a Π^0_1 class containing z. Clearly the function $g = 1 - \mathbf{1}_{\mathcal{C}}$ is an integral test. Since z is a Lebesgue point for g, $\mathcal{C}$ has dyadic density one at z. Then, by Theorem 2.4, z is a density-one point.

(i) $\Rightarrow$ (ii). Let g be an integral test. Then $\sigma \to E(g, \sigma)$ is a left-c.e. martingale. By Theorem 3.3, $\lim_n E(g, Z \upharpoonright n)$ exists, whence z is a dyadic weak Lebesgue point for g. By Lemma 5.6, z is a dyadic Lebesgue point for g.

(ii) $\Rightarrow$ (iii). Let g be an integral test. The function $f(x) = \int_{[0,x]} g \, d\lambda$ is interval-c.e. by the aforementioned result of Miyabe [31, Lemmas 4.6, 4.8]. The real z is density random by (ii)$\to$(i), so $f'(z)$ exists by Theorem 4.2. In particular, $\lim_{Q \to z} \lambda(Q)^{-1} \int_Q f \, d\lambda$ exists and equals $\lim_n 2^n \int_{(Z \upharpoonright n)} f \, d\lambda = f(z)$. Hence, z is a Lebesgue point for g.

The implication (iii) $\Rightarrow$ (ii) holds by definition. □

6. Birkhoff's theorem

We give an effective version, in the c.e. setting, of Birkhoff's Theorem 1.2. Franklin and Towsner [17] considered the case of a not necessarily ergodic measure-preserving operator T on Cantor space $2^{\mathbb{N}}$ with the uniform measure, and a lower semicomputable function f. They showed that the limit of the averages in the sense of Theorem 1.2 exists for each weakly 2-random point z. Under an additional, hypothetical assumption, in [17, Thm. 5.6] they were able to obtain convergence on the weaker assumption that z is balanced random in the sense of [15].

We work in the more general setting of Cantor space $2^{\mathbb{N}}$ with a computable probability measure μ. That is, $\mu[\sigma]$ is a left-c.e. real uniformly in a string σ. For background see Hoyrup and Rojas [20]

Bienvenu, Greenberg, Kučera, Nies, and Turetsky [3, Def. 2.5] introduced a randomness notion that implies density randomness. A *left-c.e. bounded test* over μ is a nested sequence $\langle \mathcal{V}_n \rangle$ of uniformly Σ^0_1 classes such that for some computable sequence of rationals $\langle \beta_n \rangle$ and $\beta = \sup_n \beta_n \leq 1$ we have $\mu(\mathcal{V}_n) \leq \beta - \beta_n$ for all n. Z fails this test if $Z \in \bigcap_n \mathcal{V}_n$. Z is *μ-Oberwolfach (OW)* random if it passes each left-c.e. bounded test.

Let $\mu = \lambda$ be the uniform measure; it is known that balanced randomness in the sense of [15] implies OW randomness, which implies density randomness. The converse of the first implication fails, as noted in [3]: some low ML-random is not balanced random [15]; on the other hand, any such set is OW random. It is unknown whether the converse of the second implication holds.

In the following let μ be a probability measure on $2^{\mathbb{N}}$ which is computable in the strong sense of [39] that $\mu[\sigma]$ is a computable real uniformly in a string σ. Note that this is equivalent to the weaker condition above that $\mu[\sigma]$ is uniformly left-c.e., in the case that the boundary of any open set is a null set.

Theorem 6.1. *Let T be a computable measure preserving operator on $(2^{\mathbb{N}}, \mu)$. Let f be a non-negative integrable lower semicomputable function on X. Let $A_n f(x)$ be the usual ergodic average*

$$\frac{1}{n}\sum_{i<n} f \circ T^i(x).$$

For every μ-Oberwolfach random point $z \in X$, $\lim_n A_n f(z)$ exists.

Note that we do not assume that the operator T is total. However, being measure preserving, its domain is conull. Since T is computable, the domain is also Π^0_2. So $T(x)$ is defined whenever x is μ-Kurtz random, namely, x is in no Π^0_1 class $\mathcal{P}$ with $\mu\mathcal{P} = 0$.

Proof. By V'yugin [39, Prop. 2], we have $f(z) = \sup_t f_t(z)$ for every $z \in X$ that is Kurtz random w.r.t. μ, where $\langle f_t \rangle$ is a computable non-decreasing sequence of simple functions (namely, there is a partition of $2^{\mathbb{N}}$ into finitely many clopen sets such that f_t has constant rational value on each of them).

Since simple functions are computable, by the main result of V'yugin [39, Thm. 2], $\lim_n A_n f_t(x)$ exists for each t and each ML-random point x. By the maximal ergodic inequality (see e.g. Krengel [23, Cor. 2.2]), for each non-negative integrable function g and each $r > 0$, we have

$$\mu\{x\colon \exists n\, A_n g(x) > r\} < \frac{1}{r}\int g d\mu.$$

Since z is weakly random, for each n the value $A_n f(z)$ exists. Thus, if $\lim_n A_n f(z)$ fails to exists, there are reals $a < b$ such that $A_n f(z) < a$ for infinitely many n, and $A_n f(z) > b$ for infinitely many n.

Let

$$\mathcal{V}_t = \{x\colon \exists k\, A_k(f - f_t) > b - a\}.$$

Then $\langle \mathcal{V}_t \rangle_{t\in\mathbb{N}}$ is a sequence of uniformly Σ^0_1 open sets in X with $\mathcal{V}_t \supseteq \mathcal{V}_{t+1}$. By the maximal ergodic inequality we have $\mu\mathcal{V}_t \leq 1/(b-a)\int(f - f_t)d\mu$. Finally, $\lim_n A_n f_t(z)$ exists for each t, and $\lim_n A_n f_t(z) \leq a$. Therefore $z \in \bigcap_t \mathcal{V}_t$. □

7. Density-one points for Π^0_n classes and Σ^1_1 classes

In this section we work again in the setting of Cantor space. So far we have looked at the density of Π^0_1 classes at points. Now we will consider classes of higher descriptional complexity. Firstly, we look at Π^0_n classes. It turns out that if Z is density random relative to $\emptyset^{(n-1)}$, then each Π^0_n class has density 1 at Z.

Thereafter we consider the density of Σ^1_1 classes at Z. This complexity forms a natural bound for our investigation because Σ^1_1 classes are measurable (Lusin; see e.g. [33, Thm. 9.1.9]), which is no longer true within ZFC for more complex classes.

7.1. **Density of Π^0_n classes at a real.** Recall that Z is n-random if Z is ML-random relative to $\emptyset^{(n-1)}$. By a $\Pi^{0,X}_1$ class we mean a Π^0_1 class relative to X. Every $\Pi^{0,\emptyset^{(n-1)}}_1$ class is Π^0_n. We show that for an n-random Z, it is sufficient to consider $\Pi^{0,\emptyset^{(n-1)}}_1$ classes in order to obtain that every Π^0_n class has density one at Z. To do so, we rely on a lemma about the approximation in terms of measure of Π^0_n classes by $\Pi^{0,\emptyset^{(n-1)}}_1$ subclasses. This can be seen as an effective form of regularity for Lebesgue measure. See [12, Thm. 6.8.3] for a recent write-up of the proof.

Lemma 7.1 (Kurtz [24], Kautz [21])**.** *From an index of a Π^0_n class P and $q \in \mathbb{Q}^+$, $\emptyset^{(n-1)}$ can compute an index of a $\Pi^{0,\emptyset^{(n-1)}}_1$ class $V \subseteq P$ such that $\lambda(P) - \lambda(V) < q$.*

Theorem 7.2. *Suppose $n \geq 1$ and $Z \in 2^{\mathbb{N}}$ is density random relative to $\emptyset^{(n-1)}$. Let P be Π^0_n class such that $Z \in P$. Then $\underline{\varrho}_2(P|Z) = 1$.*

Proof. Let $P = \bigcap_s U_s$ where $\langle U_s : s \in \omega\rangle$ is a nested sequence of uniformly Σ^0_{n-1} classes. It suffices to show that there exists a $\Pi^{0,\emptyset^{(n-1)}}_1$ class $Q \subseteq P$ such that $Z \in Q$.

We define a Solovay test relative to $\emptyset^{(n-1)}$. By Lemma 7.1, effectively in $\emptyset^{(n-1)}$ we obtain an index of a $\Pi^{0,\emptyset^{(n-1)}}_1$ class $Q_s \subseteq U_s$ such that

$$\lambda(U_s) - \lambda(Q_s) < 2^{-n}.$$

The sequence of uniformly $\Sigma^{0,\emptyset^{(n-1)}}_1$ classes

$$\langle U_s \backslash Q_s : s \in \mathbb{N}\rangle$$

is a Solovay test relative to $\emptyset^{(n-1)}$ since $\lambda(U_s \backslash Q_s) \leq 2^{-s}$. Notice $Z \in P \subseteq U_s$ for each $s \in \mathbb{N}$. Since Z is Martin-Löf random relative to $\emptyset^{(n-1)}$, there exists $k \in \mathbb{N}$ such that for all $j \geq k$, $Z \in Q_j$. Since $\langle Q_j : j \geq k\rangle$ is a uniform sequence of $\Pi^{0,\emptyset^{(n-1)}}_1$ classes, $V = \bigcap_{j\geq k} Q_j$ is itself a $\Pi^{0,\emptyset^{(n-1)}}_1$ class. Also $V \subseteq \bigcap_{i\in\mathbb{N}} U_i = P$ because $Q_j \subseteq U_j$. We have found a $\Pi^{0,\emptyset^{(n-1)}}_1$ class $V \subseteq P$ that contains Z. $\square$

Relativizing Theorem 3.3 to $\emptyset^{(n-1)}$ we obtain:

Corollary 7.3. *An $n-$random set Z is a density one point for Π^0_n classes if and only if every left-$\emptyset^{(n-1)}$-c.e. martingale converges along Z.*

7.2. **Higher randomness.** The adjective "higher" indicates that algorithmic tools are replaced by tools from effective descriptive theory. See e.g. [33, Ch. 9] for background. The work of the Madison group described in Section 3 can be adapted to this setting. For a higher version of density randomness, instead of Π^0_1 classes we now look at Σ^1_1 classes containing the real in question. Similar to the foregoing case of Π^0_n classes, it does not matter whether the Σ^1_1 class is closed.

We use the following fact due to Greenberg (personal communication). It is a higher analog of the original weaker version of Prop. 2.7(i) proved in Bienvenu et al. [3, Prop. 5.4]. The hypothesis on Z could be weakened to a higher notion of partial computable randomness as well.

Proposition 7.4 (Greenberg, 2013)**.** *Let $\mathcal{C} \subseteq 2^{\mathbb{N}}$ be Σ^1_1. Let $Z \in \mathcal{C}$ be Π^1_1-ML-random. Then $\overline{\varrho}_2(\mathcal{C} \mid Z) = 1$.*

Proof. If $\overline{\varrho}_2(\mathcal{C} \mid Z) < 1$ then there is a positive rational $q < 1$ and n^* such that for all $n \geq n^*$ we have $\lambda_{Z\upharpoonright n}(\mathcal{C}) < q$. Choose a rational r with

$q < r < 1$. We define Π^1_1-antichains in $U_n \subseteq 2^{<\omega}$, uniformly in n. Let $U_0 = \{\langle Z \upharpoonright_{n^*} \rangle\}$. Suppose U_n has been defined. For each $\sigma \in U_n$, at a stage α such that $\lambda_\sigma(\mathcal{C}_\alpha) < q$, we obtain effectively a hyper-arithmetical antichain V of extensions of σ such that $\mathcal{C}_\alpha \cap [\sigma] \subseteq [V]^\prec$ and $\lambda_\sigma([V]^\prec) < r$. Put V into U_{n+1}.

Clearly $\lambda[U_n]^\prec \leq r^n$ for each n. Also, $Z \in \bigcap_n [U_n]^\prec$, so Z is not Π^1_1-ML-random. □

A martingale $L\colon 2^{<\omega} \to \mathbb{R}^+_0$ is called left-Π^1_1 if $L(\sigma)$ is a left-Π^1_1 real uniformly in σ. We provide a higher analog of Theorem 3.3.

Theorem 7.5. *Let Z be Π^1_1-ML-random. The following are equivalent.*

(i) $\underline{\varrho}_2(\mathcal{C} \mid Z) = 1$ *for each Σ^1_1 class $\mathcal{C}$ containing Z.*

(ii) $\underline{\varrho}_2(\mathcal{C} \mid Z) = 1$ *for each* closed *Σ^1_1 class $\mathcal{C}$ containing Z.*

(iii) *Each left-Π^1_1 martingale converges along Z to a finite value.*

Proof. (iii) → (i): The measure of a Σ^1_1 set is left-Σ^1_1 in a uniform way (see e.g. [33, Thm. 9.1.10]). Therefore $M(\sigma) = 1 - \lambda_\sigma(\mathcal{C})$ is a left-Π^1_1 martingale. Since M converges along Z, and since by Prop. 7.4 $\liminf_n M(Z \upharpoonright_n) = 0$, it converges along Z to 0. This shows that $\underline{\varrho}_2(\mathcal{C} \mid Z) = 1$.

(ii) → (iii). We follow the proof of the Madison group's Theorem 3.3 given above. All stages s are now interpreted as computable ordinals. Computable functions are now functions $\omega_1^{CK} \to L_{\omega_1^{CK}}$ with Σ_1 graph. Constructions are now assignments of recursive ordinals to instructions.

Definition 7.6. A Π^1_1-*Madison test* is a Σ_1 over $L_{\omega_1^{CK}}$ function $\langle U_s \rangle_{s<\omega_1^{CK}}$ mapping ordinals to (hyperarithmetical) subsets of $2^{<\omega}$ such that $U_0 = \emptyset$, for each stage s we have $\mathsf{wt}(U_s) \leq c$ for some constant c, and for all strings σ, τ,

(a) $\tau \in U_s - U_{s+1} \to \exists \sigma \prec \tau\, [\sigma \in U_{s+1} - U_s]$

(b) $\mathsf{wt}(\sigma^\prec \cap U_s) > 2^{-|\sigma|} \to \sigma \in U_s$.

Also $U_t(\sigma) = \lim_{s<t} U_s(\sigma)$ for each limit ordinal t.

The following well-known fact can be proved similar to [33, 1.9.19].

Lemma 7.7. *Let $\mathcal{A} \subseteq 2^{\mathbb{N}}$ be a hyperarithmetical open set. Given a rational q with $q > \lambda A$, we can effectively determine from $\mathcal{A}, q$ a hyperarithmetical open $\mathcal{S} \supseteq \mathcal{A}$ with $\lambda\mathcal{S} = q$.*

We provide an analog of Lemma 3.5. Its proof is a variant of the former argument.

Lemma 7.8. *Let Z be a Π^1_1 ML-random such that $\underline{\varrho}_2(\mathcal{C} \mid Z) = 1$ for each* closed *Σ^1_1 class $\mathcal{C}$ containing Z. Then Z passes each Π^1_1-Madison test.*

The sets $\mathcal{A}^k_{\sigma,s}$ are now hyperarithmetical open sets computed from k, σ, s. Suppose $\sigma \in U_{s+1} - U_s$. The set $\widetilde{\mathcal{A}}^k_{\sigma,s}$ is defined as before. To effectively obtain $\mathcal{A}^k_{\sigma,s+1}$, we apply Lemma 7.7 to add mass from $[\sigma]$ to $\widetilde{\mathcal{A}}^k_{\sigma,s+1}$ in order to ensure that $\lambda(\mathcal{A}^k_{\sigma,s+1}) = 2^{-|\sigma|-k}$.

As before, let $\mathcal{S}^k_t = \bigcup_{\sigma \in U_t} \mathcal{A}^k_{\sigma,t}$. Then $\mathcal{S}^k_t \subseteq \mathcal{S}^k_{t+1}$ by condition (a) on Π^1_1-Madison tests. Clearly $\lambda\mathcal{S}^k_t \leq 2^{-k}\mathsf{wt}(U_t) \leq 2^{-k}$. So $\mathcal{S}^k = \bigcup_{t<\omega_1^{CK}} \mathcal{S}^k_t$ determines a Π^1_1 ML-test.

By construction $\underline{\varrho}_2(2^{\mathbb{N}} - \mathcal{S}^k \mid Z) \leq 1 - 2^{-k}$. Since Z is ML-random we have $Z \notin \mathcal{S}^k$ for some k. So $\underline{\varrho}_2(\mathcal{C} \mid Z) < 1$ for the closed Σ^1_1 class $\mathcal{C} = 2^{\mathbb{N}} - \mathcal{S}_k$ containing Z.

The analog of Lemma 3.8 also holds.

Lemma 7.9. *Suppose that Z passes each Π^1_1-Madison test. Then every left-Π^1_1 martingale L converges along Z.*

We wrote the proof of Lemma 3.8 in such a way that this works. If $L\colon 2^{<\omega} \to \mathbb{R}$ is a left-Π^1_1 martingale, then $L(\sigma) = \sup_s L_s(\sigma)$ for a non-decreasing sequence $\langle L_s \rangle$ of hyperarithmetical martingales computed uniformly from $s < \omega_1^{CK}$. The labelling functions $\gamma_s\colon U_s \to \omega_1^{CK}$ are now uniformly hyperarithmetical.

We may assume that $L_t(\sigma) = \lim_{s<t} L_s(\sigma)$ for each limit ordinal t. This implies $U_t(\sigma) = \lim_{s<t} U_s(\sigma)$ for each limit ordinal t as required in the definition of higher Madison tests. □

References

[1] L. Bienvenu, A. Day, N. Greenberg, A. Kučera, J. Miller, A. Nies, and D. Turetsky. Computing K-trivial sets by incomplete random sets. *Bull. Symb. Logic*, 20:80–90, 2014. 1

[2] L. Bienvenu, A. R. Day, M. Hoyrup, I. Mezhirov, and A. Shen. A constructive version of Birkhoff's ergodic theorem for Martin-Löf random points. *Inf. Comput.*, 210:21–30, 2012. 1

[3] L. Bienvenu, N. Greenberg, A. Kučera, A. Nies, and D. Turetsky. Coherent randomness tests and computing the K-trivial sets. To appear in J. European Math. Society, 2015. 2.4, 3, 3.6, 3, 6, 7.2

[4] L. Bienvenu, R. Hölzl, J. Miller, and A. Nies. Denjoy, Demuth, and Density. *J. Math. Log.*, 1450004, 2014. 35 pages. 1, 2.1, 2.1, 4.1, 4.4, 4.2

[5] G. Birkhoff. The mean ergodic theorem. *Duke Math. J.*, 5(1):19–20, 1939. 1.2, 1

[6] V.I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007. 4.2

[7] V. Brattka, J. Miller, and A. Nies. Randomness and differentiability. *Transactions of the AMS*, 368:581–605, 2016. arXiv version at `http://arxiv.org/abs/1104.4465`. 1, 1.3, 1, 4.1, 4.3

[8] N.L. Carothers. *Real analysis.* Cambridge University Press, 2000. 1

[9] A. R. Day and J. S. Miller. Cupping with random sets. *Proceedings of the American Mathematical Society*, 142(8):2871–2879, 2014. 1, 2.5

[10] A. R. Day and J. S. Miller. Density, forcing and the covering problem. *Mathematical Research Letters*, 22(3):719–727, 2015. 2.1, 4.2

[11] O. Demuth. The differentiability of constructive functions of weakly bounded variation on pseudo numbers. *Comment. Math. Univ. Carolin.*, 16(3):583–599, 1975. (Russian). 1

[12] R. Downey and D. Hirschfeldt. *Algorithmic randomness and complexity.* Springer-Verlag, Berlin, 2010. 855 pages. 1, 2.4, 3, 3, 7.1

[13] R. Downey, A. Nies, R. Weber, and L. Yu. Lowness and Π^0_2 nullsets. *J. Symbolic Logic*, 71(3):1044–1052, 2006. 2.3, 2.3

[14] R. Durrett. *Probability: theory and examples.* Duxbury Press, Belmont, CA, second edition, 1996. 1, 3, 3, 3.6

[15] S. Figueira, D. Hirschfeldt, J. Miller, Selwyn Ng, and A Nies. Counting the changes of random Δ^0_2 sets. *J. Logic Computation*, 25:1073–1089, 2015. Journal version of conference paper at CiE 2010. 6

[16] J. Franklin, N. Greenberg, J. S. Miller, and K. M. Ng. Martin-Löf random points satisfy Birkhoff's ergodic theorem for effectively closed sets. *Proc. Amer. Math. Soc.*, 140(10):3623–3628, 2012. 1

[17] J. Franklin and H. Towsner. Randomness and non-ergodic systems. *Moscow Mathematical Journal*, 14:711–714, 2014. 1, 6

[18] C. Freer, B. Kjos-Hanssen, A. Nies, and F. Stephan. Algorithmic aspects of lipschitz functions. *Computability*, 3(1):45–61, 2014. 1, 3, 3, 4, 5.1, 5.1

[19] P. Gács, M. Hoyrup, and C. Rojas. Randomness on computable probability spaces - a dynamical point of view. *Theory Comput. Syst.*, 48(3):465–485, 2011. 1

[20] M. Hoyrup and C. Rojas. Computability of probability measures and Martin-Löf randomness over metric spaces. *Inform. and Comput.*, 207(7):830–847, 2009. 6

[21] S. Kautz. *Degrees of random sets*. Ph.D. Dissertation, Cornell University, 1991. 7.1

[22] M. Khan. Lebesgue density and Π^0_1-classes. J. Symb. Logic, To appear. 2, 2.4

[23] U. Krengel. *Ergodic Theorems*. W. de Gruyter, 1985. 1, 6

[24] S. Kurtz. *Randomness and genericity in the degrees of unsolvability*. Ph.D. Dissertation, University of Illinois, Urbana, 1981. 7.1

[25] H. Lebesgue. *Leçons sur l'Intégration et la recherche des fonctions primitives*. Paris: Gauthier-Villars, 1904. 5, 5.2

[26] H. Lebesgue. Sur les intégrales singulières. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (3)*, 1:25–117, 1909. 1.1, 1, 2.2

[27] H. Lebesgue. Sur l'intégration des fonctions discontinues. *Ann. Sci. École Norm. Sup. (3)*, 27:361–450, 1910. 1, 5

[28] M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Graduate Texts in Computer Science. Springer-Verlag, New York, second edition, 1997. 5.2

[29] J. S. Miller. Lebesgue density in Π^0_1-classes. Slides, Available at `http://www-2.dc.uba.ar/ccr/talks/miller0201.pdf`, February 2013. 3

[30] J. S. Miller and A. Nies. Randomness and computability: Open questions. *Bull. Symbolic Logic*, 12(3):390–410, 2006. 1

[31] K. Miyabe. Characterization of Kurtz Randomness by a Differentiation Theorem. *Theory of Computing Systems*, 52(1):113–132, 2013. 5.2, 5.3

[32] M. Morayne and S. Solecki. Martingale proof of the existence of Lebesgue points. *Real Anal. Exchange*, 15(1):401–406, 1989/90. 4.3

[33] A. Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009. 444 pages. Paperback version 2011. 1, 2.4, 3, 3, 3, 7, 7.2, 7.2, 7.2

[34] A. Nies. Differentiability of polynomial time computable functions. In Ernst W. Mayr and Natacha Portier, editors, *31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014)*, volume 25 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 602–613, Dagstuhl, Germany, 2014. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. 1, 4, 4.2, 4.4

[35] N. Pathak, C. Rojas, and S. G. Simpson. Schnorr randomness and the Lebesgue differentiation theorem. *Proc. Amer. Math. Soc.*, 142(1):335–349, 2014. 1, 5.1, 5.1

[36] M. Pour-El and J. Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989. 5.1

[37] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987. 5

[38] C.P. Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 218. 1

[39] V. V'yugin. Ergodic theorems for individual random sequences. *Theor. Comput. Sci.*, 207(2):343–361, 1998. 1, 5.2, 6, 6

[40] K. Weihrauch. *Computable Analysis*. Springer, Berlin, 2000. 1