# Nonmeasurable sets and unions with respect to tree ideals

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ABSTRACT. In this paper we consider a notion of nonmeasurablity with respect to Marczewski and Marczewski-like tree ideals  $s_0, m_0, l_0$ , and  $cl_0$ . We show that there exists a subset A of the Baire space  $\omega^{\omega}$  which is s-, l-, and m-nonmeasurable, that forms dominating m.e.d. family. We introduce and investigate a notion of T-Bernstein sets sets that intersect but does not containt any body of a tree from a given family of trees  $\mathbb{T}$ . We also acquire some results on  $\mathcal{I}$ -Luzin sets, namely we prove that there are no  $m_0$ -,  $l_0$ -, and  $cl_0$ -Luzin sets and that if  $\mathfrak{c}$  is a regular cardinal, then the algebraic sum (considered on the real line  $\mathbb{R}$ ) of a generalized Luzin set and a generalized Sierpiński set belongs to  $s_0, m_0, l_0$  and  $cl_0$ .

# 1. Introduction and preliminaries

We will use standard set-theoretic notation following e.g. [4]. For a set X, P(X)denotes the power set of X and |X| denotes the cardinality of X. If  $\kappa$  is a cardinal number then we denote:

Let X be an uncountable Polish space and  $\mathcal{I} \subseteq P(X)$  be a  $\sigma$ -ideal. Let us recall some cardinal coefficients from Cichoń's Diagram:

- $\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land \bigcup \mathcal{A} \notin \mathcal{I}\},\$
- $\operatorname{non}(\mathcal{I}) = \min\{|A| : A \subseteq X \land A \notin \mathcal{I}\},\$
- $\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land \bigcup \mathcal{A} = X\},\$
- $\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land (\forall A \in \mathcal{I}) (\exists B \in \mathcal{A}) (A \subseteq B)\},\$
- $\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^{\omega} \land (\forall x \in \omega^{\omega}) (\exists f \in \mathcal{F}) (\exists^{\infty} n) (x(n) < f(n))\},\$
- $\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^{\omega} \land (\forall x \in \omega^{\omega}) (\exists f \in \mathcal{F}) (\forall^{\infty} n) (x(n) < f(n))\}.$

We call  $\mathfrak{b}$  a bounding number and  $\mathfrak{d}$  a dominating number. A family  $\mathcal{F} \subseteq \omega^{\omega}$  is dominating, if  $\mathcal{F}$  has a property described in the definition of domintaing number (it doesn't have to be of minimal cardinality).

We say that T is a tree on a set A if  $T \subseteq A^{<\omega}$  and whenever  $\tau \in T$  then  $\tau \upharpoonright n \in T$  for each natural n.

DEFINITION 1. Let T be a tree on a set A. Then

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- for each  $t \in T$  succ $(t) = \{a \in A : t \cap a \in T\};$
- $split(T) = \{t \in T : |succ(t)| \ge 2\};$
- $\omega$ -split $(T) = \{t \in T : |succ(t)| = \aleph_0\};$
- for  $s \in T$  Succ<sub>T</sub> $(s) = \{t \in split(T) : s \subsetneq t, (\forall t' \in T)(s \subsetneq t' \subsetneq t \longrightarrow t' \notin split(T))\};$
- for  $s \in T$   $\omega$ -Succ<sub>T</sub> $(s) = \{t \in \omega$ -split $(T) : s \subsetneq t, (\forall t' \in T) (s \subsetneq t' \subsetneq t \longrightarrow t' \notin \omega$ -split $(T))\};$
- $stem(T) \in T$  is a node  $\tau$  such that for each  $s \subsetneq \tau |succ(s)| = 1$  and  $|succ(\tau)| > 1$ .

Let us now recall definitions of families of trees.

DEFINITION 2. A tree T on  $\omega$  is called

- Sacks tree or perfect tree, denoted by  $T \in S$ , if for each node  $s \in T$  there is  $t \in T$  such that  $s \subseteq t$  and  $|succ(t)| \ge 2$ ;
- Miller tree or superperfect tree, denoted by  $T \in \mathbb{M}$ , if for each node  $s \in T$  exists  $t \in T$  such that  $s \subseteq t$  and  $|succ(t)| = \aleph_0$ ;
- Laver tree, denoted by  $T \in \mathbb{L}$ , if for each node  $t \supseteq stem(T)$  we have  $|succ(t)| = \aleph_0$ ;
- complete Laver tree, denoted by  $T \in \mathbb{CL}$ , if T is Laver and stem $(T) = \emptyset$ ;
- Hechler tree, denoted by  $T \in \mathbb{H}$ , if for each node  $t \supseteq stem(T)$  we have that a set  $\{n \in \omega : t \frown n \notin T\}$  is finite;
- complete Hechler, denoted by  $T \in \mathbb{CH}$  tree, if T is Hechler and  $stem(T) = \emptyset$ .

The notion of complete Laver trees was defined and investigated in [11], although Miller in [10] defines Laver trees *de facto* as complete Laver trees and Hechler trees as complete Hechler trees.

For every tree  $T \subseteq \omega^{<\omega}$  let [T] be the set of all infinite branches of T, i.e.

$$[T] = \{ x \in \omega^{\omega} : \ (\forall n \in \omega) \ x \upharpoonright n \in T \}.$$

DEFINITION 3 (Tree ideal). Let  $\mathbb{T}$  be a family of trees. We say that  $A \in P(\omega^{\omega})$  is in  $t_0$  iff

$$(\forall P \in \mathbb{T})(\exists Q \in \mathbb{T}) \ Q \subseteq P \land [Q] \cap A = \emptyset.$$

DEFINITION 4 (t-measurability). Let  $\mathbb{T}$  be a family of trees. We say that  $A \in P(\omega^{\omega})$  is t-measurable iff

$$(\forall P \in \mathbb{T})(\exists Q \in \mathbb{T}) \ Q \subseteq P \land ([Q] \subseteq A \lor [Q] \cap A = \emptyset).$$

 $s_0$  tree ideal is simply a classic Marczewski ideal (see [8]).

It is well known due to Judah, Miller, Shelah (see [5]) and Repický (see [12]) that  $add(s_0) \leq cov(s_0) \leq cof(\mathfrak{c}) \leq non(s_0) = \mathfrak{c} < cof(s_0) \leq 2^{\mathfrak{c}}$ . Moreover, in [2] Brendle, Khomskii and Wohofsky have shown that also  $\mathfrak{c} < cof(m_0)$  and  $\mathfrak{c} < cof(l_0)$ . Clearly  $\omega_1 \leq add(l_0) \leq cov(l_0) \leq \mathfrak{c}$  holds. In [3], Goldstern, Repický, Shelah and Spinas showed that it is relatively consistent with ZFC that  $add(l_0) < cov(l_0)$ .

Let us notice that the families  $s_0, l_0, m_0$  form  $\sigma$ -ideals. On the other hand  $cl_0$  is not a  $\sigma$ -ideal. To see that it is enough to consider sets of the form  $C_n = \{x \in \omega^{\omega} : x(0) = n\}$ . Then  $C_n \in cl_0$  for each n, but  $\bigcup_n C_n = \omega^{\omega}$ . Using the fact that  $s_0$  is a  $\sigma$ -ideal we may give another proof of the following well known result.

PROPOSITION 5 (Essentially a joke).  $cf(\mathfrak{c}) > \aleph_0$ .

PROOF. Suppose that  $cf(\mathfrak{c}) = \aleph_0$  and let  $\mathbb{R} = \bigcup_{n \in \omega} A_n$ ,  $|A_n| < \mathfrak{c}$  for each  $n \in \omega$ . Sets of cardinality lesser than  $\mathfrak{c}$  belong to  $s_0$ , so  $\mathbb{R} = \bigcup_{n \in \omega} A_n \in s_0$ , a contradiction.

### 2. Tree ideals and measurability

In [1] the following result was obtained.

THEOREM 6 (Brendle). If  $i_0, j_0 \in \{s_0, l_0, m_0\}$  and  $i_0 \neq j_0$  then  $i_0 \not\subseteq j_0$ .

First we will compare the ideal  $cl_0$  with ideals  $s_0, m_0, l_0$ .

FACT 7.  $cl_0 \not\subseteq (l_0 \cup m_0 \cup s_0)$ .

PROOF. To show the assertion let us take  $C_0 = \{x \in \omega^{\omega} : x(0) = 0\}$ . By  $\mathbb{CL} \subseteq \mathbb{L} \subseteq \mathbb{M} \subseteq \mathbb{S}, [C_0] \notin l_0 \cup m_0 \cup s_0$ . On the other hand  $[C_0] \in cl_0$ , which finishes the proof.  $\Box$ 

THEOREM 8. The following statements are true:

(i)  $m_0 \not\subseteq cl_0$ .

(*ii*)  $s_0 \not\subseteq cl_0$ .

**PROOF.** To prove that  $m_0 \setminus cl_0 \neq \emptyset$  we will slightly modify the proof of Theorem 2.1 from [1]. We will use the notions of apple trees and pear trees.

First, let us recall that each Miller tree contains an apple tree and each apple tree is a special kind of a Miller tree (apple trees forms a dense subfamily in all Miller trees).

Second, each complete Laver tree C contains a pear tree  $P_C$ . A pear tree is not a complete Laver tree, it is only a special kind of Sacks tree. Pear trees  $P_C$  have the following property: for every apple tree A and pear tree  $P_C |[A] \cap [P_C]| \leq 1$ .

Let us now enumerate all apple trees  $\{A_{\alpha} : \alpha < \mathfrak{c}\}$  and all complete Laver trees  $\{C_{\alpha} : \alpha < \mathfrak{c}\}$ . Having the above two propositions we can proceed by induction and construct a sequence  $(x_{\alpha})_{\alpha < \mathfrak{c}}$  such that for every  $\alpha < \mathfrak{c}$ :

$$x_{\alpha} \in [P_{C_{\alpha}}] \setminus \bigcup_{\beta < \alpha} [A_{\beta}]$$

Finally, we set  $X = \{x_{\alpha} : \alpha < \mathfrak{c}\}$ . Let us notice that  $X \in m_0 \setminus cl_0$ , which finishes the first part of the proof.

To prove that  $s_0 \setminus cl_0 \neq \emptyset$  we use slight modification of the proof of Theorem 2.2 from [1], which fits a similar pattern from the first case.

QUESTION 9. Is it true that  $l_0 \not\subseteq cl_0$ ?

As a consequence we obtain the following result.

COROLLARY 10. The following statements are true:

- (i) There exists a cl-nonmeasurable set which is m-measurable.
- (ii) There exists a cl-nonmeasurable set which is s-measurable.

Let us introduce a notion of T-Bersntein sets.

DEFINITION 11. Let  $\mathbb{T}$  a family of trees. We say that a set B is an  $\mathbb{T}$ -Bernstein set if for every  $T \in \mathbb{T}$   $B \cap [T] \neq \emptyset$  and  $B \setminus [T] \neq \emptyset$ .

Observe that a classic Bernstein set is an S-Bernstein set. If  $\mathbb{T} \subseteq \mathbb{T}'$  are families of trees, then  $\mathbb{T}'$ -Bersntein sets are  $\mathbb{T}$ -Bernstein sets. No  $\mathbb{T}$ -Bernstein set is in  $t_0$  (or *t*measurable), and if  $\mathbb{T} \subseteq \mathbb{T}'$  then  $\mathbb{T}'$ -Bernstein sets don't belong to  $t_0$ . Also note that if  $\mathbb{T} \subsetneq \mathbb{T}'$  then a  $\mathbb{T}$ -Bernstein set may be not a  $\mathbb{T}'$ -Bernstein set (e.g. one may fix a tree from  $\mathbb{T}' \setminus \mathbb{T}$  which body will be always omitted). The following theorem slightly generalizes Theorems 2.1 and 2.2 from [1].

THEOREM 12. The following statements are true:

(i) There exists an  $\mathbb{L}$ -Bernstein set which belongs to  $m_0$ .

(ii) There exists an  $\mathbb{M}$ -Bernstein set which belongs to  $s_0$ .

PROOF. As in in the proof of Theorem 8 we will use notions established in [1]. To prove (i) let us enumerate all Laver trees  $\{L_{\alpha} : \alpha < \mathfrak{c}\}$  and all apple trees  $\{A_{\alpha} : \alpha < \mathfrak{c}\}$ . Let us construct two sequences:  $(b_{\alpha})_{\alpha < \mathfrak{c}}$  and  $(x_{\alpha})_{\alpha < \mathfrak{c}}$  such that for each  $\alpha < \mathfrak{c}$ :

$$b_{\alpha} \in [L_{\alpha}] \setminus (\bigcup_{\beta < \alpha} [A_{\beta}] \cup \{x_{\xi} : \xi < \alpha\}),$$
$$x_{\alpha} \in [L_{\alpha}] \setminus (\{b_{\beta} : \beta \le \alpha\} \cup \{x_{\beta} : \beta < \alpha\}).$$

It can be done, since for each Laver tree  $L_{\alpha}$  there is a pear tree  $P_{L_{\alpha}}$  for which  $|[P_{L_{\alpha}}] \cap [A]| \leq 1$  for every apple tree A, so the set  $[L_{\alpha}] \setminus (\bigcup_{\beta < \alpha} [A_{\beta}] \cup \{x_{\xi} : \xi < \alpha\})$  is nonempty at each step  $\alpha$ . Then  $B = \{b_{\alpha} : \alpha < \mathfrak{c}\}$  is the desired set.

To prove (ii) we use a similar modification of Theorem 2.2 from [1].

Analogously to the Question 9 we may ask the following question.

QUESTION 13. Is there a  $\mathbb{CL}$ -Bernstein set which belongs to  $l_0$ ?

Let us invoke a theorem by Miller from [10].

THEOREM 14 (Miller). Let  $A \in \Sigma_1^1$ . Either A contains body of some complete Laver tree or  $A^c$  contains a body of some complete Hechler tree.

THEOREM 15. The following is true:

- (i)  $\mathcal{B} \cap s_0$  is an ideal of Borel sets that don't contain a perfect subset (so it's an ideal of countable sets).
- (ii)  $\mathcal{B} \cap m_0$  is an ideal of Borel sets which don't contain a body of any Miller tree.
- (iii)  $\mathcal{B} \cap l_0$  is an ideal of Borel sets that don't contain a body of any Laver tree.

**PROOF.** (i) is evident.

(ii) follows by the fact that any analytic set is either  $\sigma$  - bounded or contains a superperfect set. If a Borel set contains a superfect set then clearly it's not in  $m_0$ . On the other hand, if for some Miller tree T and  $\sigma$  - bounded Borel a set  $B[T] \setminus B$  didn't contain a superperfect set, then [T] would be  $\sigma$  - bounded too. A contradiction.

(iii): If a Borel set B contains a body of some Laver tree, then clearly  $B \notin l_0$ . If it doesn't contain a Laver tree, but there is a Laver L for which each body of Laver subtree of L has a nonempty intersection with B, then let us trim B and L in the following way:

$$B' = \{x \in \omega^{\omega} : stem(L) \cap x \in B\},\$$
  
$$L' = \{x \in \omega^{\omega} : stem(L) \cap x \in L\}.$$

A function  $f: \omega^{\omega} \to \omega^{\omega}$  given by the formula  $f(x) = stem(L) \frown x$  is continuous. Clearly,  $B' = f^{-1}[B]$ , so B' is Borel, and  $[L'] = f^{-1}[[L]]$  is a body of a complete Laver tree L'. B'still doesn't contain a body of any Laver tree, so by Theorem 14 there is a Hechler tree H which body is contained in  $B'^c$ .  $H \cap L'$  contains (in fact - is) a Laver tree, body of which B' should intersect - a contradiction.

DEFINITION 16. We say that a set A is  $\mathcal{I}$ -nonmeasurable if  $A \notin \sigma(\mathcal{B} \cup \mathcal{I})$ . A is completely  $\mathcal{I}$ -nonmeasurable if  $A \cap B$  is  $\mathcal{I}$ -nonmeasurable for each Borel set  $B \notin \mathcal{I}$ , or equivalently - A intersects each, but doesn't contain any, Borel  $\mathcal{I}$ -positive set.

COROLLARY 17. Let  $(\mathbb{T}, t_0) \in \{(\mathbb{S}, s_0), (\mathbb{M}, m_0), (\mathbb{L}, l_0)\}$ . Then a set B is a  $\mathbb{T}$ -Bernstein iff it is completely  $t_0 \cap \mathcal{B}$ -nonmeasurable.

PROOF. By Theorem 15 a set A is  $t_0 \cap \mathcal{B}$ -positive Borel set if and only if it contains a body of some tree from  $\mathbb{T}$ , so a set B is  $\mathbb{T}$ -Bernstein if and only if it intersects each each, but does not contain any, Borel set containing a body of a tree from  $\mathbb{T}$ .  $\Box$ 

## 3. $\mathcal{I}$ -Luzin sets and algebraic properties

Let us recall the notion of  $\mathcal{I}$ -Luzin sets. Let X be a Polish space and  $\mathcal{I}$  be an ideal.

DEFINITION 18. We say that a set L is an  $\mathcal{I}$ -Luzin set if  $(\forall A \in \mathcal{I})(|A \cap L| < |L|)$ .

For classic ideals of Lebesgue measure zero sets  $\mathcal{N}$  and meager sets  $\mathcal{M}$  we will call  $\mathcal{M}$ -Luzin sets generalized Luzin sets and  $\mathcal{N}$ -Luzin sets generalized Sierpiński sets.

In [14] the following result was proven.

THEOREM 19 (Wohofsky). There is no  $s_0$ -Luzin set.

We will show that similar results can be obtained for other tree ideals.

THEOREM 20. The following statements are true.

- (i) There is no  $l_0$ -Luzin set.
- (ii) There is no  $cl_0$ -Luzin set.
- (iii) There is no  $m_0$ -Luzin set.

PROOF. Let us consider  $l_0$  case. We will prove that for every set X of cardinality  $\mathfrak{c}$  there exists a set  $A \subseteq X$  such that  $A \in l_0$  and  $|A| = \mathfrak{c}$ . Indeed, let us assume that  $X \notin l_0$ . Then there exists  $L \in \mathbb{L}$  such that for every  $L' \subseteq L$ ,  $L' \in \mathbb{L}$  we have  $|[L] \cap X| = \mathfrak{c}$ . Let us now fix a maximal antichain  $\{L_\alpha : \alpha < \mathfrak{c}\}$  of Laver trees contained in L such that  $|[L_\alpha] \cap X| = \mathfrak{c}$ . Let us construct a sequence  $(a_\alpha)_{\alpha < \mathfrak{c}}$  such that for each  $\alpha < \mathfrak{c}$ :

$$a_{\alpha} \in X \setminus \bigcup_{\xi < \alpha} [L_{\alpha}].$$

Then  $A = \{a_{\alpha} : \alpha < \mathfrak{c}\}$  is the set. Proofs of the other cases are almost identical.

Now we will consider  $\mathcal{I}$ -Luzin sets in a context of algebraic properties and tree ideals. We will work on the real line  $\mathbb{R}$  with addition. Since  $\mathbb{R}$  is  $\sigma$ -compact, it does not contain even superperfect sets. We will tweak the definition a bit by saying that  $A \subseteq \mathbb{R}$  belongs to  $t_0$  if  $h^{-1}[A]$  belongs to  $t_0$  in  $\omega^{\omega}$ , where h is a homeomorphism between  $\omega^{\omega}$  and a subspace of irrational numbers (see [7] for a similar modification in the case of  $2^{\omega}$ ). Having this in mind we will usually mean by  $[\tau], \tau \in \omega^{<\omega}$ , an open interval of rational endpoints on  $\mathbb{R}$ .

Before we proceed let us define a non-standard kind of fusion of Miller and Laver trees, that we will use later. Let T be a Miller tree. Let  $\tau_{\emptyset} \in \omega$ -split(T) and let  $T_0$  be any Miller subtree of T such that  $\tau_{\emptyset}$  remains an infinitely splitting node in  $T_0$ . Suppose we have a Miller subtree  $T_n$  and a set of nodes  $B_n = \{\tau_{\sigma} : \sigma \in n^{\leq n}\}$  such that

- (i)  $\tau_{\sigma} \in \omega$ -split $(T_n)$  for every  $\sigma \in n^{\leq n}$ ;
- (ii)  $\tau_{\sigma \frown k} \supseteq \tau_{\sigma}$  for every k < n and  $\sigma \in n^{< n}$ ;

(iii)  $\tau_{\sigma \frown k} \cap \tau_{\sigma \frown j} = \tau_{\sigma}$  for every  $\sigma \in n^{< n}$  and distinct k, j < n.

We extend the set of nodes  $B_n$  to  $B_{n+1} = \{\tau_{\sigma} : \sigma \in (n+1)^{\leq n+1}\}$  in a way that preserves above conditions, so we gonna have n+1 levels of infinitely splitting nodes with fixed n+1splits. The only  $\sigma \in (n+1)^0$  is  $\emptyset$ , and  $\tau_{\emptyset}$  is an old node. It is  $\omega$ -splitting in  $T_n$  and  $T_n$ is a Miller tree, so we may find  $\tau_n \supseteq \tau_{\emptyset}$ , which is  $\omega$ -splitting and  $\tau_n \cap \tau_j = \tau_{\emptyset}$  for j < n. If we already have  $\tau_{\sigma}$ 's with desired properties for  $\sigma \in (n+1)^{\leq k}$ , k < n+1, then for  $\tau_{\sigma}$ ,  $\sigma \in n^k$  (old node), we add  $\tau_{\sigma \frown n}$  such that conditions (i) - (iii) are still met. For a new node  $\tau_{\sigma}, \sigma \in (n+1)^k \setminus n^k$ , we find  $\tau_{\sigma \frown j}$  for each j < n+1 such that conditions (i) - (iii) are satisfied too. Then let  $T_{n+1}$  be any Miller subtree of  $T_n$  for which nodes from  $B_{n+1}$  are still infinitely splitting.

We will call a sequence of trees  $(T_n)_{n \in \omega}$  (or, interchangeably, their bodies  $[T_n]$ ) derived that way a *Miller fusion sequence*.

Similarly we define a *Laver fusion sequence*. The only difference would be that if  $\tau_{\sigma} \subseteq \tau_{\sigma \frown k}$ , then actually  $\tau_{\sigma \frown k} = \tau_{\sigma} \frown j$  for some  $j \in \omega$ .

PROPOSITION 21. For every Miller (resp. Laver) fusion sequence  $(T_n)_{n\in\omega}$  a set  $\bigcap_{n\in\omega} T_n$  is a Miller (resp. Laver) tree.

LEMMA 22. For every sequence of intervals  $(I_n)_{n\in\omega}$  and a Miller (resp. Laver) tree T there is a Miller (resp. Laver) fusion sequence  $(T_n)_{n\in\omega}$  such that for all n > 0:

$$\lambda([T_n] + I_n) < (1 + \sum_{k=0}^{n-1} (n-1)^k)\lambda(I_n).$$

PROOF. Let us focus on a little more complicated "Miller" case. Let  $I_0$  be an interval,  $\lambda(I_0) = \epsilon_0$ , T a Miller tree. We proceed by induction on n. Let  $\tau_{\emptyset} \in \omega$ -split(T) such that  $\lambda([\tau_{\emptyset}]) < \epsilon_0$ . Then  $\lambda([\tau_{\emptyset}] + I_0) = \lambda([\tau_{\emptyset}]) + \lambda(I_0) < 2\epsilon_0$ . Let  $T_0$  be Miller subtree of T such that that  $\tau_{\emptyset} = stem(T_0)$  and  $\tau_{\emptyset} \in \omega$ -split( $T_0$ ). Clearly, we have  $\lambda([T_0] + I_0) < 2\epsilon_0$ .

Now assume that we have a tree  $T_n$  that is an element of the emerging Miller fusion sequence, and associated with it set  $B_n$  of fixed nodes satisfying conditions (i) - (iii). Let  $\lambda(I_{n+1}) = \epsilon_{n+1}$ . Let us denote for each  $\sigma \in \omega^{<\omega}$  and interval  $I_{\sigma}$  a set

$$N(I_{\sigma}) = \{ \tau_{\sigma} \widehat{k} \in T_n : [\tau_{\sigma} \widehat{k}] \subseteq I_{\sigma} \land (\forall j < n) (\tau_{\sigma} \widehat{j} \not\supseteq \tau_{\sigma} \widehat{k}) \}$$

At each level k < n for every  $\sigma \in n^k$  let  $I_{\sigma}$  be an interval with  $\lambda(I_{\sigma}) < \frac{\epsilon_{n+1}}{(n+1)^n}$  such that a set  $N(I_{\sigma})$  is infinite and choose  $\tau_{\sigma \frown n} \in \omega$ -split $(T_n)$  such that  $\tau_{\sigma \frown n} \supseteq \tau_{\sigma} \frown l$  for some  $\tau_{\sigma} \frown l \in N(I_{\sigma})$ . At the level n let us fix an intervals  $I_{\sigma}, \lambda(I_{\sigma}) < \frac{\epsilon_{n+1}}{(n+1)^n}$ , for  $\sigma \in n^n$  such that sets  $N(I_{\sigma})$  are infinite and pick  $\tau_{\sigma \frown 0}, \tau_{\sigma \frown 1}, ..., \tau_{\sigma \frown n}$  which are extensions of some nodes  $\tau_{\sigma} \frown k_0, \tau_{\sigma} \frown k_1, ..., \tau_{\sigma} \frown k_n \in N(I_{\sigma})$  respectively. Finally we pick remaining nodes to complete a set  $B_{n+1}$  in the gist of our definition of Miller fusion sequence however we like. We take as  $T_{n+1}$  any Miller subtree of  $T_n$  for which nodes from  $B_{n+1}$  are infinitely splitting and which body is covered by intervals  $I_{\sigma}, \sigma \in n^{\leq n}$  (which is possible by infiniteness of each  $N(I_{\sigma})$ ).

Let us approximate  $\lambda([T_{n+1}] + I_{n+1})$ :

$$\lambda([T_{n+1}] + I_{n+1}) \leq \lambda(\bigcup \{I_{\sigma} + I_{n+1} : \sigma \in n^{\leq n}\} \leq \Sigma_{\sigma \in n^{\leq n}}(\lambda(I_{\sigma}) + \lambda(I_{n+1})) <$$
  
$$< \Sigma_{\sigma \in n^{\leq n}}(\frac{\epsilon_{n+1}}{(n+1)^n} + \epsilon_{n+1}),$$

and since the count of intervals  $I_{\sigma}$  is  $|n^{\leq n}| = \sum_{k=0}^{n} n^k \leq (n+1)^n$ , we have:

$$\lambda([T_{n+1}] + I_{n+1}) \le \sum_{k=0}^{n} n^k (\frac{\epsilon_{n+1}}{(n+1)^n} + \epsilon_{n+1}) \le (n+1)^n \frac{\epsilon_{n+1}}{(n+1)^n} + \sum_{k=0}^{n} n^k \epsilon_{n+1} = \epsilon_{n+1} + \sum_{k=0}^{n} n^k \epsilon_{n+1} = (1 + \sum_{k=0}^{n} n^k) \epsilon_{n+1}.$$

REMARK 23. In the above Lemma in the case of a Laver tree we may demand that  $stem(T) = stem(\bigcap_{n \in \omega} T_n)$ , if stem(T) is nonempty.

PROOF. The major difference is at the first step of the induction. Instead of picking a suitable "far enough" node  $\tau_{\emptyset} \in T$  such that  $\lambda([\tau_{\emptyset}] + I_0) < 2\lambda(I_0)$ , we already restrict the choice of nodes at the stem level by picking an interval  $I_{\emptyset}$  of measure  $\lambda(I_{\emptyset}) < \lambda(I_0)$ such that a set

$$N(I_{\emptyset}) = \{stem(T)^{\frown}k \in T : [stem(T)^{\frown}k] \subseteq I_{\emptyset}\}$$

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is infinite. It can be done since  $stem(T) \neq \emptyset$ , so all clopens  $[stem(T)^{k}], k \in \omega$ , are contained in an interval. We take a Laver subtree  $T_0$  of T for which  $[T] \subseteq I_{\emptyset}$  and  $stem(T) = stem(T_0)$  (so all nodes extending  $stem(T_0)$  come from  $I_{\emptyset}$ ). Then we continue analogously to the proof of the Lemma 22.

LEMMA 24. There exists a dense  $G_{\delta}$  set G such that for each Miller (resp. Laver or complete Laver) tree T there exists a Miller (resp. Laver or complete Laver) subtree  $T' \subseteq T$  such that  $G + [T'] \in \mathcal{N}$ .

PROOF. Let  $D = \{d_n : n \in \omega\}$  be a countable dense set,  $G = \bigcap_{n \in \omega} \bigcup_{k > n} I_k$ , where  $I_k$  is an interval with center  $d_k$  and  $\lambda(I_k) < \frac{1}{(k)^{k-1}2^k}$ . Proofs are almost identical in cases of Miller and Laver trees so let T be a Miller tree. By the Lemma 22 there is a Miller fusion sequence  $(T_n)_{n \in \omega}$  such that

$$\lambda([T_n] + I_n) < (1 + \sum_{k=0}^{n-1} (n-1)^k)\lambda(I_n) \le n^{n-1} \frac{1}{n^{n-1}2^n} = \frac{1}{2^n}.$$

 $T' = \bigcap_{n \in \omega} T_n$  is a Miller tree containing all  $T_n$ 's, so we may replace  $[T_n]$  with [T'] in the above formula and it still holds. Then for fixed  $n \in \omega$ :

$$\lambda(\bigcup_{k>n} I_k + [T']) = \lambda(\bigcup_{k>n} ([T'] + I_k)) \le \Sigma_{k>n} \lambda([T'] + I_k) \le \Sigma_{k>n} \frac{1}{2^k} = \frac{1}{2^n},$$

so, given that  $[T'] + \bigcap_{n \in \omega} \bigcup_{k > n} I_k \subseteq \bigcap_{n \in \omega} \bigcup_{k > n} ([T'] + I_k)$ , we have:

$$\lambda(G + [T']) \le \lambda(\bigcap_{n \in \omega} \bigcup_{k > n} ([T'] + I_k)) \le \lim_{n \to \infty} \frac{1}{2^n} = 0.$$

In the case of a complete Laver tree T let us observe that  $T = \bigcup_{n \in \omega} T_n$ , where  $T_n = \{\sigma \in T : (n) \subseteq \sigma \lor \sigma \subseteq (n)\}$  is a Laver tree with a nonempty stem. Let us notice that  $[T] = \bigcup_{n \in \omega} [T_n]$ . By the Lemma 22, Remark 23, and using the first part of the proof we find for each (nonempty)  $T_n$  a Laver subtree  $T'_n$  which shares the stem with  $T_n$  and for which we have:

$$[T'_n] + G \in \mathcal{N}.$$

Then  $T' = \bigcup_{n \in \omega} T'_n$  is a complete Laver subtree of T and:

$$[T'] + G = [\bigcup_{n \in \omega} T'_n] + G = \bigcup_{n \in \omega} [T'_n] + G = \bigcup_{n \in \omega} ([T'_n] + G) \in \mathcal{N}$$

as a countable union of null sets.

Before we proceed to the main theorem of this section let us recall a generalized version of Rothberger's theorem (see [13]).

THEOREM 25. (Essentially Rothberger) Assume that generalized Luzin set L and generalized Sierpiński set S exist. Then, if  $\kappa = \max\{|L|, |S|\}$  is a regular cardinal,  $|L| = |S| = \kappa$ .

PROOF. Assume that  $\kappa = |L| > |S|$  and  $\kappa$  is a regular cardinal. Let M be a meager set of full measure (the Marczewski decomposition). Then

$$\kappa = |L \cap \mathbb{R}| = |L \cap (M + S)| = |\bigcup_{s \in S} (L \cap (M + s))| < \kappa,$$

by regularity of  $\kappa$ . In the case of  $\kappa = |S| > |L|$  the proof is almost the same.

The following theorem extends the result obtained in [9].

THEOREM 26. Let  $\mathfrak{c}$  be a regular cardinal and  $t_0 \in \{s_0, m_0, l_0, cl_0\}$ . Then for every generalized Luzin set L and generalized Sierpiński set S we have  $L + S \in t_0$ .

PROOF. Let L and S be a generalized Luzin set and generalized Sierpiński set respectively. If  $|L| < \mathfrak{c}$  and  $|S| < \mathfrak{c}$ , then  $L + S \in t_0$ , since every set of cardinality less than  $\mathfrak{c}$  belongs to  $t_0$ . So, without a loss of generality (Theorem 25), let us assume that  $|L| = |S| = \mathfrak{c}$ .

We will proceed with the proof in the case  $t_0 = m_0$ , the other cases are almost identical. Let T be a Miller tree. By the virtue of Lemma 24 let G be a dense  $G_{\delta}$  set and  $T' \subseteq T$  a Miller tree such that  $[T'] + G \in \mathcal{N}$ . Let A = -G and  $B = ([T'] + G)^c$ . Then  $[T'] \subseteq (A+B)^c$ . We will show that there is a Miller tree  $T'' \subseteq T'$  which body is contained in  $(L+S)^c$ . We have:

$$L + S = ((L \cap A) \cup (L \cap A^{c})) + ((S \cap B) \cup (S \cap B^{c}))$$
  
=  $((L \cap A) + (S \cap B)) \cup ((L \cap A) + (S \cap B^{c})) \cup$   
 $\cup ((L \cap A^{c}) + (S \cap B)) \cup ((L \cap A^{c}) + (S \cap B^{c})).$ 

 $(L \cap A) + (S \cap B) \subseteq A + B$  and sets  $(L \cap A) + (S \cap B^c)$ ,  $(L \cap A^c) + (S \cap B)$  and  $(L \cap A^c) + (S \cap B^c)$  are generalized Luzin, generalized Sierpiński and of size less than  $\mathfrak{c}$ , so their intersection with [T'] has a cardinality less than  $\mathfrak{c}$ . It follows that indeed there exists a Miller tree  $T'' \subseteq T'$  such that  $(L + S) \cap [T''] = \emptyset$  and therefore L + S belongs to  $m_0$ .

Let us remark that the assumption that  $\mathfrak{c}$  is regular cannot be omitted due to the following result ([9]).

THEOREM 27. It is consistent that there exist generalized Luzin set L and generalized Sierpiński set S such that  $L + S = \mathbb{R}^n$ , and  $\mathfrak{c} = \aleph_{\omega_1}$ .

## 4. Eventually different families and t-measurablity

Two members  $f, g \in \omega^{\omega}$  of the Baire space are *eventually different* (briefly: e.d.) iff  $f \cap g$  is a finite subset of  $\omega \times \omega$ . Maximal eventually different families with respect to inclusion are called *m.e.d. families*.

Every e.d. family is a meager subset of the Baire space. It is natural to ask whether the existence of m.e.d. families that are either s-measurable or s-nonmeasurable can be proven in ZFC. It is relatively consistent with ZFC that there is a m.e.d. family  $\mathcal{A}$  of cardinality smaller then  $\mathfrak{c}$  (see [6]). In such a case  $\mathcal{A} \in s_0$ . On the other hand there exists a perfect e.d. family and therefore not all m.e.d. families are in  $s_0$ . The following two theorems answer this question positively.

THEOREM 28. There exists an s-nonmeasurable m.e.d. family in the Baire space.

PROOF. Let us fix a perfect tree  $T \subseteq \omega^{<\omega}$  such that [T] is e.d. in  $\omega^{\omega}$ . Let  $\{T_{\alpha} : \alpha < \mathfrak{c}\}$  be an enumeration of  $\mathbb{S}(T)$  - a family of all perfect subtrees of T. By transfinite reccursion we define:

$$\{(a_{\alpha}, d_{\alpha}, x_{\alpha}) \in [T] \times [T] \times \omega^{\omega} : \alpha < \mathfrak{c}\}$$

such that for any  $\alpha < \mathfrak{c}$  we have:

(1)  $a_{\alpha}, d_{\alpha} \in [T_{\alpha}],$ (2)  $\{a_{\xi} : \xi < \alpha\} \cap \{d_{\xi} : \xi < \alpha\} = \emptyset,$ (3)  $\{a_{\xi} : \xi < \alpha\} \cup \{x_{\xi} : \xi < \alpha\}$  is e.d., (4)  $\forall^{\infty}n \ x_{\alpha}(n) = d_{\alpha}(n)$  but  $x_{\alpha} \neq d_{\alpha}.$ 

Assume that we are at the step  $\alpha < \mathfrak{c}$  of the construction and we have already defined the sequence:

$$\{(a_{\xi}, d_{\xi}, x_{\xi}) \in [T]^2 \times \omega^{\omega} : \xi < \alpha\}.$$

We can choose  $a_{\alpha}, d_{\alpha} \in [T_{\alpha}]$  ( $[T_{\alpha}]$  has cardinality  $\mathfrak{c}$ ) which fulfills conditions (1), (2). Then choose any  $x_{\alpha} \in \omega^{\omega}$  distinct from  $d_{\alpha}$  but  $(\forall^{\infty} n)d_{\alpha}(n) = x_{\alpha}(n)$ . Then  $x_{\alpha} \in \omega^{\omega} \setminus [T]$  and

$$\{a_{\xi}:\xi<\alpha\}\cup\{x_{\xi}:\xi<\alpha\}$$

forms an e.d. family in  $\omega^{\omega}$ . This completes the construction. Now let us set  $A_0 = \{a_{\alpha} : \alpha < \mathfrak{c}\} \cup \{x_{\alpha} : \alpha < \mathfrak{c}\}$  and let us extend it to m.e.d. family A. It is easy to check that A is the desired *s*-nonmeasurable m.e.d. family.

In [11] it was shown that if  $\mathfrak{d} = \omega_1$  then there exists a *s*-nonmeasurable m.e.d. family  $\mathcal{A}$  and  $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$  which is dominating in  $\omega^{\omega}$ . Here *s*-nonmeasurability can be replaced by *l*-, *m*- or *cl*-nonmeasurability.

In the same paper it was proved that the following statement is relatively consistent with ZFC: " $\omega_1 < \mathfrak{d}$  and there exists *cl*-nonmeasurable m.e.d. family  $\mathcal{A}$  and a dominating family  $\mathcal{A}' \subseteq \mathcal{A}$  of the cardinality equal to  $\mathfrak{d}$ ".

The next theorem generalizes the result obtained in [11].

THEOREM 29. There exists a m.e.d. family  $\mathcal{A} \subseteq \omega^{\omega}$  such that  $\mathcal{A}$  is not s-, l- and m-measurable, with a dominating subfamily  $\mathcal{D} \in [\mathcal{A}]^{\leq \mathfrak{d}}$ .

PROOF. By definition there is a dominating family  $\mathcal{D}_0 \subseteq \omega^{\omega}$  of size  $\mathfrak{d}$ . We will show that there is an a.d. dominating family  $\mathcal{D}$  of the same size. Let  $\mathcal{P} = \{A_m \in [\omega]^{\omega} : m \in \omega\}$  be a partition of  $\omega$  into infinite subsets. Let us construct a tree as follows:  $T_{-1} = \{\emptyset\}$ , next  $T_0 = \{(0,n) : n \in \omega\}$ . Now assume that we have defined  $T_n$  for a fixed  $n \in \omega$  and let us enumerate  $T_n = \{s_k : k \in \omega\}$  then for every  $m \in \omega$  let us set  $A_m = \{k_{m,i} : i \in \omega\}$  as an increasing sequence with *i* running through  $\omega$  and *m* fixed. Define  $T_{n+1,m} = \{s_m \cup \{(n+1, k_{m,i})\} : i \in \omega\}$  and then let  $T_{n+1} = \bigcup_{m \in \omega} T_{n+1,m}$  and finally  $T = \bigcup_{n \in \omega \cup \{-1\}} T_n$ . It is easy to observe that [T] forms an a.d. family in  $\omega^{\omega}$ .

Now let us define an embedding  $f: \mathcal{D}_0 \to [T]$  as follows: pick an arbitrary element  $d \in \mathcal{D}_0$  which is an union  $\bigcup \{d \upharpoonright n : n \in \omega\}$  then assign to  $d \upharpoonright 0 = \emptyset \in T_{-1}$  and to  $d \upharpoonright 1 t_0 = d \upharpoonright 1 = \{(0, d(0))\}$ . Now let us assume that we have assigned for a fixed  $d \upharpoonright n t_n \in T_n$  for  $n \in \omega$ . Then there is unique  $m \in \omega$  such that  $t_n \in T_{n,m}$  but  $A_m = \{k_{m,i} : i \in \omega\}$  is represented by the increasing sequence  $(k_{m,i})_{i\in\omega} \in \omega^{\omega}$  then  $d \upharpoonright n+1$  is assigned to  $t_{n+1} = t_n \cup \{(n+1,w)\}$  where  $w = k_{m,d(n+1)}$  which is a greater than d(n+1). From the construction we see that  $t_{n+1} \in T_{n+1}$  and for any  $n \in \omega t_n \subseteq t_{n+1}$ . Now let  $f(d) = \bigcup \{t_n \in T_n : n \in \omega :\} \in [T]$ . It easy to see that this ensures that f is one to one mapping and for any  $d \in \mathcal{D}_0 d \leq f(d)$ . Now let  $\mathcal{D} = \{4f(d) : d \in \mathcal{D}_0\} \subseteq (4\mathbb{N})^{\omega}$  which forms a dominating family in  $\omega^{\omega}$  of size equal to  $\mathfrak{d} = |\mathcal{D}_0|$ .

Now let us choose a.d. trees  $S \subseteq (4\mathbb{N}+1)^{<\omega}$ ,  $M \subseteq (4\mathbb{N}+2)^{<\omega}$  and  $L \subseteq (4\mathbb{N}+3)^{<\omega}$ where S is a perfect tree, M is Miller and L is Laver.

Let us enumerate  $S(S) = \{S_{\alpha} : \alpha < \mathfrak{c}\}$  - a family of all perfect subtrees of S, analogously  $\mathbb{M}(M) = \{M_{\alpha} : \alpha < \mathfrak{c}\}$ , and  $\mathbb{L}(L) = \{L_{\alpha} : \alpha < \mathfrak{c}\}$ . By transfinite recursion let us define

$$\{w_{\alpha} \in [S]^{2} \times \omega^{\omega} \times [M]^{2} \times \omega^{\omega} \times [L]^{2} \times \omega^{\omega} : \alpha < \mathfrak{c}\}$$

where  $w_{\alpha} = (a_{\xi}^{s}, d_{\xi}^{s}, x_{\xi}^{s}, a_{\xi}^{m}, d_{\xi}^{m}, x_{\xi}^{m}, a_{\xi}^{l}, d_{\xi}^{l}, x_{\xi}^{l})$  for any  $\alpha < \mathfrak{c}$ , and such that for any  $\alpha < \mathfrak{c}$  we have:

 $(1) \ a_{\alpha}^{s}, d_{\alpha}^{s} \in [S_{\alpha}],$   $(2) \ \{a_{\xi}^{s}:\xi < \alpha\} \cap \{d_{\xi}^{s}:\xi < \alpha\} = \emptyset,$   $(3) \ \{a_{\xi}^{s}:\xi < \alpha\} \cup \{x_{\xi}^{s}:\xi < \alpha\} \text{ is e.d.},$   $(4) \ \forall^{\infty}n \ x_{\alpha}^{s}(n) = d_{\alpha}^{s}(n) \text{ but } x_{\alpha}^{s} \neq d_{\alpha}^{s}.$   $(5) \ a_{\alpha}^{m}, d_{\alpha}^{m} \in [M_{\alpha}],$   $(6) \ \{a_{\xi}^{m}:\xi < \alpha\} \cap \{d_{\xi}^{m}:\xi < \alpha\} = \emptyset,$ 

$$\begin{array}{l} (7) \ \left\{a_{\xi}^{m}:\xi<\alpha\right\}\cup\left\{x_{\xi}^{m}:\xi<\alpha\right\} \text{ is e.d.,}\\ (8) \ \forall^{\infty}n\ x_{\alpha}^{m}(n)=d_{\alpha}^{m}(n) \ \text{but}\ x_{\alpha}^{m}\neq d_{\alpha}^{m}.\\ (9) \ a_{\alpha}^{l},d_{\alpha}^{l}\in[L_{\alpha}],\\ (10) \ \left\{a_{\xi}^{l}:\xi<\alpha\right\}\cap\left\{d_{\xi}^{l}:\xi<\alpha\right\}=\emptyset,\\ (11) \ \left\{a_{\xi}^{l}:\xi<\alpha\right\}\cup\left\{x_{\xi}^{l}:\xi<\alpha\right\} \text{ is e.d.,}\\ (12) \ \forall^{\infty}n\ x_{\alpha}^{l}(n)=d_{\alpha}^{l}(n) \ \text{but}\ x_{\alpha}^{l}\neq d_{\alpha}^{l}. \end{array}$$

Now assume that we are at the step  $\alpha < \mathfrak{c}$  of the construction and we have a partial sequence:

 $\{w_{\alpha}: \xi < \alpha\}$ 

which has a length at most  $\omega \cdot |\alpha| < \mathfrak{c}$ . In the case of the perfect part we can choose in  $[S_{\alpha}]$  (of size  $\mathfrak{c}$ )  $a^s_{\alpha}, d^s_{\alpha} \in [S_{\alpha}]$  which fulfills the first condition. Then choose any  $x^s_{\alpha} \in \omega^{\omega}$  different than  $d^s_{\alpha}$  but  $(\forall^{\infty}n)d_{\alpha}(n) = x_{\alpha}(n)$  then  $x^s_{\alpha} \in \omega^{\omega} \setminus [S]$  and

$$\{a_{\xi}:\xi\leq\alpha\}\cup\{x_{\xi}:\xi\leq\alpha\}$$

forms an e.d. family in  $\omega^{\omega}$ . In the same way we can choose other points of our tuple for Miller and Laver trees. The construction is complete. Now let us set:

$$\mathcal{A}_{s} = \mathcal{D} \cup \{a_{\alpha}^{s} : \alpha < \mathfrak{c}\} \cup \{x_{\alpha}^{s} : \alpha < \mathfrak{c}\},\$$
$$\mathcal{A}_{m} = \mathcal{D} \cup \{a_{\alpha}^{m} : \alpha < \mathfrak{c}\} \cup \{x_{\alpha}^{m} : \alpha < \mathfrak{c}\}$$

and

$$\mathcal{A}_l = \mathcal{D} \cup \{a_\alpha^l : \alpha < \mathfrak{c}\} \cup \{x_\alpha^l : \alpha < \mathfrak{c}\}.$$

Let us extend the family  $\mathcal{D} \cup \mathcal{A}_s \cup \mathcal{A}_m \cup \mathcal{A}_l$  to any m.e.d. family  $\mathcal{A}$ . It is easy to check that  $\mathcal{A}$  is required *s*-, *m*- and *l*-nonmeasurable m.e.d. family in  $\omega^{\omega}$  with a dominating subfamily of size  $\mathfrak{d}$ , which completes the proof.

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