Ramsey Classes: Examples and Constructions

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Abstract

This article is concerned with classes of relational structures that are closed under taking substructures and isomorphism, that have the joint embedding property, and that furthermore have the *Ramsey property*, a strong combinatorial property which resembles the statement of Ramsey's classic theorem. Such classes of structures have been called *Ramsey classes*. Nešetřil and Rödl showed that they have the *amalgamation property*, and therefore each such class has a homogeneous Fraïssé limit. Ramsey classes have recently attracted attention due to a surprising link with the notion of extreme amenability from topological dynamics. Other applications of Ramsey classes include reduct classification of homogeneous structures.

We give a survey of the various fundamental Ramsey classes and their (often tricky) combinatorial proofs, and about various methods to derive new Ramsey classes from known Ramsey classes. Finally, we state open problems related to a potential classification of Ramsey classes.

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1   Introduction

Let $\mathcal{C}$ be a class of finite relational structures. Then $\mathcal{C}$ has the *Ramsey property* if it satisfies a property that resembles the statement of Ramsey's theorem: for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ there exists $\mathfrak{C} \in \mathcal{C}$ such that for every colouring of the embeddings of $\mathfrak{A}$ into $\mathfrak{C}$ with finitely many colours there exists a 'monochromatic copy' of $\mathfrak{B}$ in $\mathfrak{C}$, that is, an embedding e of $\mathfrak{B}$ into $\mathfrak{C}$ such that all embeddings of $\mathfrak{A}$ into the image of e have the same colour. An example of a class of structures with the Ramsey property is the class of all finite linearly ordered sets; this is Ramsey's theorem [45]. Another example of a class with the Ramsey property is the class of all ordered finite graphs, that is, structures $(V; E, \preceq)$ where V is a finite set, E the undirected edge relation, and $\preceq$ a linear order on V; this result has been discovered by Nešetřil and Rödl [41], and, independently, Abramson and Harrington [1].

In this article we will be concerned exclusively with classes $\mathcal{C}$ that are closed under taking substructures and isomorphism, and that have the *joint embedding property*: whenever $\mathfrak{A}, \mathfrak{B} \in C$, then there exists a $\mathfrak{C} \in \mathcal{C}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ embed into $\mathfrak{C}$. These are precisely the classes $\mathcal{C}$

for which there exists a countably infinite structure Γ such that a structure belongs to $\mathcal{C}$ if and only if it embeds into Γ. This statement also holds when the relational signature of $\mathcal{C}$ is infinite, but here we additionally require that the class $\mathcal{C}$ has only countably many non-isomorphic members. Following Fraïssé's terminology, we say that $\mathcal{C}$ is the *age* of Γ.

A class $\mathcal{C}$ will be called a *Ramsey class* [38] if it has the Ramsey property, and is the age of a countable structure. It is an open research problem, raised in [38], whether Ramsey classes can be *classified* in some sense that needs to be specified.

It has been shown by Nešetřil [38] that Ramsey classes have the *amalgamation property*, a central property in model theory. A class of structures $\mathcal{C}$ has the amalgamation property if for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ with embeddings e_i of $\mathfrak{A}$ into $\mathfrak{B}_i$, for $i \in \{1, 2\}$, there exist $\mathfrak{C} \in \mathcal{C}$ and embeddings f_i of $\mathfrak{B}_i$ into $\mathfrak{C}$ such that $f_1(e_1(a)) = f_2(e_2(a))$ for all elements a of $\mathfrak{A}$. A class of finite relational structures $\mathcal{C}$ is an *amalgamation class* if it is closed under induced substructures, isomorphism, has countably many non-isomorphic members, and the amalgamation property. By Fraïssé's theorem (which will be recalled in Section 2.5) for every amalgamation class $\mathcal{C}$ there exists a countably infinite structure Γ of age $\mathcal{C}$ which is *homogeneous*, that is, any isomorphism between finite substructures of Γ can be extended to an automorphism of Γ. The structure Γ is in fact unique up to isomorphism, and called the *Fraïssé limit of* $\mathcal{C}$. In our example above where $\mathcal{C}$ is the class of all finite linearly ordered sets $(V; <)$, the Fraïssé limit is isomorphic to $(\mathbb{Q}; <)$, that is, the linear order of the rationals.

The age of a homogeneous structure with a finite relational signature is in general not Ramsey. However, quite surprisingly, homogeneous structures with finite relational signature typically have a homogeneous *expansion* by finitely many relations such that the age of the resulting structure is Ramsey. The question whether we can replace in the previous sentence the word 'typically' by the word 'always' appeared in discussions of the author with Michael Pinsker and Todor Tsankov in 2010, and has been asked, first implicitly in a conference publication [11], then explicitly in the journal version. The question motivates much of the material present in this article, so we prominently state it here as follows.

Conjecture 1.1 (Ramsey expansion conjecture) *Let Γ be a homogeneous structure with finite relational signature. Then Γ has a homogeneous expansion by finitely many relations whose age has the Ramsey property.*

This conjecture has explicitly been confirmed for all countable homogeneous directed graphs in [31] (those graphs have been classified by Cherlin [17]), and other homogeneous structures of interest [26]. The Ramsey

expansion conjecture has several variants that are formally unrelated, but related in spirit; we will come back to this in the final section of the article. There we also discuss that the conjecture can be translated into questions in topological dynamics which are of independent interest.

This text has its focus on the combinatorial aspects of the theory, rather than the links with topological dynamics. What we do find convenient, though, is the usage of concepts from model theory to present the results: instead of manipulating amalgamation classes $\mathcal{C}$ it is often more convenient to directly manipulate the homogeneous structures of age $\mathcal{C}$.

Outline of the article. In Section 2.2 we give a self-contained introduction to the basics of Ramsey classes, including the proofs of some well-known and easy observations about them. In Section 3 we show how to derive new Ramsey classes from known ones; this section contains various facts or proofs that have not explicitly appeared in the literature yet.

- In Section 3.3 we have basic results about the Ramsey properties of interpreted structures that have not been formulated previously in this form, but that are not difficult to show via variations of the so-called *product Ramsey theorem*.

- In Section 3.4 we present a new non-topological proof, due to Miodrag Sokic, of a known fact from [11] about expanding Ramsey classes with constants.

- In Section 3.5 and 3.6 we present generalisations of results from [6] about the Ramsey properties of model-companions and model-complete cores of ω-categorical structures.

Some fundamental Ramsey classes cannot be constructed by the general construction principles from Section 3. The most powerful tool that we have to prove Ramsey theorems from scratch is the *partite method*, developed in the 70s and 80s, most notably by Nešetřil and Rödl, which we present in Section 4. With this method we will show that the following classes are Ramsey: the class of all ordered graphs, the class of all ordered triangle-free graphs, or more generally the class of all ordered structures given by a set of homomorphically forbidden irreducible substructures.

There are also Ramsey classes with finite relational signature where it is not clear how to show the Ramsey property with the partite method, to the best of my knowledge. We will see such an example, based on Ramsey theorems for tree-like structures, in Section 5.

When we want to make progress on Conjecture 1.1, we need a better understanding of the type of expansion needed to turn a homogeneous

structure in a finite language into a Ramsey structure. Very often, this can be done by adding a linear ordering to the signature (a partial explanation for this is given in Section 2.8). But not any linear ordering might do the job; a crucial property for finding the right ordering is the so-called *ordering property*, which is a classical notion in structural Ramsey theory. We will present in Section 6 a powerful condition that implies that a Ramsey class has the ordering property with respect to some given ordering.

Finally, in Section 7, we discuss the mentioned link between Ramsey theory and topological dynamics, then present an application of Ramsey theory for classifying reducts of homogeneous structures, and conclude with some open problems related to Conjecture 1.1.

2   Ramsey classes: definition, examples, background

The definition of Ramsey classes is inspired by the statement of the classic theorem of Ramsey, which we therefore recall in the next subsection, before defining the Ramsey property in Section 2.2 and Ramsey classes in Section 2.3.

There are two important necessary conditions for a class to be Ramsey: *rigidity* (Section 2.4) and *amalgamation* (Section 2.5). We will see examples that show that these two conditions are not sufficient (Section 2.6). The Ramsey property of a Ramsey class $\mathcal{C}$ can be seen as a property of the automorphism group of the Fraïssé limit of $\mathcal{C}$; this perspective is discussed in Sections 2.7 and 2.8.

2.1   Ramsey's theorem

The set of positive integers is denoted by $\mathbb{N}$, and the set $\{1, \ldots, n\}$ is denoted by $[n]$. For $M, S \subseteq \mathbb{N}$ we write $\binom{M}{S}$ for the set of all order-preserving maps from S into M. When f is a map, and $\mathcal{S}$ is a set of maps whose range equals the domain of f, then $f \circ \mathcal{S}$ denotes the set $\{f \circ e \mid e \in \mathcal{S}\}$. A proof of Ramsey's theorem can be found in almost any textbook on combinatorics.

Theorem 2.1 (Ramsey's theorem [45]**)** *For all $r, m, k \in \mathbb{N}$ there is a positive integer g such that for every $\chi \colon \binom{[g]}{[k]} \to [r]$ there exists an $f \in \binom{[g]}{[m]}$ such that $|\chi(f \circ \binom{[m]}{[k]})| \leq 1$.*

2.2   The Ramsey property

In this section we define the Ramsey property for classes of structures. All structures in this article have an at most countable domain, and have

an at most countable signature. Typically, the signature will be relational and even finite; but many results generalise to signatures that are infinite and also contain function symbols. In Section 3.4 it will be useful to consider signatures that also contain constant symbols (i.e., function symbols of arity zero).

Let τ be a relational signature, let $\mathfrak{B}$ be a τ-structure. For $R \in \tau$, we write $R^{\mathfrak{B}}$ for the corresponding relation of $\mathfrak{B}$. Typically, the domain of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ will be denoted by A, B, C, respectively. Let A be a subset of the domain B of $\mathfrak{B}$. Then the *substructure* of $\mathfrak{B}$ *induced by* A is the τ-structure $\mathfrak{A}$ with domain A such that for every relation symbol $R \in \tau$ of arity k we have $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^k$.

If τ is not a purely relational signature, but also contains constant symbols, then every substructure $\mathfrak{A}$ of $\mathfrak{B}$ must contain for every constant symbol c in τ the element $c^{\mathfrak{B}}$, and $c^{\mathfrak{A}} = c^{\mathfrak{B}}$. An *embedding* of $\mathfrak{B}$ into $\mathfrak{A}$ is a mapping f from B to A which is an isomorphism between $\mathfrak{B}$ and the substructure induced by the image of f in $\mathfrak{B}$. This substructure will also be called a *copy* of $\mathfrak{A}$ in $\mathfrak{B}$. We write $\binom{\mathfrak{B}}{\mathfrak{A}}$ for the set of all embeddings of $\mathfrak{A}$ into $\mathfrak{B}$.

Definition 2.2 (The partition arrow) When $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are τ-structures, and $r \in \mathbb{N}$, then we write $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ if for all $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \to [r]$ there exists an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| \leq 1$.

We would like to mention that in some papers, the partition arrow is defined for the situation where $\binom{\mathfrak{B}}{\mathfrak{A}}$ does not denote the set of embeddings of $\mathfrak{A}$ into $\mathfrak{B}$, but the set of copies of $\mathfrak{A}$ in $\mathfrak{B}$. These two definitions are closely related; the article [36] is specifically about this difference. Also [27] and [55] treat the relationship between the two definitions.

In analogy to the statement of Ramsey's theorem, we can now define the Ramsey property for a class of relational structures.

Definition 2.3 (The Ramsey property) A class $\mathcal{C}$ of finite structures has the *Ramsey property* if for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $k \in \mathbb{N}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_k$.

Example 2.4 The class of all finite linear orders, denoted by $\mathcal{LO}$, has the Ramsey property. This is a reformulation of Theorem 2.1. $\square$

The following well-known fact shows that we can always work with 2-colourings instead of general colourings when we want to prove that a certain class has the Ramsey property.

Lemma 2.5 *Let $\mathcal{C}$ be a class of structures, and $\mathfrak{A} \in \mathcal{C}$. Then for every $\mathfrak{B} \in \mathcal{C}$ and $r \in \mathbb{N}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ if and only if for every $\mathfrak{B} \in \mathcal{C}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_2$.*

Proof Suppose that for every $\mathfrak{B} \in \mathcal{C}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_2$. We inductively define a sequence $\mathfrak{C}_1, \ldots, \mathfrak{C}_{r-1}$ of structures in $\mathcal{C}$ as follows. Let $\mathfrak{C}_1$ be such that $\mathfrak{C}_1 \to (\mathfrak{B})^{\mathfrak{A}}_2$. For $i \in \{2, \ldots, r-1\}$, let $\mathfrak{C}_i$ be such that $\mathfrak{C}_i \to (\mathfrak{C}_{i-1})^{\mathfrak{A}}_2$. We leave it to the reader to verify that $\mathfrak{C}_{r-1} \to (\mathfrak{B})^{\mathfrak{A}}_r$. $\square$

2.3   The joint embedding property and Ramsey classes

We say that a class of structures $\mathcal{C}$ is *closed under substructures* if for every $\mathfrak{B} \in \mathcal{C}$, all substructures of $\mathfrak{B}$ are also in $\mathcal{C}$. The class $\mathcal{C}$ is *closed under isomorphism* if for every $\mathfrak{B} \in \mathcal{C}$, all structures that are isomorphic to $\mathfrak{B}$ are also in $\mathcal{C}$. In this article, we will focus on classes of finite structures that are closed under induced substructures and isomorphism, and that have the joint embedding property. Recall from the introduction that $\mathcal{C}$ has the joint embedding property if for every $\mathfrak{A}, \mathfrak{B} \in C$, there exists a $\mathfrak{C} \in \mathcal{C}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ embed into $\mathfrak{C}$. Such classes of structures naturally arise as follows; see e.g. [25].

Proposition 2.6 *A class of finite relational structures $\mathcal{C}$ is closed under substructures, isomorphism, has the joint embedding property, and has countably many non-isomorphic members if and only if there exists a countable structure Γ whose age equals $\mathcal{C}$.*

Proposition 2.6 is the main motivation why we exclusively work with classes of structures that are closed under substructures; however, as demonstrated in a recent paper by Zucker [55], several Ramsey results and techniques can meaningfully be extended to isomorphism-closed classes that only satisfy the joint embedding property and amalgamation, but that are not necessarily closed under substructures.

Definition 2.7 (Ramsey class) Let τ be an at most countable relational signature. A class of finite τ-structures is called a *Ramsey class* if it is closed under substructures, isomorphism, has countably many non-isomorphic members, the joint embedding, and the Ramsey property.

Examples of Ramsey classes will be presented below, in Example 2.11, or more generally, in Example 2.12. The following can be shown by a simple compactness argument.

Proposition 2.8 *Let Γ be a structure of age $\mathfrak{C}$. Then $\mathfrak{C}$ is a Ramsey class if and only if for all $\mathfrak{A}, \mathfrak{B} \in \mathfrak{C}$ and $r \in \mathbb{N}$ we have that $\Gamma \to (\mathfrak{B})^{\mathfrak{A}}_r$.*

Proof Let $\mathfrak{A}, \mathfrak{B} \in \mathfrak{C}$, and $r \in \mathbb{N}$ an integer. When k is the cardinality of $\binom{\mathfrak{B}}{\mathfrak{A}}$, then for any structure $\mathfrak{C}$ the fact that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ can equivalently be expressed in terms of r-colourability of a certain k-uniform hypergraph, defined as follows. Let $G = (V; E)$ be the structure whose vertex set V is $\binom{\mathfrak{C}}{\mathfrak{A}}$, and where $(e_1, \ldots, e_k) \in E$ if there exists an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $f \circ \binom{\mathfrak{B}}{\mathfrak{A}} = \{e_1, \ldots, e_k\}$. Let $H = ([r]; E)$ be the structure where E contains all tuples except for the tuples $(1, \ldots, 1), \ldots, (r, \ldots, r)$. Then $\mathfrak{C} \not\to (\mathfrak{B})^{\mathfrak{A}}_r$ if and only if G does not homomorphically map to H. An easy and well-known compactness argument (see Lemma 3.1.5 in [5]) shows that this is the case if and only if some finite substructure of G does not homomorphically map to H. Thus, $\Gamma \to (\mathfrak{B})^{\mathfrak{A}}_r$ if and only if $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ for all finite substructures $\mathfrak{C}$ of Γ. $\qquad\square$

2.4 Ramsey degrees and rigidity

Let $\mathfrak{C}$ be a class of structures with the Ramsey property. In this section we will see that each structure in $\mathfrak{C}$ must be *rigid*, that is, it has no automorphism other than the identity.

Definition [Ramsey degrees] Let $\mathfrak{C}$ be a class of structures and let $\mathfrak{A} \in \mathfrak{C}$. We say that $\mathfrak{A}$ has *Ramsey degree k (in $\mathfrak{C}$)* if $k \in \mathbb{N}$ is least such that for any $\mathfrak{B} \in \mathfrak{C}$ and for any $r \in \mathbb{N}$ there exists a $\mathfrak{C} \in \mathfrak{C}$ such that for any r-colouring χ of $\binom{\mathfrak{C}}{\mathfrak{A}}$ there is an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| \le k$.

Hence, by definition, $\mathfrak{C}$ has the Ramsey property if every $\mathfrak{A} \in \mathfrak{C}$ has Ramsey degree one.

Lemma 2.9 *Let $\mathfrak{C}$ be a class of finite structures. Then for every $\mathfrak{A} \in \mathfrak{C}$, the Ramsey degree of $\mathfrak{A}$ in $\mathfrak{C}$ is at least $|\operatorname{Aut}(\mathfrak{A})|$.*

Proof We have to show that for some $\mathfrak{B} \in \mathfrak{C}$ and $r \in \mathbb{N}$, every $\mathfrak{C} \in \mathfrak{C}$ can be r-coloured such that for all $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ we have $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| \ge |\operatorname{Aut}(\mathfrak{A})|$. We choose $\mathfrak{B} := \mathfrak{A}$ and $r := |\operatorname{Aut}(\mathfrak{A})|$.

Let $\mathfrak{C} \in \mathfrak{C}$ be arbitrary. Define an equivalence relation $\sim$ on $\binom{\mathfrak{C}}{\mathfrak{A}}$ by setting $f \sim g$ if there exists an $h \in \operatorname{Aut}(\mathfrak{A})$ such that $f = g \circ h$. Let $f_1, \ldots, f_t$ be a list of representatives for the equivalence classes of $\sim$. Define $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \to \operatorname{Aut}(\mathfrak{A})$ as follows. For $f \in \binom{\mathfrak{C}}{\mathfrak{A}}$, let i be the unique i such that $f_i \sim f$. Define $\chi(f) = h$ if $f = f_i \circ h$. Now let $e \in \binom{\mathfrak{C}}{\mathfrak{A}}$ be arbitrary. Then $|\chi(e \circ \binom{\mathfrak{A}}{\mathfrak{A}})| = |\operatorname{Aut}(\mathfrak{A})|$. $\qquad\square$

Corollary 2.10 *Let $\mathcal{C}$ be a class with the Ramsey property. Then all $\mathfrak{A}$ in $\mathcal{C}$ are rigid.*

It follows that in particular the class of all finite graphs does *not* have the Ramsey property. Frequently, a class without the Ramsey property can be made Ramsey by expanding its members appropriately with a linear ordering (the expanded structures are clearly rigid).

Example 2.11 Abramson and Harrington [1] and independently Nešetřil and Rödl [39] showed that for any relational signature τ, the class $\mathcal{C}$ of all finite *linearly ordered* τ-structures has the Ramsey property. That is, the members of $\mathcal{C}$ are finite structures $\mathfrak{A} = (A; \preceq, R_1, R_2, \dots)$ for some fixed signature $\tau = \{\preceq, R_1, R_2, \dots\}$ where $\preceq$ denotes a linear order of A.

A shorter and simpler proof of this substantial result, based on the *partite method*, can be found in [40] and [37] and will be presented in Section 4. $\square$

For a class of finite τ-structures $\mathcal{N}$, we write $\mathrm{Forb}(\mathcal{N})$ for the class of all finite τ-structures that do not admit a homomorphism from any structure in $\mathcal{N}$.

Example 2.12 The classes from Example 2.11 have been further generalised by Nešetřil and Rödl [39] as follows. Suppose that $\mathcal{N}$ is a (not necessarily finite) class of structures $\mathfrak{F}$ with finite relational signature τ such that for all elements u, v of $\mathfrak{F}$ there is a tuple in a relation $R^{\mathfrak{F}}$ for $R \in \tau$ that contains both u and v. Such structures have been called *irreducible* in the Ramsey theory literature. Then the class of all expansions of the structures in $\mathcal{C} := \mathrm{Forb}(\mathcal{N})$ by a linear order has the Ramsey property. Again, there is a proof based on the partite method, which will be presented in Section 4. This is indeed a generalization since we obtain the classes from Example 2.11 by taking $\mathcal{N} = \emptyset$. $\square$

2.5   The amalgamation property

The Ramsey classes we have seen so far will look familiar to model theorists. As mentioned in the introduction, the fact that all of the above Ramsey classes could be described as the age of a homogeneous structure is not a coincidence.

Theorem 2.13 ([38]) *Let τ be a relational signature, and let $\mathcal{C}$ be a class of finite τ-structures that is closed under isomorphism, and has the joint embedding property. If $\mathcal{C}$ has the Ramsey property, then it also has the amalgamation property.*

Proof Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be members of $\mathcal{C}$ such that there are embeddings $e_i \in \binom{\mathfrak{B}_i}{\mathfrak{A}}$ for $i = 1$ and $i = 2$. Since $\mathcal{C}$ has the joint embedding property, there exists a structure $\mathfrak{C} \in \mathcal{C}$ with embeddings f_1, f_2 of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ into $\mathfrak{C}$. If $f_1 \circ e_1 = f_2 \circ e_2$, then $\mathfrak{C}$ shows that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ amalgamate over $\mathfrak{A}$, so assume otherwise.

Let $\mathfrak{D} \in \mathcal{C}$ be such that $\mathfrak{D} \to (\mathfrak{C})^{\mathfrak{A}}_2$. Define a colouring $\chi \colon \binom{\mathfrak{D}}{\mathfrak{A}} \to [2]$ as follows. For $g \in \binom{\mathfrak{D}}{\mathfrak{A}}$, let $\chi(g) = 1$ if there is a $t \in \binom{\mathfrak{D}}{\mathfrak{C}}$ such that $g = t \circ f_1 \circ e_1$, and $\chi(g) = 0$ otherwise. Since $\mathfrak{D} \to (\mathfrak{C})^{\mathfrak{A}}_2$, there exists a $t_0 \in \binom{\mathfrak{D}}{\mathfrak{C}}$ such that $|\chi(t_0 \circ \binom{\mathfrak{C}}{\mathfrak{A}})| = 1$. Note that $\chi(t_0 \circ f_1 \circ e_1) = 1$ by the definition of χ. It follows that $\chi(t_0 \circ h) = 1$ for all $h \in \binom{\mathfrak{C}}{\mathfrak{A}}$. In particular $\chi(t_0 \circ f_2 \circ e_2) = 1$, because $f_2 \circ e_2 \in \binom{\mathfrak{C}}{\mathfrak{A}}$. Thus, by the definition of χ, there exists a $t_1 \in \binom{\mathfrak{D}}{\mathfrak{C}}$ such that $t_1 \circ f_1 \circ e_1 = t_0 \circ f_2 \circ e_2$ (here we use that the structure $\mathfrak{A}$ must be rigid, by Corollary 2.10). This shows that $\mathfrak{D}$ together with the embeddings $t_1 \circ f_1 \colon \mathfrak{B}_1 \to \mathfrak{D}$ and $t_0 \circ f_2 \colon \mathfrak{B}_2 \to \mathfrak{D}$ is an amalgam of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ over $\mathfrak{A}$. $\qquad\square$

Definition 2.14 (Amalgamation Class) An isomorphism-closed class of finite structures with an at most countable relational signature that contains at most countably many non-isomorphic structures, has the amalgamation property (defined in the introduction), and that is closed under taking induced substructures, is called an *amalgamation class*.

Theorem 2.15 (Fraïssé [20,21]; see [25]) *Let τ be a countable relational signature and let $\mathcal{C}$ be an amalgamation class of τ-structures. Then there is a homogeneous and at most countable τ-structure $\mathfrak{C}$ whose age equals $\mathcal{C}$. The structure $\mathfrak{C}$ is unique up to isomorphism, and called the* Fraïssé limit *of $\mathcal{C}$.*

Example 2.16 The Fraïssé limit of the class of all finite linear orders is isomorphic to $(\mathbb{Q}; <)$, the order of the rationals. The Fraïssé limit of the class of all graphs is the so-called random graph (or Rado graph); see e.g. [15].

We also have the following converse of Theorem 2.15.

Theorem 2.17 (Fraïssé; see [25]) *Let Γ be a homogeneous relational structure. Then the age of Γ is an amalgamation class.*

As we have seen, there is a close connection between amalgamation classes and homogeneous structures, and we therefore make the following definition.

Definition 2.18 (Ramsey structure) A homogeneous structure Γ is called *Ramsey* if the age of Γ has the Ramsey property.

2.6 Counterexamples

We have so far seen two important necessary conditions for a class $\mathcal{C}$ to be a Ramsey class: rigidity of the members of $\mathcal{C}$ (Corollary 2.10) and amalgamation (Theorem 2.13). As we will see in the examples in this section, these conditions are not sufficient for being Ramsey.

Example 2.19 Let $\mathcal{C}$ be the class of all finite $\{E, <\}$-structures where E denotes an equivalence relation and $<$ denotes a linear order. It is easy to verify that $\mathcal{C}$ has the amalgamation property. Moreover, all automorphisms of structures in $\mathcal{C}$ have to preserve $<$ and hence must be the identity. But $\mathcal{C}$ does not have the Ramsey property: let $\mathfrak{A}$ be the structure with domain $\{u, v\}$ such that $<^{\mathfrak{A}} = \{(u, v)\}$, and such that u and v are not E-equivalent. Let $\mathfrak{B}$ be the structure with domain $\{a, b, c, d\}$ such that $b <^{\mathfrak{B}} c <^{\mathfrak{B}} a <^{\mathfrak{B}} d$ and such that $\{a, b\}$ and $\{c, d\}$ are the equivalence classes of $E^{\mathfrak{B}}$. There are four copies of $\mathfrak{A}$ in $\mathfrak{B}$.

Suppose for contradiction that there is $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_2$. Let $\prec$ be a *convex* linear ordering of the elements of C, that is, a linear ordering such that $E(x, z)$ and $x < y < z$ implies that $E(x, y)$ and $E(y, z)$. Let $g \in \binom{\mathfrak{C}}{\mathfrak{A}}$. Define $\chi(g) = 1$ if $g(u) \prec g(v)$, and $\chi(g) = 2$ otherwise. Note that there are only two convex linear orderings of $\mathfrak{B}$, and that $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| = 2$ for all $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$. $\square$

However, the class of all equivalence relations with a *convex* linear order is Ramsey; see [27]. Moreover, as we will see in Example 3.25 in Section 3.7, the Fraïssé limit of the class $\mathcal{C}$ from Example 2.19 can be expanded by a convex linear order $\prec$ so that the resulting structure is homogeneous and Ramsey.

Example 2.20 The class of finite trees is not closed under taking substructures. If we close it under substructures, we obtain the class of all finite forests, a class which does not have the amalgamation property. The solution for a proper model-theoretic treatment of trees and forests is to use the concept of C-relations.

Formally, a ternary relation C is said to be a *C-relation*[2] on a set L if for all $a, b, c, d \in L$ the following conditions hold:

C1 $C(a; b, c) \to C(a; c, b);$

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[2]Terminology of Adeleke and Neumann [2].

C2 $C(a; b, c) \rightarrow \neg C(b; a, c)$;

C3 $C(a; b, c) \rightarrow C(a; d, c) \vee C(d; b, c)$;

C4 $a \neq b \rightarrow C(a; b, b)$.

A C-relation on a set L is called *binary branching* if for all pairwise distinct $a, b, c \in L$ we have $C(a; b, c)$ or $C(b; a, c)$ or $C(c; a, b)$.

The intuition here is that the elements of L denote the leaves of a rooted binary tree, and $C(a; b, c)$ holds if in the tree, the shortest path from b to c does not intersect the shortest path from a to the root; see Figure 1. For finite L, this property is actually equivalent to the axiomatic definition above [2].

The class of structures $(L; C)$ where L is a finite set and C is a binary branching C-relation on L is of course not a Ramsey class, since $(L; C)$ has nontrivial automorphisms, unless $|L| = 1$. The same argument does not work for the class $\mathcal{C}$ of all structures $(L; C, <)$ where L is finite set, C is a binary branching C-relation on L, and $<$ is a linear ordering of L. In fact, $\mathcal{C}$ is an amalgamation class (a well-known fact; for a proof, see [5]), but not a Ramsey class. To see how the Ramsey property fails, consider the structure $\mathfrak{B} \in \mathcal{C}$ with domain $\{a, b, c, d\}$ where $a < c < b < d$ such that $C(a; c, d), C(b; c, d), C(d; a, b), C(c; a, b)$, and the structure $\mathfrak{A} \in \mathcal{C}$ with domain $\{u, v\}$ where $u < v$. Now let $\mathfrak{C} \in \mathcal{C}$ be arbitrary. Let $\prec$ be a *convex* ordering of $\mathfrak{C}$, that is, a linear ordering such that for all $u, v, w \in L$, if $C(u; v, w)$ and $v \prec w$, then either $u \prec v \prec w$ or $v \prec w \prec u$. Define $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \rightarrow [2]$ as follows. For $g \in \binom{\mathfrak{C}}{\mathfrak{A}}$ define $\chi(g) = 1$ if $g(u) \prec g(v)$, and $\chi(g) = 2$ otherwise. Note that for every convex ordering $\prec$ of B there exists an $e_1 \in \binom{\mathfrak{B}}{\mathfrak{A}}$ such that $e_1(u) \prec e_1(v)$, and an $e_2 \in \binom{\mathfrak{B}}{\mathfrak{A}}$ such that $e_2(v) \prec e_2(u)$. Hence, for every $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ we have $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| = 2$.

Again, the class of all convexly ordered binary branching C-relations over a finite set is an amalgamation class (Theorem 5.1). Moreover, by the results from Section 3.7, the Fraïssé limit of the class $\mathcal{C}$ from Example 2.20 can be expanded by a convex linear order so that the resulting structure is homogeneous and Ramsey; see Example 5.3.

2.7   Automorphism groups

Let $f \colon D \rightarrow D$ be a function and $t \in D^m$ a tuple. Then $f(t)$ denotes the tuple $(f(t_1), \ldots, f(t_m))$. We say that a relation $R \subseteq D^m$ is *preserved* by a function $f \colon D \rightarrow D$ if $f(t) \in R$ for all $t \in R$. An automorphism of a structure Γ with domain D is a permutation α such that both α and α^{-1} preserve all relations (and if the signature contains constant symbols, α must fix the constants).
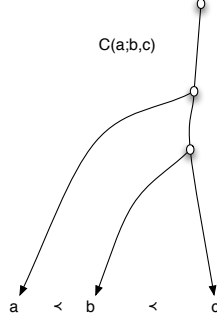
C(a;b,c)

a   ≺   b   ≺   c

Figure 1: Illustration of a convexly ordered C-relation.

The equivalent formulation of the Ramsey property in Proposition 2.21 will be useful later, for instance to prove that for every homogeneous Ramsey structure Γ there exists a linear order on the domain of Γ that is preserved by all automorphisms of Γ.

Proposition 2.21 *Let Γ be a homogeneous structure. Then the following are equivalent.*

1. *Γ is Ramsey.*

2. *For every finite substructure $\mathfrak{B}$ of Γ and $r \in \mathbb{N}$ there exists a finite substructure $\mathfrak{C}$ of Γ such that for all substructures $\mathfrak{A}_1, \ldots, \mathfrak{A}_\ell$ of $\mathfrak{B}$ and all $\chi_i \colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to [r]$ there exists an $e \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi_i(e \circ \binom{\mathfrak{B}}{\mathfrak{A}_i})| = 1$ for all $i \in [\ell]$.*

Proof $(1) \Rightarrow (2)$. We only show the forward implication, the backward implication being trivial. Let $\mathfrak{B}$ be a finite substructure of Γ and $r \in \mathbb{N}$. Let $\mathfrak{A}_1, \ldots, \mathfrak{A}_\ell$ be an enumeration of the substructures of $\mathfrak{B}$. We are going to construct a sequence of structures $\mathfrak{C}_1, \ldots, \mathfrak{C}_\ell$. Since Γ is Ramsey, there exists a substructure $\mathfrak{C}_1$ of Γ such that $\mathfrak{C}_1 \to (\mathfrak{B})^{\mathfrak{A}_1}_r$. Inductively, for $i \in \{2, \ldots, \ell\}$ there exists a substructure $\mathfrak{C}_i$ of Γ such that $\mathfrak{C}_i \to (\mathfrak{C}_{i-1})^{\mathfrak{A}_i}_r$. Define $\mathfrak{C} := \mathfrak{C}_\ell$.

For all $i \in [\ell]$, let $\chi_i \colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to [r]$ be arbitrary. Since $\mathfrak{C}_\ell \to (\mathfrak{C}_{\ell-1})^{\mathfrak{A}_\ell}_r$, there exists an $e_\ell \in \binom{\mathfrak{C}_\ell}{\mathfrak{C}_{\ell-1}}$ with $|\chi(e_\ell \circ \binom{\mathfrak{C}_{\ell-1}}{\mathfrak{A}_\ell})| \leq 1$. Inductively, suppose we have already defined $e_i \in \binom{\mathfrak{C}_\ell}{\mathfrak{C}_{i-1}}$ for an $i \in \{2, \ldots, \ell\}$ such that for all $j \in \{i, \ldots, \ell\}$ we have $|\chi(e_i \circ \binom{\mathfrak{C}_{i-1}}{\mathfrak{A}_j})| \leq 1$. Then there exists an $e_{i-1} \in$

$\binom{\mathfrak{C}_{i-1}}{\mathfrak{C}_{i-2}}$ such that $|\chi(e_{i-1} \circ \binom{\mathfrak{C}_{i-2}}{\mathfrak{A}_{i-1}})| \leq 1$. Hence, for all $j \in \{i-1, \ldots, \ell\}$ we have $|\chi(e_{i-1} \circ \binom{\mathfrak{C}_{i-2}}{\mathfrak{A}_j})| \leq 1$. Then the map $e_1 \in \binom{\mathfrak{C}_k}{\mathfrak{B}}$ has the desired properties from the statement of the proposition. $\qquad\square$

Proposition 2.22 *Let Γ be a homogeneous Ramsey structure with domain D. Then there exists a linear order on D that is preserved by all automorphisms of Γ.*

Proof Let $d_1, d_2, \ldots$ be an enumeration of D, and let $<$ be the linear order on D given by this enumeration, that is, $d_i < d_j$ if and only if $i < j$. Let $\mathfrak{T}$ be a tree whose vertices on level n are linear orders $\prec$ of $D_n := \{d_1, \ldots, d_n\}$ with the property that for all $a, b \in D_n$ and $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(a), \alpha(b) \in D_n$, we have that $a \prec b$ if and only if $\alpha(a) \prec \alpha(b)$. Note that when a linear order satisfies this condition, then also restrictions of the linear order to subsets satisfy this condition. Adjacency in $\mathfrak{T}$ is defined by restriction. Clearly, $\mathfrak{T}$ is finitely branching. We will show that $\mathfrak{T}$ has vertices on each level. By König's lemma, there is an infinite path in $\mathfrak{T}$, which defines a linear ordering on D that is preserved by $\mathrm{Aut}(\Gamma)$.

To show that there is a linear order $\prec$ on D_n that satisfies the condition, let $\mathfrak{B}$ be the structure induced by D_n in Γ, and let $\mathfrak{A}_1, \ldots, \mathfrak{A}_\ell$ list the substructures of Γ that are induced by the two-element subsets of D_n. By Proposition 2.21, there exists a finite substructure $\mathfrak{C}$ of Γ such that for all $\chi_i \colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to [r]$ there exists an $e \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi_i(e \circ \binom{\mathfrak{B}}{\mathfrak{A}_i})| = 1$ for all $i \in [l]$. Let $\chi_i \colon \binom{\mathfrak{C}}{\mathfrak{A}_i} \to [2]$ be defined as follows. For $e \in \binom{\mathfrak{C}}{\mathfrak{A}_i}$, we define $\chi_i(e) = 1$ if e preserves $<$, and $\chi_i(e) = 2$ otherwise. By the property of $\mathfrak{C}$, there is an $e \in \binom{\mathfrak{C}}{\mathfrak{B}}$ be such that $|\chi_i(e \circ \binom{\mathfrak{B}}{\mathfrak{A}_i})| = 1$ for all $i \in [l]$.

Let $\prec$ be the linear order on D_n given by $a \prec b$ if $e(a) < e(b)$. Suppose now that $a, b \in D_n$ and $\alpha \in \mathrm{Aut}(\Gamma)$ such that $\alpha(a), \alpha(b) \in D_n$. Let $i \in [l]$ be such that $\{a, b\}$ induce $\mathfrak{A}_i$ in Γ. Let f_1 be the identity on $\{a, b\}$, and let f_2 be the restriction of α to $\{a, b\}$; then $f_1, f_2 \in \binom{\mathfrak{B}}{\mathfrak{A}_i}$, and $\chi_i(e \circ f_1) = \chi_i(e \circ f_2)$. By the definition of χ_i, we have that $e(a) < e(b)$ if and only if $e(\alpha(a)) < e(\alpha(b))$. By the definition of $\prec$ we obtain that $a \prec b$ if and only if $\alpha(a) \prec \alpha(b)$. $\qquad\square$

2.8 Countably categorical structures

In this subsection we present a generalization of the class of all homogeneous structures with a finite relational signature that still satisfies a certain finiteness condition, namely the class of all countable ω-*categorical* structures.

Definition 2.23 A countable structure is said to be *ω-categorical* if all countable structures that satisfy the same first-order sentences as Γ are isomorphic to Γ.

Theorem 2.25 below explains why ω-categoricity can be seen as a finiteness condition. It will be easy to see from Theorem 2.25 that all structures that are homogeneous in a finite relational signature are ω-categorical. But we first show an example where ω-categoricity can be seen directly.

Example 2.24 All countably infinite vector spaces $\mathfrak{V}$ over a fixed finite field $\mathbb{F}$ are isomorphic. Since the isomorphism type of $\mathbb{F}$, the axioms of vector spaces, and having infinite dimension can be expressed by first-order sentences it follows that $\mathfrak{V}$ is ω-categorical. These structures are homogeneous; however, their signature is not relational. The relational structure with the same domain that contains all relations that are first-order definable over $\mathfrak{V}$ is homogeneous, too (this follows from Theorem 2.25 below). It is easy to see that all relational structures obtained from those examples by dropping all but finitely many relations, but have the same automorphism group as $\mathfrak{V}$, are *not* homogeneous. For example, the structure that just contains the ternary relation defined by $x = y + z$ has the same automorphism group as $\mathfrak{V}$, but is *not* homogeneous. The Ramsey properties of those examples are beyond the scope of this survey, but are discussed in [27].

The following theorem of Engeler, Ryll-Nardzewski, and Svenonius shows that whether a structure is ω-categorical can be seen from the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ (as a permutation group).

Theorem 2.25 (see e.g. [25]**)** *Let Γ be a countably infinite structure with a countably infinite signature. Then the following are equivalent.*

1. *Γ is ω-categorical;*

2. *$\mathrm{Aut}(\Gamma)$ is* oligomorphic*, that is, for all $n \geq 1$, the componentwise action of $\mathrm{Aut}(\Gamma)$ on n-tuples from Γ has finitely many orbits;*

3. *all orbits of n-tuples in Γ are first-order definable in Γ;*

4. *all relations preserved by $\mathrm{Aut}(\Gamma)$ are first-order definable in Γ.*

The following is a direct consequence of Proposition 2.22 and Theorem 2.25.

Corollary 2.26 *Let Γ be an ω-categorical Ramsey structure. Then there is a linear order with a first-order definition in Γ.*

Proof Let Γ^* be the homogeneous expansion of Γ by all first-order definable relations. By Proposition 2.22, there exists a linear ordering of the domain of Γ^* which is preserved by all automorphisms of Γ. By ω-categoricity of Γ and Γ^*, Theorem 2.25, this linear order is first-order definable in Γ^*. Since all first-order definable relations of Γ^* are first-order definable in Γ, they are present in the signature of Γ^*, and the statement follows. $\square$

Theorem 2.25 implies that when Γ is ω-categorical, then the expansion Γ' of Γ by all first-order definable relations is homogeneous. We therefore make the following definition.

Definition 2.27 An ω-categorical structure Γ is called *Ramsey* if the expansion of Γ by all relations with a first-order definition in Γ is Ramsey (as a homogeneous structure).

This definition is compatible with Definition 2.18, since expansions by first-order definable relations do not change the automorphism group, and since the Ramsey property only depends on the automorphism group, as reflected in the next proposition. For subsets S and M of the domain of an ω-categorical structure Γ, we write $\binom{M}{S}$ for the set of all maps from S to M that can be extended to an automorphism of Γ. The following is immediate from the definitions, Theorem 2.25, and Proposition 2.8.

Proposition 2.28 *Let Γ be an ω-categorical structure with domain D. Then the following are equivalent.*

1. *Γ is Ramsey;*

2. *For all $r \in \mathbb{N}$ and finite $M \subset D$ and $S \subset M$ there exists a finite $L \subseteq D$ such that for every map χ from $\binom{L}{S}$ to $[r]$ there exists $f \in \binom{L}{M}$ such that $|\chi(f \circ \binom{M}{S})| = 1$.*

3. *For all $r \in \mathbb{N}$ and finite $M \subset D$ and $S \subset M$ and every map χ from $\binom{D}{S}$ to $[r]$ there exists $f \in \binom{D}{M}$ such that $|\chi(f \circ \binom{M}{S})| = 1$.*

3   New Ramsey classes from old

The class of ω-categorical Ramsey structures is remarkably robust with respect to basic model-theoretic constructions. We will consider the following model-theoretic constructions to obtain new structures from given structures $\Gamma, \Gamma_1, \Gamma_2$:

- disjoint unions and products of Γ_1 and Γ_2;

- structures with a first-order interpretation in Γ;

- expansions of Γ by finitely many constants;

- the model companion of Γ;

- the model-complete core of Γ;

- superpositions of Γ_1 and Γ_2.

If the structures $\Gamma, \Gamma_1, \Gamma_2$ we started from are ω-categorical (or homogeneous in a finite relational signature), the structure we thus obtain will be again ω-categorical (or homogeneous in a finite relational signature). In this section we will see that if the original structures have good Ramsey properties, then the new structures also do.

3.1   Disjoint unions

One of the simplest operations on structures is the formation of disjoint unions: when $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are structures with the same relational signature τ and disjoint domains, then the disjoint union of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ is the structure $\mathfrak{B}$ with domain $B := A_1 \cup A_2$ where for each $R \in \tau$ we set $R^{\mathfrak{B}} := R^{\mathfrak{A}_1} \cup R^{\mathfrak{A}_2}$. The disjoint union of two ω-categorical structures is always ω-categorical. The disjoint union of two homogeneous structures Γ_1 and Γ_2 might not be homogeneous; but it clearly becomes homogeneous when we add an additional new unary predicate P to the disjoint union which precisely contains the vertices from Γ_1. We denote the resulting structure by $\Gamma_1 \uplus_P \Gamma_2$. The transfer of the Ramsey property is a triviality in this case.

Lemma 3.1 *Let Γ_1 and Γ_2 be ω-categorical Ramsey structures. Then $\Gamma := \Gamma_1 \uplus_P \Gamma_2$ is an ω-categorical Ramsey structure, too. If Γ_1 and Γ_2 are homogeneous with finite relational signature, then so is Γ.*

While this lemma looks innocent, it still has interesting applications in combination with the other constructions that we present; see Example 3.26.

3.2   Products

When G_1 and G_2 are permutation groups acting on the sets D_1 and D_2, respectively, then the direct product $G_1 \times G_2$ of G_1 and G_2 naturally acts on $D_1 \times D_2$: the element (g_1, g_2) of $G_1 \times G_2$ maps (x_1, x_2) to $(g_1(x_1), g_2(x_2))$. When G_1 and G_2 are the automorphism groups of relational structures Γ_1 and Γ_2, then the following definition yields a structure whose automorphism group is precisely $G_1 \times G_2$. (The direct product $\Gamma_1 \times \Gamma_2$ does not have this property.)

Definition 3.2 (Full Product) Let $\Gamma_1, \ldots, \Gamma_d$ be structures with domains $D_1, \ldots, D_d$ and pairwise disjoint signatures $\tau_1, \ldots, \tau_d$. Then the *full product structure* $\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_d$ is the structure with domain $D_1 \times \cdots \times D_d$ that contains for every $i \leq d$ and m-ary $R \in (\tau_i \cup \{=\})$ the relation defined by $\left\{ ((x_1^1, \ldots, x_1^d), \ldots, (x_m^1, \ldots, x_m^d)) : (x_1^i, \ldots, x_m^i) \in R^{\Gamma_i} \right\}$.

The following proposition is known as the *product Ramsey theorem* to combinatorists.

Proposition 3.3 *Let $\Gamma_1, \ldots, \Gamma_d$ be ω-categorical Ramsey structures with pairwise disjoint signatures. Then $\Gamma := \Gamma_1 \boxtimes \cdots \boxtimes \Gamma_d$ is ω-categorical and Ramsey. If $\Gamma_1, \ldots, \Gamma_d$ are homogeneous with finite relational signature, then so is Γ.*

Proof It follows from Theorem 2.25 that if $\Gamma_1, \ldots, \Gamma_d$ are ω-categorical, then $\Gamma_1 \boxtimes \cdots \boxtimes \Gamma_d$ is ω-categorical.

For the homogeneity of $\Gamma_1 \boxtimes \Gamma_2$, let $u_1 := (u_1^1, \ldots, u_1^d)$, $\ldots$, $u_m := (u_m^1, \ldots, u_m^d)$ and $v_1 := (v_1^1, \ldots, v_1^d)$, $\ldots$, $v_m := (v_m^1, \ldots, v_m^d)$ be elements of Γ such that the map a that sends $(u_1, \ldots, u_m)$ to $(v_1, \ldots, v_m)$ is an isomorphism between substructures of Γ. For $i \leq d$, define a_i as the map that sends u_j^i to v_j^i for all $j \leq m$; this is well-defined since a preserves the relation $\{(x^1, \ldots, x^d, y^1, \ldots, y^d) : x^i = y^i\}$. By homogeneity of Γ_i, there exists an extension α_i of a_i to an automorphism of Γ_i. Then the map α given by $\alpha(x_1, \ldots, x_d) := (\alpha_1(x_1), \ldots, \alpha_d(x_d))$ is an automorphism of Γ and extends α.

To prove that Γ is Ramsey, we show the statement for $d = 2$; the general case then follows by induction on d. Let $\mathfrak{A}, \mathfrak{B}$ be substructures of $\Gamma = \Gamma_1 \boxtimes \Gamma_2$ and $r \in \mathbb{N}$ be arbitrary. We will show that $\Gamma \to (\mathfrak{B})_r^{\mathfrak{A}}$, so let $\chi \colon \binom{\mathfrak{B}}{\mathfrak{A}} \to [r]$ be arbitrary. If $\binom{\mathfrak{B}}{\mathfrak{A}}$ is empty, then the statement is trivial, so in the following we assume that $\mathfrak{A}$ embeds into $\mathfrak{B}$. For $i \in \{1, 2\}$, let $\mathfrak{A}_i$ be the structure induced in Γ_i by $\{a_i : (a_1, a_2) \in A\}$, and define $\mathfrak{B}_i$ analogously with B instead of A. Since Γ_2 is Ramsey there exists a finite substructure $\mathfrak{C}_2$ of Γ_2 such that $\mathfrak{C}_2 \to (\mathfrak{B}_2)_r^{\mathfrak{A}_2}$. Define $s := |\binom{\mathfrak{C}_2}{\mathfrak{A}_2}|$. Since Γ_1 is Ramsey there exists a finite substructure $\mathfrak{C}_1$ of Γ_1 such that $\mathfrak{C}_1 \to (\mathfrak{B}_1)_{r^s}^{\mathfrak{A}_1}$. We identify the elements of $[r^s]$ with functions from $\binom{\mathfrak{C}_2}{\mathfrak{A}_2}$ to $[r]$. Define $\chi_1 \colon \binom{\mathfrak{C}_1}{\mathfrak{A}_1} \to [r^s]$ as follows. Let $e_1 \in \binom{\mathfrak{C}_1}{\mathfrak{A}_1}$, let $e_2 \in \binom{\mathfrak{C}_2}{\mathfrak{A}_2}$, and let $e \in \binom{\mathfrak{B}}{\mathfrak{A}}$ be the embedding such that $e(a_1, a_2) = (e_1(a_1), e_2(a_2))$. Let $\xi \colon \binom{\mathfrak{C}_2}{\mathfrak{A}_2} \to [r]$ be the function that maps $e_2 \in \binom{\mathfrak{C}_2}{\mathfrak{A}_2}$ to $\chi(e)$. Define $\chi_1(e_1) = \xi$. Then there exists an $f_1 \in \binom{\mathfrak{C}_1}{\mathfrak{B}_1}$ such that $\chi_1(f_1 \circ \binom{\mathfrak{B}_1}{\mathfrak{A}_1}) = \{\chi_2\}$ for some $\chi_2 \in \binom{\mathfrak{C}_2}{\mathfrak{A}_2} \to [r]$. As $\mathfrak{C}_2 \to (\mathfrak{B}_2)_r^{\mathfrak{A}_2}$, there exists an $f_2 \in \binom{\mathfrak{C}_2}{\mathfrak{B}_2}$ such that $|\chi_2(f_2 \circ \binom{\mathfrak{B}_2}{\mathfrak{A}_2})| = 1$. Let $f \in \binom{\mathfrak{C}_1 \boxtimes \mathfrak{C}_2}{\mathfrak{B}}$ be given by $b \mapsto (f_1(b), f_2(b))$.

We claim that $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| = 1$. Arbitrarily choose $e, e' \in \binom{\mathfrak{B}}{\mathfrak{A}}$. Then there are $e_i, e_i' \colon \binom{\mathfrak{B}_i}{\mathfrak{A}_i}$ for $i \in \{1, 2\}$ such that $e(A) \subseteq (e_1(A_1), e_2(A_2))$ and $e'(A) \subseteq (e_1'(A_1), e_2'(A_2))$. Then $\chi_1(f_1 \circ e_1) = \chi_1(f_1 \circ e_1') = \chi_2$, and $\chi_2(f_2 \circ e_1) = \chi_2(f_2 \circ e_2')$. Then $\chi(e) = \chi_2(f_2 \circ e_2) = \chi_2(f_2 \circ e_2') = \chi(e')$, which is what we had to show. $\qquad\square$

The special case of Proposition 3.3 where $\Gamma_1 = \cdots = \Gamma_d = (\mathbb{Q}; <)$ can be found in [22] (page 97). The general case can also be shown inductively, see e.g. [11]. One may also derive it using the results in Kechris-Pestov-Todorcevic [27], since the direct product of extremely amenable groups is extremely amenable (also see [6]).

3.3   Interpretations

The concept of *first-order interpretations* is a powerful tool to construct new structures. A simple example of an interpretation is the line graph of a graph G, which has a first-order interpretation over G. By passing to the age of the constructed structure, they are also a great tool to define new *classes* of structures.

Definition 3.4 A relational σ-structure $\mathfrak{B}$ has a *(first-order) interpretation* I in a τ-structure $\mathfrak{A}$ if there exists a natural number d, called the *dimension* of I, and

- a τ-formula $\delta_I(x_1, \ldots, x_d)$ – called the *domain formula*,

- for each atomic σ-formula $\phi(y_1, \ldots, y_k)$ a τ-formula

$$\phi_I(y_{1,1}, \ldots, y_{1,d}, y_{2,1}, \ldots, y_{2,d}, \ldots, y_{k,1}, \ldots, y_{k,d})$$

  – the *defining formulas*;

- a surjective map h from $\{\bar{a} : \mathfrak{A} \models \delta_I(\bar{a})\}$ to B – called the *coordinate map*,

such that for all atomic σ-formulas ϕ and all elements $a_{1,1}, \ldots, a_{k,d}$ with $\mathfrak{A} \models \delta_I(a_{i,1}, \ldots, a_{i,d})$ for all $i \leq k$

$$\mathfrak{B} \models \phi(h(a_{1,1}, \ldots, a_{1,d}), \ldots, h(a_{k,1}, \ldots, a_{k,d}))$$
$$\Leftrightarrow \quad \mathfrak{A} \models \phi_I(a_{1,1}, \ldots, a_{k,d}) \, .$$

We give illustrating examples.

Example 3.5 When $(V; E)$ is an undirected graph, then the *line graph* of $(V; E)$ is the undirected graph $(E; F)$ where $F := \big\{\{u, v\} : |u \cap v| = 1\big\}$. Undirected graphs can be seen as structures where the signature contains a single binary relation denoting a symmetric irreflexive relation. Then the line graph of $(V; E)$ has the following 2-dimensional interpretation I over $(V; E)$: the domain formula $\delta_I(x_1, x_2)$ is $E(x_1, x_2)$, the defining formula for the atomic formula $y_1 = y_2$ is

$$(y_{1,1} = y_{2,1} \land y_{1,2} = y_{2,2}) \lor (y_{1,1} = y_{2,2} \land y_{1,2} = y_{2,1}),$$

and the defining formula for the atomic formula $F(y_1, y_2)$ is

$$\big((y_{1,1} \neq y_{2,1} \land y_{1,1} \neq y_{2,2}) \lor (y_{1,2} \neq y_{2,2} \land y_{1,2} \neq y_{2,2})\big)$$
$$\land (y_{1,1} = y_{2,1} \lor y_{1,1} = y_{2,2} \lor y_{1,2} = y_{2,1} \lor y_{1,2} = y_{2,2}).$$

The coordinate map is the identity.

Example 3.6 A poset $(P; \leq)$ has *poset dimension at most k* if there are k linear extensions $\leq_1, \ldots, \leq_k$ of $\leq$ such that $x \leq y$ if and only if $x \leq_i y$ for all $i \in [k]$. The class of all finite posets of poset dimension at most k is the age of $(\mathbb{Q}; \leq)^k$, which clearly has a k-dimensional interpretation in $(\mathbb{Q}; <)$.

Lemma 3.7 (Theorem 7.3.8 in [24]**)** *Let $\mathfrak{A}$ be an ω-categorical structure. Then every structure $\mathfrak{B}$ that is first-order interpretable in $\mathfrak{A}$ is countably infinite ω-categorical or finite.*

Note that in particular all reducts (defined in the introduction) of an ω-categorical structure Γ have an interpretation in Γ and are thus again ω-categorical. On the other hand, being homogeneous with finite relational signature is not inherited by the interpreted structures. An example of a structure which is not interdefinable with a homogeneous structure in a finite relational signature, but which has a first-order interpretation over $(\mathbb{N}; =)$, has been found by Cherlin and Lachlan [16].

Proposition 3.8 *Suppose that Γ is ω-categorical Ramsey. Then every structure with a first-order interpretation in Γ has an ω-categorical Ramsey expansion Δ. Furthermore, if Γ is homogeneous with a finite relational signature, then we can choose Δ to be homogeneous in a finite relational signature, too.*

Corollary 3.9 *Conjecture 1.1 is true for countable stable[3] homogeneous structures with finite relational signature.*

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[3]For the definition of stability we refer to any text book in model theory.

Proof Lachlan [30] proved that every stable homogeneous structure with a finite relational signature has a first-order interpretation over $(\mathbb{Q}; <)$. The statement follows from the fact that $(\mathbb{Q}; <)$ is Ramsey, and Proposition 3.8. □

3.4 Adding constants

Let Γ be homogeneous. It is clear that the expansion $(\Gamma, d_1, \ldots, d_n)$ by finitely many constants $d_1, \ldots, d_n$ is again homogeneous. Similarly, if Γ is ω-categorical, then $(\Gamma, d_1, \ldots, d_n)$ is ω-categorical, as a consequence of Theorem 2.25. We will show here that if Γ is Ramsey, then $(\Gamma, d_1, \ldots, d_n)$ remains Ramsey. The original proof [11] went via a more general fact from topological dynamics (open subgroups of extremely amenable groups are extremely amenable). We give an elementary proof here, due to Miodrag Sokic.

Theorem 3.10 *Let Γ be homogeneous and Ramsey. Let $d_1, \ldots, d_n$ be elements of Γ. Then $(\Gamma, d_1, \ldots, d_n)$ is also Ramsey.*

Proof Let τ be the signature, and D the domain of Γ. We write d for $(d_1, \ldots, d_n)$. Let $\mathfrak{A}^*, \mathfrak{B}^*$ be two finite substructures of Γ^*, let $r \in \mathbb{N}$, and let $\chi^* \colon \binom{\Gamma^*}{\mathfrak{A}^*} \to [r]$ be arbitrary. We have to show that there exists an $f \in \binom{\Gamma^*}{\mathfrak{B}^*}$ such that $|\chi(f \circ \binom{\mathfrak{B}^*}{\mathfrak{A}^*})| = 1$. We write $\mathfrak{A}$ and $\mathfrak{B}$ for the τ-reducts of $\mathfrak{A}^*$ and $\mathfrak{B}^*$, respectively.

Define $\chi \colon \binom{\Gamma}{\mathfrak{A}} \to [r]$ as follows. First, we fix for each tuple $a \in D^n$ that lies in the same orbit as d in $\mathrm{Aut}(\Gamma)$ an automorphism α_a of Γ such that $\alpha_a(a) = d$. Let $e \in \binom{\Gamma}{\mathfrak{A}}$, and $a := e(d)$. By the homogeneity of Γ, the tuples a and d lie in the same orbit of $\mathrm{Aut}(\Gamma)$. Note that $\alpha_a \circ e$ fixes d and is an embedding of $\mathfrak{A}^*$ into Γ^*. Define $\chi(e) := \chi^*(\alpha_a \circ e)$.

Since Γ is Ramsey, there is an $f \in \binom{\Gamma}{\mathfrak{B}}$ such that $\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}}) = \{c\}$ for some $c \in [r]$. Let b be $f(d)$. By the homogeneity of Γ, the tuples b and d lie in the same orbit of $\mathrm{Aut}(\Gamma)$. Observe that $f' := \alpha_b \circ f$ fixes d and is an embedding of $\mathfrak{B}^*$ into Γ^*.

We claim that $|\chi^*(f' \circ \binom{\mathfrak{B}^*}{\mathfrak{A}^*})| = 1$. To prove this, let $g \in \binom{\mathfrak{B}^*}{\mathfrak{A}^*}$ be arbitrary. Since g is in particular from $\binom{\mathfrak{B}}{\mathfrak{A}}$ we have $\chi(f \circ g) = c$. By the definition of χ we have that $\chi(f \circ g) = \chi^*(\alpha_b \circ f \circ g) = \chi^*(f' \circ g)$. Hence, $\chi^*(f' \circ g) = c$, which proves the claim. □

In this article, we work mostly with relational signatures. It is therefore important to note that the relational structure $(\Gamma, \{d_1\}, \ldots, \{d_n\})$ is in general *not* homogeneous even if Γ is. Consider for example the Fraïssé limit $\Gamma = (\mathbb{V}; E)$ of the class of all finite graphs, and an arbitrary $d_1 \in \mathbb{V}$.

Let $p \in \mathbb{V} \setminus \{d_1\}$ be such that $E(p, d_1)$ and $q \in \mathbb{V} \setminus \{d_1\}$ be such that $\neg E(d_1, q)$. Then the mapping that sends p to q is an isomorphism between (one-element) substructures of $(\Gamma, \{d_1\})$ which cannot be extended to an automorphism of $(\Gamma, \{d_1\})$. (The difference to (Γ, d_1) is that all substructures of (Γ, d_1) must contain d_1.) Note, however, that $(\Gamma, d_1, \ldots, d_n)$ and $(\Gamma, \{d_1\}, \ldots, \{d_n\})$ have the same automorphism group.

The solution to stating the result about expansions of homogeneous structures with constants in the relational setting is linked to the following definition.

Definition 3.11 Let Γ be a relational structure with signature τ, and $d_1, \ldots, d_n$ elements of Γ. Then $\Gamma_{d_1, \ldots, d_n}$ denotes the expansion of Γ which contains for every $R \in (\tau \cup \{=\})$ of arity $k \geq 2$, every $i \in [k]$ and $j \in [n]$, the $(k-1)$-ary relation $\{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k) : (x_1, \ldots, x_k) \in R$ and $x_i = d_j\}$.

Note that if the signature of Γ is finite, then the signature of $\Gamma_{d_1, \ldots, d_n}$ is also finite, and the maximal arity is unaltered. Also note that $\Gamma_{d_1, \ldots, d_n}$ has in particular the unary relations $\{d_1\}, \ldots, \{d_n\}$.

Lemma 3.12 *Let Γ be a homogeneous relational structure, and $d_1, \ldots, d_n$ elements of Γ. Then $\Gamma_{d_1, \ldots, d_n}$ is homogeneous.*

Proof Let a be an isomorphism between two finite substructures A_1, A_2 of $\Gamma_{d_1, \ldots, d_n}$. Since $\Gamma_{d_1, \ldots, d_n}$ contains for all $i \leq n$ the relation $\{d_i\}$ which is preserved by a, it follows that if A_1 or A_2 contains c_i, then both A_1 and A_2 must contain d_i, and $a(d_i) = d_i$. If d_i is contained in neither A_1 nor A_2, then a can be extended to a partial isomorphism a' of $\Gamma_{d_1, \ldots, d_n}$ with domain $A_1 \cup \{d_i\}$ by setting $a(d_i) = d_i$: this follows directly from the definition of the signature of $\Gamma_{d_1, \ldots, d_n}$. By the homogeneity of Γ, the map a' can be extended to an automorphism of Γ. This automorphism fixes $d_1, \ldots, d_n$ pointwise, and hence is an automorphism of $\Gamma_{d_1, \ldots, d_n}$. $\square$

Corollary 3.13 *Let Γ be homogeneous, ω-categorical, and Ramsey, and let $d_1, \ldots, d_n$ be elements of Γ. Then $\Gamma_{d_1, \ldots, d_n}$ is also Ramsey.*

Proof The statement follows from Theorem 3.10 from the observation that $\Gamma_{d_1, \ldots, d_n}$ and $(\Gamma, d_1, \ldots, d_n)$ have the same automorphism group, and that whether an ω-categorical structure has the Ramsey property only depends on its automorphism group (Proposition 2.28). $\square$

3.5   Passing to the model companion

A structure Γ is called *model-complete* if all embeddings between models of the first-order theory of Γ preserve all first-order formulas. It is well-known that this is equivalent to every first-order formula being equivalent to an existential formula over Γ (see e.g. [25]). It is also known (see Theorem 3.6.7 in [5]) that an ω-categorical structure Γ is model-complete if and only if for every finite tuple t of elements of Γ and for every self-embedding e of Γ into Γ there exists an automorphism α of Γ such that $e(t) = \alpha(t)$.

A *model companion* of Γ is a model-complete structure Δ with the same age as Γ. If Γ has a model companion, then the model companion is unique up to isomorphism [25]. Every ω-categorical structure has a model companion, and the model companion is again ω-categorical [46].

Example 3.14 We write $\mathbb{Q}_0^+$ for $\{q \in \mathbb{Q} : q \geq 0\}$. The structure $\Gamma :=$ $(\mathbb{Q}_0^+; <)$ is ω-categorical, but not model-complete: for instance the map $x \mapsto x + 1$ is an embedding of Γ into Γ which does not preserve the unary relation $\{0\}$ with the first-order definition $\forall y (y \geq x)$ over Γ. The model companion of Γ is $(\mathbb{Q}; <)$.

In this subsection we prove the following.

Theorem 3.15 *Let Γ be ω-categorical and Ramsey, and let Δ be the model companion of Γ. Then Δ is also Ramsey.*

Proof Let e be an embedding of Γ into Δ, and let i be an embedding of Δ into Γ; such embeddings exist by ω-categoricity of Δ and Γ, see Section 3.6.2 in [5]. We will work with the equivalent characterisation of the Ramsey property given in item 2 of Proposition 2.28.

Let S and M be finite subsets of the domain D of Δ and $r \in \mathbb{N}$, and let $\chi \colon \binom{D}{S} \to [r]$ be arbitrary. Let D' be the domain of Γ. We define a map $\chi' \colon \binom{D'}{i(S)} \to [r]$ as follows. For $q' \in \binom{D'}{i(S)}$, note that $e \circ q' \circ i \in \binom{D}{S}$. We define $\chi'(q') := \chi(e \circ q' \circ i)$.

Since Γ is Ramsey, there exists an $f' \in \binom{D'}{i(M)}$ and $c \in [r]$ such that for all $g' \in \binom{i(M)}{i(S)}$ we have $\chi'(f' \circ g') = c$. Let $\alpha' \in \mathrm{Aut}(\Gamma)$ be an extension of f'. Note that $e \circ \alpha' \circ i$ is an embedding of Δ into Δ, and since Δ is model-complete there exists an $\alpha \in \mathrm{Aut}(\Delta)$ that extends the restriction f of $e \circ \alpha' \circ i$ to M.

Let $g \in \binom{M}{S}$ be arbitrary. We claim that $\chi(f \circ g) = c$. Since $e \circ i$ is an embedding of Δ into Δ and Δ is model-complete, there exists an automorphism β of Δ such that $\beta(e(i(x))) = x$ for all $x \in S$. Note that

$g' := i \circ g \circ \beta \circ e \in \binom{i(M)}{i(S)}$, and hence $\chi'(f' \circ g') = c$. Also note that by the definition of χ' we have

$$\chi'(f' \circ g') = \chi(e \circ f' \circ g' \circ i) = \chi(e \circ f' \circ i \circ g \circ \beta \circ e \circ i) = \chi(f \circ g) \,.$$

Hence, $\chi(f \circ g) = c$, and $|\chi(f \circ \binom{M}{S}))| \leq 1$, and thus Δ is Ramsey.   □

3.6   Passing to the model-complete core

Cores play an important role in finite combinatorics. The concept of model-complete cores can be seen as an existential-positive analog of model-companions, where embeddings are replaced by homomorphisms and self-embeddings are replaced by endomorphisms. We state here results that are analogous to the results for model companions that we have seen in the previous section.

Definition 3.16 Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures with domain A and B, respectively, and the same relational signature τ. Then a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a function $f\colon A \to B$ such that for all $(a_1, \ldots, a_n) \in R^{\mathfrak{A}}$ we have $(f(a_1), \ldots, f(a_n)) \in R^{\mathfrak{B}}$. An *endomorphism* of a structure Γ is a homomorphism from Γ to Γ. A structure Γ is called a *core* if every endomorphism of Γ is an embedding.

An ω-categorical structure Γ is a model-complete core if and only if for every finite tuple t of elements of Γ and for every endomorphism e of Γ there exists an automorphism α of Γ such that $e(t) = \alpha(t)$ (Theorem 3.6.11 in [5]). The following has been shown in [4] (also see [7]). Two structures Γ and Δ are *homomorphically equivalent* if there is a homomorphism from Γ to Δ and a homomorphism from Δ to Γ.

Theorem 3.17 *Every ω-categorical structure is homomorphically equivalent to a model-complete core Δ, which is unique up to isomorphism, and again countably infinite ω-categorical or finite. The expansion of Δ by all existential positive definable relations is homogeneous.*

The structure Δ in Theorem 3.17 will be called *the model-complete core of* Γ.

Theorem 3.18 *Let Γ be ω-categorical and Ramsey, and let Δ be the model-complete core of Γ. Then Δ is also Ramsey.*

Proof   The proof is similar to the proof of Theorem 3.15.   □

3.7 Superimposing signatures

An amalgamation class $\mathcal{C}$ is called a *strong amalgamation class* if, informally, we can amalgamate structures $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ over $\mathfrak{A} \in \mathcal{C}$ in such a way that no points of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ other than the elements of $\mathfrak{A}_1$ will be identified in the amalgam. Formally, we require that for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ and embeddings $e_i \colon \mathfrak{A} \to \mathfrak{B}_i$, $i \in \{1, 2\}$, there exists a structure $\mathfrak{C} \in \mathcal{C}$ and embeddings $f_i \colon \mathfrak{B}_i \to \mathfrak{C}$ such that $e_1(f_1(x)) = e_2(f_2(x))$ for all $x \in A$, and additionally $f_1(B_1) \cap f_2(B_2) = f_1(e_1(A)) = f_2(e_2(A))$. When an amalgamation class $\mathcal{C}$ even has strong amalgamation, then this can be seen from the automorphism group of the Fraïssé limit of $\mathcal{C}$.

Definition 3.19 ([14]) We say that a permutation group has *no algebraicity* if for every finite tuple $(a_1, \ldots, a_n)$ of the domain the set of all permutations of the group that fix each of $a_1, \ldots, a_n$ fixes no other elements of the domain.

For automorphism groups of ω-categorical structures Γ, having no algebraicity coincides with the model-theoretic notion of Γ having no algebraicity (see, e.g., [25]).

Lemma 3.20 (see (2.15) in [14]) *Let $\mathcal{C}$ be an amalgamation class of relational structures and Γ its Fraïssé limit. Then $\mathcal{C}$ has strong amalgamation if and only if Γ has no algebraicity.*

For strong amalgamation classes there is a powerful construction to obtain new strong amalgamation classes from known ones.

Definition 3.21 Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of finite structures with disjoint relational signatures τ_1 and τ_2, respectively. Then the *free superposition of $\mathcal{C}_1$ and $\mathcal{C}_2$*, denoted by $\mathcal{C}_1 * \mathcal{C}_2$, is the class of $(\tau_1 \cup \tau_2)$-structures $\mathfrak{A}$ such that the τ_i-reduct of $\mathfrak{A}$ is in $\mathcal{C}_i$, for $i \in \{1, 2\}$.

The following lemma has a straightforward proof by combining amalgamation in $\mathcal{C}_1$ with amalgamation in $\mathcal{C}_2$.

Lemma 3.22 *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are strong amalgamation classes, then $\mathcal{C}_1 * \mathcal{C}_2$ is also a strong amalgamation class.*

When Γ_1 and Γ_2 are homogeneous structures with no algebraicity, then $\Gamma_1 * \Gamma_2$ denotes the (up to isomorphism unique) Fraïssé limit of the free superposition of the age of Γ_1 and the age of Γ_2.

Example 3.23 For $i \in \{1, 2\}$, let $\tau_i = \{<_i\}$, let $\mathcal{C}_i$ be the class of all finite τ_i-structures where $<_i$ denotes a linear order, and let Γ_i be the Fraïssé limit of $\mathcal{C}_i$. Then $\Gamma_1 * \Gamma_2$ is known as the *random permutation* (see e.g. [12, 33, 47]).

We have the following result about free superpositions.

Theorem 3.24 ([6]) *Let Γ_1 and Γ_2 be homogeneous ω-categorical structures with no algebraicity such that both Γ_1 and Γ_2 are Ramsey. Then $\Gamma_1 * \Gamma_2$ is Ramsey.*

We mention that the proof of Theorem 3.24 from [6] uses Theorem 3.18 about model-complete cores. An alternative proof can be found in [49].

Example 3.25 Recall from Example 2.19 that the amalgamation class of all finite structures $(V; E, <)$ where E denotes an equivalence relation and $<$ denotes a linear order, is *not* Ramsey. In the light of Conjecture 1.1 for the Fraïssé limit Γ of this class, we therefore look for a homogeneous Ramsey expansion of Γ. Let $\mathcal{C}$ be the class of all finite structures $(V; E, \prec)$ where E is an equivalence relation and $\prec$ is a linear order that is convex with respect to E. We have mentioned before that $\mathcal{C}$ is Ramsey, and by Theorem 3.24 the class $\mathcal{C} * \mathcal{LO}$ is Ramsey. Then the Fraïssé limit of $\mathcal{C} * \mathcal{LO}$ is isomorphic to a homogeneous Ramsey expansion of Γ.

Example 3.26 The directed graph $S(2)$ is one of the homogeneous directed graphs that figures in the classification of all homogeneous directed graphs of Cherlin [17]. In fact, it is a homogeneous tournament and therefore already appeared in the classification of homogeneous tournaments of Lachlan [29]. It has many equivalent definitions, one of them being the following: the vertices of $S(2)$ are a countable dense set of points on the unit circle without antipodal points. We add an edge from x to y if and only if the line from x to y has the origin on the left; that is, x, y, and $(0,0)$ lie in clockwise order in the plane.

We will show that $S(2)$ has a Ramsey expansion which is homogeneous and has a finite relational signature. This can be derived from general principles and Ramsey's theorem as follows. In the following, $(\mathbb{Q}; <_1)$ and $(\mathbb{Q}; <_2)$ both denote the order of the rationals, but have disjoint signature. Let Γ be the disjoint union $(\mathbb{Q}; <_1) \uplus_P (\mathbb{Q}; <_1)$, which has the Ramsey property by Theorem 2.1 and Lemma 3.1 (Example 2.4). Then the free superposition Δ of $(\mathbb{Q}; <_2)$ with Γ is Ramsey by Theorem 3.24, and homogeneous with finite relational signature. The structure $S(2)$ is a reduct of Δ: for elements $x, y \in S(2)$, we define $x \prec y$ if

$$\big(x <_2 y \wedge (P(x) \Leftrightarrow P(y))\big) \vee \big(y <_2 x \wedge (P(x) \not\Leftrightarrow P(y))\big) .$$

Let Γ_1 and Γ_2 be two ω-categorical Ramsey structures. Note that since there is a linear order with a first-order definition in Γ_1, and a first-order definition of a linear order in Γ_2, the structure $\Gamma_1 * \Gamma_2$ must carry two independent linear orders.

To prove the Ramsey property for structures that do not have a second independent linear order, we have the following variant.

Theorem 3.27 ([6]) *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be classes of structures such that $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{LO}$ have pairwise disjoint signatures. Also suppose that $\mathcal{C}_1$ and $\mathcal{LO} * \mathcal{C}_2$ are Ramsey classes with strong amalgamation and ω-categorical Fraïssé limits. Then $\mathcal{C}_1 * \mathcal{C}_2$ is also a Ramsey class.*

4   The partite method

There are some homogeneous Ramsey structures where no proof of the Ramsey property from general principles is known. One of the most powerful methods to prove the Ramsey property is such situations is the *partite method*. The first result that we see in this section is that for any finite relational signature τ, the class of all finite ordered τ-structures is a Ramsey class. This is due to Nešetřil and Rödl [39] and independently to Abramson and Harrington [1]; in these original papers, the statement is made for hypergraphs only, but it holds for relational structures in general. We then apply the partite method to classes that are characterised by forbidding finite structures as induced substructures.

4.1   The class of all ordered structures

We will prove the following theorem, due to Nešetřil and Rödl, and, independently, Abramson and Harrington.

Theorem 4.1 ([1,39]) *For every relational signature τ the class of all $\tau \cup \{\preceq\}$-structures, where $\preceq$ denotes a linear order, is a Ramsey class.*

The construction to prove the Ramsey property is due to Nešetřil and Rödl [40], with only minor modifications in the presentation. It relies on the concept of *n-partite structures*. We formalize this slightly differently than Nešetřil and Rödl in [40].

Definition 4.2 Let $n \in \mathbb{N}$, and τ a relational signature. An *n-partite structure* is a finite $(\tau \cup \{\preceq\})$-structure $(\mathfrak{A}, \preceq)$ where $\preceq$ is a weak linear order (that is, a linear quasi-order) such that the equivalence relation $\approx$ on A defined by $x \approx y \Leftrightarrow (x \preceq y \wedge y \preceq x)$ has n equivalence classes. An

n-partite structure is called a *transversal* if each equivalence class of $\approx$ has size one.

Note that the elements of a finite n-partite structure $(\mathfrak{A}, \preceq)$ are partitioned into *levels* $A_1, \ldots, A_n$ which are uniquely given by the property that for $u \in A_i$ and $v \in A_j$ we have $u \preceq v$ if and only if $i \leq j$.

Lemma 4.3 (Partite Lemma) *Let $\mathfrak{A}$ be an n-partite transversal, $\mathfrak{B}$ an arbitrary n-partite structure, and $r \in \mathbb{N}$. Then there exists an n-partite structure $\mathfrak{C}$ such that $\mathfrak{C} \to (\mathfrak{B})_r^{\mathfrak{A}}$.*

The idea of the proof of Lemma 4.3 is to use the theorem of Hales-Jewett (see [22]), which we quickly recall here to fix some terminology.

Definition 4.4 Let $m, d \in \mathbb{N}$. A *combinatorial line* is a set $L \subseteq [m]^d$ of the form
$$\{(\alpha_1^1, \ldots, \alpha_d^1), \ldots, (\alpha_1^m, \ldots, \alpha_d^m)\}$$
such that there exists a non-empty set $P_L \subseteq [d]$ satisfying

- $\alpha_p^k = \alpha_p^l$ for all $k, l \in [m]$ and $p \in [d] \setminus P_L$, and

- $\alpha_p^k = k$ for all $k \in [m]$ and $p \in P_L$.

Note that for every $k \in [m]$ there exists exactly one $\alpha = (\alpha_1, \ldots, \alpha_d) \in L$ with $\alpha_p = k$ for all $p \in P_L$; we write $L(k)$ for this α.

Theorem 4.5 (Hales-Jewett; see [22]) *For any $m, r \in \mathbb{N}$ there exists $d \in \mathbb{N}$ such that for every function $\xi \colon [m]^d \to [r]$ there exists a combinatorial line L such that ξ is constant on L.*

We write $HJ(m, r)$ for the smallest $d \in \mathbb{N}$ that satisfies the condition in Theorem 4.5. See Figure 4.1 for an illustration that shows that $HJ(2, 2) = 2$: if we colour the vertices of $[2]^2$ with two colours, we always find a monochromatically coloured combinatorial line.

Proof [of Lemma 4.3] We assume that every vertex of $\mathfrak{B}$ is contained in a copy of $\mathfrak{A}$ in $\mathfrak{B}$. This is without loss of generality: if $\mathfrak{B}^*$ is the substructure of $\mathfrak{B}$ induced by the elements of the copies of $\mathfrak{A}$ in $\mathfrak{B}$, and $\mathfrak{C}^*$ is such that $\mathfrak{C}^* \to (\mathfrak{B}^*)_r^{\mathfrak{A}^*}$, then we can construct $\mathfrak{C}$ such that $\mathfrak{C} \to (\mathfrak{B})_r^{\mathfrak{A}}$ from $\mathfrak{C}^*$ by amalgamating at every copy of $\mathfrak{B}^*$ in $\mathfrak{C}^*$ a copy of $\mathfrak{B}$. So assume in the following that $\mathfrak{B} = \mathfrak{B}^*$.

Let $g_1, \ldots, g_m$ be an enumeration of $\binom{\mathfrak{B}}{\mathfrak{A}}$. Let d be $HJ(m, r)$ (according to Theorem 4.5). The idea of the construction in the proof of Lemma 4.3
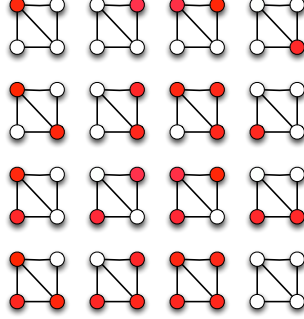
Figure 2: Illustration for $HJ(2, 2) = 2$.

is to construct $\mathfrak{C}$ in such a way that for every element of $[m]^d$ there exists a copy of $\mathfrak{A}$ in $\mathfrak{C}$ such that monochromatically coloured lines in $[m]^d$ correspond to monochromatic copies of $\mathfrak{B}$ in $\mathfrak{C}$. The direct product $\mathfrak{B}^d$ has many copies of $\mathfrak{A}$, but in general does not have enough copies of $\mathfrak{B}$. The following ingenious construction, named after the initials of its inventors, is a modification of the direct product that overcomes the mentioned problem by creating sufficiently many copies of $\mathfrak{B}$.

Definition 4.6 (The NR-power) Let $\mathfrak{A}$, $\mathfrak{B}$ be n-partite structures with signature τ. Then the *d-th NR-power of $\mathfrak{B}$ over $\mathfrak{A}$* is the n-partite structure $\mathfrak{C}$ defined as follows. Write B_i for the i-th level of B, for $i \in [n]$. The domain of $\mathfrak{C}$ is $C_1 \cup \cdots \cup C_n$ where $C_i := (B_i)^d$. For $R \in \tau$ of arity h, and $u^1, \ldots, u^h \in C$, we define $(u^1, \ldots, u^h) \in R^{\mathfrak{C}}$ iff

- there is a non-empty set $P \subseteq [d]$ and $(w^1, \ldots, w^h) \in R^{\mathfrak{B}}$ such that $u^s_q = w^s$ for $q \in P$ and $s \in [h]$, and

- for $q \in [d] \setminus P$, all of $u^1_q, \ldots, u^h_q$ lie in the same copy of $\mathfrak{A}$ in $\mathfrak{B}$.

For an illustration of the NR-power, see Figure 3.

Let $\mathfrak{C}$ be the d-th NR-power of $\mathfrak{B}$ over $\mathfrak{A}$. To prove the partite lemma, it suffices to show that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$. Let $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \to [r]$ be arbitrary. We are going to define a function $\xi \colon [m]^d \to [r]$.

Claim 1. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in [m]^d$, the map $g_\alpha \colon A \to C$ given by $a \mapsto (g_{\alpha_1}(a), \ldots, g_{\alpha_d}(a))$ is an embedding of $\mathfrak{A}$ into $\mathfrak{C}$.
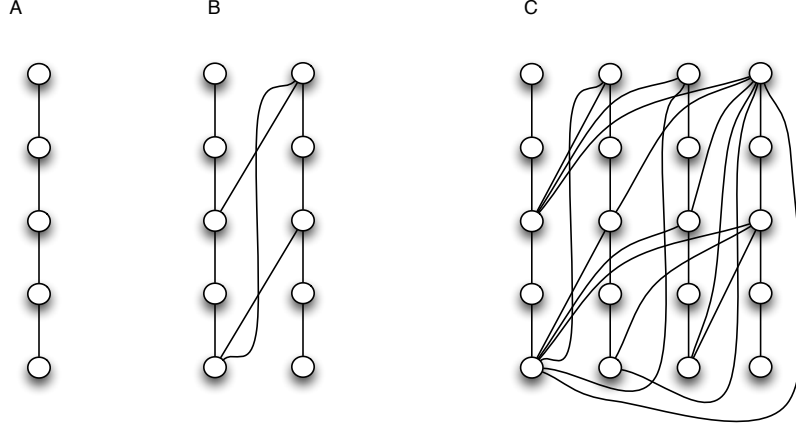
Figure 3: Illustration of two five-partite graphs $\mathfrak{A}$, $\mathfrak{B}$, where $\mathfrak{A}$ is transversal. On the right, we see the second NR-power of $\mathfrak{B}$ over $\mathfrak{A}$.

Proof [of Claim 1.] Suppose that $(a_1, \ldots, a_h) \in R^{\mathfrak{A}}$. Then

$$(g_{\alpha_p}(a_1), \ldots, g_{\alpha_p}(a_h)) \in R^{\mathfrak{B}}$$

for all $p \in [d]$ since g_{α_p} preserves R. By the definition of $R^{\mathfrak{C}}$ we have that $(g_\alpha(a_1), \ldots, g_\alpha(a_h)) \in R^{\mathfrak{C}}$ (arbitrarily choose $i \in [d]$ and verify Definition 4.6 for $P = \{i\}$). Conversely, suppose that $(g_\alpha(a_1), \ldots, g_\alpha(a_h)) \in R^{\mathfrak{C}}$. Then there exists a non-empty set $P \subseteq [d]$ and $(w_1, \ldots, w_h) \in R^{\mathfrak{B}}$ such that for all $q \in P$ and $s \in [h]$ we have $(g_\alpha(a_s))_q = g_{\alpha_q}(a_s) = w_s$. Since g_{α_q} is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, we obtain in particular that $(a_1, \ldots, a_h) \in R^{\mathfrak{A}}$, proving the claim. $\qquad\square$

Define $\xi(\alpha) := \chi(g_\alpha)$. By the theorem of Hales-Jewett (Theorem 4.5), there exists a combinatorial line $L \subseteq [m]^d$ and $c \in [r]$ such that $\xi(\alpha) = c$ for all $\alpha \in L$. We describe how L gives rise to an embedding g_L of $\mathfrak{B}$ into $\mathfrak{C}$. For $u \in B$, we write $\pi(u)$ for the unique element of A that lies on the same level as u. Observe that $\pi(g_k(a)) = a$ for all $k \in [m]$ and for all $a \in A$ since $\mathfrak{A}$ is transversal. Recall our assumption that every $u \in B$ appears in a copy of $\mathfrak{A}$ in $\mathfrak{B}$, and hence there exists a $k \in [m]$ such that $u \in g_k(A)$.

Claim 2. The map $g_L \colon B \to C$ given by $g_L(u) := g_{L(k)}(\pi(u))$, for some $k \in [m]$ such that $u \in g_k(A)$, is well-defined, and an embedding of $\mathfrak{B}$ into $\mathfrak{C}$.

Proof [of Claim 2.] In order to show that the value of g_L does not depend on the choice of k, we have to show that if there are $k, l \in [m]$ such that $u \in B$ appears in both $g_k(A)$ and in $g_l(A)$, then $g_{L(k)}(\pi(u)) = g_{L(l)}(\pi(u))$, that is, $g_{L(k)_p}(\pi(u)) = g_{L(l)_p}(\pi(u))$ for all $p \in [d]$. This is clear when $p \in [d] \setminus P_L$ since we then have $L(k)_p = L(l)_p$. So consider the case $p \in P_L$. Then

$$g_{L(k)_p}(\pi(u)) = g_k(\pi(u)) = u = g_l(\pi(u)) = g_{L(l)_p}(\pi(u))$$

where the equation $g_k(\pi(u)) = u = g_l(\pi(u))$ holds since A is transversal and $u \in g_k(A) \cap g_l(A)$.

   To show that g_L is an embedding, let $R \in \tau$ be of arity h, and let $u_1, \ldots, u_h \in B$ be arbitrary. Let $s \in [h]$ and k be such that $u_s \in g_k(A)$. Let $p \in P_L$ be arbitrary. Then

$$(g_L(u_s))_p = (g_{L(k)}(\pi(u_s)))_p = g_{L(k)_p}(\pi(u_s)) = g_k(\pi(u_s)) = u_s \ . \quad (4.1)$$

Hence, if $(u_1, \ldots, u_s) = ((g_L(u_1))_p, \ldots, (g_L(u_h))_p) \in R^{\mathfrak{B}}$, then by the definition of $R^{\mathfrak{C}}$ for $P := P_L$ and $w^s := u_s$ for all $s \in [h]$ we have that $(g_L(u_1), \ldots, g_L(u_h)) \in R^{\mathfrak{C}}$, and g_L preserves R.

   Conversely, suppose that $(g_L(u_1), \ldots, g_L(u_h)) \in R^{\mathfrak{C}}$. Then there is a non-empty set $P \subseteq [d]$ and $(w^1, \ldots, w^h) \in R^{\mathfrak{B}}$ such that for $q \in P$ and $s \in [h]$ we have $g_L(u_s)_q = w^s$, and for $q \in [d] \backslash P$, all of $g_L(u_1)_q, \ldots, g_L(u_h)_q$ lie in the same copy of $\mathfrak{A}$ in $\mathfrak{B}$. For $p \in P$ we have $w^s = (g_L(u_s))_p = u_s$, and thus $(u_1, \ldots, u_h) \in R^{\mathfrak{B}}$. Applied to the case where R is the equality relation (for proving Ramsey results, we can assume without loss of generality that the signature contains a symbol for equality), this also shows injectivity of g_L. Hence, g_L is an embedding, which concludes the proof of the claim. $\square$

   Since $|L| = m$ and since the embeddings g_α, g_β are distinct whenever α, β are distinct elements of L, we conclude that all of the m copies of $\mathfrak{A}$ in the structure induced by $h(B)$ in $\mathfrak{C}$ have the same colour under χ, which concludes the proof. $\square$

   To finally prove Theorem 4.1, we combine the partite lemma (Lemma 4.3) with the so-called *partite construction*; again, we follow [41].

Proof [Proof of Theorem 4.1] Let $\mathfrak{A}, \mathfrak{B}$ be $\tau \cup \{\preceq\}$-structures where $\preceq$ denotes a linear order, and $r \in \mathbb{N}$ be arbitrary. Set $a := |A|$ and $b := |B|$. We view $\mathfrak{A}$ as an a-partite transversal and $\mathfrak{B}$ as a b-partite transversal. Let $p \in \mathbb{N}$ be such that $([p], <) \to ([b], <)_r^{([a],<)}$ which exists since $\mathcal{LO}$ is a Ramsey class (Example 2.4). Let $q := \binom{p}{q}$, and $\binom{([p],<)}{([a],<)} = \{g_1, \ldots, g_q\}$.

Construct p-partite $\tau \cup \{\preceq\}$-structures $\mathfrak{P}_0, \mathfrak{P}_1, \ldots, \mathfrak{P}_q$ inductively as follows. Let $\mathfrak{P}_0$ be such that for any b parts $P_{0,i_1}, \ldots, P_{0,i_b}$ of $\mathfrak{P}_0$ there is an embedding of $\mathfrak{B}$ into the substructure of $\mathfrak{P}_0$ induced by those parts. It is clear that such a $(\tau \cup \{\preceq\})$-structure $\mathfrak{P}_0$ exists; one may for instance take an appropriate quasi-ordering $\preceq$ on a disjoint union of the τ-reduct of $\mathfrak{B}$.

Now suppose that we have already constructed the p-partite structure $\mathfrak{P}_{k-1}$, with parts $P_{k-1,1}, \ldots, P_{k-1,p}$; to construct $\mathfrak{P}_k$, let $\mathfrak{D}_{k-1}$ be the a-partite system induced in $\mathfrak{P}_{k-1}$ by $\bigcup_{i \in [a]} P_{k-1, g_k(i)}$. By the partite lemma (Lemma 4.3) there exists an a-partite structure $\mathfrak{E}_k$ such that $\mathfrak{E}_k \to (\mathfrak{D}_{k-1})_r^{\mathfrak{A}}$. We construct the p-partite structure $\mathfrak{P}_k$ by amalgamating $\mathfrak{E}_k$ with $\mathfrak{P}_{k-1}$ over $\mathfrak{D}_{k-1}$, for each occurrence of $\mathfrak{D}_{k-1}$ in $\mathfrak{E}_k$.

Finally, let $\mathfrak{C}$ be the structure obtained from $\mathfrak{P}_q$ by replacing the linear quasi-order $\preceq$ by a (total) linear extension. We claim that $\mathfrak{C} \to (\mathfrak{B})_r^{\mathfrak{A}}$. Let $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \to [r]$ be arbitrary. For $k \in \{0, \ldots, q\}$ and $l \in \{k, \ldots, q\}$, we will construct embeddings $h_{l,k} \in \binom{\mathfrak{P}_l}{\mathfrak{P}_k}$ such that for all $m \in \{k, \ldots, l\}$

- $h_{l,m} \circ h_{m,k} = h_{l,k}$, and

- $|\chi(h_{q,m} \circ \binom{\mathfrak{D}_m}{\mathfrak{A}}))| \leq 1$.

Our construction is by induction on k, starting with $k = q$. For $k = l = q$ we can choose $h_{q,q}$ to be the identity. Now suppose that $h_{l',k'}$ has already been defined for all k' such that $k \leq k' \leq l' \leq q$. We want to define $h_{k,k-1}$. Since $\mathfrak{E}_k \to (\mathfrak{D}_{k-1})_c^{\mathfrak{A}}$, there exists an $e_{k-1} \in \binom{\mathfrak{E}_k}{\mathfrak{D}_{k-1}}$ such that $|\chi(h_{q,k} \circ e_{k-1} \circ \binom{\mathfrak{D}_{k-1}}{\mathfrak{A}}))| \leq 1$. By construction of $\mathfrak{P}_k$, the embedding e_{k-1} can be extended to an embedding $h_{k,k-1} \in \binom{\mathfrak{P}_k}{\mathfrak{P}_{k-1}}$. For $m \in \{k, \ldots, l\}$, we define $h_{m,k-1} := h_{m,k} \circ h_{k,k-1}$, completing the inductive construction.

For all $m \in [q]$ there exists a $c_m \in [r]$ such that for all $f \in \binom{\mathfrak{D}_m}{\mathfrak{A}}$ we have $\chi(h_{q,m} \circ f) = c_m$. Define $\xi(g_m) := c_m$. Since $([p], <) \to ([b], <)_r^{([a],<)}$, there exists an $h \in \binom{([p],<)}{([a],<)}$ and $c \in [r]$ such that for all $h' \in \binom{([b],<)}{([a],<)}$ we have $\xi(h \circ h') = c$. By construction of $\mathfrak{P}_0$, there exists a $g \in \binom{\mathfrak{P}_0}{\mathfrak{B}}$ such that $g(B) \subseteq \bigcup_{i \in [a]} P_{0,h(i)}$. To show the claim it suffices to prove that $\chi(g_{k,0} \circ g \circ \binom{\mathfrak{B}}{\mathfrak{A}}) \leq 1$. Let $g' \in \binom{\mathfrak{B}}{\mathfrak{A}}$ be arbitrary. Note that $g_{k,0} \circ g \circ g' \in \binom{\mathfrak{D}_k}{\mathfrak{A}}$ for some $k \in [q]$. Hence, $\chi(g_{q,0} \circ g \circ g') = \chi(g_{q,k} \circ g_{k,0} \circ g \circ g') = c$, finishing the proof of the claim. $\square$

4.2 Irreducible homomorphically forbidden structures

For every $n \geq 2$, the class of all ordered K_n-*free* graphs is Ramsey. In fact, something more general is true; in order to state the result in full generality, we need the following concept.
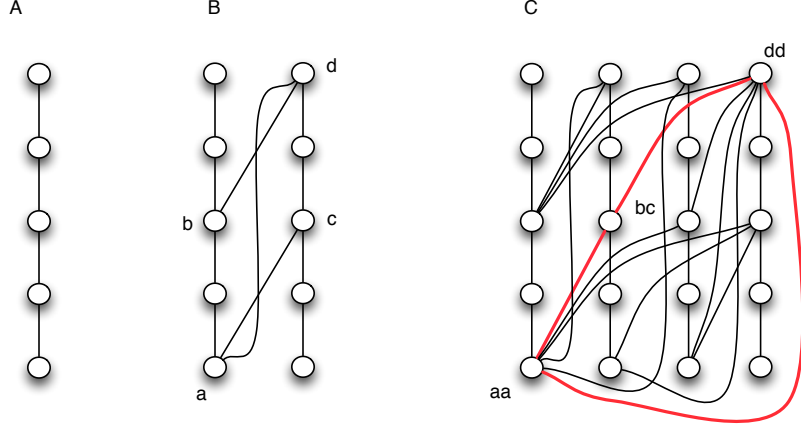
Figure 4: The partite lemma (Lemma 4.3) can create triangles from triangle-free 5-partite A and B.

A structure $\mathfrak{F}$ is called *irreducible* (in the terminology of [40]) if for any pair of distinct elements $x, y \in F$ there exists an $R \in \tau$ and $z_1, \ldots, z_h \in F$ such that $(z_1, \ldots, z_h) \in R^{\mathfrak{F}}$ and $x, y \in \{z_1, \ldots, z_h\}$. It is straightforward to verify that for a set $\mathcal{F}$ of irreducible structures with finite relational signature τ, the class $\mathrm{Forb}(\mathcal{F})$ has (strong) amalgamation, is closed under substructures, isomorphism, and has the joint embedding property, and therefore is an amalgamation class.

Theorem 4.7 (Nešetřil-Rödl) *Let $\mathcal{F}$ be a set of finite irreducible τ-structures. Then $\mathcal{C} := \mathrm{Forb}(\mathcal{F}) * \mathcal{LO}$ is a Ramsey class.*

This theorem can be shown by a variant of the partite method as presented in the previous section. However, it is important to note that the proof from the previous section cannot be applied without an important modification. More concretely, already for the class of triangle-free graphs, NR-powers of $\mathfrak{B}$ over $\mathfrak{A}$ might contain triangles even if the n-partite structures $\mathfrak{A}$ and $\mathfrak{B}$ are triangle-free; see Figure 4. To overcome this problem, we need the following definition. Let $\mathfrak{A}, \mathfrak{B}$ be two n-partite $\tau \cup \{\preceq\}$-structures, and suppose that $\mathfrak{A}$ is transversal. Recall that for $u \in \mathfrak{B}$, we write $\pi(u)$ for the unique element of $\mathfrak{A}$ that lies on the same level as u.

Definition 4.8 We say that $\mathfrak{A}$ is a *template for* $\mathfrak{B}$ if for all $R \in \tau$, $(b_1, \ldots, b_h) \in R^{\mathfrak{B}}$ implies that $(\pi(b_1), \ldots, \pi(b_h)) \in R^{\mathfrak{A}}$.

We state an important property of the NR-powers of $\mathfrak{B}$ over $\mathfrak{A}$ when $\mathfrak{A}$ is a template for $\mathfrak{B}$.

Lemma 4.9 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be n-partite structures such that $\mathfrak{A}$ is transversal and $\mathfrak{A}$ is a template for $\mathfrak{B}$, and let $r \in \mathbb{N}$. Then every irreducible structure $\mathfrak{F}$ that homomorphically maps into an NR-power of $\mathfrak{B}$ over $\mathfrak{A}$ also homomorphically maps into $\mathfrak{A}$.*

Proof   Let $\mathfrak{C}$ be the d-th NR-power of $\mathfrak{B}$ over $\mathfrak{A}$ for some $d \in \mathbb{N}$. Suppose that e is a homomorphism from $\mathfrak{F}$ to $\mathfrak{C}$. Let $(z_1, \ldots, z_h) \in R^{\mathfrak{F}}$. Since $(e(z_1), \ldots, e(z_h)) \in \mathfrak{C}$ and by the definition of the NR-power of $\mathfrak{B}$ over $\mathfrak{A}$, there exists a non-empty set $P \subseteq [d]$ and $(w_1, \ldots, w_h) \in R^{\mathfrak{B}}$ such that $(e(z_s))_q = w_s$ for all $q \in P$ and $s \in [h]$. Note that $\pi(w_s) = \pi(z_s)$. Since $\mathfrak{A}$ is a template for $\mathfrak{B}$, it follows that $(\pi(w_1), \ldots, \pi(w_h)) \in R^{\mathfrak{A}}$. Hence, $\pi \circ e$ is a homomorphism from $\mathfrak{F}$ to $\mathfrak{A}$. $\square$

We can now modify the partite construction from Section 4.1 as follows.

Proof   [of Theorem 4.7] Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $r \in \mathbb{N}$ be arbitrary. By Theorem 4.1, there exists a $\tau \cup \{\preceq\}$-structure $\mathfrak{C}$ where $\preceq$ denotes a linear order (but which need not be from $\mathcal{C}$) such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$. Let $q := |\binom{\mathfrak{C}}{\mathfrak{A}}|$, and $\binom{\mathfrak{C}}{\mathfrak{A}} = \{g_1, \ldots, g_q\}$. Let $p := |\binom{\mathfrak{C}}{\mathfrak{B}}|$, and $\binom{\mathfrak{C}}{\mathfrak{B}} = \{f_1, \ldots, f_p\}$. Let $\mathfrak{C}_i$ be the substructure of $\mathfrak{C}$ induced by $f_i(B)$. We inductively construct a sequence of $|C|$-partite τ-structures $\mathfrak{P}_0, \mathfrak{P}_1, \ldots, \mathfrak{P}_q$. Let $\mathfrak{P}_0$ be the $(\tau \cup \{\preceq\})$-structure obtained as follows: define the relation $\preceq$ on the disjoint union of all the $\mathfrak{C}_i$ by setting $x \preceq y$ if x is a copy of a vertex x' in $\mathfrak{C}$, y is a copy of a vertex y' in $\mathfrak{C}$, and $x' \preceq y'$ in $\mathfrak{C}$. Note that $\mathfrak{C}$ is a template for $\mathfrak{P}_0$.

The construction of $\mathfrak{P}_k$ for $k > 0$ is as in the partite construction in the proof of Theorem 4.1: suppose that we have already constructed $\mathfrak{P}_{k-1}$; to construct $\mathfrak{P}_k$, let $\mathfrak{D}_{k-1}$ be the $|A|$-partite system induced in $\mathfrak{P}_{k-1}$ by $\bigcup_{i \in [a]} P_{k, g_k(i)}$. By the partite lemma (Lemma 4.3) there exists an $|A|$-partite structure $\mathfrak{E}_k$ such that $\mathfrak{E}_k \to (\mathfrak{D}_{k-1})^{\mathfrak{A}}_r$. Note that $\mathfrak{A}$ is a template for $\mathfrak{D}_{k-1}$, and hence, by Lemma 4.9, none of the structures from $\mathcal{F}$ embeds into $\mathfrak{E}_k$. We construct the p-partite structure $\mathfrak{P}_k$ by amalgamating $\mathfrak{E}_k$ with $\mathfrak{P}_{k-1}$ over $\mathfrak{D}_{k-1}$, for each occurrence of $\mathfrak{D}_{k-1}$ in $\mathfrak{E}_k$. The proof that $\mathfrak{P}_q \to (\mathfrak{B})^{\mathfrak{A}}_r$ is as in the proof of Theorem 4.1. $\square$

For a recent application of the partite method to prove the Ramsey property for classes of structures given by homomorphically forbidden trees, see [19].

5 An inductive proof

In this section we present a Ramsey class with finite relational signature for which (to the best of my knowledge) no proof with the partite method is known. Recall the definition of C-relations, and of convex linear orders of C-relations from Example 2.20.

Theorem 5.1 (see $[8, 32, 35]$**)** *The class of all finite binary branching convexly ordered C-relations is a Ramsey class.*

This is a consequence of a more powerful theorem due to Milliken [35], and follows also from results of Leeb [32]. A weaker version of this theorem has been shown by Deuber [18] (my academic grand-father). A direct proof for the statement in the above form can be found in [8].

Throughout this section, $\mathcal{C}$ denotes the class of all finite binary branching convexly ordered C-relations. Recall that the members of $\mathcal{C}$ are in one-to-one correspondence to rooted binary trees, and in the proof it will be convenient to use this perspective.

If $\mathfrak{T}$ is a tree with more than one vertex, then the root of $\mathfrak{T}$ has exactly two children; we denote the subtree $\mathfrak{T}$ rooted at the left child (with respect to the convex linear ordering) by $\mathfrak{T}_{\swarrow}$, and the subtree of $\mathfrak{T}$ rooted at the right child by $\mathfrak{T}_{\searrow}$ (and we speak of the *left subtree of* $\mathfrak{T}$ and the *right subtree of* $\mathfrak{T}$, respectively). Finally, suppose that $e_1 \in \binom{\mathfrak{T}_{\swarrow}}{\mathfrak{A}_{\swarrow}}$ and $e_2 \in \binom{\mathfrak{T}_{\searrow}}{\mathfrak{A}_{\searrow}}$, then $\langle e_1, e_2 \rangle$ is the embedding e of $\mathfrak{A}$ into $\mathfrak{T}$ defined by $e(a) := e_1(a)$ if $a \in \mathfrak{A}_{\swarrow}$ and $e(a) := e_2(a)$ if $a \in \mathfrak{A}_{\searrow}$. We write $\bullet$ for the up to isomorphism unique structure from $\mathcal{C}$ with one element.

Proof [of Theorem 5.1] Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$, and $r \in \mathbb{N}$; we have to show that there is a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$. We prove the statement by induction over the size of $\mathfrak{A}$. For $\mathfrak{A} = \bullet$, the proof of the statement is easy and left to the reader.

Claim 5.2 *For all $\mathfrak{D} \in \mathcal{C}$ there exists an $\mathfrak{F} \in \mathcal{C}$ such that for any $\chi \colon \binom{\mathfrak{F}}{\mathfrak{A}} \to [r]$ there are $f_1 \in \binom{\mathfrak{F}_{\swarrow}}{\mathfrak{D}}$, $f_2 \in \binom{\mathfrak{F}_{\searrow}}{\mathfrak{D}}$, and $c \in [r]$ such that for all $e_1 \in \binom{\mathfrak{D}}{\mathfrak{A}_{\swarrow}}$ and $e_2 \in \binom{\mathfrak{D}}{\mathfrak{A}_{\searrow}}$ we have $\chi(\langle f_1 \circ e_1, f_2 \circ e_2 \rangle) = c$.*

Proof By the inductive assumption, there are structures $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$ such that $\mathfrak{F}_2 \to (\mathfrak{D})^{\mathfrak{A}_{\searrow}}_r$ and $\mathfrak{F}_1 \to (\mathfrak{D})^{\mathfrak{A}_{\swarrow}}_s$ where $s := \left| [r]^{\binom{\mathfrak{F}_2}{\mathfrak{A}_{\searrow}}} \right|$. Let $\mathfrak{F}$ be such that $\mathfrak{F}_{\swarrow} = \mathfrak{F}_1$ and $\mathfrak{F}_{\searrow} = \mathfrak{F}_2$. For a given $\chi \colon \binom{\mathfrak{F}}{\mathfrak{A}} \to [r]$, define $\psi \colon \binom{\mathfrak{F}_{\swarrow}}{\mathfrak{A}_{\swarrow}} \to [r]^{\binom{\mathfrak{F}_{\searrow}}{\mathfrak{A}_{\searrow}}}$ as follows.

$$\psi(e_1) := \big(e_2 \mapsto \chi(\langle e_1, e_2 \rangle) \big)$$

By the choice of $\mathfrak{F}_{\swarrow} = \mathfrak{F}_1$ there exists an $f_1 \in \binom{\mathfrak{F}_{\swarrow}}{\mathfrak{D}}$ and $\phi\colon \binom{\mathfrak{F}_{\searrow}}{\mathfrak{A}_{\searrow}} \to [r]$ such that $\psi(f_1 \circ \binom{\mathfrak{D}}{\mathfrak{A}_{\swarrow}}) = \{\phi\}$. By the choice of $\mathfrak{F}_{\searrow} = \mathfrak{F}_2$ there exists an $f_2 \in \binom{\mathfrak{F}_2}{\mathfrak{D}}$ and a $c \in [r]$ such that $\phi(f_2 \circ \binom{\mathfrak{D}}{\mathfrak{A}_{\searrow}}) = \{c\}$. Let $g_1 \in \binom{\mathfrak{D}}{\mathfrak{A}_{\swarrow}}$ and $g_2 \in \binom{\mathfrak{D}}{\mathfrak{A}_{\searrow}}$. Note that $\psi(f_1 \circ g_1) = \phi$ and $\phi(f_2 \circ g_2) = c$. By definition, $\phi(f_2 \circ g_2) = \chi(\langle f_1 \circ g_1, f_2 \circ g_2 \rangle)$, and hence $\chi(\langle f_1 \circ g_1, f_2 \circ g_2 \rangle) = c$ as desired. $\qquad\square$

Let h be the height of B (that is, the maximal distance from the root of $\mathfrak{B}$ to one of its leaves), and let n be h^r. Define $\mathfrak{C}_1, \mathfrak{C}_2, \ldots$ inductively as follows. Set $\mathfrak{C}_1 := \bullet$, and for $i \geq 2$ let $\mathfrak{C}_i$ be the structure $\mathfrak{F}$ that has been constructed for $\mathfrak{D} := \mathfrak{C}_{i-1}$ in Claim 5.2. Set $\mathfrak{C} := \mathfrak{C}_n$.

We claim that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$. So let $\chi\colon \binom{\mathfrak{C}}{\mathfrak{A}} \to [r]$ be given. For all words w over the alphabet $[2]$ of length $i \in \{0, \ldots, n-1\}$ we define $g_w \in \binom{\mathfrak{C}}{\mathfrak{C}_{n-i}}$, $f_{w1} \in \binom{(\mathfrak{C}_i)_{\swarrow}}{\mathfrak{C}_{i-1}}$, $f_{w2} \in \binom{(\mathfrak{C}_i)_{\searrow}}{\mathfrak{C}_{i-1}}$, and $c_w \in [r]$ as follows. For $i = 0$ and $w = \epsilon$, the empty word of length 0, Claim 5.2 asserts the existence of $f_1 \in \binom{(\mathfrak{C}_n)_{\swarrow}}{\mathfrak{C}_{n-1}}$, $f_2 \in \binom{(\mathfrak{C}_n)_{\searrow}}{\mathfrak{C}_{n-1}}$, and $c \in [r]$ such that for all $e_1 \in \binom{\mathfrak{C}_{n-1}}{\mathfrak{A}_{\swarrow}}$ and $e_2 \in \binom{\mathfrak{C}_{n-1}}{\mathfrak{A}_{\searrow}}$ we have $\chi(\langle f_1 \circ e_1, f_2 \circ e_2 \rangle) = c_\epsilon$. Set $g_1 := f_1$ and $g_2 := f_2$.

Now suppose that f_w and g_w are already defined for a word w of length $i \in \{1, \ldots, n-1\}$. Let $\psi\colon \binom{\mathfrak{C}_{n-i}}{\mathfrak{A}} \to [r]$ be the map defined by $\psi(e) := \chi(g_w \circ e)$ for all $e \in \binom{\mathfrak{C}_{n-i}}{\mathfrak{A}}$. Then Claim 5.2 asserts the existence of $f_{w1} \in \binom{(\mathfrak{C}_{n-i})_{\swarrow}}{\mathfrak{C}_{n-i-1}}$, $f_{w2} \in \binom{(\mathfrak{C}_{n-i})_{\searrow}}{\mathfrak{C}_{n-i-1}}$, and $c_w \in [r]$ such that

$$\psi(\langle f_{w1} \circ e_1, f_{w2} \circ e_2 \rangle) = c_w \tag{5.1}$$

for all $e_1 \in \binom{\mathfrak{C}_{n-i-1}}{\mathfrak{A}_{\swarrow}}$ and $e_2 \in \binom{\mathfrak{C}_{n-i-1}}{\mathfrak{A}_{\searrow}}$. Set $g_{w1} := g_w \circ f_{w1}$ and $g_{w1} := g_w \circ f_{w2}$.

We claim that there exists an injection $\beta\colon B \to [2]^{h^r}$ and a $c \in [r]$ such that

- the map m given by $x \mapsto g_{\beta(x)}(\bullet)$ is from $\binom{\mathfrak{C}}{\mathfrak{B}}$, and

- for all $b_1, b_2 \in B$ we have that $c_w = c$ when w is the longest common prefix of $\beta(b_1)$ and $\beta(b_2)$.

We show this claim by induction on r. The statement is true if $r = 1$ since we can certainly find an injection $\beta\colon B \to [2]^h$ such that $x \mapsto g_{\beta(x)}$ is from $\binom{\mathfrak{C}}{\mathfrak{B}}$, since h is the height of $\mathfrak{B}$.

Otherwise we distinguish two possibilities. We write S_w for the set of words of length at most $|w| + h^{r-1}$ that start with w. Then either

1. for every word w of length at most $n' := n - h^{r-1} = (h-1)h^{r-1}$ there exists a word $u_w \in S_w$ such that $c_{u_w} = r$. In this case, we construct the desired map β recursively as follows. If $\mathfrak{B} = \bullet$ then define $\beta(b_1) = \epsilon 1 \cdots 1 \in [2]^n$ (where ϵ denotes the empty word). Otherwise, $h \geq 1$, and there exists a word $v := u_\epsilon$ with $c_v = r$. We repeat this procedure for $\mathfrak{B}_{\nearrow}$, with $v1$ instead of ϵ, and for $\mathfrak{B}_{\searrow}$, with $v2$ instead of ϵ, until β is defined on all elements of $\mathfrak{B}$.

2. there exists a word w of length at most n' such that $\{c_u \mid u \in S_w\}$ does not contain the colour r. In this case we find an injection $\beta \colon B \to S_w$ with the desired properties by the inductive hypothesis (since we only consider $r-1$ colours instead of r, and S_w still has size h^{r-1}).

We claim that $\chi(m \circ e) = c$ for all $e \in \binom{\mathfrak{B}}{\mathfrak{A}}$. Since $|A| \geq 2$, there are $a_1 \in \mathfrak{A}_{\nearrow}$ and $a_2 \in \mathfrak{A}_{\searrow}$. Let w be the longest common prefix of $\beta(e(a_1))$ and $\beta(e(a_2))$. We then have $c_w = c$. Write e as $\langle e_1, e_2 \rangle$ where e_1 is the restriction of e to $\mathfrak{A}_{\nearrow}$ and e_2 is the restriction of e to $\mathfrak{A}_{\searrow}$. Let $k_1 \in \binom{\mathfrak{C}_{n-i-1}}{\mathfrak{A}_{\nearrow}}$ be

$$x \mapsto g_{w1}^{-1} g_{\beta(e_1(x))}(\bullet) \ .$$

Similarly, let $k_2 \in \binom{\mathfrak{C}_{n-i-1}}{\mathfrak{A}_{\searrow}}$ be $x \mapsto g_{w2}^{-1} g_{\beta(e_2(x))}(\bullet)$. We then have

$$\chi(m \circ e) = \chi(\langle g_w \circ f_{w1} \circ k_1, g_w \circ f_{w2} \circ k_2 \rangle) = c_w = c$$

due to Equation (5.1). $\square$

In Example 2.20, we have seen that the class of finite ordered binary branching C-relations does not have the Ramsey property. In the context of Conjecture 1.1, we want to show how to expand the class to make it Ramsey.

Example 5.3 The class $\mathcal{C}$ of all finite structures $(L; C, <, \prec)$, where $<$ is an arbitrary linear order, and $\prec$ is convex with respect to C, is a Ramsey class. This is an immediate consequence of Theorem 3.24: the class $\mathcal{C}$ can be described as the superposition of the Ramsey class $\mathcal{LO}$ with the class of all convexly ordered C-relations, which is Ramsey by Theorem 5.1.

6   The ordering property

There are strong links between the Ramsey property and the *ordering property* (as defined in [27, 38]).

Definition 6.1 (Ordering Property) Let $\mathcal{C}'$ be a class of finite structures over the signature $\tau \cup \{\prec\}$ where $\prec$ denotes a linear order, and let $\mathcal{C}$ be the class of all τ-reducts of structures from $\mathcal{C}'$. Then $\mathcal{C}'$ has the *ordering property with respect to* $\prec$ if for every $\mathfrak{X} \in \mathcal{C}$ there exists a $\mathfrak{Y} \in \mathcal{C}$ such that for all expansions $\mathfrak{X}' \in \mathcal{C}'$ of $\mathfrak{X}$ and $\mathfrak{Y}' \in \mathcal{C}'$ of $\mathfrak{Y}$ there is an embedding of $\mathfrak{X}'$ into $\mathfrak{Y}'$.

Many examples of classes with the ordering property can be obtained from Theorem 6.4 below, so we rather start with an example of a Ramsey class *without* the ordering property.

Example 6.2 Let $\mathcal{C}$ be the class of all finite sets that are linearly ordered by two linear orders $<_1$ and $<_2$ (see Example 3.23). Then $\mathcal{C}$ does *not* have the ordering property with respect to $<_1$. Indeed, let $\mathfrak{A} \in \mathcal{C}$ be the structure $(\{0, 1, 2\}; \{(0,1), (1,2), (0,2)\}, \{(1,0), (0,2), (1,2)\})$, and let $\mathfrak{B}$ be an arbitrary $\{<_1\}$-reduct of a structure from $\mathcal{C}$, that is, an arbitrary finite linearly ordered set. Then the expansion of $\mathfrak{B}$ where $<_2$ denotes the same relation as $<_1$ is in $\mathcal{C}$, but certainly contains no copy of $\mathfrak{A}$.

Proposition 6.3 *Let Γ be a homogeneous relational τ-structure with domain D, and suppose that Γ has an ω-categorical homogeneous expansion Γ' with signature $\tau \cup \{\prec\}$ where $\prec$ denotes a linear order. Then the following are equivalent.*

- *the age of Γ' has the ordering property with respect to $\prec$;*

- *for every finite $X \subseteq D$ there exists a finite $Y \subseteq D$ such that for every $\beta \in \mathrm{Aut}(\Gamma)$ there exists an $\alpha \in \mathrm{Aut}(\Gamma')$ such that $\alpha(X) \subseteq \beta(Y)$.*

Proof   First suppose that the age $\mathcal{C}'$ of Γ' has the ordering property. Let $X \subset D$ be finite, and let $\mathfrak{X}$ be the structure induced by X in Γ. Then there exists a $\mathfrak{Y}$ in $\mathrm{Age}(\Gamma)$ such that for all expansions $\mathfrak{X}' \in \mathcal{C}'$ of $\mathfrak{X}$ and $\mathfrak{Y}' \in \mathcal{C}'$ of $\mathfrak{Y}$ there exists an embedding of $\mathfrak{X}'$ into $\mathfrak{Y}'$. Suppose without loss of generality that $\mathfrak{Y}$ is a substructure of Γ with domain Y. Let $\beta \in \mathrm{Aut}(\Gamma)$ be arbitrary. Let $\mathfrak{X}'$ be the structure induced by $\beta(X)$ in Γ', and $\mathfrak{Y}'$ the structure induced by $\beta(Y)$ in Γ'. Since $\beta \in \mathrm{Aut}(\Gamma)$, $\mathfrak{X}'$ is isomorphic to an expansion of $\mathfrak{X}$, and $\mathfrak{Y}'$ is isomorphic to an expansion of $\mathfrak{Y}$. By assumption, $\mathfrak{X}'$ embeds into $\mathfrak{Y}'$. By homogeneity of Γ', this embedding can be extended to an automorphism α of Γ', and α has the desired property.

For the converse, let $\mathfrak{X}$ be an arbitrary structure in $\mathrm{Age}(\Gamma)$. Let $Z \subseteq D$ be inclusion-wise minimal with the property that for every embedding e of $\mathfrak{X}$ into Γ there exists an automorphism α of Γ' such that $\alpha(e(X)) \subseteq Z$. Since Γ' is ω-categorical, it has a finite number m of orbits of $|X|$-tuples,

and therefore Z has cardinality at most $m|X|$. Let $Y \subseteq D$ be such that for every $\beta \in \mathrm{Aut}(\Gamma)$ there exists an $\alpha \in \mathrm{Aut}(\Gamma')$ such that $\alpha(Z) \subseteq \beta(Y)$. Let $\mathfrak{Y}$ be the structure induced by Y in Γ. Now let $\mathfrak{X}' := (\mathfrak{X}, \prec) \in \mathcal{C}'$ and $\mathfrak{Y}' := (\mathfrak{Y}, \prec) \in \mathcal{C}'$ be order expansions of $\mathfrak{X}$ and $\mathfrak{Y}$. Let g be an embedding of $\mathfrak{Y}$ into Γ'. By the definition of Z, there is an embedding ρ of $\mathfrak{X}'$ into the substructure induced by Z in Γ'. By homogeneity of Γ, there is a $\beta \in \mathrm{Aut}(\Gamma)$ that maps Y to $g(Y)$. By the choice of Y there exists an $\alpha \in \mathrm{Aut}(\Gamma')$ such that $\alpha(Z) \subseteq \beta(Y)$. Now, $\beta^{-1} \circ \alpha \circ \rho$ is an embedding of $\mathfrak{X}'$ into $\mathfrak{Y}'$, which concludes the proof of the ordering property for $\mathcal{C}'$ with respect to $\prec$. $\square$

Our next theorem gives a sufficient condition for ω-categorical structures to have the ordering property with respect to a given ordering; this condition covers most structures of interest and generalises many previous isolated results [27, 38, 44, 49].

An *orbital* of a permutation group G on a set D is an orbit O of the componentwise action of G on D^2. An *O-cycle* is a sequence of pairs $(u_1, u_2), (u_2, u_3), \ldots, (u_n, u_1)$ from O, for some n. We say that O is *cyclic* if it contains an O-cycle, and *acyclic* otherwise.

Theorem 6.4 *Let Γ be a homogeneous τ-structure with domain D, and $\prec$ an order on D such that $\Gamma' := (\Gamma, \prec)$ is ω-categorical homogeneous Ramsey. Suppose furthermore that every acyclic orbital of $\mathrm{Aut}(\Gamma)$ is also an orbital of $\mathrm{Aut}(\Gamma')$. Then $\mathrm{Age}(\Gamma')$ has the ordering property with respect to $\prec$.*

Proof Let $X \subset D$ be finite. By Proposition 6.3 we have to show that there exists a finite $Y \subset D$ such that for all $\beta \in \mathrm{Aut}(\Gamma)$ there exists an $\alpha \in \mathrm{Aut}(\Gamma')$ such that $\alpha(X) \subseteq \beta(Y)$. Since Γ' is ω-categorical, it has a finite number m of orbits of $|X|$-tuples, and hence there exists a finite $Z \subset D$ with the following properties:

- for every $\gamma \in \mathrm{Aut}(\Gamma)$ there is a $\delta \in \mathrm{Aut}(\Gamma')$ such that $\delta(\gamma(X)) \subseteq Z$;

- for every cyclic orbital O of $\mathrm{Aut}(\Gamma)$, Z contains an O-cycle.

Since Γ' is Ramsey, there exists by Proposition 2.21 a finite set $L \subset D$ such that for all 2-element subsets $S_1, \ldots, S_\ell$ of Z and all $\chi_i \colon \binom{L}{S_i} \to [2]$ there exists a $\theta \in \mathrm{Aut}(\Gamma')$ such that $|\chi_i(\theta \circ \binom{Z}{S_i})| = 1$ for all $i \in [\ell]$.

Let $\beta \in \mathrm{Aut}(\Gamma)$ be arbitrary. Define the map $\chi_i \colon \binom{L}{S_i} \to [2]$ as follows. For $g \in \binom{L}{S_i}$, put $\chi_i(g) := 0$ if $\beta|_{g(S_i)}$ preserves $\prec$, and $\chi_i(g) := 1$ otherwise. Let $\theta \in \mathrm{Aut}(\Gamma')$ be the automorphism that exists for these colourings $\chi_1, \ldots, \chi_\ell$ according to the choice of L.

We claim that $Y := \theta(Z)$ has the desired properties, that is, we show that there is an $\alpha \in \mathrm{Aut}(\Gamma')$ mapping X into $\beta(Y)$. By the definition of Z, there exists a $\delta_1 \in \mathrm{Aut}(\Gamma')$ that maps X into Z. By the definition of Z, there also exists a $\delta_2 \in \mathrm{Aut}(\Gamma')$ that maps $\beta(\theta(\delta_1(X)))$ into Z.

We claim that the restriction g of $\beta \circ \theta \circ \delta_2 \circ \beta \circ \theta \circ \delta_1$ to X can be extended to an automorphism α of Γ'. Since $\beta, \theta, \delta_1, \delta_2 \in \mathrm{Aut}(\Gamma)$, and by homogeneity of Γ', it suffices to show that g preserves $\prec$. So let $x_1, x_2 \in X$ be such that $x_1 \prec x_2$. Let i be such that $S_i = \delta_1(\{x_1, x_2\})$, and let T be $\theta \circ \delta_1(\{x_1, x_2\})$.

- If $\chi_i(\theta \circ \delta_1) = 0$, then $\beta|_T$ preserves $\prec$. It follows that the restriction of $\theta \circ \delta_2 \circ \beta$ to T can be extended to an automorphism η of Γ'. By the property of θ, this means that $\chi_i(\theta \circ \delta_2 \circ \beta \circ \theta \circ \delta_1) = \chi_i(\theta \circ \delta_1) = 0$. By the definition of χ_i, it follows that $\beta|_{\theta \circ \delta_2 \circ \beta(T)}$ preserves $\prec$, and so does the restriction of g to $\{x_1, x_2\}$.

- Otherwise, if $\chi_i(\theta \circ \delta_1) = 1$, then $\beta|_T$ reverses $\prec$. In this case the orbital O of (x_1, x_2) in $\mathrm{Aut}(\Gamma)$ cannot be acyclic: it contains the orbital O_1 of (x_1, x_2) and the orbital O_2 of $(\beta(\theta(\delta_1(x_1))), \beta(\theta(\delta_1(x_2))))$ in $\mathrm{Aut}(\Gamma')$, which are distinct, contrary to our assumption for acyclic orbitals. Therefore, Z contains an O-cycle, and so does $\theta(Z)$ since θ preserves O. Let $(u_0, u_1), (u_1, u_2), \ldots, (u_{n-1}, u_0)$ be this O-cycle in $\theta(Z)$. Suppose for contradiction that $\chi_i(\theta \circ \delta_2 \circ \beta \circ \theta \circ \delta_1) = 0$. We claim that $\beta(u_{i+1}) \prec \beta(u_i)$ for all $i \in \{0, \ldots, n-1\}$ where the indices are modulo n. Then $(\beta(u_{n-1}), \ldots, \beta(u_1), \beta(u_0), \beta(u_{n-1}))$ is a directed cycle in $\prec$, a contradiction since $\prec$ is a linear order. To see the claim, observe that if $u_i \prec u_{i+1}$ (this is, $(u_i, u_{i+1}) \in O_1$), then $\beta(u_{i+1}) \prec \beta(u_i)$ since $\beta|_T$ reverses $\prec$. On the other hand, if $u_{i+1} \prec u_i$ (this is, $(u_i, u_{i+1}) \in O_2$), then $\beta(u_{i+1}) \prec \beta(u_i)$ since $\beta|_{\theta(\delta_2(\beta(T)))}$ preserves $\prec$.

  We conclude that $\chi_i(\theta \circ \delta_2 \circ \beta \circ \theta \circ \delta_1) = 1$, and thus $\beta|_{\theta(\delta_2(\beta(T)))}$ also reverses $\prec$. Reversing $\prec$ twice means preserving $\prec$, and so we conclude that the restriction of $g = \beta \circ \theta \circ \delta_2 \circ \beta \circ \theta \circ \delta_1$ to $\{x_1, x_2\}$ preserves $\prec$.

So g indeed preserves $\prec$ on all of X, which proves the claim about the existence of $\alpha \in \mathrm{Aut}(\Gamma')$. Note that by the properties of g we also have that $g(X) \subseteq \beta(Y)$, and this concludes the proof. $\qquad\square$

Corollary 6.5 *The following classes have the ordering property with respect to $\prec$:*

- *the class of all finite $\prec$-ordered graphs;*

- *the class of all C-relations over finite sets which are convexly ordered by $\prec$;*

- *the class of all finite $\prec$-ordered directed graphs;*

- *the class of all finite partially ordered sets with a linear order $\prec$ that extends the partial order.*

Proof The Fraïssé limit Γ of the first three classes do not have acyclic orbitals. The Ramsey property for those classes has been established earlier in this text, so the statement follows from Theorem 6.4.

Now, let $\Gamma' = (\Gamma, \prec)$ be the Fraïssé limit of the class from the last item. A proof of the Ramsey property for this class can be found in [47, 48]. There is one acyclic orbital in Γ, namely the strict order relation of the poset. But since $\prec$ is a linear extension of the poset relation, this orbital is also an orbital of Γ'. The statement therefore follows again from Theorem 6.4.   □

The following example shows that the sufficient condition for the ordering property that we gave in Theorem 6.4 is not necessary.

Example 6.6 Let $(D; \prec)$ be any countable dense linear order without endpoints. By Theorem 3.24, the structure $\Gamma := (\mathbb{Q}; \mathrm{Betw}, <) * (D; \prec)$ is Ramsey. Note that the reduct of Γ with signature $\{\mathrm{Betw}, <\}$ is isomorphic to $(\mathbb{Q}; \mathrm{Betw}, <)$ (this can be shown by a simple back-and-forth argument), so we assume that Γ has domain $\mathbb{Q}$. Observe that $<$ is certainly an acyclic orbital in $(\mathbb{Q}; \mathrm{Betw}, <)$, but it splits into the orbital $\{(x,y) : x \prec y \wedge x < y\}$ and the orbital $\{(x,y) : y \prec x \wedge x < y\}$ of $\mathrm{Aut}(\Gamma)$. Hence, the condition from Theorem 6.4 does not apply. But nonetheless, the age $\mathcal{C}$ of $(\mathbb{Q}; \mathrm{Betw}, \prec)$ has the ordering property with respect to $<$. To see this, note that for every finite substructure $\mathfrak{A}$ of $(\mathbb{Q}; \mathrm{Betw}, \prec)$ the only two expansions of $\mathfrak{A}$ by a linear order such that the expansion is isomorphic to a structure from $(\mathbb{Q}; \mathrm{Betw}, \prec, <)$ are $(\mathfrak{A}, <)$ and $(\mathfrak{A}, >)$. With this observation it is straightforward to adapt the proof given in Theorem 6.4 to show the ordering property of $\mathcal{C}$.

7 Concluding remarks and open problems

7.1 An application

Ramsey classes are an important tool in classifications of *reducts* of structures. When Γ is a structure, a *reduct* of Γ is a relational structure Δ with the same domain as Γ such that all the relations of Δ have a *first-order definition* over Γ, that is, for every relation R of Δ there exists a

first-order formula ϕ over the signature of Γ (without parameters) such that a tuple t is in R if and only $\phi(t)$ holds in Γ. We say that two reducts are *interdefinable* if they are reducts of each other. Quite surprisingly, countable structures Γ that are homogeneous in a finite relational language tend to have finitely many reducts, up to interdefinability (see e.g. [3, 10, 13, 42, 43, 52, 53]), and Thomas [52] conjectured that this is always the case. If the age of Γ is Ramsey, or a homogeneous expansion of the structure is Ramsey, then this helps in classifying reducts; we refer to the survey article [9] for the technical details. Note that this application provides another motivation to study Conjecture 1.1: if the conjecture is true, then this means that Ramsey classes can be used to attack Thomas' conjecture in general.

7.2   Link with topological dynamics

Whether an amalgamation class $\mathcal{C}$ has the Ramsey property only depends on the (topological) automorphism group $\mathrm{Aut}(\Gamma)$ of the Fraïssé limit Γ of $\mathcal{C}$: by a theorem of Kechris, Pestov, and Todorcevic [27], the class $\mathcal{C}$ is Ramsey if and only if $\mathrm{Aut}(\Gamma)$ is *extremely amenable*, that is, if every continuous action of $\mathrm{Aut}(\Gamma)$ on a compact Hausdorff space has a fixed point. This result has attracted considerable attention [23, 36, 50, 51, 54]. We would like to mention that recently, Melleray, Van Thé, and Tsankov [34] showed that a variant of Conjecture 1.1 (namely Question 7.1 in Section 7) is equivalent to the so-called *universal minimal flow* of $\mathrm{Aut}(\Gamma)$ being metrizable and having a G_δ orbit. Even more recently, Zucker proved that $\mathrm{Aut}(\Gamma)$ does have a G_δ orbit [55] provided that it is metrisable. Hence, the task that remains to prove Conjecture 1.1 with the topological approach is to prove that the universal minimal flow of $\mathrm{Aut}(\Gamma)$ is metrisable (also see Theorem 8.14 in [55]).

These developments in topological dynamics are promising, but so far every single combinatorial result about Ramsey classes that can be proved using topological dynamics also has a direct combinatorial proof. The converse is not true: we have seen in this introductory article several combinatorial proofs where no topological proof is known.

7.3   Variants of the Ramsey expansion conjecture

We have seen that many classes of homogeneous structures can be expanded so that the class of expanded structures becomes Ramsey. Note that if we are allowed to use any expansion, we can trivially turn every class into a Ramsey class, simply by adding unary predicates such that for every element of every structure in the class there is a unary predicate that

just contains this element of the structure. This is why it is important to require in Conjecture 1.1 that the expansion has a finite signature.

A weaker finiteness condition than being homogeneous in a finite relational language is the requirement that the expansion is ω-categorical (recall Theorem 2.25). Therefore, a natural variant of Conjecture 1.1 is the following.

Question 7.1 *Is it true that every ω-categorical structure has an ω-categorical expansion which is Ramsey? Equivalently, is it true that every closed oligomorphic subgroup of S_ω has an extremely amenable closed oligomorphic subgroup?*

Formally, Conjecture 1.1 and Question 7.1 are unrelated, since both the hypothesis and the conclusion are stronger in Conjecture 1.1. A common weakening is the question whether every structure which is homogeneous in a finite relational language has an ω-categorical Ramsey expansion. Time will show for which set of hypotheses we can obtain which positive results.

Also in Question 7.1, the assumption that the expansion be ω-categorical is important, since otherwise the answer is trivially positive since the trivial group that just consists of the identity element is extremely amenable. It is also true (Theorem 4.5 in [28]) that every closed oligomorphic subgroup of S_ω has a non-trivial extremely amenable subgroup (which is not ω-categorical, though).

7.4   More Ramsey classes from old?

We do not know the answer to the following question about model companions and model-complete cores in the context of Ramsey classes.

Question 7.2 *Suppose that Γ is a relational structure with a homogeneous Ramsey expansion with finite relational signature, and let Δ be the model companion of Γ. Is it true that Δ has a homogeneous Ramsey expansion with finite relational signature?*

A positive answer would be a strengthening of Theorem 3.15. We can ask the same question for the model-complete core Δ of Γ. Note that a positive answer to Conjecture 1.1 implies a positive answer to Question 7.2.

We also have a variant for ω-categorical expansions instead of homogeneous expansions in a finite relational language.

Question 7.3 *Suppose that Γ is a relational structure with an ω-categorical Ramsey expansion, and let Δ be the model companion of Γ. Is it true that Δ has an ω-categorical Ramsey expansion?*

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