# Some old and new problems in combinatorial geometry I: Around Borsuk's problem

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#### Abstract

Borsuk [16] asked in 1933 if every set of diameter 1 in  $\mathbf{R}^d$  can be covered by d+1 sets of smaller diameter. In 1993, a negative solution, based on a theorem by Frankl and Wilson [42], was given by Kahn and Kalai [65]. In this paper I will present questions related to Borsuk's problem.

#### 1 Introduction

The title of this paper is borrowed from Paul Erdős who used it (or a similar title) in many lectures and papers, e.g., [39]. I will describe several open problems in the interface between combinatorics and geometry, mainly convex geometry. In this part, I describe and pose questions related to the Borsuk conjecture. The selection of problems is based on my own idiosyncratic tastes. For a fuller picture, the reader is advised to read the review papers on Borsuk's problem and related questions by Raigorodskii [105, 106, 107, 110, 112]. Among other excellent sources are [13, 14, 18, 89, 99].

Karol Borsuk [16] asked in 1933 if every set of diameter 1 in  $\mathbf{R}^d$  can be covered by d+1 sets of smaller diameter. That the answer is positive was widely believed and referred to as the *Borsuk conjecture*. However, some people, including Rogers, Danzer, and Erdős, suggested that a counterexample might be obtained from some clever combinatorial configuration. In hindsight, the problem is related to several questions that Erdős asked and its solution was a great triumph for Erdősian mathematics.

#### 2 Better lower bounds to Borsuk's problem

#### 2.1 The asymptotics

Let f(d) be the smallest integer such that every set of diameter one in  $\mathbf{R}^d$  can be covered by f(d) sets of smaller diameter. The set of vertices of a regular simplex of diameter one demonstrates that f(d) > d + 1. The

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famous Borsuk–Ulam theorem [16] asserts that the d-dimensional ball of diameter 1 cannot be covered by d sets of smaller diameter. The Borsuk–Ulam theorem has many important applications in many areas of mathematics. See Matousek's book [90] for applications and connections to combinatorics. In the same paper [16] Borsuk asked if  $f(d) \leq d+1$ . This was proved for d=2,3. It was shown by Kahn and Kalai [65] that  $f(d) \geq 1.2^{\sqrt{d}}$ , by Lassak [84] that  $f(d) \leq 2^{d-1}+1$  and by Schramm [116] that  $f(d) \leq (\sqrt{3/2}+o(1))^d$ .

#### **Problem 2.1** Is f(d) exponential in d?

The best shot (in my opinion) at an example leading to a positive answer is:

- (a) Start with binary linear codes of length n (based on algebraic-geometry codes) with the property that the number of maximal-weight codewords is exponential in n.
- (b) Show that the code cannot be covered by less than an exponential number of sets that do not realize the maximum distance.

Part (a) should not be difficult, given that it is known that for certain AG-codes the number of minimal-weight codewords is exponential in n [5]. Part (b) can be difficult, but the algebraic techniques used for the Frankl and Wilson theorem may apply.

The Kahn–Kalai counterexample and many of the subsequent results depend on the Frankl–Wilson [42] theorem or on some related algebraically-based combinatorial results. (One can rely also on the Frankl–Rödl theorem [41], which allows much greater generality but not as good quantitative estimates.) We will come back to these results later on.

Let g(d) be the smallest integer such that every *finite* set of diameter one in  $\mathbf{R}^d$  can be covered by g(d) sets of smaller diameter.

#### **Problem 2.2** *Is* f(d) = g(d)?

I am not aware of any reduction from infinite sets to finite sets, and indeed the proof of Borsuk's conjecture for d=2,3 is easier if one considers only finite sets. On the other hand, the counterexamples are based on finite configurations. Perhaps one can demonstrate a gap between the finite and infinite behavior for some extension or variation of the problem, e.g., for arbitrary metric spaces. (Our knowledge of f(d) does not seem accurate enough to hope to prove that such a gap exists for the original problem.)

#### 2.2 Larman's conjecture

The counterexample to Borsuk's conjecture is based on the special case where the set consists of 0-1 vectors of fixed weight. Here, the conjecture has an appealing combinatorial formulation.

**Problem 2.3 (Larman's conjecture)** Let  $\mathcal{F}$  be a t-intersecting family of k sets from [n]. Then  $\mathcal{F}$  can be covered by n (t+1)-intersecting subfamilies.

We now know that Larman's conjecture does not hold in general. However:

#### **Problem 2.4** *Is Larman's conjecture true for* t = 1?

For more discussion of the combinatorics of Larman's conjecture and related combinatorial questions on the packing and coloring of graphs and hypergraphs, see [63]. We can sort of "dualize" the t=1 case of Larman's conjecture by replacing "intersecting" (i.e., "every pair of sets has at least one common element") by "nearly disjoint" (namely, "every pair of distinct sets has at most one common elements") and thus recover the famous:

Conjecture 2.5 (Erdős–Faber–Lovász) Let  $\mathcal{F}$  be a family of nearly disjoint k-sets from [n]. Then  $\mathcal{F}$  is the union of n matchings.

While the Erdős–Faber–Lovász conjecture is still open it is known that n is the right number for the fractional version of the problem [66], and that (1 + o(1))n matchings suffice [62]. Such results are not available for the t = 1 case of Larman's conjecture but a counterexample to a certain strong form of the conjecture is known [64].

#### 2.3 Embedding the elliptic metric into an Euclidean one

Let  $\mathcal{E}_n$  be the *elliptic* metric space of lines through the origin in  $\mathbf{R}^n$  where the distance between two lines is the (smallest) angle between them. So the diameter of  $\mathcal{E}_n$  is  $\pi/2$  and the famous Frankl–Wilson theorem implies that  $\mathcal{E}_n$  cannot be covered by less than an exponential number (in d) of sets of diameter smaller than  $\pi/2$ . The proof by Kahn and Kalai can be seen as adding a single simple fact:  $\mathcal{E}$  can be embedded into an Euclidean space  $\mathbf{R}^{n(n+1)/2}$  by the map

The distance between  $x \otimes x$  and  $y \otimes y$  is a simple monotonic function of the distance between x and y. (Here  $||x||_2 = 1$  and we note that x and -x are mapped onto the same point.)

The original counterexample,  $C_1$ , is the image under this map for (normalized)  $\pm 1$  vectors of length n with n/2 '1's (n divisible by 4). Another example,  $C_2$ , is the image of all  $\pm 1$  vectors, and we can also look at  $C_3$  the image of all unit vectors. All these geometric objects are familiar:  $C_3$  is the unit vectors in the cone of rank-one positive semi-definite matrices,  $C_2$  is called the cut polytope, and  $C_1$  is the polytope of balanced cuts [33].

We can ask if there are more economic embeddings of the elliptic space into a Euclidean space. Namely, is there an embedding  $\varphi: \mathcal{E}_n \to \mathbf{R}^m$ ,  $m = o(n^2)$ , such that  $\|\varphi(x) - \varphi(y)\|_2 = \varphi(d(x, y))$  for some (strictly) monotone function  $\varphi$ ?

The answer to this question is negative by an important theorem of de Caen from 2000 [24].

**Theorem 2.6 (de Caen)** There are quadratically many equiangular lines in  $\mathcal{E}_n$ .

Weaker forms of embeddings of  $\mathcal{E}_n$  into Euclidean spaces possibly with some symmetry-breaking may still lead to improved lower bounds for f(d), and are of independent interest.

**Problem 2.7** Is there a continuous map  $\varphi : \mathcal{E}_n \to \mathbf{R}^m$ ,  $m = o(n^2)$  so that  $\varphi$  preserves the set of diameters of  $\mathcal{E}_n$ ?

#### 2.4 Spherical sets without pairwise orthogonal vectors

Regarding  $\mathcal{E}_n$  itself, Witsenhausen asked in 1974 [129] what is the maximum volume  $\mu(A)$  of an *n*-dimensional spherical set A without a pair of orthogonal vectors. Witsenhausen proved that:

$$\mu(A) \le 1/(n+1).$$

The following natural conjecture is very interesting:

Conjecture 2.8 Let A be a measurable subset of  $S^n$  and suppose that A does not contain two orthogonal vectors. Then the volume of A is at most twice the volume of two spherical caps of radius  $\pi/4$ .

Asymptotically this conjecture asserts that a subset of the n sphere of measure  $(1/\sqrt{2}+o(1))^n$  must contain a pair of orthogonal vectors. If true, this can replace the Frankl-Wilson bound and will show that  $C_3$  defined

above is a counterexamples to Borsuk's conjecture for d > 70 or so. The Frankl–Wilson theorem gives that if  $\mu(A) > 1.203...^{-n}$  then A contains two orthogonal vectors. It seems that the main challenge is to extend the linear algebra/polynomial method from 0-1 vectors to general vectors. One important step was taken by Raigorodskii [108] who improved the bound to  $1.225^{-n}$ .

Remarkably, the upper bound of 1/3 for the two-dimensional case stood unimproved for 40 years until very recently DeCorte and Pikhurko improved it to 0.31.. -[26]! The proof uses Delsartes' linear programming method [31] combined with a combinatorial argument.

#### 2.5 Borsuk's problem for spherical sets

Borsuk's problem itself has an important extension to spherical sets. Consider a set of Euclidean diameter 1 on a d-dimensional sphere  $S_r^{d-1}$  of radius r.

**Problem 2.9** What is the maximum number  $f_r(d)$  of parts one needs to partition any set of diameter 1 on  $S_r^d$ ?

Obviously, one has  $f_r(d) \leq f(d)$  for any r, and we as well have  $f_{1/2}(d) = d+1$  due to Borsuk–Ulam theorem.

Kupavskii and Raigorodskii [76] proved the following theorem:

**Theorem 2.10** Given  $k \in \mathbb{N}$ , if  $r > \frac{1}{2}\sqrt{\frac{2k+1}{2k}}$ , then there exists c > 1 such that  $f_r(d) \ge (c+o(1))^{\frac{2k}{d}}$ . Moreover, there exist a c > 0 such that if  $r > 1/2 + c\frac{\log\log d}{\log d}$ , then for all sufficiently large d we have  $f_r(d) > d+1$ .

The proof is based on mappings involving multiple tensor products. We note that embeddings of similar nature via multiple tensor-products play a role also in the disproof of Khot and Vishnoi [69] of the Goemans–Linial conjecture.

**Problem 2.11** Is it the case that for every r < 1/2 there is a constant  $C_r > 1$  such that

$$f_r(d) \geq C_r^d$$
?

#### 2.6 Low dimensions and two-distance sets

The initial counterexample showed that the Borsuk conjecture is false for n = 1325 and all n > 2014 and there were gradual improvements over the years down to 946 (Nilli [96]), 903 (Weissbach), 561 (Raigorodski

[109]), 560 (Weissbach [126]), 323 (Hinrichs [57]), 321 (Pikhurko [102]), 298 (Hinrichs and Richter [58]). The construction of Hinrichs and the subsequent ones remarkably rely on the Leech lattice.

A two-distance set is a set of vectors in  $\mathbf{R}^d$  that attain only two distances. Larman asked early on and asked again recently:

#### **Problem 2.12** *Is the Borsuk conjecture correct for two-distance sets?*

This has proven to be a very fruitful question. In 2013 Bondarenko [15] found a two-distance set with 416 points in 65 dimensions that cannot be partitioned into less than 83 parts of smaller diameter. Remarkable! Jenrich [61] pushed the dimension down to 64. These constructions beatifully relies on known strongly-regular graphs.

**Problem 2.13** What is the smallest dimension for which Borsuk's conjecture fails? Is Borsuk's conjecture correct in dimension 4?

In dimensions 2 and 3 Borsuk's conjecture is correct. Eggleston gave the first proof for dimension 3 [35], which was followed by simpler proofs by Grünbaum [47] and Heppes [54]. A simple proof for finite sets of points in 3-space was found by Heppes and Revesz [56]. For dimension 2 it follows from an earlier 1906 result that every set of diameter one can be embedded into a regular hexagon whose opposite edges are distance one apart. For a simpler argument see Pak's book [99]. Here too, for finite configurations the proof is very simple.

# 3 Upper bounds for Borsuk's problem and sets of constant width

#### 3.1 Improving the upper bound

Lassak [84] proved that for every d,  $f(d) \leq 2^{d-1} + 1$  (and this still gives the best-known bound when the dimension is not too large). Schramm [116] proved that every convex body of constant width 1 can be covered by  $(\sqrt{3/2} + o(1))^d$  smaller homothets. It is a well-known fact [36] that every set of diameter one is contained in a set of constant width 1, and, therefore, for proving an upper bound on f(d) it is enough to consider sets of constant width. Bourgain and Lindenstrauss [17] showed that every convex body in  $\mathbf{R}^d$  of diameter 1 can be covered by  $(\sqrt{3/2} + o(1))^d$  balls of diameter 1. Both these results show that  $f(d) \leq (\sqrt{3/2} + o(1))^d$ .

**Problem 3.1** Prove that  $f(d) \leq C^d$  for some  $C < \sqrt{3/2}$ .

We note that Danzer constructed a set of diameter 1 that requires exponentially many balls to cover. Danzer constructed a set for which  $(1.003)^d$  balls needed, and Bourgain and Lindenstrauss in 1991 [17] found much better bound,  $(1.064)^n$ .

As for covering by smaller homothets we recall the famous:

Conjecture 3.2 (Hadwiger [52, 53]) Every convex body K in  $\mathbb{R}^d$  can be covered by  $2^d$  smaller homothets of K.

The case of sets of constant width is of particular interest:

**Problem 3.3** Are there  $\epsilon > 0$  and sets of constant width in  $\mathbf{R}^n$  that require at least  $(1 + \epsilon)^n$  smaller homothets to cover?

Note that a positive answer neither implies nor follows from a  $(1 + \epsilon)^d$  lower bound for the Borsuk number f(d).

#### 3.2 Volumes of sets of constant width

Let us denote the volume of the *n*-ball of radius 1/2 by  $V_n$ .

**Problem 3.4 (Schramm [117])** Is there some  $\epsilon > 0$  such that for every d > 1 there exists a set  $K_d$  of constant width 1 in dimension d whose volume satisfies  $VOL(K_d) \leq (1 - \epsilon)^d V_d$ ?

Schramm raised a similar question for spherical sets of constant width and pointed out that a negative answer for spherical sets will push the  $(3/2)^{d/2}$  upper bound for f(d) to  $(4/3)^{d/2}$ .

#### 4 Saving the Borsuk conjecture

#### 4.1 Borsuk's conjecture under transversality

I would like to examine the possibility that Borsuk's conjecture is correct except for some "coincidental" sets. The question is how to properly define "coincidental," and we will now give it a try!

Let K be a set of points in  $\mathbf{R}^d$  and let A be a set of pairs of points in K. We say that the pair (K,A) is general if for every continuous deformation of the distances on A there is a deformation K of K which realizes the deformed distances.

**Remark** This condition is related to the "strong Arnold property" (a.k.a. "transversality") in Colin de Verdiére's theory of invariants of graphs [21].

**Conjecture 4.1** If D is the set of diameters in K and (K, D) is general then K can be partitioned into d + 1 sets of smaller diameter.

We further propose (somewhat more strongly) that this conjecture holds even when "continuous deformation" is replaced with "infinitesimal deformation."

The finite case is of special interest. A graph embedded in  $\mathbf{R}^d$  is stress-free if we cannot assign not-all-zero weights to the edges such that the weighted sum of the edges containing any vertex v (regarded as vectors from v) is zero for every vertex v. Here we embed the vertices and regard the edges as straight line segments. (Edges may intersect.) Such a graph is called a "geometric graph." When we restrict the conjecture to finite configurations of points we get:

**Conjecture 4.2** If G is a stress-free geometric graph of diameters in  $\mathbb{R}^d$  then G is (d+1)-colorable.

**Remark** A stress-free graph for embeddings into  $\mathbb{R}^d$  has at most  $dn - \binom{d+1}{2}$  edges and therefore its chromatic number is at most 2d - 1.

#### 4.2 A weak form of Borsuk's conjecture

Conjecture 4.3 Every polytope P with m facets can be covered by m sets of smaller diameter.

This conjecture was motivated by recent important works on projections of polytopes [40]. A positive answer will give an alternative path for showing that the cut polytope cannot be described as a projection of a polytope with only polynomially many facets.

#### 4.3 Classes of bodies for which Borsuk's conjecture holds

Perhaps the most natural way to "save" Borsuk's conjecture is given by:

**Problem 4.4** Find large and interesting classes of convex bodies for which Borsuk's conjecture holds!

Borsuk's conjecture is known to be true for centrally symmetric bodies, Hadwiger proved it for smooth convex bodies [53], and Boris Dekster proved the conjecture both for bodies of revolution [29] and for convex bodies with a belt of regular points [30].

#### 4.4 Partitioning the unit ball and diametric codes

The unit ball in  $\mathbf{R}^d$  can be covered by d+1 convex sets of smaller diameter. But how much smaller? We do not know the answer. Let u(d) be the minimum value of t such that the unit ball in  $\mathbf{R}^d$  can be covered by d+1 sets of diameter at most t.

#### **Problem 4.5** Determine the behavior of u(d)!

The motivation for this question comes from an even stronger form of Borsuk's conjecture asserting that every set of diameter 1 can be covered by d+1 sets of diameter u(d). It was also conjectured that the optimal covering for the sphere is described by a partition based on the Voronoi regions of a regular simplex that gives  $u(n) \leq 1 - \Omega(1)/n$ . This is known to be optimal in dimensions two and three and is open in higher dimensions. Larman and Tamvakis [81] showed by a volume argument that  $u(n) \geq 1 - 3/2 \log n/n + O(1/n)$ . See also [28].

It will be interesting to close the logarithmic gap for u(d). I don't know what one should expect for the answer, and it will be quite exciting if the standard example is *not* optimal.

We can more pose general questions:

**Problem 4.6** (i) What is the smallest number of sets of diameter t that are needed to cover the unit sphere?

(ii) What is the largest number of convex sets of width  $\geq t$  that can be packed into the unit sphere? (The width of a convex set is the minimum distance between opposite supporting hyperplanes.)

Let  $\Omega_n = \{0, 1\}^n$ . We can ask the analogous questions about the binary cube.

**Problem 4.7** (i) What is the smallest number of sets of diameter t that are needed to cover  $\Omega_n$ ?

(ii) What is the largest number of sets of width  $\geq t$  that can be packed into  $\Omega_n$ ?

Here by "diameter" and "width" we refer to the Euclidean notions (for which, for Problem 4.7, "diameter" essentially coincides with the Hamming diameter).

#### 5 Unit-distance graphs and complexes

For a subset A of  $\mathbb{R}^n$  the *unit-distance graph* is a graph whose set of edges consists of pairs of points of A of distance 1. If all pairwise distances

are at most 1, we call the unit-distance graph a *diameter graph*. If all the pairwise distances are at least 1 we call it a *kissing graph*. Borsuk's question is a question about coloring diameter graphs.

**Problem 5.1** What is the maximum number of edges, the maximum chromatic number, and the maximum minimal degree for the diameter graph, kissing graph, and unit-distance graph for a set of n points in  $\mathbf{R}^d$ ?

Finding the maximum number of edges in a planar unit distance graphs is a famous problem by Erdős [37]. Another famous problem by Hadwiger and Nelson is about the chromatic number of the planar unit distance graph and yet another famous question is if the minimal kissing number of a set of n points in  $\mathbf{R}^d$  is exponential in d, see [22, 4].

We can define also the unit-distance complex to be the simplicial complex of cliques in the unit-distance graphs or, alternatively, the simplicial complex whose faces are sets of points in A that form regular simplices of diameter 1. And again when the diameter of A is 1 we call it the  $diameter\ complex$  and when the minimum distance is 1 we call it the  $kissing\ complex$ .

**Problem 5.2** What is the maximum number of r-faces for the diameter complex, kissing complex, and the unit-distance complex for a set of n points in  $\mathbb{R}^d$ ?

For the chromatic number of the unit-distance graph it makes a difference if we demand further that each color class be measurable. (This is referred to as the *measurable chromatic number*.) For progress on the chromatic number of unit-distance graphs, see [44, 72, 73, 74, 75]. For progress on the measurable chromatic number and related questions, see [8, 9, 10, 27].

Rosenfeld asked (see [115]):

**Problem 5.3** Does the graph whose vertex set is the set of points in the plane and whose edges represent points whose distance is an odd integer have a bounded chromatic number?

For measurable chromatic numbers the answer is negative as follows from a theorem of Furstenberg, Katznelson and Weiss that asserts that every planar set of positive measure realizes all sufficiently large distances. See also [121] for a simple direct proof.

#### 5.1 Schur's conjecture

A conjecture by Schur deals with an interesting special case:

**Conjecture 5.4 (Schur)** The number of (d-1)-faces of every diameter complex for a set of n points in  $\mathbb{R}^d$  is at most n.

The planar case is an old result and it implies a positive answer to Borsuk's problem for finite planar sets. The proof is based on an observation that sets the metric aside: the edges of the diameter graph are pairwise intersecting and therefore we need to show that every geometric graph with n vertices and n+1 edges must have two disjoint edges. This result by Hopf and Pannwitz [59] from 1934 can be seen as the starting point of "geometric graph theory" [98]. Zvi Schur was a high school teacher who did research in his spare time. He managed to prove his conjecture in dimension 3 (see [118]) but in his writing he mentioned that "the power of my methods diminishes as the dimension goes up." The paper [118] includes also a proof that in any dimension the number of d-faces of the diameter complex is at most one. Schur's conjecture has recently been proven by Kupavskii and Polyanski [77]! (For d = 4 it was proved by Bulankina, Kupavskii, and Polyanskii [19].) A key step in Kupavskii and Polyanskii's work is proving the k = m = d - 1 case of the following additional conjecture by Schur, still open in the general case.

**Conjecture 5.5 (Schur)** Let  $S_1$  and  $S_2$  be two regular simplices of dimensions k and m in  $\mathbf{R}^d$  such that their union has diameter 1. Then  $S_1$  and  $S_2$  share at least  $\min(0, k + 2m - 2d + 1)$  vertices for  $k \ge m$ .

Heppes and Révész proved that the number of edges in the diameter graph of n points in space is 2n-2. This gives an easy proof of Borsuk's conjecture for finite sets of points in  $\mathbb{R}^3$ .

A natural weakening of Borsuk's conjecture is:

**Problem 5.6** What is the smallest r = r(d) such that every set of diameter 1 in  $\mathbf{R}^d$  can be covered by d+1 sets, none of which contains an r-dimensional simplex of diameter 1?

Unit-distance graphs and especially diameter graphs and complexes are closely related to the study of *ball polytopes*. Those are convex bodies that can be described as the intersection of unit balls. A systematic study of ball polytopes was initiated by Cároly Bezdek around 2004 and they were also studied by Kupitz, Martini, and Perles, see [11, 12, 79]. Ball polytopes are also related to sets of constant width.

### 5.2 Tangent graphs and complexes for collections of balls (of different radii)

We can try further to adjust the problems discussed in this section to the case where we have a collection A of points in  $\mathbf{R}^d$  and a close ball centered around each point. Two balls can be in three mutual positions (that we care about): They can be disjoint, they can have intersecting interiors, or they can be tangential.

The  $tangent\ graph$  is a graph whose set of vertices is A and a pair of vertices are adjacent if the corresponding balls are tangential. Note that if all balls have the same radius 1/2, then the tangent graph is the unit-distance graph. As before we can consider also the  $tangent\ complex$  - the simplicial complex described by cliques in the tangent graph.

**Problem 5.7** (i) What is the maximum number of edges in a tangent graph (especially in the plane)? What is its maximum chromatic number (especially in the plane)?

- (ii) If every two balls intersect, then the tangent graph is a generalization of the diameter graph. Again we can ask for the maximum number of edges, cliques of size r, and the chromatic number. Again we can ask if when the graph is stress-free the chromatic number is at most d+1.
- (iii) If every two balls have disjoint interiors then the tangent graph is a generalization of the kissing graph. Again we can ask for the maximum number of edges, cliques of size r, the maximum minimal degree, and the chromatic number.

In the plane we can find n points and n lines with  $n^{C4/3}$  incidences and the famous Szemeredi–Trotter theorem (see, e.g., [124]) asserts that this is best possible. Now, we can replace each point by a small circle, arrange for the lines incident to the points to be tangential to them, and regard the lines as circles as well. This shows that tangent graphs with n vertices in the plane can have as much as  $n^{C4/3}$  edges. It is conjectured by Pinchasi, Sharir, and others that

Conjecture 5.8 (i) Planar tangent graphs with n vertices can have at most

- $n^{4/3}$  polylog(n) edges.
- (ii) More generally, m red discs and n blue discs (special case: n blue points), can touch at most  $((mn)^{2/3} + m + n)$ polylog(m, n) times.

This conjecture proposes a profound extension of the Szemeredi–Trotter theorem. The best known upper bound  $n^{3/2} \log n$  is by Markus and Tardos [88], following an earlier argument by Pinchási and Radoicić [104]. This

particular approach based on a certain "forbidden configurations" – a self crossing 4-cycle – cannot lead to better exponents. Sharir found a beautiful connection with Erdos's distinct distances problem [51] which also shows that the polylog(n,m) term cannot be eliminated. Indeed, assume you have n points with just x distances. Then draw around each point x circles whose radii are the x possible distances and then you get a collection of m = nx circles and n points with  $n^2$  incidences (because every point lies on n circles exactly). Therefore:  $n^2 \leq polylog(xn)(n(xn))^2/3 + n + xn$  which implies  $x \geq npolylog(n)$ .

For circles that pairwise intersect, Pinchasi [103] proved a Gallai–Sylvester conjecture by Bezdek asserting that (for more than 5 circles) there is always a circle tangential to at most two other circles. This was the starting point of important studies [6, 1] concerning arrangement of circles and pseudo-circles in the plane. Alon, Last, Pinchasi, and Sharir [6] showed an upper bound of 2n-2 for the number of edges in the tangent graph for pairwise intersecting circles.

The problem considered in this section can be asked under greater generality in at least two ways: one important generalization is to consider two circles adjacent if their intersection is an *empty lens*, that is, not intersected by the boundary of another disc. Another generalization is for pseudocircles (where both notions of adjacency essentially coincide).

Let me end with the following problem:

Conjecture 5.9 (Ringle circle problem) Tangent graphs for finite collections of circles in the plane such that no more than two circles pass through a point have bounded chromatic numbers.

Recently, Pinchasi proved that without the assumption that no more than two circles pass through a point, the chromatic number is  $O(\log^2 n)$ , where n is the number of vertices of the graphs. Pinchasi also gave an example where (again, dropping the extra assumption) you need  $\log n$  colors.

#### 6 Other metric spaces

#### 6.1 Very symmetric spaces

We already discussed Borsuk's problem for spherical sets. We can also ask

**Problem 6.1** Study the Borsuk problem, and other questions considered above, for very symmetric spaces like the hyperbolic space, the Grassmanian, and GL(n).

The Grassmanian, the space of k-dimensional linear spaces of  $\mathbf{R}^n$  is of special interest. The "distance" between two vector spaces can be seen as a vector of k angles, and there may be several interesting ways to extend the questions considered here. (The case k=1 brings us back to the Elliptic space).

#### 6.2 Normed spaces

Given a metric space X and a real number t we can consider the Borsuk number b(X,t) defined as the smallest integer such that every subset of diameter t in X can be covered by b(X,t) sets of smaller diameter. There are interesting results and questions regarding Borsuk's numbers of various metric spaces. Let a(X,t) be the maximum cardinality of an equilateral subset  $Y \subset X$  of diameter t (namely, a set so that every pairwise distance between distinct points in Y is t). Of course,  $a(X,t) \leq b(X,t)$ . Understanding a(X,t) for various metric spaces is of great interest. Kusner conjectured that an equilateral set in  $\ell_1^n$  has at most 2n elements and an equilateral set in  $\ell_p^n$  has size at most n+1 for n+1 for

**Theorem 6.2 (Alon and Pudlak)** For an odd integer p, an equilateral set in  $\ell_p^n$  has at most  $c_p n \log n$  points.

When we move to general normed spaces there are very basic things we do not know. It is widely conjectured that:

**Conjecture 6.3** Every normed n-dimensional space has an equilateral set of n + 1 points.

For more on this conjecture see Swanepoel [122].

Petty [101] proved the n=3 case and his proof is based on the topological fact that a Jordan curve in the plane enclosing the origin cannot be contracted without passing through the origin at some stage. Makeév proved the four-dimensional case using more topology. Brass and Dekster proved independently a  $(\log n)^{1/3}$  lower bound and a major improvement by Swanepoel and Villa [123] improved the lower bound to  $\exp(c \log n)^{1/2}$ . I would not be surprised if Conjecture 6.3 is false. It is known that  $2^n$  is an upper bound for the size of an equilateral set for a normed n-dimensional space.

Let me end this section with a beautiful result of Matoušek about unit distances in normed space. One of the most famous problems in geometry is Erdős' unit distance problem of finding the maximum number of unit distances among n points in the plane. This question can be asked with respect to every planar norm with unit ball K. It is known that for every norm the number of edges can be as large as  $\theta(n \log n)$  and here we state a breakthrough theorem by Matoušek [91]:

**Theorem 6.4** There are norms (in fact, for most norms in a Baire category sense) for which the maximum number of unit distances on n points is  $O(n \log n \log \log n)$ .

#### 7 Around Frankl-Wilson and Frankl-Rödl

#### 7.1 The combinatorics of cocycles and Turán numbers

The original counterexamples to Larman's conjecture (and Borsuk's conjecture) were based on cuts: we consider the family of edges of complete bipartite graphs with 4n vertices. (In one variant we consider balanced bipartite graphs, and in another, arbitrary bipartite graphs.) We now consider high-dimensional generalization of cuts in graphs.

A ((k-1)-dimensional) cocycle is a k-uniform hypergraph G such that every k+1 vertices contains an even number of edges. Equivalently, you can start with an arbitrary (k-1) uniform hypergraph H and consider the k-uniform hypergraph G of all k-sets that contain an odd number of edges from H. Cocycles are familiar objects from simplicial cohomology and they have also been studied by combinatorialists and mainly by Seidel [119].

For even k, let f(n,k) be the largest number of edges in a (k-1)-dimensional cocycle with n vertices. (Note that when k is odd, the complete k-uniform hypergraph is a cocycle.) Let T(n,k,k+1) be the maximum number of edges in a k-uniform hypergraph without having a complete sub-hypergraph with (k+1) vertices.

#### Conjecture 7.1 ([68]) When k is even, T(n, k, k+1) = f(n, k).

The best constructions for Turán numbers T(n, 2k, 2k+1) are obtained by cocycles. Let me just consider the case where k=2. For a while the best example was based on a planar drawing of  $K_n$  with the minimum number of crossings. For every such drawing the set of 4-sets of points without a crossing is an example for Turán's (5,4) problem because  $K_5$ is non-planar. It is easy to see that this non-crossing hypergraph is also a cocycle. In 1988 de Caen, Kreher, and Wiseman [25] found a better, beautiful example: consider a n/2 by n/2 matrix M with  $\pm 1$  entries. Your hypergraph vertices will correspond to rows and columns of M. It will include all 4-tuples with 3 rows or with 3 columns and also all sets

with 2 rows and 2 columns such that the product of the four matrix entries is -1. The expected number of edges in the hypergraph for a random  $\pm 1$  matrix is  $(11/16 + o(1))\binom{n}{4}$ .

As for upper bounds, the best-known upper bounds are stronger for cocycles. Peled [100] used a flag-algebras technique to show that  $f(n,4) \le \binom{n}{4}(0.6916 + o(1))$ .

### 7.2 High-dimensional versions of the cut cone and the cone of rank-one PSD matrices

The counterexamples for Borsuk's conjecture were very familiar geometric objects [33]. The example based on bipartite graphs (where the number of edges is arbitrary) is the cut-polytope. The image of the elliptic space under the map  $x \to x \otimes x$  is simply the set of unit vectors in the cone of rank-one positive semidefinite matrices. The unit vectors in the cone of cocycles is an interesting generalization of the cut polytope since for graphs (1-dimensional complexes) it gives us the cut-polytope.

**Problem 7.2** Find and study a "high-dimensional" extension of the cone of rank one PSD matrices (analogous to the cone of cocycles).

One possibility is the following: start with an arbitrary real-valued function g on  $\binom{[n]}{k-1}$  and derive a real-valued function on  $\binom{[n]}{k}$  by:

$$f(T) = \prod \{g(S): S \subset T, |S| = k-1\}.$$

Let  $U_{k,n}$  be the cone of all such g's.

Speculative application to Borsuk's problem is given by:

Conjecture 7.3 (i) The set of unit vectors in the cone of 3-cocycles with n vertices demonstrates a Euclidean set in  $\mathbf{R}^d$  that cannot be covered by less than  $\exp(d^{4/5})$  sets of smaller diameter.

(ii) The set of norm-1 vectors in  $U_{4,n}$  demonstrates a Euclidean set in  $\mathbf{R}^d$  that cannot be covered by less than  $\exp(d^{4/5})$  sets of smaller diameter.

#### 7.3 The Frankl-Wilson and Frankl-Rödl theorems

We conclude this paper with the major technical tool needed for the disproof of Borsuk's conjecture, which is the Frankl–Wilson (or Frankl–Rödl) forbidden intersection theorem. Most of the counterexamples to Borsuk's conjecture in low dimensions are based on algebraic techniques "the polynomial method" (or some variant) which seem related to the technique used for the proof of Frankl–Wilson's theorem. (The only exception are

the new examples based on strongly regular graphs.) The Frankl–Wilson theorem [42] is wonderful and miraculous and the Frankl–Rödl theorem [41] is great - it allows many extensions (but not with sharp constants). The proof of Frankl–Wilson is a terrific demonstration of the linear-algebra method. The proof of Frankl–Rödl is an ingenious application (bootstrapping of a kind) of isoperimetric results. Recently Keevash and Long [67] found a new proof of Frankl–Rödl's theorem based on the Frankl–Wilson theorem.

**Problem 7.4** Is there a proof of Frankl–Rödl's theorem based on Delsarte's linear-programming method [31]?

The work of Evan and Pikhurko [26] mentioned above suggests that applying the linear-programming method with input coming from other combinatorial methods can lead to improved result.

It is time to state the Frankl-Rödl theorem.

**Theorem 7.5 (Frankl–Rödl)** For every  $\alpha, \beta, \gamma, \epsilon > 0$ , there is  $\delta > 0$  with the following property. Let  $\mathcal{U}_1$  be the family of  $[\alpha n]$ -subsets of [n], let  $\mathcal{U}_2$  be the family of  $[\beta n]$ -subsets of [n], and let X be the number of pairs of sets  $A \in \mathcal{U}_1$ ,  $B \in \mathcal{U}_2$  whose intersection is of size  $[\gamma n]$ .

Let  $\mathcal{F}, \mathcal{G}$  be two subfamilies of  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , respectively, with  $|\mathcal{F}||\mathcal{G}| \geq (1-\delta)^n |\mathcal{U}_1| \cdot |\mathcal{U}_2|$ . Then the number of pairs (A,B),  $A \in \mathcal{F}$  and  $B \in \mathcal{G}$  whose intersection has  $[\gamma n]$  elements is at least  $(1-\epsilon)^n X$ .

An important special case is where  $\alpha=\beta=1/2$  and  $\gamma=1/4$ . The Frankl–Rödl paper contains generalizations in various directions. We could have assumed that, e.g.,  $\mathcal F$  and  $\mathcal G$  are families of partitions of [n] into r parts instead of families of sets. It also contains interesting geometric applications. We will propose here two extensions of the Frankl–Rödl theorem.

#### 7.4 Frankl-Rödl/Frankl-Wilson with sum restrictions

For  $S \subset [n]$ , we write  $||S|| = \sum \{s : s \in S\}$ . Let  $\alpha_1, \alpha_2, \beta_1, \beta_2$  be reals such that  $0 < \alpha_1, \alpha_2 < 1, \ 0 < \beta_1, \beta_2 < 1$ . Consider the family  $\mathcal{G}$  of subsets of [n] such that for every  $S \in \mathcal{G}$  we have  $|S| = [\alpha_1 n]$ , and  $||S|| = [\alpha_2 {n \choose 2}]$ .

Let X be the number of pairs A and B in  $\mathcal{G}$  with the properties:

- (\*) The intersection C of A and B has precisely  $[\beta_1 n]$  elements.
- (\*\*) The sum of elements in C is precisely  $[\beta_2(\binom{n}{2})]$ .

Conjecture 7.6 (Frankl–Rödl/Frankl–Wilson with sum restrictions) For every  $\epsilon > 0$ , there is  $\delta > 0$  such that if you have a subfamily F of G of size  $> (1 - \delta)^n |G|$ , then the number of pairs of sets in F satisfying (\*) and (\*\*) is at least  $(1 - \epsilon)^n X$ .

**Remark** (February 2015): Eoin Long has recently reduced many cases of this conjecture to the original Frankl–Rödl theorem.

#### 7.5 Frankl-Rödl/Frankl-Wilson for cocycles

Conjecture 7.7 (Frankl–Rödl/Frankl–Wilson theorem for cocycles) For every  $\epsilon, \gamma > 0$ , there is  $\delta > 0$  with the following property. Let  $\mathcal{F}$  be the family of 3-cocycles. Let X be the number of pairs of elements in  $\mathcal{F}$  whose symmetric difference has precisely  $m = \left[\gamma\binom{n}{4}\right]$  sets. Then for every  $\mathcal{G} \subset \mathcal{F}$  if  $|\mathcal{G}| \geq (1-\delta)^{\binom{n}{4}}|\mathcal{F}|$ , the number of pairs of elements in  $\mathcal{G}$  whose symmetric difference has precisely m sets is at least  $(1-\epsilon)^{\binom{n}{4}}X$ .

The case of 1-cocycles is precisely the conclusion of Frankl–Wilson/Frankl–Rödl needed for Borsuk's conjecture, and a Frankl–Rödl theorem for 4-cycles may also be a way to push up the asymptotic lower bounds for Borsuk's problem via Conjecture 7.3.

# 8 Paul Erdős' way with people and with mathematical problems

There is a saying in the ancient Hebrew scriptures:

Do not scorn any person and do not dismiss any thing, for there is no person who has not his hour, and there is no thing that has not its place.

Paul Erdős had an amazing way of practicing this saying, when it came to people, and likewise when it came to his beloved "things," - mathematical problems. And his way accounts for some of our finest hours.

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