A sharp lower bound for choosing the maximum of an independent sequence

Pieter C. Allaart and José A. Islas^{*}

July 19, 2018

Abstract

This paper considers a variation of the full-information secretary problem where the random variables to be observed are independent but not necessary identically distributed. The main result is a sharp lower bound for the optimal win probability. Precisely, if X_1, \ldots, X_n are independent random variables with known continuous distributions and $V_n(X_1, \ldots, X_n) := \sup_{\tau} P(X_{\tau} = M_n)$, where $M_n := \max\{X_1, \ldots, X_n\}$ and the supremum is over all stopping times adapted to X_1, \ldots, X_n , then

$$V_n(X_1,\ldots,X_n) \ge \left(1-\frac{1}{n}\right)^{n-1},$$

and this bound is attained. The method of proof consists in reducing the problem to that of a sequence of random variables taking at most two possible values, and then applying Bruss' sum-the-odds theorem (2000). In order to obtain a sharp bound for each n, we improve Bruss' lower bound (2003) for the sum-the-odds problem.

AMS 2010 subject classification: 60G40 (primary)

Key words and phrases: Choosing the maximum, Sum-the-odds theorem, Stopping time

^{*}Address: Department of Mathematics, University of North Texas, 1155 Union Circle #311430, Denton, TX 76203-5017, USA; E-mail: allaart@unt.edu, JoseIslas@my.unt.edu

1 Introduction

In the classical secretary problem or best-choice problem, a known number n of applicants for a single position are interviewed one by one in a random order, and after each interview a manager must decide whether or not to hire the applicant just interviewed, based solely on his or her relative rank among the applicants interviewed thus far. The objective is to hire the best applicant. It is well known that under the optimal strategy, the probability of success decreases monotonically to $1/e \approx .3679$ as $n \to \infty$. An entertaining account of the history of the secretary problem and some of its many variations can be found in Ferguson [3]; see also Samuels [5].

One natural extension is the full-information best-choice problem, in which each applicant, independent of the others, can be assigned a numerical score whose distribution is known in advance, and the objective is to hire the applicant with the highest score. The mathematical framework for this problem is as follows. Here and throughout this paper, let X_1, \ldots, X_n be independent random variables with known continuous distributions, let $M_n := \max\{X_1, \ldots, X_n\}$, and define

$$V_n(X_1,\ldots,X_n) := \sup_{\tau} \mathcal{P}(X_\tau = M_n), \tag{1}$$

where the supremum is over all stopping times adapted to the natural filtration of X_1, \ldots, X_n . Gilbert and Mosteller [4] solved this optimal stopping problem in the case when X_1, \ldots, X_n are independent and identically distributed (i.i.d.). They showed that the optimal win probability is independent of the distribution of the X_i 's, and decreases monotonically to .5802.

It is natural to ask how much lower the optimal win probability can be if we drop the assumption that the random variables are identically distributed. In this case, it should be fairly clear that the optimal win probability depends on the distributions of X_1, \ldots, X_n , and we aim to find a sharp lower bound. Our main result is

Theorem 1.1. For each $n \in \mathbb{N}$, we have

$$V_n(X_1,\ldots,X_n) \ge \left(1-\frac{1}{n}\right)^{n-1},\tag{2}$$

and this bound is attained.

Remark 1.2. The bound in the theorem decreases to $1/e \approx .3679$ as $n \to \infty$. This is quite a bit smaller than the limiting win probability .5802 in the i.i.d. case, which should not come as a surprise. (Intuitively, when the distributions become progressively more spread out – as they do in the extremal case; see Lemma 3.2 below – there is more uncertainty about the future than when all distributions are identical, resulting in a lower optimal win probability.) Comparing our bound with the classical secretary problem, we see moreover that, in the worst case and for large n, a gambler who has full information can do no better than a gambler who observes only relative ranks.

Remark 1.3. One might ask whether the continuity assumption about the X_i 's is needed in Theorem 1.1. The way our objective function is defined in (1) implies that we win if we stop with a value that is at least tied for the overall maximum. Thus, it seems that the possibility of ties should only make it easier to win and consequently, the lower bound (2) should continue to hold when the random variables X_1, \ldots, X_n are permitted to have atoms. However, it is not clear to us how to prove this formally.

Our method of proof consists of two steps. First, we construct a sequence Y_1, \ldots, Y_n of simple random variables (each of which, in fact, takes on at most two possible values) such that $V_n(X_1, \ldots, X_n) \geq V_n(Y_1, \ldots, Y_n)$. Next, we use Bruss' "sum-the-odds" theorem [1] to obtain an explicit expression for $V_n(Y_1, \ldots, Y_n)$, which we then minimize by establishing a lower bound for the "sum-the-odds" problem which improves on that given by Bruss [2]. This lower bound, though not difficult to obtain, is interesting in its own right.

We have not found a more direct way to prove the main result, though we do not rule out the possibility that one exists. However, we believe that the technique of proof used here is of independent interest and may be applicable to other optimal stopping problems of a similar nature.

2 Bruss' "sum-the-odds" theorem

Bruss [1] considered the problem of stopping at the last success in a sequence of independent Bernoulli trials. Specifically, let A_1, \ldots, A_n be independent events with $P(A_i) = p_i, i = 1, \ldots, n$, and let $I_i := I_{A_i}$, the indicator random variable corresponding to A_i , for $i = 1, \ldots, n$. If we think of the value 1 as representing a success and 0 as representing a failure, the problem of stopping at the last success comes down to finding a stopping time τ that maximizes $P(I_{\tau} = 1, I_{\tau+1} = \cdots = I_n = 0)$.

Theorem 2.1 (Sum-the-odds theorem, [1]). Let I_1, I_2, \ldots, I_n be a sequence of independent indicator random variables with $p_j = E(I_j)$. Let $q_j = 1 - p_j$ and $r_j = p_j/q_j$. Consider the problem of stopping at the last success; that is, the optimal stopping problem

$$v := v(p_1, \dots, p_n) := \sup_{\tau} P(I_{\tau} = 1, I_{\tau+1} = \dots = I_n = 0).$$

Then the optimal rule is to stop on the first index (if any) k with $I_k = 1$ and $k \ge s$, where

$$s := \sup\left\{1, \sup\left\{1 \le k \le n : \sum_{j=k}^n r_j \ge 1\right\}\right\},\$$

with $\sup\{\emptyset\} := -\infty$. Moreover, the optimal win probability is given by

$$v = v(p_1, \dots, p_n) = \left(\prod_{j=s}^n q_j\right) \left(\sum_{k=s}^n r_k\right).$$

To see how this may be applied to the best-choice problem, let A_i be the event that the *i*th applicant is best so far. In the classical secretary problem, where only relative ranks are observed and all n! orderings of the applicants are equally likely, the events A_1, \ldots, A_n are independent, so Bruss' theorem applies to give the optimal stopping rule. However, in the full-information case we are considering here, we have $A_i = \{X_i \ge \max\{X_1, \ldots, X_{i-1}\}\}$ for $i = 1, \ldots, n$, and these events are generally *not* independent when X_1, \ldots, X_n are not identically distributed. Consider, for instance, the case when n = 3, $X_1 \equiv 1$, $X_2 = 0$ or 3 each with probability 1/2, and $X_3 = 0, 2$ or 4 each with probability 1/3; one checks easily that A_2 and A_3 are not independent.

To overcome this issue, we will reduce the problem to that of a sequence of random variables which is simple enough that the sum-the-odds theorem *does* apply to it; see the next section. But first we need a sharp lower bound for the optimal win probability $v(p_1, \ldots, p_n)$.

Bruss [2] showed that if $\sum_{k=1}^{n} r_k \geq 1$, then $v(p_1, \ldots, p_n) > e^{-1}$, and this is the best *uniform* lower bound. However, for fixed n we can do somewhat better. Let b_n denote the lower bound in Theorem 1.1, that is,

$$b_n := \left(1 - \frac{1}{n}\right)^{n-1}.$$

Theorem 2.2. If $\sum_{k=1}^{n} r_k \ge 1$, then

$$v = v(p_1, \dots, p_n) \ge b_{n+1} = \left(1 - \frac{1}{n+1}\right)^n.$$

The bound is attained when $p_1 = p_2 = \cdots = p_n = 1/(n+1)$.

Lemma 2.3. The sequence (b_n) is strictly decreasing, and $\lim_{n\to\infty} b_n = e^{-1}$.

Proof. This follows since $b_{n+1}^{-1} = \left(1 + \frac{1}{n}\right)^n$, which increases in n and has limit e.

Proof of Theorem 2.2. We initially follow Bruss' proof [2] of the bound e^{-1} . For convenience, we reindex the p_k 's, and hence the q_k 's and r_k 's, by redefining $p_i := p_{n-i+1}$, $q_i := q_{n-i+1}$ and $r_i := r_{n-i+1}$, for i = 1, 2, ..., n. In this new notation, let $R_k := r_1 + \cdots + r_k$ and let $t := \inf\{k : R_k \ge 1\}$. Then we can write v as

$$v = R_t \prod_{k=1}^t q_k.$$
(3)

If t = 1, then $r_1 \ge 1$, and hence $v = r_1q_1 = p_1 \ge 1/2$. Thus, the statement is true for t = 1. Assume therefore $t \ge 2$. Use $q_j = (1 + r_j)^{-1}$ to rewrite v in the form

$$v = \frac{R_t}{(1+r_t)\prod_{j=1}^{t-1}(1+r_j)}.$$
(4)

Using the geometric mean-arithmetic mean inequality we have

$$\left(\prod_{j=1}^{t-1} (1+r_j)\right)^{\frac{1}{t-1}} \le \frac{\sum_{j=1}^{t-1} (1+r_j)}{t-1} = 1 + \frac{R_{t-1}}{t-1}$$

and substituting this into (4) gives

$$v \ge \frac{R_t}{(1+r_t)\left(1+\frac{R_{t-1}}{t-1}\right)^{t-1}}$$

We now refine the analysis from the second part of Bruss' proof. It is simpler to consider v^{-1} :

$$v^{-1} \le \frac{(1+r_t)\left(1+\frac{R_{t-1}}{t-1}\right)^{t-1}}{R_{t-1}+r_t} = g_t(R_{t-1},r_t),$$

where

$$g_t(x,y) := \frac{1+y}{x+y} \left(1+\frac{x}{t-1}\right)^{t-1}.$$

In view of

$$R_{t-1} = \sum_{k=1}^{t-1} r_k < 1 \le R_t = R_{t-1} + r_t,$$

we need to maximize $g_t(x, y)$ over $0 \le x < 1$, $1 - x \le y$. For fixed x in this range, $g_t(x, y)$ is clearly decreasing in y, so that

$$g_t(x,y) \le g_t(x,1-x) = (2-x)\left(1+\frac{x}{t-1}\right)^{t-1}.$$

Elementary calculus shows that this function is maximized at $x = x^* := (t - 1)/t$, and so

$$v^{-1} \le g_t(x^*, 1 - x^*) = \left(\frac{t+1}{t}\right)^t.$$

Hence,

$$v \ge \left(\frac{t}{t+1}\right)^t = b_{t+1} \ge b_{n+1},$$

where the last inequality follows from Lemma 2.3. When $p_1 = p_2 = \cdots = p_n = 1/(n+1)$, we have $q_k = n/(n+1)$ and $r_k = 1/n$ for each k, so that t = n and $R_t = 1$, and (3) yields $v = b_{n+1}$.

3 Proof of the main theorem

We first introduce some notation and terminology. For j = 1, ..., n, let F_j denote the distribution function of X_j . By our assumption, each F_j is continuous. For k = 1, ..., n, we shall call X_k a *candidate* if $X_k = \max\{X_1, ..., X_k\}$. If we observe a candidate $X_k = x$ and we stop, we win if and only if none of the future observations $X_{k+1}, ..., X_n$ exceeds x. This happens with probability

$$U_k(x) := P(X_{k+1} \le x, \dots, X_n \le x) = \prod_{j=k+1}^n F_j(x).$$
(5)

On the other hand, if we have observed $X_1 = x_1, \ldots, X_k = x_k$ and we continue, we win if and only if we stop at some time $k < \tau \le n$ and $X_{\tau} = \max\{m, X_{k+1}, \ldots, X_n\}$, where $m := \max\{x_1, \ldots, x_k\}$. Thus, the optimal win probability if we continue is

$$W_k(m) := \sup_{k < \tau \le n} P(X_{\tau} = M_n | \max\{X_1, \dots, X_k\} = m)$$

=
$$\sup_{k < \tau \le n} P(X_{\tau} \ge \max\{m, X_{k+1}, \dots, X_n\}).$$

It is straightforward to verify, by conditioning on the value of X_{k+1} , the recursion

$$W_k(m) = F_{k+1}(m)W_{k+1}(m) + \int_m^\infty \max\{U_{k+1}(x), W_{k+1}(x)\}dF_{k+1}(x), \qquad (6)$$

for k = 1, ..., n - 1, since an observation smaller than m at time k + 1 would force us to continue.

The following definition identifies a class of sequences of random variables which will turn out to be extremal for our problem. **Definition 3.1.** A sequence Y_1, \ldots, Y_n of independent random variables will be called a V-sequence if there exist constants a_1, a_2, \ldots, a_n and b_2, \ldots, b_n with $a_1 < a_2 < \cdots < a_n$ and $a_1 > b_2 > b_3 > \cdots > b_n$, and

- (i) $P(Y_1 = a_1) = 1$, and
- (*ii*) $P(Y_i = a_i \text{ or } b_i) = 1 \text{ for } i = 2, ..., n.$

We will show first that, if Y_1, \ldots, Y_n is any V-sequence, then the optimal win probability $V_n(Y_1, \ldots, Y_n)$ for this sequence is greater than or equal to the bound in Theorem 1.1. Then, for a given sequence X_1, \ldots, X_n of independent continuous random variables, we will construct a specific V-sequence Y_1, \ldots, Y_n (whose distributions depend on those of X_1, \ldots, X_n , of course) such that $V_n(X_1, \ldots, X_n) \ge V_n(Y_1, \ldots, Y_n)$. This will clearly establish the lower bound of Theorem 1.1.

Lemma 3.2. If Y_1, \ldots, Y_n is a V-sequence, then

$$V_n(Y_1,\ldots,Y_n) \ge \left(1-\frac{1}{n}\right)^{n-1},$$

and this bound is attained.

Proof. For k = 1, ..., n - 1, we define the indicator random variables

$$I_k := \begin{cases} 1 & \text{if } Y_{k+1} > \max\{Y_1, \dots, Y_k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $p_k := E(I_k) = P(Y_{k+1} = a_{k+1}), q_k := 1 - p_k$ and $r_k := p_k/q_k$. We have that

$$V_n(Y_1, \dots, Y_n) = \max\{U_1(a_1), W_1(a_1)\}.$$
(7)

Notice that if we decide to continue after observing Y_1 , we win if and only if we choose the last success in the sequence I_1, \ldots, I_{n-1} , which is an independent sequence since for each $k \ge 1$, $I_k = 1$ if and only if $Y_{k+1} = a_{k+1}$. Thus, $W_1(a_1) = v(p_1, \ldots, p_{n-1})$, so if $\sum_{k=1}^{n-1} r_k \ge 1$, Theorem 2.2 gives

$$W_1(a_1) \ge \left(1 - \frac{1}{n}\right)^{n-1}.$$

Suppose now that $\sum_{k=1}^{n-1} r_k \leq 1$. Stopping at the first observation gives us win probability

$$U_1(a_1) = q_1 \cdots q_{n-1} = \prod_{i=1}^{n-1} (1+r_i)^{-1}.$$

Using the arithmetic mean-geometric mean inequality we have

$$\left(\prod_{i=1}^{n-1} (1+r_i)\right)^{\frac{1}{n-1}} \le \frac{\sum_{k=1}^{n-1} (1+r_k)}{n-1} \le \frac{n}{n-1}$$

Thus,

$$U_1(a_1) \ge \left(\frac{n}{n-1}\right)^{-(n-1)} = \left(1 - \frac{1}{n}\right)^{n-1}.$$

In both cases, (7) yields the desired lower bound.

To see that the bound is attained, let $p_i := P(Y_{i+1} = a_{i+1}) = 1/n$, $q_i := 1 - p_i$ and $r_i := p_i/q_i = 1/(n-1)$, for i = 1, ..., n-1. If we skip the first observation and play optimally from then on, we win if and only if we stop at the last success of $I_1, ..., I_{n-1}$, that is, $W_1(a_1) = v(p_1, ..., p_{n-1})$. Note that $\inf\{s : \sum_{j=s}^n r_j \ge 1\} = 2$. Theorem 2.1 gives

$$v(p_1, \dots, p_{n-1}) = \sum_{j=2}^n \frac{1}{n} \left(\frac{n-1}{n}\right)^{n-2} = \left(1 - \frac{1}{n}\right)^{n-1}.$$

If, on the other hand, we choose to stop at the first observation, the win probability is $U_1(a_1) = q_1 \cdots q_{n-1} = \left(1 - \frac{1}{n}\right)^{n-1}$ as well. Hence

$$V_n(Y_1,\ldots,Y_n) = \left(1-\frac{1}{n}\right)^{n-1}$$

as we wanted to show.

Lemma 3.3. For the optimal stopping problem (1), there exists a sequence of critical values x_1^*, \ldots, x_{n-1}^* such that at observation $1 \le k < n$ it is optimal to stop if and only $X_k \ge x_k^*$ and X_k is a candidate. In other words, the optimal stopping rule is

$$\tau^* := \min \left\{ k \le n : X_k \ge \max\{X_1, \dots, X_{k-1}, x_k^*\} \right\},\$$

or $\tau^* = n$ if no such k exists.

Proof. For each i = 1, ..., n - 1, $U_i(x)$ is continuous and increasing, since it is the product of continuous and increasing functions. Moreover, $\lim_{x\to\infty} U_i(x) = 0$ and $\lim_{x\to\infty} U_i(x) = 1$. For each i = 1, ..., n - 1, we have

$$W_i(x) = \sup_{i < \tau \le n} P\left(X_{\tau} \ge \max\{x, X_{i+1}, \dots, X_n\}\right).$$

L		

This shows that W_i is continuous and decreasing. We claim that

$$\lim_{x \to -\infty} W_i(x) > 0 \tag{8}$$

and

$$\lim_{x \to \infty} W_i(x) < 1.$$
(9)

It then follows that the graphs of U_i and W_i intersect each other at some (not necessarily unique) point x_i^* .

To see (8), choose $k \in \{i + 1, ..., n\}$ such that $P(X_k = \max\{X_{i+1}, ..., X_n\}) > 0$. Then

$$L := \lim_{x \to -\infty} P(X_k \ge x, X_k = \max\{X_{i+1}, \dots, X_n\})$$

= $P(X_k = \max\{X_{i+1}, \dots, X_n\})$
> 0,

so, by considering the stopping rule $\tau \equiv k$,

$$W_i(x) \ge P(X_k \ge \max\{x, X_{i+1}, \dots, X_n\})$$

= $P(X_k \ge x, X_k = \max\{X_{i+1}, \dots, X_n\})$
 $\rightarrow L > 0, \quad \text{as } x \rightarrow -\infty.$

This gives (8). Next, choose x such that $P(\max\{X_{i+1},\ldots,X_n\} \ge x) < 1$. Then

$$W_i(x) \le \sup_{i < \tau \le n} P(X_\tau \ge x) \le P(\max\{X_{i+1}, \dots, X_n\} \ge x) < 1,$$

which implies (9).

When X_1, \ldots, X_n are i.i.d., the critical values x_1^*, \ldots, x_{n-1}^* form a decreasing sequence; see [4]. This remains the case if X_1, \ldots, X_n are merely assumed to be independent.

Lemma 3.4. We have $x_1^* \ge x_2^* \ge \cdots \ge x_{n-1}^*$.

Proof. It suffices to show that

$$W_{k+1}(x) \ge U_{k+1}(x) \implies W_k(x) \ge U_k(x), \qquad x \in \mathbb{R}, \quad k = 1, \dots, n-2.$$

But this is almost obvious, since assuming the inequality on the left, (6) gives

$$W_k(x) \ge F_{k+1}(x)W_{k+1}(x) \ge F_{k+1}(x)U_{k+1}(x) = U_k(x),$$

where the last equality is a consequence of (5).

Lemma 3.5. For each $i \in \{2, 3, ..., n-1\}$, there exists a constant $c_i > x_1^*$ such that

$$\int_{x_1^*}^{\infty} U_i(x) dF_i(x) = U_i(c_i) P(X_i > x_1^*).$$
(10)

Proof. If $P(X_i > x_1^*) = 0$, we can choose $c_i > x_1^*$ arbitrarily, and both sides of (10) will be zero. So assume $P(X_i > x_1^*) > 0$. Since U_i is increasing and bounded by 1, we have

$$U_i(x_1^*)P(X_i > x_1^*) \le \int_{x_1^*}^{\infty} U_i(x)dF_i(x) \le P(X_i > x_1^*),$$

 \mathbf{SO}

$$U_i(x_1^*) \le \frac{1}{P(X_i > x_1^*)} \int_{x_1^*}^{\infty} U_i(x) dF_i(x) \le 1.$$

If the inequality on the left is strict, then, since $U_i(x)$ is continuous and tends to 1 as $x \to \infty$, we can apply the intermediate value theorem to obtain $c_i > x_1^*$ such that

$$U_i(c_i) = \frac{1}{P(X_i > x_1^*)} \int_{x_1^*}^{\infty} U_i(x) dF_i(x) dF_i(x)$$

Otherwise, it must be the case that U_i is constant (and hence equal to 1) on $[x_1^*, \infty)$, and we can choose c_i to be any number strictly greater than x_1^* . In either case, we get (10).

Lemma 3.6 (Reduction lemma). There exists a V-sequence X'_1, \ldots, X'_n such that $V_n(X_1, \ldots, X_n) \ge V_n(X'_1, \ldots, X'_n)$.

Proof. Let c_2, \ldots, c_{n-1} be the constants from Lemma 3.5. Choose numbers a_2, \ldots, a_n and b_2, \ldots, b_n such that $x_1^* < a_2 < a_3 < \cdots < a_n$ and $a_j \leq c_j$ for $j = 2, \ldots, n-1$, and $x_1^* > b_2 > \cdots > b_n$. Set $X'_1 :\equiv x_1^*$ and for $2 \leq i \leq n$, define

$$X'_{i} := \begin{cases} a_{i}, & \text{if } X_{i} > x_{1}^{*} \\ b_{i}, & \text{if } X_{i} \le x_{1}^{*}. \end{cases}$$

Note that the random variables X'_1, \ldots, X'_n are independent. The idea of the proof is to replace the random variables X_1, \ldots, X_n by their counterparts X'_1, \ldots, X'_n one at a time, starting with X_1 , and to show that each such replacement does not increase the optimal win probability. In order to do so, we need to introduce analogs of the functions U_j and W_j for the sequence $X'_1, \ldots, X'_k, X_{k+1}, \ldots, X_n$, where $k = 0, 1, \ldots, n$. First, introduce random variables

$$Y_j^{(k)} := \begin{cases} X'_j & \text{if } 1 \le j \le k, \\ X_j & \text{if } k < j \le n, \end{cases} \qquad k = 0, 1, \dots, n, \quad j = 1, 2, \dots, n.$$

Now define

$$U_{j}^{(k)}(x) := \mathcal{P}(Y_{j+1}^{(k)} \le x, \dots, Y_{n}^{(k)} \le x) = \mathcal{P}(Y_{j+1}^{(k)} \le x) \cdots \mathcal{P}(Y_{n}^{(k)} \le x),$$
$$W_{j}^{(k)}(x) := \sup_{j < \tau \le n} \mathcal{P}\left(Y_{\tau}^{(k)} = \max\{x, Y_{j+1}^{(k)}, \dots, Y_{n}^{(k)}\}\right),$$

for $k = 0, 1, \ldots, n$ and $j = 1, 2, \ldots, n$. We also write

$$V_n^{(k)} := V_n\left(Y_1^{(k)}, \dots, Y_n^{(k)}\right), \qquad k = 0, 1, \dots, n,$$

so $V_n^{(0)} = V_n(X_1, \ldots, X_n)$. Observe that

$$k \le j \implies W_j^{(k)} = W_j \text{ and } U_j^{(k)} = U_j.$$
 (11)

And since

$$P(X'_i \le x_1^*) = P(X'_i = b_i) = P(X_i \le x_1^*), \quad i = 2, 3, ..., n_i$$

we also have

$$U_j^{(k)}(x_1^*) = U_j(x_1^*), \qquad j = 1, \dots, n-1, \quad k = 0, 1, \dots, n.$$
 (12)

We will show that for all $0 \le k \le n-1$, $V_n^{(k)} \ge V_n^{(k+1)}$. First,

$$V_n^{(0)} = V_n(X_1, X_2, \dots, X_n) = E[\max\{U_1(X_1), W_1(X_1)\}]$$

= $\int_{-\infty}^{\infty} \max\{U_1(x), W_1(x)\} dF_1(x)$
= $\int_{-\infty}^{x_1^*} W_1(x) dF_1(x) + \int_{x_1^*}^{\infty} U_1(x) dF_1(x)$
 $\geq \int_{-\infty}^{x_1^*} W_1(x_1^*) dF_1(x) + \int_{x_1^*}^{\infty} U_1(x_1^*) dF_1(x)$
= $U_1(x_1^*) = W_1(x_1^*)$
= $V_n(X_1', X_2, \dots, X_n) = V_n^{(1)}$,

where the inequality follows since W_1 is decreasing and U_1 is increasing. Next, we show for every $1 \le k \le n-1$ that $V_n^{(k)} \ge V_n^{(k+1)}$. In view of (12), this will follow if we prove that

$$W_j^{(k)}(x_1^*) \ge W_j^{(k+1)}(x_1^*), \qquad j \le k < n,$$
(13)

for j = 1, ..., n - 1; we can then take j = 1. We prove (13) by backward induction on j. First, (13) holds for j = n - 1 since

$$W_{n-1}^{(n-1)}(x_1^*) = P(X_n > x_1^*) = P(X'_n > x_1^*) = W_{n-1}^{(n)}(x_1^*).$$

Suppose (13) holds with j replaced by j + 1; that is,

$$W_{j+1}^{(k)}(x_1^*) \ge W_{j+1}^{(k+1)}(x_1^*), \qquad j < k < n.$$
 (14)

We first claim that

$$U_{j+1}^{(k)}(a_{j+1}) \ge W_{j+1}^{(k)}(a_{j+1}), \qquad j < k < n.$$
(15)

This follows since $a_{j+1} > x_1^* \ge x_{k+1}^*$, $U_{j+1}^{(k)}$ is increasing and $W_{j+1}^{(k)}$ is decreasing, and so, for j < k < n,

$$U_{j+1}^{(k)}(a_{j+1}) \ge U_{j+1}^{(k)}(x_1^*) = U_{j+1}(x_1^*) \ge W_{j+1}(x_1^*)$$

= $W_{j+1}^{(j+1)}(x_1^*) \ge W_{j+1}^{(k)}(x_1^*) \ge W_{j+1}^{(k)}(a_{j+1}),$

where we used (12) in the first equality, (11) in the second equality, and the induction hypothesis (14) in the third inequality.

We will also need that

$$U_{j+1}^{(k)}(a_{j+1}) \ge U_{j+1}^{(k+1)}(a_{j+1}), \qquad j < k < n.$$
(16)

To see this, note that

$$\frac{U_{j+1}^{(k)}(a_{j+1})}{U_{j+1}^{(k+1)}(a_{j+1})} = \frac{P(X_{k+1} \le a_{j+1})}{P(X'_{k+1} \le a_{j+1})},$$

and

$$P(X'_{k+1} \le a_{j+1}) = P(X'_{k+1} = b_{k+1}) = P(X_{k+1} \le x_1^*) \le P(X_{k+1} \le a_{j+1}),$$

since k > j implies $a_{k+1} > a_{j+1}$. Now using (15), the induction hypothesis (14), (16), and finally (15) again, we obtain for k > j,

$$W_{j}^{(k)}(x_{1}^{*}) = P(X_{j+1}' = b_{j+1})W_{j+1}^{(k)}(x_{1}^{*}) + P(X_{j+1}' = a_{j+1})U_{j+1}^{(k)}(a_{j+1})$$

$$\geq P(X_{j+1}' = b_{j+1})W_{j+1}^{(k+1)}(x_{1}^{*}) + P(X_{j+1}' = a_{j+1})U_{j+1}^{(k+1)}(a_{j+1})$$

$$= W_{j}^{(k+1)}(x_{1}^{*}).$$

It remains to verify (13) for k = j. Here we use (10) and the fact (by our choice of a_{j+1}) that $a_{j+1} \leq c_{j+1}$ to obtain

$$W_{j}^{(j)}(x_{1}^{*}) = W_{j}(x_{1}^{*}) = P(X_{j+1} \le x_{1}^{*})W_{j+1}(x_{1}^{*}) + \int_{x_{1}^{*}}^{\infty} U_{j+1}(x)dF_{k+1}(x)$$

$$\geq P(X_{j+1} \le x_{1}^{*})W_{j+1}(x_{1}^{*}) + P(X_{j+1} > x_{1}^{*})U_{j+1}(a_{j+1})$$

$$= P(X_{j+1}' = b_{j+1})W_{j+1}^{(j+1)}(x_{1}^{*}) + P(X_{j+1}' = a_{j+1})U_{j+1}^{(j+1)}(a_{j+1})$$

$$= W_{j}^{(j+1)}(x_{1}^{*}),$$

where we used (11) in the third equality, and (15) in the last equality. This completes the backward induction. Finally, setting j = 1 in (13) and using (12) and

$$V_n^{(k)} = \max\{U_1^{(k)}(x_1^*), W_1^{(k)}(x_1^*)\}, \qquad k = 1, 2, \dots, n\}$$

it follows that $V_n^{(k)} \ge V_n^{(k+1)}$ for k = 1, ..., n-1. This proves the lemma.

Proof of Theorem 1.1. The inequality (2) is an immediate consequence of Lemma 3.6 and Lemma 3.2. The bound is attained by replacing the V-sequence that attains the bound in Lemma 3.2 with a continuous sequence, as follows. Let X_1, \ldots, X_n be independent random variables such that X_1 has the uniform distribution on (0, 1), and for $i = 2, \ldots, n, X_i$ has density function

$$f_i(x) := \begin{cases} \frac{1}{n}, & \text{if } i < x < i+1, \\ \frac{n-1}{n}, & \text{if } -i < x < -i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, since the supports of X_1, \ldots, X_n do not overlap, the optimal win probability $V_n(X_1, \ldots, X_n)$ is the same as the optimal win probability for the V-sequence that attains the bound in Lemma 3.2. That is, $V_n(X_1, \ldots, X_n) = (1 - \frac{1}{n})^{n-1}$.

Acknowledgment

The authors thank the anonymous referee for several helpful suggestions that led to an improved presentation of the paper.

References

[1] F. T. BRUSS, Sum the odds to one and stop. Ann. Probab. 28 (2000), 1384–1391.

- [2] F. T. BRUSS, A note on bounds for the odds theorem of optimal stopping. Ann. Probab. 31 (2003), no. 4, 1859–1861.
- [3] T. S. FERGUSON, Who solved the secretary problem? (With comments and a rejoinder by the author.) *Statist. Sci.* 4 (1989), no. 3, 282–296.
- [4] J. P. GILBERT and F. MOSTELLER, Recognizing the maximum of a sequence. J. Amer. Statist. Assoc. 61 (1966), 35–73.
- [5] S. M. SAMUELS, Secretary problems. In Handbook of Sequential Analysis (Eds., B. K. Ghosh and P. K. Sen), pp. 381-406, Marcel Dekker, New York, 1991.