Path to survival for the critical branching processes in a random environment*

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Abstract

A critical branching process $\{Z_k, k = 0, 1, 2, ...\}$ in a random environment is considered. A conditional functional limit theorem for the properly scaled process $\{\log Z_{pu}, 0 \leq u < \infty\}$ is established under the assumptions $Z_n > 0$ and $p \ll n$. It is shown that the limiting process is a Levy process conditioned to stay nonnegative. The proof of this result is based on a limit theorem describing the distribution of the initial part of the trajectories of a driftless random walk conditioned to stay nonnegative.

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1 Introduction

We consider a branching process in a random environment specified by a sequence of independent identically distributed random laws. Denote by Δ the space of probability measures on $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Equipped with the metric of total variation, Δ becomes a Polish space. Let Q be a random variable taking values in Δ. Then, an infinite sequence

$$\Pi = (Q_1, Q_2, \ldots) \tag{1}$$

of i.i.d. copies of Q is said to form a *random environment*. A sequence of $\mathbb{N}_0$-valued random variables $Z_0, Z_1, \ldots$ is called a *branching process in the random environment* Π, if Z_0 is independent of Π and given Π the process $\mathcal{Z} = (Z_0, Z_1, \ldots)$ is a Markov chain with

$$\mathcal{L}\left(Z_n \mid Z_{n-1} = z, \Pi = (q_1, q_2, \ldots)\right) \;=\; \mathcal{L}\left(\xi_{n1} + \cdots + \xi_{nz}\right) \tag{2}$$

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for every $n \geq 1$, $z \in \mathbb{N}_0$ and $q_1, q_2, \ldots \in \Delta$, where $\xi_{n1}, \xi_{n2}, \ldots$ are i.i.d. random variables with distribution q_n.

In the language of branching processes Z_n is the nth generation size of the population and Q_n is the distribution of the number of children of an individual at generation $n-1$. We assume that $Z_0 = 1$ a.s. for convenience and denote the corresponding probability measure on the underlying probability space by $\mathbf{P}$. (If we refer to other probability spaces, then we use notation $\mathbb{P}$, $\mathbb{E}$ and $\mathbb{L}$ for the respective probability measures, expectations and laws.)

As it turns out the properties of $\mathcal{Z}$ are first of all determined by its associated random walk $\mathcal{S} := \{S_n, n \geq 0\}$. This random walk has initial state $S_0 = 0$ and increments $X_n = S_n - S_{n-1}$, $n \geq 1$ defined as

$$X_n \ := \ \log \left(\sum_{y=0}^{\infty} y \ Q_n(\{y\}) \right),$$

which are i.i.d. copies of the logarithmic mean offspring number

$$X \ := \ \log \left(\sum_{y=0}^{\infty} y \ Q(\{y\}) \right).$$

Following [6] we call the process $\mathcal{Z} := \{Z_n, \, n \geq 0\}$ *critical* if and only if the random walk $\mathcal{S}$ is oscillating, that is,

$$\limsup_{n \to \infty} S_n = \infty \quad \text{and} \quad \liminf_{n \to \infty} S_n = -\infty.$$

It is shown in [6] that the extinction moment of the critical branching process in a random environment is finite with probability 1. For this reason it is natural to study the asymptotic behavior of the survival probability $\mathbf{P}(Z_n > 0)$ as $n \to \infty$. This has been done in [6]: If

$$\lim_{n \to \infty} \mathbf{P}\left(S_n > 0\right) = \rho \in (0, 1), \tag{3}$$

then (under some mild additional assumptions to be specified later on)

$$\mathbf{P}(Z_n > 0) \sim \theta \mathbf{P}(\min\left(S_0, S_1, ..., S_n\right) \geq 0) = \theta \frac{l(n)}{n^{1-\rho}}, \tag{4}$$

where $l(n)$ is a slowly varying function and θ is a known positive constant whose explicit expression is given by formula (25) below.

Let

$$\mathcal{A} = \{0 < \alpha < 1; \, |\beta| < 1\} \cup \{1 < \alpha < 2; |\beta| < 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable X write $X \in \mathcal{D}\left(\alpha, \beta\right)$ if the distribution of X belongs to the domain of attraction of a stable law with characteristic function

$$\mathcal{H}_{\alpha,\beta}(t) := \exp\left\{ -c|t|^{\,\alpha} \left(1 + i\beta \frac{t}{|t|} \tan \frac{\pi\alpha}{2} \right) \right\}, \ c > 0, \tag{5}$$

and, in addition, $\mathbf{E}X = 0$ if this moment exists.

Denote $\mathbb{N}_+ := \{1, 2, ...\}$ and let $\{c_n, n \geq 1\}$ be a sequence of positive integers specified by the relation

$$c_n := \inf\left\{u \geq 0 : G(u) \leq n^{-1}\right\}, \tag{6}$$

where

$$G(u) := \frac{1}{u^2}\int_{-u}^{u} x^2 \mathbf{P}(X \in dx).$$

It is known (see, for instance, [17, Ch. XVII, §5]) that, for every $X \in \mathcal{D}(\alpha, \beta)$ the function $G(u)$ is regularly varying with index $-\alpha$. This implies that $\{c_n, n \geq 1\}$ is a regularly varying sequence with index α^{-1}, i.e., there exists a function $l_1(n)$, slowly varying at infinity, such that

$$c_n = n^{1/\alpha} l_1(n). \tag{7}$$

In addition, the scaled sequence $\{S_n/c_n,\, n \geq 1\}$ converges in distribution, as $n \to \infty$, to the stable law given by (5).

Observe that if $X \in \mathcal{D}(\alpha, \beta)$, then (see, for instance, [28]) the quantity ρ in (3) is calculated by the formula

$$\rho = \left\{ \begin{array}{l} \frac{1}{2}, \text{ if } \ \alpha = 1 \text{ or } 2 \\ \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta \tan\frac{\pi\alpha}{2}\right), \text{ otherwise.} \end{array} \right. \tag{8}$$

In particular, $\rho \in (0, 1)$.

Denote

$$M_n := \max(S_1, ..., S_n), \quad L_{k,n} := \min_{k \leq j \leq n} S_j, \quad L_n := L_{0,n} = \min(S_0, S_1, ..., S_n)$$

and introduce a right-continuous renewal function

$$V(x) := 1 + \sum_{k=1}^{\infty} \mathbf{P}(-S_k \leq x, M_k < 0), \ x \geq 0, \tag{9}$$

and 0 elsewhere. In particular, $V(0) = 1$.

The fundamental property of V is the identity

$$\mathbf{E}[V(x + X); x + X \geq 0] \ = \ V(x), \ x \geq 0 \ , \tag{10}$$

which holds for any oscillating random walk.

It follows from (10) that V gives rise to further probability measures $\mathbf{P}_x^+, x \geq 0$, specified by corresponding expectations $\mathbf{E}_x^+$. The construction procedure of this measure is explained in [6] in detail. We only recall that if the random walk $\mathcal{S} = (S_n, n \geq 0)$ with $S_0 = x \geq 0$ is adapted to some filtration $\mathcal{F} = (\mathcal{F}_n)$ and $\zeta_0, \zeta_1, \ldots$ is a sequence of random variables, adapted to $\mathcal{F}$, then for each fixed n and a bounded and measurable function $g_n : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$\mathbf{E}_x^+[g_n(\zeta_0, \ldots, \zeta_n)] \ := \ \frac{1}{V(x)}\mathbf{E}_x[g_n(\zeta_0, \ldots, \zeta_n)V(S_n); L_n \geq 0],$$

where $\mathbf{E}_x$ is the expectation corresponding to the probability measure $\mathbf{P}_x$ which is generated by $\mathcal{S}$. Under the measure $\mathbf{P}^+ = \mathbf{P}_0^+$ the sequence $S_0, S_1, \ldots$ is a Markov chain with state space $[0,\infty)$ and transition probabilities

$$\mathbf{P}^+(x,dy) \;:=\; \frac{1}{V(x)}\mathbf{P}\left(x+X\in dy\right)V(y), \quad x\geq 0\ .$$

It is the random walk conditioned never to enter $(-\infty,0)$.

We now describe in brief a construction of Levy processes conditioned to stay positive following basically the definitions given in [13] and [14].

Let $\Omega := D\left([0,\infty),\mathbb{R}\right)$ be the space of real-valued càdlàg paths on the real half-line $[0,\infty)$ and let $\mathcal{B} := \{B_t, t\geq 0\}$ be the coordinate process defined by the equality $B_t(\omega) = \omega_t$ for $\omega\in\Omega$. In the sequel we consider also the spaces $\Omega_U := D\left([0,U],\mathbb{R}\right), U>0$.

We endow the spaces Ω and Ω_U with Skorokhod topology and denote by $\mathcal{F} = \{\mathcal{F}_t, t\geq 0\}$ and by $\mathcal{F}^U\!=\!\{\mathcal{F}_t, t\in[0,U]\}$ (with some misuse of notation) the natural filtrations of the processes $\mathcal{B}$ and $\mathcal{B}^U = \{B_t, t\in[0,U]\}$.

Let $\mathbb{P}_x$ be the law on Ω an $\alpha-$stable process $\mathcal{B}$, $\alpha\in(0,2]$ started at x and let $\mathbb{P}=\mathbb{P}_0$. Denote by $\rho = \mathbb{P}\left(B_1\geq 0\right)$ the positivity parameter of the process $\mathcal{B}$ (in fact, this quantity is the same as in (8)). We now introduce an analogue of the measure $\mathbf{P}^+$ for Levy processes. Namely, following [12] we specify for all $t>0$, $\mathcal{A}\in\mathcal{F}_t$ the law $\mathbb{P}_x^+$ on Ω of the Levy process starting at point $x>0$ and conditioned to stay positive by the equality

$$\mathbb{P}_x^+\left(A\right) := \frac{1}{x^{\alpha(1-\rho)}}\mathbb{E}_x\left[B_t^{\alpha(1-\rho)}I\left\{\mathcal{A}\right\}I\left\{\inf_{0\leq u\leq t}B_u\geq 0\right\}\right],$$

where $I\left\{\mathcal{C}\right\}$ is the indicator of the event $\mathcal{C}$.

Thus, $\mathbb{P}_x^+$ is an $h-$transform of the Levy process killed when it first enters the negative half-line. The corresponding positive invariant function is $H(x) = x^{\alpha(1-\rho)}$.

This definition has no sense for $x=0$. However, it is shown in [13] that it is possible to construct a law $\mathbb{P}^+ := \mathbb{P}_0^+$ and a càdlàg Markov process with the same semigroup as $(\mathcal{B},\{\mathbb{P}_x^+, x>0\})$ and such that $\mathbb{P}^+\left(B_0=0\right)=1$. Moreover,

$$\mathbb{P}_x^+ \Longrightarrow \mathbb{P}^+,\ \text{as } x\downarrow 0,$$

where here and in what follows $\Longrightarrow$ means weak convergence.

Let $\mathbb{P}^{(m)}$ be the law on Ω_1 of the meander of length 1 associated with $\left(\mathcal{B},\mathbb{P}\right)$, i.e.

$$\mathbb{P}^{(m)}\left(\cdot\right) := \lim_{x\downarrow 0}\mathbb{P}_x\left(\cdot\middle|\inf_{0\leq u\leq 1}B_u\geq 0\right). \tag{11}$$

Thus, the law $\mathbb{P}^{(m)}$ may be viewed as the law of the Levy process $(\mathcal{B},\mathbb{P})$ conditioned to stay nonnegative on the time-interval $(0,1)$ while the law $\mathbb{P}^+$ corresponds to the law of the Levy process conditioned to stay nonnegative on the whole real half-line $(0,\infty)$.

It is proved in [13] that $\mathbb{P}^{(m)}$ and $\mathbb{P}^{+}$ are absolutely continuous with respect to each other: for every event $\mathcal{A} \in \mathcal{F}_1$

$$\mathbb{P}^{+}(A) = C_0 \mathbb{E}^{(m)} \left[I\{\mathcal{A}\} B_1^{\alpha(1-\rho)} \right], \tag{12}$$

where (see, for instance, formulas (3.5), (3.6), and (3.11) in [14])

$$C_0 := \lim_{n\to\infty} V(c_n)\mathbf{P}(L_n \geq 0) \in (0,\infty). \tag{13}$$

Hence,

$$C_0^{-1} = \mathbb{E}^{(m)} \left[B_1^{\alpha(1-\rho)} \right]. \tag{14}$$

In fact, one may extend the absolute continuity given in (12) to an arbitrary interval $[0, U]$ be considering the respective space Ω_U instead of Ω_1 and conditioning by the event $\inf_{0\leq u\leq U} B_u \geq 0$ in (11).

Set

$$\zeta(a) := \frac{\sum_{y=a}^{\infty} y^2 Q(\{y\})}{\left(\sum_{y=0}^{\infty} y\, Q(\{y\})\right)^2}, \ a \in \mathbb{N}_0.$$

In what follows we say that

1) *Condition* $A1$ *is valid* if $X \in \mathcal{D}(\alpha,\beta)$;

2) *Condition* $A2$ *is valid* if

$$\mathbf{E}\left(\log^{+} \zeta(a)\right)^{\alpha+\varepsilon} < \infty \tag{15}$$

for some $\varepsilon > 0$ and $a \in \mathbb{N}_0$;

3) *Condition* A *is valid* if Conditions $A1$ and $A2$ hold true and, in addition, the parameter $p = p(n)$ tends to infinity as $n \to \infty$ in such a way that

$$\lim_{n\to\infty} n^{-1}p = \lim_{n\to\infty} n^{-1}p(n) = 0. \tag{16}$$

Introduce two processes

$$\mathcal{H}^p := \left\{ \frac{\log Z_{[pu]}}{c_p}, 0 \leq u < \infty \right\}, \quad \mathcal{G}^n := \left\{ \frac{\log Z_{[nt]}}{c_n}, 0 \leq t \leq 1 \right\}.$$

We are now ready to formulate two main results of the paper.

The first theorem describes the initial stage of the trajectories of the critical branching process in a random environment that provide survival of the process for a long time:

Theorem 1 *If Condition A is valid, then, as $n \to \infty$*

$$\mathcal{L}\left(\mathcal{H}^p \middle| Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathbb{L}^{+}(\mathcal{B}),$$

where the symbol $\Longrightarrow$ stands for the weak convergence in the space $D([0,\infty),\mathbb{R})$ of càdlàg functions in $[0,\infty)$ endowed with the Skorokhod topology. In particular,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\log Z_p}{c_p} \leq z \middle| Z_n > 0, Z_0 = 1 \right) = \mathbb{P}^{+}(B_1 \leq z) = C_0 \mathbb{E}^{(m)}\left[I\{B_1 \leq z\} B_1^{\alpha(1-\rho)} \right]$$

for any $z > 0$.

Remark 1. This theorem complements Corollary 1.6 in [6], which states that under Conditions $A1$ and $A2$

$$\mathcal{L}\left(\mathcal{G}^n \Big| Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathbb{L}^{(m)}\left(\mathcal{B}^1\right)$$

as $n \to \infty$, where the symbol $\Longrightarrow$ stands for the weak convergence in the space $D([0,1],\mathbb{R})$ of càdlàg functions in $[0,1]$ endowed with the Skorokhod topology. In particular,

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\log Z_n}{c_n} \le z \Big| Z_n > 0, Z_0 = 1\right) = \mathbb{P}^{(m)}\left(B_1 \le z\right) = \mathbb{E}^{(m)}\left[I\left\{B_1 \le z\right\}\right]$$

for any $z > 0$.

Let, for $U > 0$

$$\mathcal{H}_U^p := \left\{\frac{\log Z_{[pu]}}{c_p}, 0 \le u \le U\right\}.$$

Corollary 2 *If Condition A is valid, then, for any $U > 0$*

$$\mathcal{L}\left(\left(\mathcal{H}_U^p, \mathcal{G}^n\right) \Big| Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathbb{L}^{+}\left(\mathcal{B}^U\right) \times \mathbb{L}^{(m)}\left(\mathcal{B}^1\right)$$

as $n \to \infty$.

Remark 2. If $\mathbf{E}X = 0$ and $VarX \in (0,\infty)$ then, for any $z > 0$

$$\mathbb{P}^{(m)}\left(B_1 \le z\right) = \int_0^z x e^{-x^2/2} dx = 1 - e^{-z^2/2}$$

and

$$\mathbb{P}^{+}\left(B_1 \le z\right) = \sqrt{\frac{2}{\pi}} \int_0^z x^2 e^{-x^2/2} dx.$$

We have seen by (4) that the asymptotic behavior of the survival probability of the process $\mathcal{Z}$ is primarily determined by the random walk $\mathcal{S}$, since only the constant θ depends on the fine structure of $\mathcal{Z}$ (see formula (25) below). However, one also has to take into account that the random walk changes its properties drastically, when conditioned on the event $\{Z_n > 0\}$. The next theorem, describing the trajectories of the random walk $\mathcal{S}$ that provide survival of the critical process in a random environment at the initial stage of the development of the population, illustrates this fact.

For $U \in (0,\infty]$ let

$$\begin{aligned} \mathcal{Q}_U^p &: = \left\{\frac{S_{[pu]}}{c_p}, 0 \le u \le U\right\}, \quad \mathcal{Q}^p = \mathcal{Q}_\infty^p, \\ \mathcal{S}_U^n &: = \left\{\frac{S_{pU+[(n-pU)t]}}{c_n}, 0 \le t \le 1\right\}, \quad \mathcal{S}^n := \mathcal{S}_0^n. \end{aligned}$$

Theorem 3 *If Conditions A is valid then, as $n \to \infty$*

$$\mathcal{L}\left(\mathcal{Q}^p \Big| Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathbb{L}^{+}\left(\mathcal{B}\right).$$

Remark 3. This theorem complements Theorem 1.5 in [6], which states that under Conditions $A1$ and $A2$

$$\mathcal{L}\left(\mathcal{S}^n \middle| Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathbb{L}^{(m)}\left(\mathcal{B}^1\right) \tag{17}$$

as $n \to \infty$.

Corollary 4 *If Condition A is valid, then for any $U > 0$*

$$\mathcal{L}\left(\left(\mathcal{Q}_U^p, \mathcal{S}^n\right) \middle| Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathbb{L}^+\left(\mathcal{B}^U\right) \times \mathbb{L}^{(m)}\left(\mathcal{B}^1\right)$$

as $n \to \infty$.

The usage of the associated random walks to study branching processes in random environment has a long history. It seems that Kozlov [19] was the first who observed that to investigate properties of the critical branching processes in random environment it is convenient to use ladder epochs of the associated random walks. This fact has been used in various situations for the case of the associated random walks with zero or negative drift and finite variance of increments (see [1], [2],[3], [4], [5] [18], [20] and [22]). The first steps to overcome the assumption of a finite variance random walk in the driftless case were taken in [16] and [24]. In recent years papers [6], [7], [8], [9], [11], [26] and some others provide a systematic approach to the study of branching processes in random environment under rather general assumptions on the properties of the associated random walk (see, surveys [23] and [25] for a detailed exposition).

2 Auxiliary results

We will use the symbols $K, K_1, K_2, ...$ to denote different constants. They are not necessarily the same in different formulas.

2.1 Properties of the associated random walk

To prove the main results of the pepar we need to know the asymptotic behavior of the function $V(x)$ as $x \to \infty$. The following lemma gives the desired asymptotics.

Lemma 5 *(compare with Lemma 13 in [27] and Corollary 8 in [15]) If $X \in \mathcal{D}(\alpha, \beta)$ then there exists a slowly varying function $l_0(x)$ such that*

$$V(x) \sim x^{\alpha(1-\rho)} l_0(x) \tag{18}$$

as $x \to \infty$.

Our next result is a combination (with a slight reformulation) of Lemma 2.1 in [6] and Corollaries 3 and 8 in [15]:

Lemma 6 *If $X \in \mathcal{D}(\alpha, \beta)$, then there exist positive constants K, K_1 and K_2 such that, as $n \to \infty$*

$$\mathbf{P}(L_n \geq -w) \sim V(w)\mathbf{P}(L_n \geq 0) \sim KV(w)n^{\rho-1}l(n) \tag{19}$$

uniformly for $0 \leq w \ll c_n$, and

$$\mathbf{P}(L_n \geq -w) \leq K_1 V(w)n^{\rho-1}l(n) \leq K_2 V(w)\mathbf{P}(L_n \geq 0), \ w \geq 0, n \geq 1. \tag{20}$$

For further references we prove the following simple statement.

Lemma 7 *Let $\mathcal{A}_n \subset \mathbb{R}$, $n \in \mathbb{N}$, be a family of subsets and let $b_n(x)$, $n \in \mathbb{N}$, be a sequence of functions such that, for any fixed sequence $\{a_n, n \in \mathbb{N}\}$ such that $a_n \in \mathcal{A}_n$ for all $n \in \mathbb{N}$*

$$\lim_{n\to\infty} b_n(a_n) = 0. \tag{21}$$

Then

$$\lim_{n\to\infty} \sup_{a\in\mathcal{A}_n} |b_n(a)| = 0.$$

Proof. Assume that the conclusion of the lemma is not true. Then, there exists $\varepsilon > 0$ such that for all N there exist $n(N) \geq N$ and $a_{n(N)} \in \mathcal{A}_{n(N)}$ such that

$$\left|b_{n(N)}\left(a_{n(N)}\right)\right| \geq \varepsilon.$$

This, clearly, contradicts (21).

The lemma is proved.

In the sequel we agree to consider the expressions of the form $\lim A(p,n)$ or $\limsup A(p,n)$ without lower indices as the lim or lim sup of the triangular array $\{A(p,n), p \geq 1, n \geq 1\}$ calculated under the assumption $pn^{-1} \to 0$ as $p, n \to \infty$. We also write $a_n \ll b_n$ if $\lim_{n\to\infty} a_n/b_n = 0$.

Let $\phi_1 : \Omega_1 \to \mathbb{R}$ be a bounded uniformly continuous functional and $\{\varepsilon_n, n \in \mathbb{N}\}$ be a sequence of positive numbers vanishing as $n \to \infty$.

Lemma 8 *If Condition $A1$ is valid then*

$$\mathbf{E}\left[\phi_1(\mathcal{S}^n) | L_n \geq -x\right] \to \mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}^1)\right] \tag{22}$$

as $n \to \infty$ uniformly in $0 \leq x \leq \varepsilon_n c_n$.

Proof of Lemma 8. It was shown in Theorem 1.1 of [14] that, given Condition $A1$ convergence (22) holds for any sequence $x = x_n$ meeting the restriction $0 \leq x_n \ll c_n$ as $n \to \infty$. This and Lemma 7 with $\mathcal{A}_n := \{0 \leq x \leq \varepsilon_n c_n\}$ imply the desired statement.

Now we are ready to demonstrate the validity of the following result.

Lemma 9 *If Conditions $A1$ and (16) are valid then, for $U > 0$ and any $r \geq 0$*

$$\mathcal{L}\left((\mathcal{Q}_U^p, \mathcal{S}^n)\Big| L_n \geq -r\right) \Longrightarrow \mathbb{L}^+\left(\mathcal{B}^U\right) \times \mathbb{L}^{(m)}\left(\mathcal{B}^1\right)$$

as $n \to \infty$.

Proof. Consider the processes $\mathcal{S}^{k,n}$ and $\tilde{\mathcal{S}}^{k,n}, 0 \le k \le n$, given by

$$S_t^{k,n} := \frac{S_{[nt]\wedge k}}{c_n}, \quad \tilde{S}^{k,n} := \frac{1}{c_n}\left(S_{[nt]} - S_{[nt]\wedge k}\right), \ 0 \le t \le 1. \tag{23}$$

Clearly,

$$\mathcal{S}^n = \mathcal{S}^{k,n} + \tilde{\mathcal{S}}^{k,n}.$$

Let $\mathcal{S}^* := \{S_n^*, n \ge 0\}$ be a probabilistic and independent copy of the random walk $\mathcal{S} = \{S_n, n \ge 0\}$ and

$$L_n^* := \min\left(S_0^*, S_1^*, ..., S_n^*\right), \quad \left(\mathcal{S}^*\right)_U^n := \left\{\frac{S^*_{[(n-pU)t]}}{c_n}, 0 \le t \le 1\right\}.$$

For a fixed $N > 0$ set

$$I_N(x) := \begin{cases} 0 & \text{if} & x \le N^{-1}, \\ Nx - 1 & \text{if} & x \in \left(N^{-1}, 2N^{-1}\right), \\ 1 & \text{if} & 2N^{-1} \le x \le N, \\ N+1-x & \text{if} & N < x \le N+1, \\ 0 & \text{if} & x > N+1, \end{cases}$$

and let

$$\phi : \Omega_U \to \mathbb{R} \text{ and } \phi_1 : \Omega_1 \to \mathbb{R}$$

be two continuous and bounded functionals.

Then, for fixed positive U and N and $pU = n\varepsilon_n$, where $\varepsilon \ge \varepsilon_n \downarrow 0$ as $n \to \infty$, we have (with a slight abuse of notation)

$$\begin{aligned} & \mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right) I_N\left(\frac{S_{pU}}{c_p}\right)\phi_1\left(\mathcal{S}^n\right); L_n \ge -r\right] \\ = \ & \mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right) I_N\left(\frac{S_{pU}}{c_p}\right) I\left\{L_{pU} \ge -r\right\} \mathbf{E}\left[\phi_1\left(\left(\mathcal{S}^*\right)_U^n + \mathcal{S}^{pU,n}\right) I\left\{L^*_{n-pU} \ge -S_{pU} - r\right\}\right]\right]. \end{aligned}$$

Here and in what follows we agree to consider pU and $n - pU$ as $[pU]$ and $[n - pU]$, respectively. Since $c_p/c_n \to 0$ as $n \to \infty$, it follows that, given $L_{pU} \ge -r$

$$\frac{S_{pU}}{c_n} I_N\left(\frac{S_{pU}}{c_p}\right) \to 0 \text{ a.s.}$$

and $\mathcal{S}^{pU,n}$ vanishes as $n \to \infty$. This observation, Lemma 8 and the continuity of ϕ_1 imply

$$\mathbf{E}\left[\phi_1\left(\left(\mathcal{S}^*\right)_U^n + \mathcal{S}^{pU,n}\right) | L^*_{n(1-\varepsilon_n)} \ge -S_{pU} - r\right] \to \mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}^1)\right]$$

as $n \to \infty$ uniformly for $0 \le S_{pU} \le Nc_p \ll c_n$. On the other hand, by (19), (13), (18) and properties of regularly varying functions (see, for instance, [21]) we deduce, as $p,n\to\infty$:

$$\begin{aligned}\mathbf{P}\left(L^{*}_{n-pU}\geq -S_{pU}-r\right)I_N\left(\frac{S_{pU}}{c_p}\right) &\sim V\left(S_{pU}\right)I_N\left(\frac{S_{pU}}{c_p}\right)\mathbf{P}\left(L_n\geq 0\right)\\ &= \frac{V\left(S_{pU}\right)}{V(c_p)}I_N\left(\frac{S_{pU}}{c_p}\right)\times V(c_p)\mathbf{P}\left(L_n\geq 0\right)\\ &\sim \left(\frac{S_{pU}}{c_p}\right)^{\alpha(1-\rho)}I_N\left(\frac{S_{pU}}{c_p}\right)\frac{C_0\mathbf{P}\left(L_n\geq 0\right)}{\mathbf{P}\left(L_p\geq 0\right)}\\ &\sim \left(\frac{S_{pU}}{c_p}\right)^{\alpha(1-\rho)}I_N\left(\frac{S_{pU}}{c_p}\right)\frac{C_0\mathbf{P}\left(L_n\geq -r\right)}{\mathbf{P}\left(L_p\geq -r\right)}.\end{aligned}$$

Hence we get after evident but awkward transformations that, as $p,n\to\infty$

$$\begin{aligned}&\mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right)\phi_1\left(\mathcal{S}^n\right)I_N\left(\frac{S_{pU}}{c_p}\right)|L_n\geq -r\right]\\ &\quad\sim C_0\mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}_1)\right]\mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right)\left(\frac{S_{pU}}{c_p}\right)^{\alpha(1-\rho)}I_N\left(\frac{S_{pU}}{c_p}\right)|L_{pU}\geq -r\right].\end{aligned}$$

By Theorem 1.1 of [14], as $p\to\infty$

$$\begin{aligned}&\mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right)\left(\frac{S_{pU}}{c_p}\right)^{\alpha(1-\rho)}I_N\left(\frac{S_{pU}}{c_p}\right)|L_{pU}\geq -r\right]\\ &\qquad\to\mathbb{E}^{(m)}\left[\phi\left(\mathcal{B}^U\right)B_U^{\alpha(1-\rho)}I_N\left(B_U\right)\right]=\mathbb{E}^{+}\left[\phi\left(\mathcal{B}^U\right)I_N\left(B_U\right)\right].\end{aligned}$$

Thus, under Conditions $A1$ and (16)

$$\begin{aligned}&\lim\mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right)I_N\left(\frac{S_{pU}}{c_p}\right)\phi_1\left(\mathcal{S}^n\right)|L_n\geq -r\right]\\ &\qquad= C_0\mathbb{E}^{+}\left[\phi\left(\mathcal{B}^U\right)I_N\left(B_U\right)\right]\times\mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}^1)\right].\end{aligned}$$

Letting now $N\to\infty$ we get

$$\mathcal{L}\left(\left(\mathcal{Q}_U^p,\mathcal{S}^n\right)\Big|L_n\geq -r\right)\Longrightarrow\mathbb{L}^{+}\left(\mathcal{B}^U\right)\times\mathbb{L}^{(m)}\left(\mathcal{B}^1\right)$$

for any $U>0$.

The lemma is proved.

Corollary 10 *If Conditions $A1$ and (16) are valid then*

$$\mathcal{L}\left(\mathcal{Q}^p\Big|L_n\geq -r\right)\Longrightarrow\mathbb{L}^{+}\left(\mathcal{B}\right)$$

as $n\to\infty$.

Proof. It follows from Lemma 9 that

$$\mathcal{L}\left(\mathcal{Q}_U^p\Big|L_n\geq -r\right)\Longrightarrow\mathbb{L}^{+}\left(\mathcal{B}^U\right)$$

for any $U>0$. This fact combined with Theorem 16.7 in [10] completes the proof of the corollary.

3 Conditional limit theorem

For convenience we introduce the notation

$$A_{u.s.} = \{Z_n > 0 \text{ for all } n \geq 0\}$$

and recall that by Corollary 1.2 in [6], (4) and (13)

$$\mathbf{P}(Z_n > 0) \sim \theta \mathbf{P}(L_n \geq 0) \sim \theta n^{-(1-\rho)} l(n) \sim \frac{\theta C_0}{V(c_n)} \tag{24}$$

as $n \to \infty$, where

$$\theta = \sum_{k=0}^{\infty} \mathbf{E}[\mathbf{P}^{+}_{Z_k}(A_{u.s.}); \tau_k = k]. \tag{25}$$

Let

$$\hat{L}_{k,n} := \min_{0 \leq j \leq n-k} (S_{k+j} - S_k)$$

and let $\tilde{\mathcal{F}}_k$ be the $\sigma-$algebra generated by the tuple $\{Z_0, Z_1, ..., Z_k; Q_1, Q_2, ..., Q_k\}$ (see (1)). For further references we formulate two statements borrowed from [6].

Lemma 11 *(see Lemma 2.5 in [6]) Assume Condition A1. Let $Y_1, Y_2, ...$be a uniformly bounded sequence of real-valued random variables adapted to the filtration $\tilde{\mathcal{F}} = \left\{\tilde{\mathcal{F}}_k, k \in \mathbb{N}\right\}$, which converges $\mathbf{P}^+$-a.s. to some random variable Y_∞. Then, as $n \to \infty$*

$$\mathbf{E}[Y_n | L_n \geq 0] \to \mathbf{E}^{+}[Y_\infty].$$

Denote

$$\tau_n := \min\{j : S_j = L_n\}. \tag{26}$$

Lemma 12 *(see Lemma 4.1 in [6]) Assume Conditions A1 and let $l \in \mathbb{N}_0$. Suppose that $\zeta_1, \zeta_2, ...$ is a uniformly bounded sequence of real-valued random variables, which, for every $k \geq 0$ meets the equality*

$$\mathbf{E}\left[\zeta_n; Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \;|\tilde{\mathcal{F}}_k\right] = \mathbf{P}(L_n \geq 0)(\zeta_{k,\infty} + o(1)), \qquad \mathbf{P}\text{-}a.s. \tag{27}$$

as $n \to \infty$ with random variables $\zeta_{1,\infty} = \zeta_{1,\infty}(l), \zeta_{k,\infty} = \zeta_{2,\infty}(l), ...$ Then

$$\mathbf{E}[\zeta_n; Z_{\tau_n + l} > 0] = \mathbf{P}(L_n \geq 0)\left(\sum_{k=0}^{\infty} \mathbf{E}[\zeta_{k,\infty}; \tau_k = k] + o(1)\right)$$

as $n \to \infty$, where the right-hand side series is absolutely convergent.

For $U > 0$ and $q \leq p, \quad pU \leq n$ let

$$\begin{aligned}
\mathcal{X}_U^{q,p} \quad &: \quad = \left\{X_u^{q,p} = e^{-S_{q+[u(p-q)]}} Z_{q+[u(p-q)]}, 0 \leq u \leq U\right\}, \\
\mathcal{X}^{q,p} \quad &: \quad = \left\{X_u^{q,p} = e^{-S_{q+[u(p-q)]}} Z_{q+[u(p-q)]}, 0 \leq u < \infty\right\}, \\
\mathcal{Y}_U^{p,n} \quad &: \quad = \left\{Y_t^{p,n} = e^{-S_{pU+[(n-pU)t]}} Z_{pU+[(n-pU)t]}, 0 \leq t \leq 1\right\}, \quad \mathcal{Y}^{p,n} := \mathcal{Y}_0^{p,n}.
\end{aligned}$$

The next statement is an evident corollary of Theorem 1.3 in [6] and we give its proof for completeness only.

Lemma 13 *Assume Conditions A1 and A2. Let* $(q_1, p_1), (q_2, p_2), \ldots$ *be a sequence of pairs of positive integers such that* $q_n \ll p_n$ *as* $n \to \infty$*. If* $p_n \ll n$ *then, for any* $U > 0$

$$\mathcal{L}\left((\mathcal{X}_U^{q_n,p_n}, \mathcal{Y}_U^{p_n,n}) \ |Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathcal{L}\left((W_u,\ 0 \le u \le U), (\breve{W}_t,\ 0 \le t \le 1)\right)$$

as $n \to \infty$*, where*

$$\mathbf{P}\left(W_u = \breve{W}_t = W,\ 0 \le u \le U, 0 \le t \le 1\right) = 1 \tag{28}$$

for some random variable W *such that*

$$\mathbf{P}(0 < W < \infty) = 1.$$

Proof. We follow (with minor changes) the line of proving Theorem 1.3 in [6]. According to Proposition 3.1 in [6] there exists a strictly positive and finite random variable W^+ such that, as $n \to \infty$

$$e^{-S_n} Z_n \to W^+ \quad \mathbf{P}^+\text{–}a.s. \tag{29}$$

and

$$\left\{W^+ > 0\right\} = \{Z_n > 0 \text{ for all } n\} \quad \mathbf{P}^+\text{–}a.s. \tag{30}$$

Fix $U > 0$ and let ϕ be a bounded continuous function on the space $\Omega_U = D([0, U], \mathbb{R})$ of càdlàg functions and let ϕ_1 be a bounded continuous function on the space Ω_1. For $s \in \mathbb{R}$ let $\mathcal{W}_U^s := \{W_u^s, 0 \le u \le U\}$ and $\breve{\mathcal{W}}^s := \left\{\breve{W}_t^s, 0 \le t \le 1\right\}$ denote the processes with constant paths coinciding (formally) within the time-interval $[0, \min\{U, 1\}]$:

$$W_u^s \ := \ e^{-s} W^+\ , \quad 0 \le u \le U, \quad \breve{W}_t^s := \ e^{-s} W^+, \quad 0 \le t \le 1.$$

It follows from (29) that, for fixed $s \in \mathbb{R}$ the two-dimensional process

$$(e^{-s} \mathcal{X}_U^{q_n,p_n}, e^{-s} \mathcal{Y}_U^{p_n,n})$$

converges, as $n, p_n \to \infty$ with $q_n \le p_n \ll n$, to $\left(\mathcal{W}_U^s, \breve{\mathcal{W}}^s\right)$ in the metric of uniform convergence and, consequently, in the Skorokhod metric on the space $\Omega_U \times \Omega_1$ $\mathbf{P}^+$–a.s., and

$$\begin{aligned} \mathcal{K}_n \ &:= \ \phi(e^{-s} \mathcal{X}_U^{q_n,p_n}) \phi_1(e^{-s} \mathcal{Y}_U^{p_n,n}) I(Z_n > 0) \\ &\to \ \mathcal{K}_\infty := \phi(\mathcal{W}_U^s) \phi_1(\breve{\mathcal{W}}^s) I\{W^+ > 0\} \quad \mathbf{P}^+ - \text{a.s.} \end{aligned}$$

For $q \le p \le n$ and $z \in \mathbb{N}_0$ define

$$\begin{aligned} \psi(z, s, q, p, n) \ &:= \ \mathbf{E}_z[\phi(e^{-s} \mathcal{X}_U^{q,p}) \phi_1(e^{-s} \mathcal{Y}_U^{p,n}); Z_n > 0, L_n \ge 0] \\ &= \ \mathbf{E}_z[\phi(e^{-s} \mathcal{X}_U^{q,p}) \phi_1(e^{-s} \mathcal{Y}_U^{p,n}) I(Z_n > 0) \,|L_n \ge 0] \mathbf{P}(L_n \ge 0). \end{aligned}$$

Since $\mathcal{K}_n \to \mathcal{K}_\infty$ $\mathbf{P}^+$–a.s. as $n \to \infty$, it follows from Lemma 11 that

$$\psi(z, s, q_n, p_n, n) \ = \mathbf{P}(L_n \ge 0) \ \left(\mathbf{E}_z^+[\phi(\mathcal{W}_U^s) \phi_1(\breve{\mathcal{W}}^s); W^+ > 0] + o(1)\right).$$

Observe now that, for $k \le q \le p \le n$

$$\mathbf{E}[\phi(e^{-s}\mathcal{X}_U^{q,p})\phi_1(e^{-s}\mathcal{Y}_U^{p,n}); Z_n > 0, \hat{L}_{k,n} \ge 0 \mid \mathcal{F}_k] \;=\; \psi(Z_k, S_k, q-k, p-k, n-k).$$

Therefore, we may apply Lemma 12 to the random variables

$$\zeta_n = \phi(e^{-s}\mathcal{X}_U^{q_n,p_n})\phi_1(e^{-s}\mathcal{Y}_U^{p_n,n})I\{Z_n > 0\}$$

and

$$\zeta_{k,\infty} = \mathbf{E}_{Z_k}^{+}[\phi(\mathcal{W}_U^{S_k})\phi_1(\check{\mathcal{W}}^{S_k}); W^+ > 0]$$

with $l = 0$.

Using (24) we get

$$\mathbf{E}[\phi(\mathcal{X}_U^{q_n,p_n})\phi_1(\mathcal{Y}_U^{p_n,n}) \mid Z_n > 0] \;\to\; \int \phi(\mathfrak{m})\phi_1(\mathfrak{n})\, \lambda(d\mathfrak{m}\times d\mathfrak{n}) \;\text{ as }\; n \to \infty,$$

where λ is the measure on the product space of càdlàg functions on $\Omega_U \times \Omega_1$ specified by

$$\lambda(d\mathfrak{m}\times d\mathfrak{n}) \;:=\; \frac{1}{\theta}\sum_{k=0}^{\infty}\mathbf{E}[\lambda_{Z_k,S_k}(d\mathfrak{m}\times d\mathfrak{n}); Z_k > 0, \tau_k = k]$$

with

$$\lambda_{z,s}(d\mathfrak{m}\times d\mathfrak{n}) \;:=\; \mathbf{P}_z^{+}[\mathcal{W}_U^s \in d\mathfrak{m}, \check{\mathcal{W}}^s \in d\mathfrak{n}, W^+ > 0].$$

By (30) the total mass of $\lambda_{z,s}$ is equal to $\mathbf{P}_z^{+}(Z_n > 0 \text{ for all } n \ge 0)$. Therefore, the representation of θ in (25) shows that λ is a probability measure. Again using (30) we see that $\lambda_{z,s}$ is concentrated on strictly positive constant functions only. Hence, the same is true for the measure λ.

Lemma 13 is proved.

Corollary 14 *Assume Conditions A1 and A2. Let $(q_1,p_1), (q_2,p_2), \ldots$ be a sequence of pairs of positive integers such that $q_n \ll p_n \ll n$ and $q_n \to \infty$ as $n \to \infty$. Then*

$$\mathcal{L}\left(\mathcal{X}^{q_n,p_n} \,|Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathcal{L}\left(\{W_u,\ 0 \le u < \infty\}\right).$$

Proof. We know that

$$\mathcal{L}\left(\mathcal{X}_U^{q_n,p_n} \,|Z_n > 0, Z_0 = 1\right) \Longrightarrow \mathcal{L}\left(\{W_u,\ 0 \le u \le U\}\right)$$

as $n \to \infty$. for any $U > 0$. This and Theorem 16.7 of [10] complete the proof of the corollary.

Proof of Theorem 3. Let $U > 0$ be fixed. Consider the processes

$$\mathcal{Q}_U^{q,p} = \left\{S_u^{q,p}, 0 \le u \le U\right\}, \tilde{\mathcal{Q}}_U^{q,p} = \left\{\tilde{S}_u^{q,p}, 0 \le u \le U\right\}, \quad 0 \le q \le pU,$$

given by

$$S_u^{q,p} := \frac{S_{[pu]\wedge q}}{c_p}, \quad \tilde{S}_u^{q,p} := \frac{1}{c_p}\left(S_{[pu]} - S_{[pu]\wedge q}\right),\ 0 \le u \le U. \tag{31}$$

Clearly,

$$\mathcal{Q}_U^p := \mathcal{Q}_U^{q,p} + \tilde{\mathcal{Q}}_U^{q,p}.$$

Take $k, l \geq 0$ with $k + l \leq pU$. We may decompose the stochastic process $\mathcal{Q}_U^p$ as

$$\mathcal{Q}_U^p := \mathcal{Q}_U^{k+l,p} + \tilde{\mathcal{Q}}_U^{k+l,p}.$$

Let ϕ be a bounded continuous functional on Ω_U. Define

$$\psi(\mathfrak{m}, r) := \mathbf{E}[\phi(\mathfrak{m} + \tilde{\mathcal{Q}}_U^{k+l,p}); \hat{L}_{k+l,n} \geq -r]$$

for $\mathfrak{m} \in D[0, U]$ and $r \geq 0$. If $p, n \to \infty$ in such a way that $pn^{-1} \to 0$ then, according to Corollary 10

$$\mathcal{L}\left(\left\{S_u^{k+l,p}, 0 \leq u < \infty\right\} \Big| \hat{L}_{k+p,n} \geq -r\right) \Longrightarrow \mathbb{L}^+\left(\{B_u, 0 \leq u < \infty\}\right)$$

for each fixed pair k and l. Hence, if the càdlàg functions $\mathfrak{m}^p \in \Omega_U$ converge uniformly to the zero function as $p \to \infty$, then, given (16)

$$\begin{aligned}\psi(\mathfrak{m}^p, r) &= \mathbf{P}\left(L_{n-(k+l)} \geq -r\right)\left(\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right] + o(1)\right) \\ &= V(r)\mathbf{P}\left(L_n \geq 0\right)\left(\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right] + o(1)\right),\end{aligned}$$

as $p, n \to \infty$, where for the second equality we have applied (19). Using the representation

$$\{\hat{L}_{k,n} \geq 0\} = \{\hat{L}_{k,k+l} \geq 0\} \cap \{\hat{L}_{k+l,n} \geq -(S_{k+l} - S_k)\} \tag{32}$$

and taking into account that $\mathcal{Q}_U^{k+l,p}$ converges uniformly to zero $\mathbf{P}$–a.s. as $p \to \infty$, we have under Condition A:

$$\begin{aligned}&\mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right); Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \mid \mathcal{F}_{k+l}\right] \\ = \ &\psi\left(\mathcal{Q}_U^{k+l,p}, S_{k+l} - S_k\right) I\left\{Z_{k+l} > 0,\ \hat{L}_{k,k+l} \geq 0\right\} \\ = \ &V(S_{k+l} - S_k)\mathbf{P}\left(L_n \geq 0\right)\left(\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right] + o(1)\right) I\left\{Z_{k+l} > 0,\ \hat{L}_{k,k+l} \geq 0\right\} \quad \mathbf{P}\text{–a.s.}\end{aligned} \tag{33}$$

This representation combined with (20) and (32) allows us to deduce the chain of estimates

$$\begin{aligned}\left|\mathbf{E}[\phi(\mathcal{Q}_U^p);\ Z_{k+l} > 0, \hat{L}_{k,n} \geq 0 \mid \mathcal{F}_{k+l}]\right| &\leq \sup|\phi|\,\mathbf{P}\left(\hat{L}_{k,n} \geq 0 \mid \mathcal{F}_{k+l}\right) \\ &= \sup|\phi|\,\mathbf{P}\left(\hat{L}_{k+l,n} \geq -(S_{k+l} - S_k) \mid \mathcal{F}_{k+l}\right) I\left\{\hat{L}_{k,k+l} \geq 0\right\} \\ &\leq K_1 V(S_{k+l} - S_k)\mathbf{P}\left(L_{n-(k+l)} \geq 0\right) I\left\{\hat{L}_{k,k+l} \geq 0\right\} \quad \mathbf{P}\text{–a.s.}\end{aligned}$$

for some $K_1 > 0$. Observe now that according to (10)

$$\mathbf{E}[V(S_{k+l} - S_k); \hat{L}_{k,k+l} \geq 0 \mid \mathcal{F}_k] = V(0) < \infty \qquad \mathbf{P} - \text{a.s.}$$

Hence, using the dominated convergence theorem, (11) and the definition of $\mathbf{P}^+$, we obtain by (33) that

$$\begin{aligned}\mathbf{E}[\phi\left(\mathcal{Q}_U^p\right); Z_{k+l}>0, \hat{L}_{k,n}\geq 0 \mid \mathcal{F}_k] &= (\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right]+o(1))\mathbf{P}\left(L_n\geq 0\right) \\ &\quad\times\mathbf{E}[V(S_{k+l}-S_k); Z_{k+l}>0, \hat{L}_{k,k+l}\geq 0 \mid \mathcal{F}_k] \\ &= (\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right]+o(1))\mathbf{P}\left(L_n\geq 0\right)\mathbf{P}^+_{Z_k}\left(Z_l>0\right) \quad \mathbf{P}\text{–a.s.}\end{aligned}$$

Applying Lemma 12 to $\zeta_n=\phi(\mathcal{Q}_U^p)$ with $n\gg p=p(n)\to\infty$ yields

$$\begin{aligned}&\mathbf{E}[\phi\left(\mathcal{Q}_U^p\right);\ Z_{\tau_n+l}>0] \\ &\quad=(\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right]+o(1))\mathbf{P}\left(L_n\geq 0\right)\sum_{k=0}^{\infty}\mathbf{E}[\mathbf{P}^+_{Z_k}\left(Z_l>0\right);\tau_k=k].\end{aligned}$$

Therefore,

$$\mathbf{P}\left(Z_{\tau_n+l}>0\right)\ \sim\ \mathbf{P}\left(L_n\geq 0\right)\sum_{k=0}^{\infty}\mathbf{E}[\mathbf{P}^+_{Z_k}\left(Z_l>0\right);\tau_k=k] \qquad (34)$$

as $n\to\infty$, where the right-hand side series is convergent. Observe that

$$\begin{aligned}&\left|\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right]\mathbf{P}\left(Z_n>0\right)-\mathbf{E}[\phi\left(\mathcal{Q}_U^p\right);Z_n>0]\right| \\ &\quad\leq\left|\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right]\mathbf{P}\left(Z_n>0\right)-\mathbf{E}[\phi\left(\mathcal{Q}_U^p\right);Z_{\tau_n+l}>0]\right| \\ &\qquad\qquad+\ \sup|\phi|\ \mathbf{E}|I\left\{Z_n>0\right\}-I\left\{Z_{\tau_n+l}>0\right\}|\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}|I\left\{Z_n>0\right\}-I\left\{Z_{\tau_n+l}>0\right\}| &\leq (\mathbf{P}\left(Z_n>0\right)-\mathbf{P}\left(Z_{n+l}>0\right)) \\ &\quad+(\mathbf{P}\left(Z_{\tau_n+l}>0\right)-\mathbf{P}\left(Z_{n+l}>0\right)).\end{aligned}$$

These estimates and (24) lead to the inequality

$$\begin{aligned}&\left|\mathbb{E}^+\left[\phi(\mathcal{B}^U)\right]-\mathbf{E}[\phi\left(\mathcal{Q}_U^p\right)\mid Z_n>0]\right| \\ &\quad\leq 2\sup|\phi|\left(\frac{1}{\theta}\sum_{k=0}^{\infty}\mathbf{E}[\mathbf{P}^+_{Z_k}\left(Z_l>0\right);\tau_k=k]\ -1\right)\ +\ \varepsilon\left(p,n\right), \quad (35)\end{aligned}$$

where $\lim\varepsilon\left(p,n\right)=0$. By the dominated convergence theorem and the definition of θ in (25) we conclude that

$$\sum_{k=0}^{\infty}\mathbf{E}[\mathbf{P}^+_{Z_k}\left(Z_l>0\right);\tau_k=k]\ \downarrow\ \theta\ \text{ as }\ l\to\infty.$$

Since the left-hand side of (35) does not depend on l, this gives the assertion of Theorem 3 for an arbitrary interval $0\leq u\leq U$. To complete the proof of the theorem it remains to apply Theorem 16.7 of [10].

Proof of Corollary 4. We use the notation of Lemma 9 and define

$$\psi^*(\mathfrak{m},\mathfrak{n},r) \ := \ \mathbf{E}[\phi(\mathfrak{m}+\tilde{\mathcal{Q}}_U^{k+l,p})\phi_1(\mathfrak{n}+\tilde{\mathcal{S}}^{k+l,p});\hat{L}_{k+l,n}\geq -r]$$

for $(\mathfrak{m},\mathfrak{n})\in\Omega_U\times\Omega_1$ and $r\geq 0$. If a two dimensional vector of càdlàg functions $(\mathfrak{m}^p,\mathfrak{n}^n)\in\Omega_U\times\Omega_1$ converges uniformly to the two dimensional vector of zero functions as $p=p(n)\to\infty$ as $n\to\infty$, and condition (16) is valid then, according to Lemma 9

$$\begin{aligned}\psi^*(\mathfrak{m}^p,\mathfrak{n}^n,r) \ &= \ \mathbf{P}\left(L_{n-(k+l)}\geq -r\right)\left(\mathbb{E}^{+}\left[\phi(\mathcal{B}^U)\right]\times\mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}^1)\right]+o(1)\right)\\ &= \ V(r)\mathbf{P}\left(L_n\geq 0\right)\left(\mathbb{E}^{+}\left[\phi(\mathcal{B}^U)\right]\times\mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}^1)\right]+o(1)\right).\end{aligned}$$

Let k and l be fixed. We know that the pair $\left(\mathcal{Q}^{k+l,p},S^{k+l,n}\right)$ uniformly converges, as $p,n\to\infty$ to the two dimensional vector of zero functions $\mathbf{P}$–a.s. Hence we obtain

$$\begin{aligned}&\mathbf{E}\left[\phi\left(\mathcal{Q}_U^p\right)\phi_1\left(S^n\right);Z_{k+l}>0,\hat{L}_{k,n}\geq 0\mid\mathcal{F}_{k+l}\right]\\ &\quad=\psi^*\left(\mathcal{Q}_U^{k+l,p},\mathcal{S}^{k+l,n},S_{k+l}-S_k\right)I\left\{Z_{k+l}>0,\,\hat{L}_{k,k+l}\geq 0\right\}\\ &\quad=V(S_{k+l}-S_k)\mathbf{P}\left(L_n\geq 0\right)\times\left(\mathbb{E}^{+}\left[\phi(\mathcal{B}^U)\right]\times\mathbb{E}^{(m)}\left[\phi_1(\mathcal{B}^1)\right]+o(1)\right)\\ &\qquad\qquad\times I\left\{Z_{k+l}>0,\,\hat{L}_{k,k+l}\geq 0\right\}\quad \mathbf{P}\text{–a.s.}\end{aligned}$$

Repeating now almost literally (with evident changes) the proof of Theorem 3 one can check the validity of Corollary 4.

Proof of Theorem 1. For each $U>0$ we have

$$\begin{aligned}&\mathcal{L}\left(\left\{\frac{\log Z_{q+up}}{c_p},0\leq u\leq U\right\}\,\Big|Z_n>0,Z_0=1\right)\\ &\qquad=\mathcal{L}\left(\left\{\frac{\log X_u^{q,p}}{c_p}+\frac{S_{pu}}{c_p},0\leq u\leq U\ \right\}\Big|Z_n>0,Z_0=1\right).\end{aligned}$$

This equality, Theorem 3 and Lemma 13 combined with Theorem 16.7 of [10] justify the desired statement.

Proof of Corollary 2. The needed statement follows from the representation

$$\begin{aligned}&\mathcal{L}\left(\left\{\frac{\log Z_{q+up}}{c_p},0\leq u\leq U;\frac{\log Z_{pU+[(n-pU)t]}}{c_n},0\leq t\leq 1\right\}\,\Big|Z_n>0,Z_0=1\right)\\ &=\mathcal{L}\left(\left\{\frac{S_{pu}+\log X_u^{q,p}}{c_p},0\leq u\leq U;\frac{S_{pU+[(n-pU)t]}+\log Y_t^{p,n}}{c_n},0\leq t\leq 1\right\}\,\Big|Z_n>0,Z_0=1\right),\end{aligned}$$

Lemma 13 and Corollary 4.

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