

Limit laws on extremes of non-homogeneous Gaussian random fields*

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Abstract: In this paper, by using the tail asymptotics derived by Dębicki, Hashorva and Ji (Ann. Probab. 2016), we prove the Gumbel limit laws for the maximum of a class of non-homogeneous Gaussian random fields. As an application of the main results, we derive the Gumbel limit law for Shepp statistics of fractional Brownian motion and Gaussian integrated process.

Key Words: Extremes, Gumbel limit law, non-homogeneous Gaussian random fields, Shepp statistics, fractional Brownian motion

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1 Introduction

The studies on the Gumbel limit law for Gaussian processes have a long history and can date back to Pickands (1969). Suppose that $\{X(t) : t \in [0, \infty)\}$ is a stationary Gaussian process with the covariance function $r(t)$ satisfying the following condition:

$$r(t) = 1 - |t|^\alpha + o(|t|^\alpha), \quad t \rightarrow 0, \quad \text{and} \quad r(t) < 1, \quad t > 0 \quad (1)$$

with $\alpha \in (0, 2]$. It is well-known (see e.g. Pickands (1969), Leadbetter et al. (1983)) that if further the so-called Berman's condition holds as follows

$$r(t) \ln t \rightarrow 0, \quad \text{as} \quad t \rightarrow \infty$$

then the Gumbel limit law

$$P \left(a_T \left(\sup_{0 \leq t \leq T} X(t) - b_T \right) \leq x \right) \rightarrow \exp(-e^{-x}) \quad (2)$$

holds for any $x \in R$, as $T \rightarrow \infty$, where

$$a_T = \sqrt{2 \ln T}, \quad b_T = \sqrt{2 \ln T} + \frac{\ln[(2\pi)^{-1/2} \mathcal{H}_\alpha (2 \ln T)^{-1/2+1/\alpha}]}{\sqrt{2 \ln T}}.$$

Here \mathcal{H}_α denotes the Pickands constant given by

$$\mathcal{H}_\alpha = \lim_{\lambda \rightarrow \infty} \mathcal{H}_\alpha[0, \lambda] / \lambda \in (0, \infty)$$

with

$$\mathcal{H}_\alpha[0, \lambda] = E \exp \left(\max_{t \in [0, \lambda]} \sqrt{2} B_\alpha(t) - t^\alpha \right)$$

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and B_α a fractional Brownian motion (fBm) with Hurst parameter $\alpha/2 \in (0, 1]$, that is, a zero mean Gaussian process with stationary increments such that $EB_\alpha^2(t) = |t|^\alpha$. To derive the Gumbel limit law (2), the following well-known Pickands asymptotics (see e.g. Pickands (1969), Berman (1974), Leadbetter et al. (1983)) plays a crucial role, i.e.,

$$P\left(\sup_{t \in [0, T]} X(t) > u\right) = T\mathcal{H}_\alpha u^{2/\alpha} \Psi(u)(1 + o(1)), \quad (3)$$

as $u \rightarrow \infty$ for some fixed $T \in (0, \infty)$, where $\Psi(\cdot)$ denotes the tail distribution of a standard normal random variable. For some recent work on the tail asymptotics for extremes, we refer to Chan and Lai (2006), Dębicki, Hashorva, Ji and Tabiś (2015), Cheng and Xiao (2016, 2017) and the references therein.

The investigation of (2) for Gaussian processes and general stochastic processes has received a lot of attention. Mittal and Ylvisaker (1975) extended (2) to the strongly dependent Gaussian case; Hülser (1990) investigated (2) for locally stationary Gaussian case, which is recently further extended to Gaussian random fields on manifolds by Qiao and Polonic (2017). We refer to McCormick (1980), Konstant and Piterbarg (1993) and Piterbarg (1996) for further extensions to Gaussian processes and fields; Leadbetter and Rootzén (1982) and Albin (1990) for stationary non-Gaussian processes. For more related extensions, we refer to Dębicki, Hashorva, Ji and Ling (2015) and the reference therein.

In many applied fields, the Gumbel limit laws for extremes of Gaussian processes play a very important role. In approximation theory, Seleznev (1991, 1996), Hülser (1999) and Hülser et al. (2003) applied the Gumbel limit law for Gaussian processes to investigate the deviation processes of some piecewise linear interpolation problems; In nonparametric statistics, the absolute deviations of many types of density estimators obey the Gumbel limit law, see e.g. Bickel and Rosenblatt (1973) and Giné et al. (2003). In applied statistics, there are also many confidence intervals and bands, which are constructed based on the Gumbel limit law of the estimators, since extremes themselves are also type of very important estimators, see e.g. Giné and Nickl (2010). For some recent studies on applications of Gumbel limit laws, we refer to Sharpnack and Arias-Castro (2016) and Qiao and Polonic (2016).

Define

$$X(s, t) = Y(s + t) - Y(s), \quad (4)$$

where $\{Y(t), t \geq 0\}$ is a Gaussian process. The process $\xi_S(t) = \sup_{0 \leq s \leq S} X(s, t)$ is referred to as the Shepp statistics in many recent works. Zholud (2008) studied the maximum of the process $\xi_S(t)$ and established the Gumbel limit law when $Y(t)$ is a Brownian motion. Hashorva and Tan (2013) and Tan and Yang (2014) extended the result to fractional Brownian motion. We refer to Piterbarg (2001), Hülser and Piterbarg (2004a) and Hashorva et al. (2013) for related work on the fractional Brownian motion.

In this paper, we generalize model (4) and impose directly some restrictions on the Gaussian random fields. We first consider the Gumbel limit law for the process $\zeta_T(s) = \sup_{0 \leq t \leq T} X(s, t)$ for some fixed $T > 0$, where $X(s, t)$ is a type of non-homogeneous Gaussian random field. Then we use the obtained results to derive the Gumbel limit law for Shepp statistics. Noting that $\zeta_T(t)$ is no longer Gaussian process, we can not derive the Gumbel limit laws from the Gaussian case directly. However, $\zeta_T(t)$ also doesn't satisfy the conditions imposed on general stochastic processes, such as those given by Leadbetter and Rootzén (1982) and Albin (1990). We will follow the method used in Chapter 12 in Leadbetter et al. (1983). The tail asymptotic result of extremes of the field $X(s, t)$ is a key tool, which has been derived by Dębicki et al. (2016).

The rest of the paper is organized as follows. In Section 2, we give some tail asymptotic results from Dębicki et al. (2016). In Section 3, we state the main results of the paper, and in Section 4, we present two applications. The technical proofs are gathered in Section 5, while in Section 6 we give two auxiliary results.

2 Preliminaries

In this section, we present the tail asymptotic result provided by Dębicki et al. (2016). Suppose that $\{X(s, t), (s, t) \in [0, \infty) \times [0, T]\}$ with fixed T is a centered Gaussian random field with variance function and correlation function $\sigma^2(s, t)$ and $r(s, t, s', t')$, respectively. Suppose the following assumptions hold.

Assumption A1: there exists some positive function $\sigma(t)$ which attains its unique maximum on $[0, T]$ at fixed T , and further

$$\sigma(s, t) = \sigma(t), \quad \forall (s, t) \in [0, \infty) \times [0, T], \quad \sigma(t) = 1 - b(T - t)^\beta(1 + o(1)), \quad t \uparrow T$$

holds for some $\beta, b > 0$.

Assumption A2: there exist constants $a_1 > 0, a_2 > 0, a_3 \neq 0$ and $\alpha_1, \alpha_2 \in (0, 2]$ such that

$$r(s, t, s', t') = 1 - (|a_1(s - s')|^{\alpha_1} + |a_2(t - t') + a_3(s - s')|^{\alpha_2})(1 + o(1))$$

holds uniformly with respect to $s, s' \in [0, L]$ with some constant $L > 0$ as $|s - s'| \rightarrow 0, \min(t, t') \uparrow T$ and further, there exists some constant $\delta_0 \in (0, T)$ such that

$$r(s, t, s', t') < 1$$

for any $s, s' \in [0, L]$ satisfying $s \neq s'$ and $t, t' \in [\delta_0, T]$.

Assumption A3: There exist positive constants $\gamma_1, \gamma_2, \gamma$ and \mathcal{C} such that

$$E(X(s, t) - X(s', t'))^2 \leq \mathcal{C}(|t - t'|^\gamma + |s - s'|^\gamma)$$

holds for all $t, t' \in [\gamma_1, T], s, s' \in [0, L]$ satisfying $|s - s'| < \gamma_2$.

To state the tail asymptotics for the maximum of the field $X(s, t)$ under assumptions **A1-A3**, we need the so-called Piterbarg constants and Pickands-Piterbarg constants, respectively. The Piterbarg constant \mathcal{P}_α^b with constant $b > 0$ is defined as

$$\mathcal{P}_\alpha^b = \lim_{\lambda \rightarrow \infty} E \exp \left(\max_{t \in [0, \lambda]} \sqrt{2} B_\alpha(t) - (1 + b)|t|^\alpha \right) \in (0, \infty).$$

For some constants $a_1 > 0, a_2 > 0, a_3 \neq 0, b > 0$, let

$$Y(s, t) = \tilde{B}_\alpha(a_1 s) + B_\alpha(a_2 t - a_3 s), \quad \sigma_Y^2(s, t) = \text{Var}(Y(s, t))$$

and

$$\mathcal{H}_Y^b[\lambda_1, \lambda_2] = E \exp \left(\max_{(s, t) \in [0, \lambda_1] \times [0, \lambda_2]} \sqrt{2} Y(s, t) - \sigma_Y^2(s, t) - b|t|^\alpha \right),$$

where \tilde{B}_α and B_α are two independent fractional Brownian motions (fBms). The Pickands-Piterbarg constant is defined as

$$\mathcal{M}_{Y, \alpha}^b = \lim_{\lambda_1 \rightarrow \infty} \lim_{\lambda_2 \rightarrow \infty} \frac{1}{\lambda_1} \mathcal{H}_Y^b[\lambda_1, \lambda_2].$$

Under the above assumptions, Dębicki et al. (2016) derived the following result.

Theorem 2.1. *Let $\{X(s, t), (s, t) \in [0, \infty) \times [0, T]\}$ with fixed T be a centered Gaussian random field with a.s. continuous sample paths. Suppose that assumptions **A1-A3** are satisfied with the parameters mentioned therein, we have as $u \rightarrow \infty$,*

$$P \left(\sup_{(s, t) \in [0, L] \times [0, T]} X(s, t) > u \right) = L\mu(u)(1 + o(1)),$$

where for $\beta > \max\{\alpha_1, \alpha_2\}$

$$\mu(u) = \Gamma(1/\beta + 1) \prod_{k=1}^2 (a_k \mathcal{H}_{\alpha_k}) b^{-\frac{1}{\beta}} u^{\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta}} \Psi(u);$$

for $\beta = \alpha_2 = \alpha_1$

$$\mu(u) = \mathcal{M}_{Y, \alpha_1}^b u^{\frac{2}{\alpha_1}} \Psi(u);$$

for $\beta = \alpha_2 > \alpha_1$

$$\mu(u) = a_1 a_2 \mathcal{P}_{\alpha_2}^{b a_2^{-\alpha_2}} \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}} \Psi(u);$$

for $\beta < \alpha_2 = \alpha_1$

$$\mu(u) = (a_1^{\alpha_1} + |a_3|^{\alpha_1})^{\frac{1}{\alpha_1}} \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}} \Psi(u);$$

for $\beta < \alpha_2$ and $\alpha_1 < \alpha_2$

$$\mu(u) = a_1 \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}} \Psi(u);$$

for $\beta = \alpha_1 > \alpha_2$

$$\mu(u) = a_1 \mathcal{P}_{\alpha_1}^{b(\frac{|a_3|}{a_1 a_2})^{\alpha_1}} \mathcal{H}_{\alpha_2} u^{\frac{2}{\alpha_2}} \Psi(u);$$

for $\beta < \alpha_1$ and $\alpha_2 < \alpha_1$

$$\mu(u) = |a_3| \mathcal{H}_{\alpha_2} u^{\frac{2}{\alpha_2}} \Psi(u).$$

This result is very powerful since it can be used to derive the exact tail asymptotics for many type of statistics, such as Shepp statistics for Gaussian processes, Brownian bridge and fBm, maximum loss and span of Gaussian processes, see Dębicki et al. (2016) for details.

3 Main Result

Note that the assumptions **A1** and **A2** are local conditions. To derive the Gumbel limit law, we need to impose the following Berman-type weak dependence condition, which is a global condition.

Assumption A4: Assume that for $c = 1 + \varepsilon I(\beta \geq \max\{\alpha_1, \alpha_2\})$ with some constant $\varepsilon > 0$ the function

$$\delta(v) := \sup\{|r(s, t, s', t')|, |s - s'| \geq v, s, s' \in [0, \infty), t, t' \in [0, T]\}$$

is such that

$$\lim_{v \rightarrow \infty} \delta(v) (\ln v)^c = 0, \quad (5)$$

where $I(\cdot)$ denotes the indicator function.

We state now the main result.

Theorem 3.1. *Let $\{X(s, t), (s, t) \in [0, \infty) \times [0, T]\}$ with fixed T be a centered Gaussian random field with a.s. continuous sample paths. Suppose that assumptions **A1-A4** are satisfied with the parameters mentioned therein. In addition, assume that $\{X(s, t), (s, t) \in [0, \infty) \times [0, T]\}$ is homogeneous with respect to the first factor s . Then*

$$\lim_{S \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| P \left(a_S \left(\sup_{(s, t) \in [0, S] \times [0, T]} X(s, t) - b_S \right) \leq x \right) - \exp(-e^{-x}) \right| = 0,$$

where $a_S = \sqrt{2 \ln S}$

$$b_S = a_S + a_S^{-1} \omega_S$$

with, for $\beta > \max\{\alpha_1, \alpha_2\}$

$$\omega_S = \ln \left((2\pi)^{-1/2} \Gamma(1/\beta + 1) \prod_{k=1}^2 (a_k \mathcal{H}_{\alpha_k}) b^{-\frac{1}{\beta}} a_S^{\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1} \right);$$

for $\beta = \alpha_2 = \alpha_1$

$$\omega_S = \ln \left((2\pi)^{-1/2} \mathcal{M}_{Y, \alpha_1}^b a_S^{\frac{2}{\alpha_1} - 1} \right);$$

for $\beta = \alpha_2 > \alpha_1$

$$\omega_S = \ln \left((2\pi)^{-1/2} a_1 a_2 \mathcal{P}_{\alpha_2}^{b a_2^{-\alpha_2}} \mathcal{H}_{\alpha_1} a_S^{\frac{2}{\alpha_1} - 1} \right);$$

for $\beta < \alpha_2 = \alpha_1$

$$\omega_S = \ln \left((2\pi)^{-1/2} (a_1^{\alpha_1} + |a_3|^{\alpha_1})^{\frac{1}{\alpha_1}} \mathcal{H}_{\alpha_1} a_S^{\frac{2}{\alpha_1} - 1} \right);$$

for $\beta < \alpha_2$ and $\alpha_1 < \alpha_2$

$$\omega_S = \ln \left((2\pi)^{-1/2} a_1 \mathcal{H}_{\alpha_1} a_S^{\frac{2}{\alpha_1} - 1} \right);$$

for $\beta = \alpha_1 > \alpha_2$

$$\omega_S = \ln \left((2\pi)^{-1/2} a_1 \mathcal{P}_{\alpha_1}^{b(\frac{|a_3|}{a_1 a_2})^{\alpha_1}} \mathcal{H}_{\alpha_2} a_S^{\frac{2}{\alpha_2} - 1} \right);$$

for $\beta < \alpha_1$ and $\alpha_2 < \alpha_1$

$$\omega_S = \ln \left((2\pi)^{-1/2} |a_3| \mathcal{H}_{\alpha_2} a_S^{\frac{2}{\alpha_2} - 1} \right).$$

Remark 3.1: Assumption A4 is a weakly dependent condition. If $\lim_{v \rightarrow \infty} \delta(v)(\ln v)^c = d > 0$, then the field $X(s, t)$ will possess some strongly dependent property with respect to the first parameter. In this case, the limit distribution will be no longer Gumbel distribution, see Mittal and Ylvisaker (1975) and Tan et al. (2012) for some related results about strongly dependent Gaussian processes.

4 Applications

In this section, we give two applications of our main results. We derive the exact tail asymptotics and Gumbel limit laws for Shepp statistics. The obtained results are of independent interest.

Throughout this section, let $\{X(t), t \geq 0\}$ be a centered Gaussian process and define

$$Z(s, t) = X(s + t) - X(s), \quad (s, t) \in [0, \infty) \times [0, T],$$

for some fixed $T > 0$. The Shepp statistic $\sup_{0 \leq s \leq S} Z(s, t)$ which was introduced by Shepp (see Shepp 1966, 1971) play a vary important role in statistics. Other important results for the Shepp statistics can be found in Cressie (1980), Deheuvels and Devroye (1987), Siegmund and Venkatraman (1995), Dumbgen and Spokoiny (2001) and Kabluchko (2011). The limit properties of extremes of Shepp statistics when $X(t)$ is a fBm have been studied by Zholud (2008) and Hashorva and Tan (2013), Tan and Yang (2015) and Tan and Chen (2016). Applying Theorem 3.1, we study the limit properties of extremes of Shepp statistics for a more general Gaussian process $X(t)$, which is a stationary Gaussian process or non-stationary Gaussian process with stationary increments.

4.1 Stationary case

Let $\{X(t), t \geq 0\}$ be a centered stationary Gaussian process. Suppose the covariance function r_X of $\{X(t), t \geq 0\}$ satisfies the following conditions:

Assumption B1: $r_X(t)$ attains its minimum on $[0, T]$ at the unique point T ;

Assumption B2: there exist positive constants α_1, a_1, a_2 and $\alpha_2 \in (0, 2)$ such that

$$r_X(t) = r_X(T) + a_1 |t - T|^{\alpha_1} (1 + o(1)), \quad t \rightarrow T, \quad \text{and} \quad r_X(t) = 1 - a_2 t^{\alpha_2} (1 + o(1)), \quad t \rightarrow 0;$$

Assumption B3: $r_X(s) < 1$ for $s > 0$.

For simplicity, write $\rho_T = \sqrt{2(1 - r_X(T))}$ and $b_i = a_i / \rho_T^2$, $i = 1, 2$.

Proposition 4.1. *Let $Z(s, t)$ be defined as above. Suppose that $r_X(t)$ satisfies conditions **B1** – **B3**. In addition, suppose that $r_X(t)$ is twice continuously differentiable on $[\tau, \infty)$ for some $\tau > 0$ and the limit of twice derivative $\lim_{t \rightarrow T} |\ddot{r}_X(t)| \in (0, \infty)$. Furthermore, if $\ddot{r}_X(t)(\ln t)^c = o(1)$ with $c = 1 + \varepsilon I(\alpha_1 \geq \alpha_2)$ and some constant $\varepsilon > 0$ as $t \rightarrow \infty$, then*

$$\lim_{S \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| P \left(a_S \left(\sup_{(s,t) \in [0,S] \times [0,T]} Z(s, t) - b_S \right) \leq x \right) - \exp\{-e^{-x}\} \right| = 0, \quad (6)$$

where $a_S = \rho_T \sqrt{2 \ln S}$,

$$b_S = a_S + a_S^{-1} \omega_S$$

with, for $\alpha_1 > \alpha_2$

$$\omega_S = \ln(\Gamma(\frac{1}{\alpha_1} + 1) \mathcal{H}_{\alpha_2}^2 b_2^{\frac{2}{\alpha_2}} b_1^{-\frac{1}{\alpha_1}} (2\pi)^{-1/2} a_S^{\frac{4}{\alpha_2} - \frac{2}{\alpha_1} - 1});$$

for $\alpha_1 = \alpha_2$

$$\omega_S = \ln(\mathcal{M}_{Y, \alpha_1}^{b_1} (2\pi)^{-1/2} a_S^{\frac{2}{\alpha_2} - 1})$$

with $Y = Y(s, t) = \tilde{B}_{\alpha_2}(b_2^{\frac{1}{\alpha_2}} s) + B_{\alpha_2}(b_2^{\frac{1}{\alpha_2}} t - b_2^{\frac{1}{\alpha_2}} s)$; for $\alpha_1 < \alpha_2$

$$\omega_S = \ln((2b_2)^{\frac{1}{\alpha_2}} \mathcal{H}_{\alpha_2} (2\pi)^{-1/2} a_S^{\frac{2}{\alpha_2} - 1}).$$

Example 4.1: The Ornstein-Uhlenbeck process with covariance function $r_X(t) = e^{-|t|^\alpha}$ and the generalized Cauchy model with covariance function $r_X(t) = (1 + |t|^\alpha)^{-\beta}$ with $\alpha \in (0, 2)$ and $\beta > 0$ satisfy the conditions of Proposition 4.1.

4.2 Non-Stationary case

Let $\{X(t), t \geq 0\}$ be a centered non-stationary Gaussian process with stationary increment and variance function $\sigma_X^2(t)$, a.s. continuous sample paths. Recall that $X(t)$ is said to have stationary increments if the law of the process $\{X(t + t_0) - X(t_0), t \in \mathbb{R}\}$ does not depend on the choice of t_0 . To study the maximum of $Z(s, t)$, we only need to impose some conditions on the variogram $\gamma(t) = \mathbb{E}(X(t) - X(0))^2$ of X . Note that for this case the variogram is $\gamma_X(t) = \sigma_X^2(t)$. Suppose that the variance function $\sigma_X^2(t)$ of $\{X(t), t \geq 0\}$ satisfies the following conditions:

Assumption C1: $\sigma_X(t)$ attains its maximum on $[0, T]$ at the unique point T , and further

$$\sigma_X(t) = 1 - b(T - t)^\beta(1 + o(1)), \quad t \uparrow T$$

holds for some $\beta, b > 0$.

Assumption C2: $\sigma_X^2(t)$ is twice continuously differentiable on $[\tau, \infty)$ for $\tau > 0$ with limit of twice derivative $\lim_{t \rightarrow T} |\ddot{\sigma}_X^2(t)| \in (0, \infty)$ and further

$$\sigma_X^2(t) = (at)^\alpha(1 + o(1)), \quad t \rightarrow 0$$

holds for some $\alpha \in (0, 2], a > 0$.

Assumption C3: $\ddot{\sigma}_X^2(t)(\ln t)^c \rightarrow 0$ with $c = 1 + \varepsilon I(\beta \geq \alpha)$ and some constant $\varepsilon > 0$ as $t \rightarrow \infty$.

Proposition 4.2. *Let $Z(s, t)$ be defined as above. Suppose that $\sigma_X(t)$ satisfies conditions **C1**, **C2**. We have for some constant $L > 0$*

$$P \left(\sup_{(s,t) \in [0,L] \times [0,T]} Z(s, t) > u \right) = L\mu(u)(1 + o(1)),$$

where for $\alpha < \beta$

$$\mu(u) = 2^{-\frac{2}{\alpha}} \Gamma(1/\beta + 1) a^2 \mathcal{H}_\alpha^2 b^{-\frac{1}{\beta}} u^{\frac{4}{\alpha} - \frac{2}{\beta}} \Psi(u);$$

for $\alpha = \beta$

$$\mu(u) = \mathcal{M}_Y^b u^{\frac{2}{\alpha}} \Psi(u)$$

with $Y = Y(s, t) = \tilde{B}_\alpha(2^{-1/\alpha}as) + B_\alpha(2^{-1/\alpha}at - 2^{-1/\alpha}as)$; for $\alpha > \beta$

$$\mu(u) = a\mathcal{H}_\alpha u^{\frac{2}{\alpha}} \Psi(u).$$

Furthermore, if condition **C3** holds, then

$$\limsup_{S \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| P \left(a_S \left(\sup_{(s,t) \in [0,S] \times [0,T]} Z(s,t) - b_S \right) \leq x \right) - \exp\{-e^{-x}\} \right| = 0, \quad (7)$$

where $a_S = \sqrt{2 \ln S}$, and

$$b_S = a_S + a_S^{-1} \omega_S$$

with for $\alpha < \beta$

$$\omega_S = \ln(2^{-\frac{2}{\alpha}} \Gamma(1/\beta + 1) a^2 \mathcal{H}_\alpha^2 b^{-\frac{1}{\beta}} (2\pi)^{-1/2} a_S^{\frac{4}{\alpha} - \frac{2}{\beta} - 1});$$

for $\alpha = \beta$

$$\omega_S = \ln(\mathcal{M}_Y^b (2\pi)^{-1/2} a_S^{\frac{2}{\alpha} - 1});$$

for $\alpha > \beta$

$$\omega_S = \ln(a\mathcal{H}_\alpha (2\pi)^{-1/2} a_S^{\frac{2}{\alpha} - 1}).$$

We illustrate Proposition 4.2 by the following two examples on the fBm and Gaussian integrated process.

Example 4.2: Let $B_{H_i}(t)$, $i = 1, 2, \dots, n$ be a sequence of independent fBms with Hurst index $H_i \in (0, 1)$ and λ_i be a positive sequence satisfying $\sum_{i=1}^n \lambda_i^2 = 1$. Since given $H = H_1 = H_2$ we have $\lambda_1 B_{H_1}(t) + \lambda_2 B_{H_2}(t) =^d \sqrt{\lambda_1^2 + \lambda_2^2} B_H(t)$, we suppose that

$$H := H_1 < H_2 < \dots < H_n.$$

Let $X(t) = \sum_{i=1}^n \lambda_i T^{-1/2} B_{H_i}(t)$ and $Z(s, t)$ be defined as above. We have for some constant $L > 0$

$$P \left(\sup_{(s,t) \in [0,L] \times [0,T]} Z(s,t) > u \right) = L\mu(u)(1 + o(1)),$$

as $u \rightarrow \infty$, where for $H \in (0, 1/2)$

$$\mu(u) = 2^{-\frac{1}{H}} \mathcal{H}_{2H}^2 \lambda_1^{\frac{2}{H}} \left(\sum_{i=1}^n \lambda_i^2 H_i \right)^{-1} u^{\frac{2}{H} - 2} \Psi(u);$$

for $H = 1/2$

$$\mu(u) = \mathcal{M}_{Y,1}^{\frac{1}{2}} u^2 \Psi(u)$$

with $Y = Y(s, t) = \tilde{B}_1(2^{-1}s) + B_1(2^{-1}(t-s))$; for $H \in (1/2, 1)$

$$\mu(u) = \lambda_1^{\frac{1}{H}} \mathcal{H}_{2H} u^{\frac{1}{H}} \Psi(u)$$

and

$$\limsup_{S \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| P \left(a_S \left(\sup_{(s,t) \in [0,S] \times [0,T]} Z(s,t) - b_S \right) \leq x \right) - \exp\{-e^{-x}\} \right| = 0, \quad (8)$$

where $a_S = \sqrt{2 \ln S}$, and

$$b_S = a_S + a_S^{-1} \omega_S$$

with for $H \in (0, 1/2)$

$$\omega_S = \ln \left((2\pi)^{-1/2} 2^{-\frac{1}{H}} \mathcal{H}_{2H}^2 \lambda_1^{\frac{2}{H}} \left(\sum_{i=1}^n \lambda_i^2 H_i \right)^{-1} a_S^{\frac{2}{H}-3} \right)$$

for any $H = 1/2$

$$\omega_S = \ln \left(2\pi^{-1/2} \mathcal{M}_{Y,1}^{\frac{1}{2}} a_S \right)$$

and for $H \in (1/2, 1)$

$$\omega_S = \ln \left((2\pi)^{-1/2} \lambda_1^{\frac{1}{H}} \mathcal{H}_{2H} a_S^{\frac{1}{H}-1} \right).$$

Next, we consider the Gaussian integrated process. For related studies, we refer to Dębicki (2002) and Hüsler and Piterbarg (2004b).

Example 4.3: Let $\{\zeta(t), t \geq 0\}$ be a centered stationary Gaussian process with variance one and suppose the covariance function $r_\zeta(t)$ of $\{\zeta(t), t \geq 0\}$ satisfying the following conditions:

Assumption D1: $r_\zeta(t) \in C([0, \infty))$ and $\int_0^t r_\zeta(s) ds > 0$ for $t \in (0, T]$;

Assumption D2: $r_\zeta(t) = 1 - t^\theta(1 + o(1))$ as $t \rightarrow 0^+$ with $\theta \in (0, 2]$;

Assumption D3: $r_\zeta(t) \ln t = o(1)$ as $t \rightarrow \infty$.

Define Gaussian integrated processes as $X(t) = \int_0^t \zeta(s) ds$ and let $Z(s, t)$ be defined as above.

If conditions **D1, D2** are satisfied, we have for some constant $L > 0$

$$P \left(\sup_{(s,t) \in [0,L] \times [0,T]} Z(s, t) > u \right) = L \frac{1}{\sqrt{\pi}} u \Psi(u) (1 + o(1)),$$

as $u \rightarrow \infty$. If further condition **D3** holds, we have

$$\lim_{S \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| P \left(a_S \left(\sup_{(s,t) \in [0,S] \times [0,T]} Z(s, t) - b_S \right) \leq x \right) - \exp\{-e^{-x}\} \right| = 0, \quad (9)$$

where $a_S = \sqrt{2 \ln S}$, and

$$b_S = a_S + a_S^{-1} \ln \left((2)^{-1/2} \pi^{-1} \right).$$

5 Proofs

We need the following lemmas to prove Theorem 3.1. For simplicity, write $u = u_S(x) = a_S^{-1}x + b_S$ in the following part.

Lemma 5.1. *Let $\delta_u = u^{-2/\beta} (\ln u)^{2/\beta}$. Under the conditions of Theorem 3.1, we have for some constant $S_0 > 0$*

$$\left| P \left(\sup_{(s,t) \in [0, S_0] \times [0, T]} X(s, t) \leq u \right) - P \left(\sup_{(s,t) \in [0, S_0] \times [T - \delta_u, T]} X(s, t) \leq u \right) \right| / P \left(\sup_{(s,t) \in [0, S_0] \times [0, T]} X(s, t) > u \right) \rightarrow 0$$

as $u \rightarrow \infty$.

Proof: It can be found in the proof of Theorem 2.2 of Dębicki et al. (2016). \square

For given $\epsilon > 0$, we divide interval $[0, S]$ into intervals of length 1, and split each of them onto subintervals I_j^ϵ, I_j of length $\epsilon, 1 - \epsilon, j = 1, 2, \dots, [S]$, respectively, where $[x]$ denotes the integral part of x . It can be easily seen that a possible remaining interval with length smaller than 1 plays no role in our consideration. We denote this interval by J .

Lemma 5.2. *Under the conditions of Theorem 3.1, we have*

$$\left| P \left(\sup_{(s,t) \in [0, S] \times [T - \delta_u, T]} X(s, t) \leq u \right) - P \left(\sup_{(s,t) \in \cup I_j \times [T - \delta_u, T]} X(s, t) \leq u \right) \right| \rightarrow 0,$$

as $u \rightarrow \infty$ and $\epsilon \rightarrow 0$.

Proof: By applying Theorem 2.1 and Lemma 5.1, we have

$$\begin{aligned}
& \left| P \left(\sup_{(s,t) \in [0,S] \times [T-\delta_u, T]} X(s,t) \leq u \right) - P \left(\sup_{(s,t) \in \cup I_j \times [T-\delta_u, T]} X(s,t) \leq u \right) \right| \\
& \leq P \left(\sup_{(s,t) \in (\cup I_j^c \cup J) \times [T-\delta_u, T]} X(s,t) > u \right) \\
& \leq \sum_{j=1}^{\lfloor S \rfloor} P \left(\sup_{(s,t) \in I_j^c \times [T-\delta_u, T]} X(s,t) > u \right) + P \left(\sup_{(s,t) \in J \times [T-\delta_u, T]} X(s,t) > u \right) \\
& \sim \sum_{j=1}^{\lfloor S \rfloor} P \left(\sup_{(s,t) \in I_j^c \times [0, T]} X(s,t) > u \right) + P \left(\sup_{(s,t) \in J \times [0, T]} X(s,t) > u \right) \\
& \leq (\lfloor S \rfloor \epsilon + 1) \mu(u).
\end{aligned}$$

Noting that by the definitions of a_S and b_S , we have $S\mu(u) = O(1)$ as $u \rightarrow \infty$, thus the result follows by letting $\epsilon \rightarrow 0$. \square

Let in the following $q_i = du^{-2/\alpha_i}$ for some $d > 0$.

Lemma 5.3. *Under the conditions of Theorem 3.1, we have for any $j = 1, 2, \dots, \lfloor S \rfloor$*

$$\left| P \left(\sup_{(s,t) \in I_j \times [T-\delta_u, T]} X(s,t) \leq u \right) - P \left(\sup_{(kq_1, lq_2) \in I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) \right| \leq K\rho(d)\mu(u)$$

as $u \rightarrow \infty$, where $\rho(d) \rightarrow 0$ as $d \rightarrow 0$.

Proof: Without loss of generality, we only show the case $j = 1$.

Case $\beta > \max(\alpha_1, \alpha_2)$: For simplicity, we only consider the case that $\alpha_1 = \alpha_2 =: \alpha$. Choose first a constant $\alpha_0 \in (\alpha, \beta)$ and denote that

$$\Delta_{ij} = \Delta_i \times \Delta_j, \quad \Delta_{ij}^T = \Delta_i \times (T - \Delta_j), \quad \text{with } \Delta_i = [iu^{-\frac{2}{\alpha_0}}, (i+1)u^{-\frac{2}{\alpha_0}}].$$

Set further

$$N_1(u) = \lfloor (1 - \epsilon)u^{\frac{2}{\alpha_0}} \rfloor + 1, \quad N_2(u) = \lfloor (\ln u)^{\frac{2}{\beta}} u^{\frac{2}{\alpha_0} - \frac{2}{\beta}} \rfloor + 1.$$

For any $\epsilon \in (0, 1)$, let $\{\eta_{\pm\epsilon}(s, t), (s, t) \in [0, \infty)^2\}$ be centered homogeneous Gaussian random fields with covariance functions

$$r_{\pm\epsilon}(s, t) = \exp \left(-(1 \pm \epsilon)^\alpha (|a_1 s|^\alpha + |a_2 t + a_3 s|^\alpha) \right), \quad (s, t) \in [0, \infty)^2$$

From the proof of case i) of Dębicki et al. (2016), it is easy to show that (letting $q = du^{-\frac{2}{\alpha}}$)

$$\begin{aligned}
& \sum_{i=0}^{N_1(u)} \sum_{j=0}^{N_2(u)} P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{+\epsilon}(s, T-t) > u_{j-} \right) \\
& \geq \sum_{i=0}^{N_1(u)} \sum_{j=0}^{N_2(u)} P \left(\sup_{(s,t) \in \Delta_{ij}} \frac{X(s, T-t)}{\sigma(s, T-t)} > u_{j-} \right) \\
& \geq \sum_{i=0}^{N_1(u)} \sum_{j=0}^{N_2(u)} P \left(\sup_{(s,t) \in \Delta_{ij}^T} X(s, t) > u \right) \\
& \geq P \left(\sup_{(s,t) \in I_1 \times [T-\delta_u, T]} X(s, t) > u \right) \\
& \geq P \left(\sup_{(kq, lq) \in I_1 \times [T-\delta_u, T]} X(kq, lq) > u \right)
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(kq,lq) \in \Delta_{ij}^T} X(kq,lq) > u \right) - \Sigma_1(u) \\
&\geq \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(kq,lq) \in \Delta_{ij}} \frac{X(kq,T-lq)}{\sigma(kq,T-lq)} > u_{j+} \right) - \Sigma_1(u) \\
&\geq \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(kq,lq) \in \Delta_{ij}} \eta_{-\varepsilon}(kq,T-lq) > u_{j+} \right) - \Sigma_1(u), \tag{10}
\end{aligned}$$

where

$$u_{j-} = u(1 + b(1 - \varepsilon)(ju^{-\frac{2}{\alpha_0}})^\beta), \quad u_{j+} = u(1 + b(1 + \varepsilon)((j+1)u^{-\frac{2}{\alpha_0}})^\beta),$$

and

$$\Sigma_1(u) = \sum_{0 \leq i, i' \leq N_1(u)-1, 0 \leq j, j' \leq N_2(u)-1} P \left(\sup_{(s,t) \in \Delta_{ij}^T} X(s,t) > u, \sup_{(s,t) \in \Delta_{i'j'}^T} X(s,t) > u \right).$$

We also can get the following results from the above mentioned paper

$$\Sigma_1(u) = o(\mu(u)) \tag{11}$$

as $u \rightarrow \infty$ and

$$\begin{aligned}
&\sum_{i=0}^{N_1(u)} \sum_{j=0}^{N_2(u)} P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{+\varepsilon}(s,T-t) > u_{j-} \right) \\
&\sim \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{-\varepsilon}(s,T-t) > u_{j+} \right) \sim \mu(u), \tag{12}
\end{aligned}$$

as $u \rightarrow \infty$ and $\varepsilon \rightarrow 0$. For the homogeneous Gaussian random fields $\eta_{\pm\varepsilon}(s,t)$, by Lemma 6.2 in the Appendix, we use the following estimate

$$\begin{aligned}
&\left| P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{\pm\varepsilon}(s,T-t) > u \right) - P \left(\sup_{(kq,lq) \in \Delta_{ij}} \eta_{\pm\varepsilon}(kq,T-lq) > u \right) \right| \\
&\leq \rho(d)a_1a_2u^{\frac{4}{\alpha}-\frac{4}{\alpha_0}}\Psi(u)(1+g(u)),
\end{aligned}$$

where $\rho(d) \rightarrow 0$ as $d \rightarrow 0$ and $g(u) \rightarrow 0$ as $u \rightarrow \infty$. Denote by $G(u) = 1 + \sup_{v \geq u} |g(v)| \rightarrow 1$ as $u \rightarrow \infty$. Then $u/u_{j\pm} \rightarrow 1$ as $u \rightarrow \infty$ uniformly in j and also

$$\begin{aligned}
&\left| P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{\pm\varepsilon}(s,T-t) > u_{j\pm} \right) - P \left(\sup_{(kq,lq) \in \Delta_{ij}} \eta_{\pm\varepsilon}(kq,T-lq) > u_{j\pm} \right) \right| \\
&\leq \rho(d)a_1a_2u_{j\pm}^{\frac{4}{\alpha}-\frac{4}{\alpha_0}}\Psi(u_{j\pm})G(u).
\end{aligned}$$

Thus, there exists $K > 0$ such that

$$\begin{aligned}
&\left| \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{-\varepsilon}(s,T-t) > u_{j+} \right) - \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(kq,lq) \in \Delta_{ij}} \eta_{-\varepsilon}(kq,T-lq) > u_{j+} \right) \right| \\
&\leq K\rho(d)u^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u). \tag{13}
\end{aligned}$$

Now it follows from (10-13) that

$$\begin{aligned}
&\left| P \left(\sup_{(s,t) \in I_1 \times [T-\delta_u, T]} X(s,t) > u \right) - P \left(\sup_{(kq,lq) \in I_1 \times [T-\delta_u, T]} X(kq,lq) > u \right) \right| \\
&\leq \sum_{i=0}^{N_1(u)} \sum_{j=0}^{N_2(u)} P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{+\varepsilon}(s,T-t) > u_{j-} \right) - \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(kq,lq) \in \Delta_{ij}} \eta_{-\varepsilon}(kq,T-lq) > u_{j+} \right) + \Sigma_1(u)
\end{aligned}$$

$$\begin{aligned} & \sim \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(s,t) \in \Delta_{ij}} \eta_{-\varepsilon}(s, T-t) > u_{j+} \right) - \sum_{i=0}^{N_1(u)-1} \sum_{j=0}^{N_2(u)-1} P \left(\sup_{(kq,lq) \in \Delta_{ij}} \eta_{-\varepsilon}(kq, T-lq) > u_{j+} \right) \\ & \leq K\rho(d)u^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u). \end{aligned}$$

Case $\beta = \alpha_1 = \alpha_2$: For simplicity, set $\alpha = \alpha_1 = \alpha_2$. Let S_0, T_0 be two positive constants and define

$$\begin{aligned} \widehat{\Delta}_i &= [iS_0u^{-\frac{2}{\alpha}}, (i+1)S_0u^{-\frac{2}{\alpha}}], \quad i = 0, 1, \dots, \widehat{N}_1(u), \quad \widetilde{\Delta}_j = [jT_0u^{-\frac{2}{\alpha}}, (j+1)T_0u^{-\frac{2}{\alpha}}], \quad j = 0, 1, \dots, \widetilde{N}_2(u), \\ \overline{\Delta}_{ij} &= \widehat{\Delta}_i \times \widetilde{\Delta}_j, \quad \overline{\Delta}_{ij}^T = \widehat{\Delta}_i \times (T - \widetilde{\Delta}_j), \end{aligned}$$

where

$$\widehat{N}_1(u) = \lfloor \frac{1-\varepsilon}{S_0} u^{\frac{2}{\alpha}} \rfloor + 1, \quad \widetilde{N}_2(u) = \lfloor \frac{(\ln u)^{\frac{2}{\beta}}}{T_0} u^{\frac{2}{\alpha}} \rfloor + 1.$$

From the proof of case ii) of Dębicki et al. (2016) again, it is easy to show that (letting $q = du^{-\frac{2}{\alpha}}$)

$$\begin{aligned} & \Sigma_2(u) + \sum_{i=0}^{\widehat{N}_1(u)} P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} X(s, t) > u \right) \\ & \geq P \left(\sup_{(s,t) \in I_1 \times [T-\delta_u, T]} X(s, t) > u \right) \\ & \geq P \left(\sup_{(kq,lq) \in I_1 \times [T-\delta_u, T]} X(kq, lq) > u \right) \\ & \geq \sum_{i=0}^{\widehat{N}_1(u)-1} P \left(\sup_{(kq,lq) \in \overline{\Delta}_{i0}^T} X(kq, lq) > u \right) - \Sigma_3(u), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Sigma_2(u) &= \sum_{i=0}^{\widehat{N}_1(u)} \sum_{j=1}^{\widetilde{N}_2(u)} P \left(\sup_{(s,t) \in \overline{\Delta}_{ij}^T} X(s, t) > u \right) = o(\mu(u)), \\ \Sigma_3(u) &= \sum_{0 < i < i' < \widehat{N}_1(u)-1} P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} X(s, t) > u, \sup_{(s,t) \in \overline{\Delta}_{i'0}^T} X(s, t) > u \right) = o(\mu(u)), \end{aligned}$$

as $u \rightarrow \infty$. We can also get the following results by Lemma 2.1 of Dębicki et al. (2016)

$$P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} X(s, t) > u \right) \sim P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} \frac{\tilde{\eta}(s, t)}{1 + bt^\beta} > u \right) \sim \mathcal{H}_{Y_1}^b[S_0, T_0]\Psi(u)$$

and

$$\sum_{i=0}^{\widehat{N}_1(u)} P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} X(s, t) > u \right) \sim \sum_{i=0}^{\widehat{N}_1(u)-1} P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} X(s, t) > u \right) \sim \frac{(1-\varepsilon)}{S_0} u^{\frac{2}{\alpha}} \mathcal{H}_{Y_1}^b[S_0, T_0]\Psi(u)$$

as $u \rightarrow \infty$, where $\{\tilde{\eta}(s, t), (s, t) \in [0, \infty)^2\}$ is a centered homogeneous Gaussian random fields with covariance functions

$$r(s, t) = \exp(-(|a_1s|^\alpha + |a_2t - a_3s|^\alpha)), \quad (s, t) \in [0, \infty)^2.$$

Since $X(s, t)$ is homogeneous with respect to s , we have

$$\begin{aligned} 0 & \leq P \left(\sup_{(s,t) \in I_1 \times [T-\delta_u, T]} X(s, t) > u \right) - P \left(\sup_{(kq,lq) \in I_1 \times [T-\delta_u, T]} X(kq, lq) > u \right) \\ & \leq \Sigma_2(u) + \sum_{i=0}^{\widehat{N}_1(u)} P \left(\sup_{(s,t) \in \overline{\Delta}_{i0}^T} X(s, t) > u \right) - \sum_{i=0}^{\widehat{N}_1(u)-1} P \left(\sup_{(kq,lq) \in \overline{\Delta}_{i0}^T} X(kq, lq) > u \right) + \Sigma_3(u) \end{aligned}$$

$$\begin{aligned}
&= \Sigma_2(u) + \widehat{N}_1(u)P\left(\sup_{(s,t)\in\overline{\Delta}_{00}^T} X(s,t) > u\right) - (\widehat{N}_1(u) - 1)P\left(\sup_{(kq,lq)\in\overline{\Delta}_{00}^T} X(kq,lq) > u\right) + \Sigma_3(u) \\
&\leq (\widehat{N}_1(u) - 1)P\left(\sup_{(s,t)\in\overline{\Delta}_{00}} \frac{\tilde{\eta}(s,t)}{1+bt^\beta} > u, \sup_{(kq,lq)\in\overline{\Delta}_{00}} \frac{\tilde{\eta}(kq,lq)}{1+b(lq)^\beta} \leq u\right) + \mathcal{H}_{Y_1}^b[S_0, T_0]\Psi(u) + o(\mu(u)). \quad (15)
\end{aligned}$$

For the constants $a_1 > 0, a_2 > 0, a_3 \neq 0, b > 0$, let (as in Section 2)

$$Y(s, t) = \tilde{B}_\alpha(a_1 s) + B_\alpha(a_2 t - a_3 s), \quad \sigma_Y^2(s, t) = \text{Var}(Y(s, t))$$

and

$$\mathcal{H}_Y^b[\lambda_1, \lambda_2](d) = E \exp\left(\max_{(kd, ld) \in [0, d\lambda_1] \times [0, d\lambda_2]} \sqrt{2}Y(kd, ld) - \sigma_Y^2(kd, ld) - b|ld|^\beta\right) \in (0, \infty),$$

where \tilde{B}_α and B_α are two independent fBms. By the same arguments as in the proof of Lemma 6.1 of Deĭbicki et al. (2016), we can show

$$\mathcal{M}_{Y, \alpha}^b(d) = \lim_{\lambda_1 \rightarrow \infty} \lim_{\lambda_2 \rightarrow \infty} \frac{1}{d\lambda_1} \mathcal{H}_Y^b[\lambda_1, \lambda_2](d) \in (0, \infty).$$

Following the arguments of Lemma 12.2.7 of Leadbetter et al. (1983), we can show that $\lim_{d \rightarrow 0} \mathcal{M}_{Y, \alpha}^b(d) = \mathcal{M}_{Y, \alpha}^b$. Now, following the arguments of Lemma 6.1 of Deĭbicki et al. (2016) (see also the proof of Lemma 6.1 of Piterbarg (1996)), we have

$$\begin{aligned}
&P\left(\sup_{(s,t)\in\overline{\Delta}_{00}} \frac{\tilde{\eta}(s,t)}{1+bt^\beta} > u, \sup_{(kq,lq)\in\overline{\Delta}_{00}} \frac{\tilde{\eta}(kq,lq)}{1+b(lq)^\beta} \leq u\right) \\
&= \Psi(u) \int_0^{+\infty} e^w P\left(\sup_{(s,t)\in[0, S_0] \times [0, T_0]} [\sqrt{2}Y(s, t) - \sigma_Y^2(s, t) - b|t|^\beta] > w, \right. \\
&\quad \left. \sup_{(kq,lq)\in[0, S_0] \times [0, T_0]} [\sqrt{2}Y(kq, lq) - \sigma_Y^2(kd, ld) - b|lq|^\beta] \leq w\right) dw (1 + o(1)) \\
&= \Psi(u) (\mathcal{H}_Y^b[S_0/d, T_0/d](d) - \mathcal{H}_Y^b[S_0, T_0]) (1 + o(1)), \quad (16)
\end{aligned}$$

as $u \rightarrow \infty$. Now, we can conclude that

$$\begin{aligned}
0 &\leq P\left(\sup_{(s,t)\in I_1 \times [T-\delta_u, T]} X(s,t) > u\right) - P\left(\sup_{(kq,lq)\in I_1 \times [T-\delta_u, T]} X(kq,lq) > u\right) \\
&\leq (1-\epsilon)u^{2/\alpha}\Psi(u) \left(\frac{\mathcal{H}_Y^b[S_0/d, T_0/d](d)}{S_0} - \frac{\mathcal{H}_Y^b[S_0, T_0]}{S_0}\right) + \mathcal{H}_{Y_1}^b[S_0, T_0]\Psi(u) + o(\mu(u)) \\
&\leq K(\mathcal{M}_{Y, \alpha}^b(d) - \mathcal{M}_{Y, \alpha}^b)\mu(u) \\
&=: K\rho(d)\mu(u), \quad (17)
\end{aligned}$$

where $\rho(d) \rightarrow 0$ as $d \rightarrow 0$.

Case $\beta = \alpha_2 > \alpha_1$: This case can be proved as case ii) by some obvious changes as follows. Let S_0, T_0 be two positive constants and define

$$\widehat{\Delta}_i = [iS_0u^{-\frac{2}{\alpha_1}}, (i+1)S_0u^{-\frac{2}{\alpha_1}}], \quad i = 0, 1, \dots, \widehat{N}_1(u), \quad \widetilde{\Delta}_j = [jT_0u^{-\frac{2}{\alpha_2}}, (j+1)T_0u^{-\frac{2}{\alpha_2}}], \quad j = 0, 1, \dots, \widetilde{N}_2(u),$$

$$\overline{\Delta}_{ij} = \widehat{\Delta}_i \times \widetilde{\Delta}_j, \quad \overline{\Delta}_{ij}^T = \widehat{\Delta}_i \times (T - \widetilde{\Delta}_j),$$

where

$$\widehat{N}_1(u) = \lfloor \frac{1-\epsilon}{S_0} u^{\frac{2}{\alpha_1}} \rfloor + 1, \quad \widetilde{N}_2(u) = \lfloor \frac{(\ln u)^{\frac{2}{\beta}}}{T_0} u^{\frac{2}{\alpha_2}} \rfloor + 1.$$

Let $q_1 = du^{-\frac{2}{\alpha_1}}, q_2 = du^{-\frac{2}{\alpha_2}}$, then repeating the proof of case ii) by replacing kq and lq by kq_1 and lq_2 , we get the desired result.

Case $\beta < \alpha_2 = \alpha_1$: For simplicity let $\alpha := \alpha_2 = \alpha_1$ and $q = du^{-\frac{2}{\alpha}}$. Let's consider the Gaussian process $X(s, T), s \geq 0$. It is easy to check that $X(s, T), s \geq 0$ is standard stationary Gaussian process, i.e., with mean 0, variance 1. For the covariance function of $X(s, T), s \geq 0$, it holds that

$$r(s, T, s', T) = 1 - (a_1^\alpha + |a_3|^\alpha)|s - s'|^\alpha(1 + o(1))$$

uniformly with respect to $s, s' \in [0, S_0]$, as $|s - s'| \rightarrow 0$. For some constant $a > 0$, let

$$\mathcal{H}_\alpha^a[0, \lambda] = E \exp \left(\max_{ak \in [0, a\lambda]} \sqrt{2} B_\alpha(ak) - (ak)^\alpha \right)$$

and define

$$\mathcal{H}_\alpha(a) = \lim_{\lambda \rightarrow \infty} \frac{\mathcal{H}_\alpha^a[0, \lambda]}{a\lambda} \in (0, +\infty).$$

Note that $\lim_{a \rightarrow 0} \mathcal{H}_\alpha(a) = \mathcal{H}_\alpha$, see e.g. Leadbetter et al. (1983). So by Lemmas 6.1 and 6.2 in the Appendix (for the one dimensional case), we have

$$P \left(\sup_{s \in I_1} X(s, T) > u \right) = (1 - \epsilon)(a_1^\alpha + |a_3|^\alpha)^{\frac{1}{\alpha}} \mathcal{H}_\alpha u^{\frac{2}{\alpha}} \Psi(u)(1 + o(1)), \quad (18)$$

$$P \left(\sup_{kq \in I_1} X(kq, T) > u \right) = (1 - \epsilon)(a_1^\alpha + |a_3|^\alpha)^{\frac{1}{\alpha}} \mathcal{H}_\alpha(d) u^{\frac{2}{\alpha}} \Psi(u)(1 + o(1)) \quad (19)$$

and

$$\left| P \left(\sup_{s \in I_1} X(s, T) > u \right) - P \left(\sup_{kq \in I_1} X(kq, T) > u \right) \right| \leq K\rho(d) u^{\frac{2}{\alpha}} \Psi(u), \quad (20)$$

as $u \rightarrow \infty$, where $\rho(d) = \mathcal{H}_\alpha(d) - \mathcal{H}_\alpha$. By repeating the proof of iv) of Dębicki et al. (2016), it is easy to show that

$$P \left(\sup_{(kq, lq) \in I_1 \times [T - \delta_u, T]} X(kq, lq) > u \right) = (1 - \epsilon)(a_1^\alpha + |a_3|^\alpha)^{\frac{1}{\alpha}} \mathcal{H}_\alpha(d) u^{\frac{2}{\alpha}} \Psi(u)(1 + o(1)). \quad (21)$$

Write

$$\begin{aligned} & \left| P \left(\sup_{(s, t) \in I_1 \times [T - \delta_u, T]} X(s, t) > u \right) - P \left(\sup_{(kq, lq) \in I_1 \times [T - \delta_u, T]} X(kq, lq) > u \right) \right| \\ & \leq \left| P \left(\sup_{(s, t) \in I_1 \times [T - \delta_u, T]} X(s, t) > u \right) - P \left(\sup_{s \in I_1} X(s, T) > u \right) \right| \\ & + \left| P \left(\sup_{s \in I_1} X(s, T) > u \right) - P \left(\sup_{kq \in I_1} X(kq, T) > u \right) \right| \\ & + \left| P \left(\sup_{kq \in I_1} X(kq, T) > u \right) - P \left(\sup_{(kq_1, lq_2) \in I_1 \times [T - \delta_u, T]} X(kq_1, lq_2) > u \right) \right| \\ & =: M_1 + M_2 + M_3, \end{aligned}$$

where $M_1 = o(\mu(u))$ by iv) of Theorem 2.1 and (18), $M_2 = K\rho(d) u^{\frac{2}{\alpha}} \Psi(u)$ by (20) and $M_3 = o(\mu(u))$ by (19) and (21) as $u \rightarrow \infty$.

Case $\beta < \alpha_2$ and $\alpha_1 < \alpha_2$: The proof is the same as that of Case $\beta < \alpha_2 = \alpha_1$.

Case $\beta = \alpha_1 > \alpha_2$ and case $\beta < \alpha_1$ and $\alpha_2 < \alpha_1$: These two cases can be proved by the same arguments as for the third and fifth cases after some time scaling as in Dębicki et al. (2016), so we omit the details. \square

Lemma 5.4. *Under the conditions of Theorem 3.1, we have*

$$\left| P \left(\sup_{(s, t) \in \cup I_j \times [T - \delta_u, T]} X(s, t) \leq u \right) - P \left(\sup_{(kq_1, lq_2) \in \cup I_j \times [T - \delta_u, T]} X(kq_1, lq_2) \leq u \right) \right| \leq K\rho(d) S\mu(u) \quad (22)$$

as $u \rightarrow \infty$.

Proof: By Lemma 5.3, we have

$$\begin{aligned} & \left| P \left(\sup_{(s,t) \in \cup I_j \times [T-\delta_u, T]} X(s,t) \leq u \right) - P \left(\sup_{(kq_1, lq_2) \in \cup I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) \right| \\ & \leq S \max_j \left| P \left(\sup_{(s,t) \in I_j \times [T-\delta_u, T]} X(s,t) \leq u \right) - P \left(\sup_{(kq_1, lq_2) \in I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) \right| \\ & \leq K\rho(d)S\mu(u), \end{aligned}$$

which completes the proof. \square

Lemma 5.5. *Under the conditions of Theorem 3.1, we have*

$$\left| P \left(\sup_{(kq_1, lq_2) \in \cup I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) - \prod_{j=1}^{\lfloor S \rfloor} P \left(\sup_{(kq_1, lq_2) \in I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) \right| \rightarrow 0,$$

as $u \rightarrow \infty$.

Proof: Applying Berman's inequality (see e.g. Piterbarg (1996)) we have

$$\begin{aligned} & \left| P \left(\sup_{(kq_1, lq_2) \in \cup I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) - \prod_{j=1}^{\lfloor S \rfloor} P \left(\sup_{(kq_1, lq_2) \in I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right) \right| \\ & = \left| P \left(\sup_{(kq_1, lq_2) \in \cup I_j \times [T-\delta_u, T]} \frac{X(kq_1, lq_2)}{\sigma(lq_2)} \leq \frac{u}{\sigma(lq_2)} \right) - \prod_{j=1}^{\lfloor S \rfloor} P \left(\sup_{(kq_1, lq_2) \in I_j \times [T-\delta_u, T]} \frac{X(kq_1, lq_2)}{\sigma(lq_2)} \leq \frac{u}{\sigma(lq_2)} \right) \right| \\ & \leq \sum_{j \neq j'} \sum_{\substack{(kq_1, lq_2) \in I_j \times [T-\delta_u, T] \\ (k'q_1, l'q_2) \in I_{j'} \times [T-\delta_u, T]}} |r(kq_1, lq_2, k'q_1, l'q_2)| \exp \left(-\frac{(\sigma^{-2}(lq_2) + \sigma^{-2}(l'q_2))u^2}{2(1 + r(kq_1, lq_2, k'q_1, l'q_2))} \right) \\ & \leq \sum_{j \neq j'} \sum_{\substack{(kq_1, lq_2) \in I_j \times [T-\delta_u, T] \\ (k'q_1, l'q_2) \in I_{j'} \times [T-\delta_u, T]}} |r(kq_1, lq_2, k'q_1, l'q_2)| \exp \left(-\frac{u^2}{1 + r(kq_1, lq_2, k'q_1, l'q_2)} \right). \end{aligned}$$

Since $|kq_1 - k'q_1| \geq \epsilon$ by definition, $r(kq_1, lq_2, k'q_1, l'q_2) \leq \delta < 1$. Set $\gamma < (1 - \delta)/(1 + \delta)$ and split the last sum into two parts W_1 and W_2 with $|kq_1 - k'q_1| < S^\gamma$ and $|kq_1 - k'q_1| \geq S^\gamma$, respectively. For the first sum there are $S^{1+\gamma}/q_2^2$ combinations of two points $kq_1, k'q_1 \in \cup_j I_j$. Together with the lq_2 combinations there are $(S^{1+\gamma}/q_1^2)(\delta_u/q_2^2)$ terms in the sum W_1 . Note that

$$S\mu(u) = O(1), \quad u \rightarrow \infty,$$

which implies for case i)

$$u^2 = 2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1 \right) \ln \ln S + O(1);$$

for case ii)-v)

$$u^2 = 2 \ln S + \left(\frac{2}{\alpha_1} - 1 \right) \ln \ln S + O(1);$$

for case vi)-vii)

$$u^2 = 2 \ln S + \left(\frac{2}{\alpha_2} - 1 \right) \ln \ln S + O(1).$$

Thus, W_1 is bounded by

$$\begin{aligned} & \delta \frac{S^{1+\gamma} \delta^2(u)}{q_1^2 q_2^2} \exp \left(-\frac{u^2}{1 + \delta} \right) \\ & \leq \delta \exp \left((1 + \gamma) \ln S + \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} \right) \ln \ln S - \frac{2(1 + o(1))}{1 + \delta} \ln S \right) \\ & = \delta \exp \left((\ln S) \left[(1 + \gamma) - \frac{2(1 + o(1))}{1 + \delta} + \frac{(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}) \ln \ln S}{\ln S} \right] \right) \rightarrow 0 \end{aligned}$$

as $S \rightarrow \infty$ since $1 + \gamma < 2/(1 + \delta)$ by the choice of γ .

For the second sum W_2 with $|kq_1 - k'q_1| \geq S^\gamma$, we use that

$$\sup_{|kq_1 - k'q_1| \geq S^\gamma} r(kq_1, lq_2, k'q_1, l'q_2)(\ln S)^c = o(1),$$

as $S \rightarrow \infty$. In this case there $(S/q_1)^2$ many combinations of two points $kq_1, k'q_1 \in \cup_i \mathbf{I}_i$. Hence W_2 is bounded by

$$\begin{aligned} R(S) &:= \frac{o(1)}{(\ln S)^c} \frac{S^2}{q_1^2} \frac{\delta^2(u)}{q_2^2} \exp\left(-\frac{u^2}{1 + o(1)/\ln S}\right) \\ &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S - (c-1) \ln \ln S - \frac{u^2}{1 + o(1)/\ln S}\right). \end{aligned}$$

For case i), by assumption **A4**, $c > 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S - (c-1) \ln \ln S \right. \\ &\quad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} [2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S]\right) \\ &\leq C \exp\left(- (c-1) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S + o(1)\right) \rightarrow 0, \end{aligned}$$

as $S \rightarrow \infty$, since $c > 1$. For cases ii)-iii), noting that $c > 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S - (c-1) \ln \ln S \right. \\ &\quad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} [2 \ln S + \left(\frac{2}{\alpha_1} - 1\right) \ln \ln S]\right) \\ &\leq C \exp\left(\left(\frac{2}{\alpha_2} - \frac{2}{\beta}\right) \ln \ln S - (c-1) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S + o(1)\right) \rightarrow 0, \end{aligned}$$

as $S \rightarrow \infty$, since $\beta = \alpha_2$ and $c > 1$. For cases iv)-v), noting that $c = 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S \right. \\ &\quad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} [2 \ln S + \left(\frac{2}{\alpha_1} - 1\right) \ln \ln S]\right) \\ &\leq C \exp\left(\left(\frac{2}{\alpha_2} - \frac{2}{\beta}\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S + o(1)\right) \rightarrow 0, \end{aligned}$$

as $S \rightarrow \infty$, since $\beta < \alpha_2$. For cases vi), we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S - (c-1) \ln \ln S \right. \\ &\quad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} [2 \ln S + \left(\frac{2}{\alpha_2} - 1\right) \ln \ln S]\right) \\ &\leq C \exp\left(\left(\frac{2}{\alpha_1} - \frac{2}{\beta}\right) \ln \ln S - (c-1) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S + o(1)\right) \rightarrow 0, \end{aligned}$$

as $S \rightarrow \infty$, since $\beta = \alpha_1$ and $c > 1$. For cases vii), noting that $c = 1$, we have

$$\begin{aligned} R(S) &\leq C \exp\left(2 \ln S + \left(\frac{2}{\alpha_1} + \frac{2}{\alpha_2} - \frac{2}{\beta} - 1\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S \right. \\ &\quad \left. - \frac{(1 + o(1))}{1 + o(1)/\ln S} [2 \ln S + \left(\frac{2}{\alpha_2} - 1\right) \ln \ln S]\right) \\ &\leq C \exp\left(\left(\frac{2}{\alpha_1} - \frac{2}{\beta}\right) \ln \ln S + \frac{4}{\beta} \ln \ln \ln S + o(1)\right) \rightarrow 0, \end{aligned}$$

as $S \rightarrow \infty$, since $\beta < \alpha_1$. \square

Proof of Theorem 3.1: Recall that $u = u(x) = a_S^{-1}x + b_S$. By the stationarity of $X(s, t)$ with respect to the first component, Lemma 5.1, Theorem 2.1 and the choice of a_S, b_S , we have

$$\begin{aligned} \prod_{j=1}^{\lfloor S \rfloor} P \left(\max_{(s,t) \in I_j \times [T-\delta_u, T]} X(s, t) \leq u \right) &\sim \exp \left(-\lfloor S \rfloor P \left(\max_{(s,t) \in I_1 \times [T-\delta_u, T]} X(s, t) > u \right) \right) \\ &\sim \exp(-\lfloor S \rfloor (1 - \epsilon) \mu(u)) \\ &\rightarrow \exp(-e^{-x}), \quad \epsilon \downarrow 0, \quad S \rightarrow \infty. \end{aligned}$$

Further, by Lemmas 5.1-5.5, it holds that as $S \rightarrow \infty$

$$\begin{aligned} P \left(\max_{(s,t) \in [0, S] \times [0, T]} X(s, t) \leq u \right) &\sim P \left(\max_{(s,t) \in \cup_j I_j \times [T-\delta_u, T]} X(s, t) \leq u \right) \\ &\sim P \left(\max_{(kq_1, lq_2) \in \cup_j I_j} X(kq_1, lq_2) \leq u \right) \\ &\sim \prod_{j=1}^{\lfloor S \rfloor} P \left(\max_{(kq_1, lq_2) \in I_j \times [T-\delta_u, T]} X(kq_1, lq_2) \leq u \right). \end{aligned}$$

Therefore, the claim follows. \square

Proof of Proposition 4.1: In the paper of Dębicki et al. (2016), it is shown that the standard deviation function of Z satisfies assumption **A1** and the correlation function of Z satisfies assumption **A2**. It is also shown that assumption **A3** holds for Z . So, in order to prove this proposition, it suffices to show assumption **A4** holds. For the correlation function $r_Z(s, t, s', t')$ of Z , we have

$$r_Z(s, t, s', t') = r_X(|s + t - s' - t'|) - r_X(|s - s' - t'|) - r_X(|s + t - s'|) + r_X(|s - s'|).$$

Since $r_X(t)$ is twice continuously differentiable in $(0, \infty)$, we have

$$|r_X(|s + t - s' - t'|) - r_X(|s - s' - t'|) - r_X(|s + t - s'|) + r_X(|s - s'|)| \leq C \ddot{r}_X(s - s')$$

for $t, t' \in [0, T]$ as $s - s' \rightarrow \infty$. Now using the condition that $\ddot{r}_X(t)(\ln t)^c \rightarrow 0$ as $t \rightarrow \infty$, we show that assumption **A4** holds. \square

Proof of Proposition 4.2: We check that assumptions **A1** – **A4** hold. Using the stationarity of the increments of $X(t)$ and **C1**, it follows that the variance $\sigma_Z^2(s, t)$ of $Z(s, t)$ attains its maximum on $[0, T]$ at the unique point T , and further

$$\sigma_Z(s, t) = \sigma_X(t) = 1 - b(T - t)^\beta(1 + o(1)), \quad t \uparrow T$$

holds for some $\beta, b > 0$.

Notice that for the process $X(t)$ with stationary increments

$$\text{Cov}(X(t), X(s)) = \frac{1}{2}[\sigma_X^2(t) + \sigma_X^2(s) - \sigma_X^2(|t - s|)].$$

Thus, using the stationarity of the increments of $X(t)$ again, we have for correlation function of $Z(s, t)$

$$r_Z(s, t, s', t') = \frac{1}{2\sigma_X(t)\sigma_X(t')}[-\sigma_X^2(|s + t - s' - t'|) + \sigma_X^2(|s - s' - t'|) + \sigma_X^2(|s - s' + t'|) - \sigma_X^2(|s - s'|)].$$

It follows from **C2** that

$$r_Z(s, t, s', t') = 1 - \frac{1}{2}[(a|s + t - s' - t'|)^\alpha + (a|s - s'|)^\alpha](1 + o(1)),$$

as $t, t' \rightarrow T$ and $|s - s'| \rightarrow 0$. **A3** holds obviously. Thus, by Theorem 2.1, the first assertion of Proposition 4.2 holds.

By Taylor expansions, it is straightforward to verify that

$$|r_Z(s, t, s', t')| \leq C \ddot{\sigma}_X^2(|s - s'|)$$

as $|s - s'| \rightarrow \infty$, which combined with **C3** implies **A4**. Thus, by Theorem 3.1, the second assertion holds. \square

6 Appendix

Let $\{\xi(\mathbf{t}) : \mathbf{t} \geq \mathbf{0}\}$ denote a two dimensional homogeneous Gaussian field with covariance function

$$r_\xi(\mathbf{t}) = \text{Cov}(\xi(\mathbf{t}), \xi(\mathbf{0})).$$

Assume that the covariance function satisfies the following conditions:

Assumption E1: There exists a non-degenerate matrix \mathbb{C} such that

$$r_\xi(\mathbb{C}\mathbf{t}) = 1 - |t_1|^{\alpha_1} - |t_2|^{\alpha_2} + o(|t_1|^{\alpha_1} + |t_2|^{\alpha_2})$$

as $\mathbf{t} \rightarrow 0$ with $\alpha_i \in (0, 2]$;

Assumption E2: $r_\xi(\mathbf{t}) < 1$ for $\mathbf{t} \neq \mathbf{0}$.

To state two key lemmas, we recall the following type of Pickands constant. For constant $a > 0$, let

$$\mathcal{H}_\alpha^a[0, \lambda] = E \exp \left(\max_{ak \in [0, a\lambda]} \sqrt{2} B_\alpha(ak) - (ak)^\alpha \right)$$

and define

$$\mathcal{H}_\alpha(a) = \lim_{\lambda \rightarrow \infty} \frac{\mathcal{H}_\alpha^a[0, \lambda]}{a\lambda}.$$

We need the following results for the proofs of our main results.

Lemma 6.1. *Let $q_i = du^{-2/\alpha_i}$ for some $d > 0$ and assume that **E1** and **E2** hold. Then for any fixed rectangle $\mathbf{I}_h = [0, h_1] \times [0, h_2]$, we have*

$$P \left(\max_{\mathbf{t} \in \mathbf{I}_h} \xi(\mathbf{t}) > u \right) = h_1 h_2 \mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2} |\det \mathbb{C}^{-1}| u^{2/\alpha_1 + 2/\alpha_2} \Psi(u) (1 + o(1))$$

and

$$P \left(\max_{\mathbf{kq} \in \mathbf{I}_h} \xi(\mathbf{kq}) > u \right) = h_1 h_2 \mathcal{H}_{\alpha_1}(d) \mathcal{H}_{\alpha_2}(d) |\det \mathbb{C}^{-1}| u^{2/\alpha_1 + 2/\alpha_2} \Psi(u) (1 + o(1))$$

as $u \rightarrow \infty$. The results also hold for the case $h_1 = u^{-2/\alpha'}$ and $h_2 = u^{-2/\alpha''}$ for $\alpha' > \alpha_1$ and $\alpha'' > \alpha_2$.

Proof: The first and second assertions can be proved following the proof of Lemma 7.1 of Piterbarg (1996) with some obvious changes, see also the proof of Lemma 1 of Dębicki, Hashorva and Soja-Kukielka (2015). The third assertion follows from the proofs of the former two by using the double sums method, see the proof of Theorem 7.2 of Piterbarg (1996). \square

Lemma 6.2. *Let $q_i = du^{-2/\alpha_i}$ for some $d > 0$ and choose two constants $\alpha' > \alpha_1$ and $\alpha'' > \alpha_2$. Assume that **E1** and **E2** hold. Then for the rectangle $\mathbf{I} = [0, u^{-2/\alpha'}] \times [0, u^{-2/\alpha''}]$, we have*

$$P \left(\max_{\mathbf{kq} \in \mathbf{I}} \xi(\mathbf{kq}) \leq u \right) - P \left(\max_{\mathbf{t} \in \mathbf{I}} \xi(\mathbf{t}) \leq u \right) \leq |\det \mathbb{C}^{-1}| \rho(d) u^{2/\alpha_1 + 2/\alpha_2 - 2/\alpha' - 2/\alpha''} \Psi(u),$$

where $\rho(d) = \mathcal{H}_{\alpha_1}(d) \mathcal{H}_{\alpha_2}(d) - \mathcal{H}_{\alpha_1} \mathcal{H}_{\alpha_2} \rightarrow 0$ as $d \rightarrow 0$.

Proof: It is an immediate consequence of Lemma 6.1. \square

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