# EXACT SIMULATION OF MULTIDIMENSIONAL REFLECTED BROWNIAN MOTION 

Jose Blanchet<br>Stanford Univesity

Karthyek Murthy<br>Columbia University


#### Abstract

We present the first exact simulation method for multidimensional reflected Brownian motion (RBM). Exact simulation in this setting is challenging because of the presence of correlated local-time-like terms in the definition of RBM. We apply recently developed so-called $\varepsilon$-strong simulation techniques (also known as Tolerance-Enforced Simulation) which allow us to provide a piece-wise linear approximation to RBM with $\varepsilon$ (deterministic) error in uniform norm. A novel conditional acceptance / rejection step is then used to eliminate the error. In particular, we condition on a suitably designed information structure so that a feasible proposal distribution can be applied.


## 1. Introduction

This paper is a contribution to the theory of exact simulation for stochastic differential equations (SDEs). In particular, we present the first exact simulation algorithm for multidimensional reflected Brownian motion (RBM).

Multidimensional RBM was introduced by Harrison and Reiman in [10] and it figures prominently in stochastic Operations Research. It turns out that RBM approximates the workload at each station in so-called generalized Jackson networks, which are comprised of single-server queues connected via Markovian routing. The approximation holds in heavy traffic (that is, as the system approaches $100 \%$ utilization) and it is applicable in great generality (assuming only a functional central limit theorem for the arrival process and the service requirements at each station, see for example [14] and [6]). Following [10], we refer a $d$-dimensional stochastic process ( $\mathbf{Y}(t): t \geq 0)$ that satisfies the following properties as a reflected Brownian motion (RBM):

1) $\mathbf{Y}(\cdot)$ is a Markov process with stationary transition probabilities, continuous sample paths taking values in the non-negative orthant of $\mathbb{R}^{d}$,
2) $\mathbf{Y}(\cdot)$ behaves in the interior of positive orthant like a $d$-dimensional Brownian motion (either standard or with a constant drift and diffusion matrix),
3) $\mathbf{Y}(\cdot)$ reflects instantaneously at the boundary of the positive orthant, and
4) the direction of reflection anywhere on the boundary surface where the $i$-th component $Y_{i}=0$ is the $i$-th column of the $d \times d$ reflection matrix $R$. It is required that $R$ is of the form $R=I-Q^{T}$, where $Q$ is a non-negative $d \times d$ matrix with zeros on the diagonal and spectral radius strictly smaller than unity.

RBM as a solution of Skorokhod problem. Let ( $\mathbf{X}(t): t \geq 0)$ denote a $d$-dimensional Brownian motion that behaves similar to $\mathrm{RBM} \mathbf{Y}(\cdot)$ in the interior of the positive orthant. Then it is well

[^0]known that the RBM $\mathbf{Y}(\cdot)$, defined above, can be represented as
\[

$$
\begin{equation*}
\mathbf{Y}(t)=\mathbf{X}(t)+R \mathbf{L}(t) \tag{1}
\end{equation*}
$$

\]

with $\mathbf{Y}(t)=\left(Y_{1}(t), \ldots, Y_{d}(t)\right)^{T}$ and $\mathbf{L}(t)=\left(L_{1}(t), \ldots, L_{d}(t)\right)^{T}$ satisfying,

1) $Y_{i}(t) \geq 0$ for all $t \geq 0$,
2) $L_{i}(t)$ is non-decreasing in $t$, and $L_{i}(0)=0$
3) $\int_{0}^{t} Y_{i}(s) d L_{i}(s)=0$,
for each $t \geq 0$ and $i=1, \ldots, d$ (see, for example, [10, 6]). We call $\mathbf{X}(\cdot)$ the driving (or free) process, and $\mathbf{Y}(\cdot)$ the reflected process. The map $S$ that takes $\mathbf{X}(\cdot)$ to $\mathbf{Y}(\cdot)$ in (1) is referred to as the Skorokhod map. Item 3) above simply states that the process $L_{i}(t)$ increases only at those times $t$ where $Y_{i}(t)=0$. Because of this property, the process $L_{i}(\cdot)$ behaves like the local time of Brownian motion at the origin. Consequently, the term $R \mathbf{L}(t)$ appearing in (1) is not a standard "drift" term, and cannot be dealt with using change of measure techniques as in [4], [2] or [12].

All the generic exact simulation techniques for diffusions are based on the acceptance / rejection, after applying Girsanov's transformation. The difficulty in applying acceptance / rejection in the multidimensional RBM setting is that there is no natural proposal distribution that can be used to "dominate" the target process directly. In particular, multidimensional RBM is not absolutely continuous with respect to any natural process that is easily simulatable. Note that in one dimension one can simulate RBM directly by keeping track of the running maximum of the driving Brownian motion, and so these challenging issues arise only in dimensions greater than one. Simulation techniques for one dimensional reflected processes have been studied in [9].

Our contributions. This paper is dedicated to the proof of the following result. Let $\mathbf{Y}(\cdot)$ denote the multi-dimensional RBM in (1).

Theorem 1. Given a deterministic time $T \in(0,1)$, it is possible to simulate $\mathbf{Y}(T)$ without any bias.

An obstacle to naively using the traditional acceptance / rejection algorithm (see, for example, [1]) in the simulation of diffusions is that the probability density from which we want to sample is typically unknown. In our setting, while the probability density of $\mathbf{Y}(T)$ itself may be unknown, we propose to simulate enough information about the RBM $\mathbf{Y}(\cdot)$, and perform acceptance / rejection sampling for the probability density of $\mathbf{Y}(T)$ conditional on the simulated filtration. Ideally, the simulated information set should be a collection of random variables such that the probability density of $\mathbf{Y}(T)$ conditional on the simulated information, denoted here by $f$, is computable. If obtaining such a computable conditional density $f$ is feasible, then one can easily perform an acceptance / rejection step of form,

$$
\begin{equation*}
V<\frac{f(\mathbf{Z})}{\operatorname{Cg}(\mathbf{Z})}=: L(\mathbf{Z}) \tag{2}
\end{equation*}
$$

where $g$ is a suitable proposal density from which proposal samples $\mathbf{Z}$ are drawn, $V$ is an independently generated random variable distributed uniformly in $[0,1]$, and $C$ is a suitable scaling constant. While this line of thought is interesting, a key difficulty arises from the fact that there is no easily simulatable information structure such that the density of $\mathbf{Y}(T)$ conditional on the simulated information is exactly computable.

To overcome this difficulty, we introduce a novel sampling scheme that we call as Refine until Accept / Reject, which relaxes the requirement that the conditional density $f$ is known exactly. The
key observation behind this algorithm is that in order to accept the proposal $\mathbf{Z}$, we simply need to decide if inequality (2) holds; we do not need to know the right-hand side of (2) exactly. So, instead of having direct access to the probability density of $\mathbf{Y}$ conditional on $\mathcal{I}$, if we can simply obtain an approximation to the right-hand side of (2) that ensures inequality (2) holds, we can accept the proposed sample without incurring any sampling error. We present this idea clearly in a stylized setting in Section 2.1 along with an outline of its applicability to the simulation of RBM in Section 2.2. We use Section 3 to fully present our algorithm for exactly simulating multidimensional RBM. Our algorithm relies on the recently developed $\varepsilon$-strong simulation (also known as ToleranceEnforced simulation) techniques in [3] to first derive an approximation of the RBM, which then is used to make one of the following decisions: Accept, reject, (or) refine the approximation of the right-hand side in (2) until either the proposal can be conclusively accepted or rejected.

We wish to finish this introduction with a critical discussion of our main result. We do believe that the conditional acceptance / rejection strategy introduced here is of significant value as it addresses an important open problem (exact sampling of multidimensional RBM). Nevertheless, we must recognize that the algorithm, in its current form, is mostly of theoretical interest. Unfortunately, in Section 4, we identify that the expected running time of the algorithm is infinite. While we are investigating strategies to mitigate this problem, we feel that the nucleus of our sampling algorithm, namely refine until accept / reject, might propel further research in exact sampling of various stochastic processes in addition to the search for efficient sampling algorithms for simulating multidimensional RBM.

## 2. Overview of the sampling scheme

We first introduce some notational conventions. Throughout the paper we consider the driving (free) stochastic process $\mathbf{X}(\cdot)$ to be a standard Brownian motion in $d$-dimensions, which write as $\mathbf{X}(\cdot)=\mathbf{B}(\cdot)$. The reflected process $\mathbf{Y}(\cdot)$ in (1) is referred to as the Reflected Brownian motion (RBM). The extension of our development to the case in which $\mathbf{X}(\cdot)$ is a Brownian motion with constant drift and diffusion coefficients is straightforward. As mentioned in the Introduction, the map $S$ that takes $\mathbf{X}(\cdot)$ to $\mathbf{Y}(\cdot)$ in (1) is referred to as the Skorokhod reflection map.

While all the variables and stochastic processes taking values in $\mathbb{R}^{d}$ for $d>1$ are typeset in boldface, their 1-dimensional counterparts are not. For example, if $\mathbf{B}(\cdot)$ denotes the Brownian motion in multiple dimensions, then $B(\cdot)$ is to be understood as 1-dimensional Brownian motion.
2.1. Refine until accept / reject sampling scheme: An introduction. In this section, let us restrict our attention to the following sampling problem to which our RBM simulation problem is later shown to be reduced: Let $\Delta$ and $Y$ be two independent random variables, and $W=Y+\Delta$. For simplicity, let us assume that the probability density of $\Delta$, denoted by $f_{\Delta}(\cdot)$, is continuous on its support which is given by the interval $[-a, a]$ for some $a>0$. Consecutively, $\sup _{x \in[-a, a]} f_{\Delta}(x) \leq C$ for some $C \in(0, \infty)$. Let us assume that $Y$ is also supported on $[-a, a]$ with an arbitrary distribution from which we do not know how to sample exactly. Our objective in this section is to obtain samples from the distribution of $W=Y+\Delta$.

Given $Y=y$, the density of $W$ evaluated at $w$ is simply $f_{\Delta}(w-y)$. Moreover, the support of such density is contained in the interval $[-2 a, 2 a]$. So, in order to simulate $W$ conditional on $y$, we could propose $W$ uniformly distributed in $[-2 a, 2 a]$, and accept if

$$
\begin{equation*}
V<C^{-1} f_{\Delta}(W-y) \tag{3}
\end{equation*}
$$

where $V \sim \operatorname{Unif}(0,1)$ (uniform in $(0,1)$ ), and independent of $W$ and $Y=y$.

The key observation is that in order to accept $W$ we simply need to decide if inequality (3) holds; we do not actually need to know the value of $y$. So, instead of having direct access to $Y$, there are settings, as we demonstrate in our RBM simulation, where we might know $\left\{Y^{\varepsilon_{n}}\right\}_{n \geq 1}$, independent of $\Delta$, which converges to $Y$; say $\left\|Y^{\varepsilon_{n}}-Y\right\| \leq \varepsilon_{n}$ for some $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$. Then under modest continuity properties of $f_{\Delta}(\cdot)$, for instance say $\left|f_{\Delta}(x)-f_{\Delta}\left(x^{\prime}\right)\right| \leq K\left\|x-x^{\prime}\right\|$, we can accept $W$ if

$$
\begin{equation*}
V \leq C^{-1} f_{\Delta}\left(W-Y^{\varepsilon_{n}}\right)-K C^{-1} \varepsilon_{n} \tag{4}
\end{equation*}
$$

or reject $W$ if

$$
\begin{equation*}
V \geq C^{-1} f_{\Delta}\left(W-Y^{\varepsilon_{n}}\right)+K C^{-1} \varepsilon_{n} \tag{5}
\end{equation*}
$$

Since $\varepsilon_{n} \rightarrow 0$ and $V=C^{-1} f_{\Delta}(W-Y)$ has zero probability of occurring, one must be able to eventually decide whether to accept or reject. As the outlined sampling procedure does one of the following - accept / reject the proposed sample of $W$, or seek for a refinement of $Y^{\varepsilon_{n}}$ until the proposal can be accepted or rejected - we call the sampling procedure as refine until accept / reject.

It is useful to remember the following requirements which are necessary for the procedure underlying (3), (4) and (5) to yield exact samples of $W=Y+\Delta$ :

R1) The probability density of $\Delta$, denoted by $f_{\Delta}(\cdot)$, has bounded support, and is Lipschitz continuous; that is, there exists $K>0$ such that $\left|f_{\Delta}(x)-f_{\Delta}\left(x^{\prime}\right)\right| \leq K\left\|x-x^{\prime}\right\|$ for all $x, x^{\prime}$.
R2) Conditional on $Y=y$ and all the information simulated to obtain $Y^{\varepsilon_{n}}$, the probability density of $W$ evaluated at $w$ is simply $f_{\Delta}(w-y)$.
2.2. An outline of the application of refine until accept / reject for multi-dimensional

RBM. Revisiting our objective of exact sampling of RBM, our plan is to apply the sampling strategy in Section 2.1 by introducing a suitable conditioning. For this purpose, we use the following key facts about multidimensional RBM. First, the fact that if the driving process is Brownian motion then, for fixed $T, \mathbb{P}\left(Y_{i}(T)=0\right)=0$ for any $i \in\{1, \ldots, d\}$. In addition, since $\mathbf{Y}(\cdot)$ is continuous, there exists a $\delta>0$ and an interval $\left(T_{\text {left }}, T_{\text {right }}\right]$ which contains $T$, satisfying $Y_{i}(s)>\delta$ for all $i \in\{1, \ldots, d\}$, and therefore,

$$
\mathbf{Y}(s)=\mathbf{Y}\left(T_{l e f t}\right)+\mathbf{X}(s)-\mathbf{X}\left(T_{l e f t}\right)
$$

for all $s \in\left(T_{\text {left }}, T_{\text {right }}\right]$. In other words, the interval $\left(T_{\text {left }}, T_{\text {right }}\right)$ is such that the RBM $\mathbf{Y}(t)$ does not hit the reflecting boundary anywhere during $t \in\left(T_{\text {left }}, T_{\text {right }}\right]$ and consecutively, $\mathbf{L}(s)-$ $\mathbf{L}\left(T_{\text {left }}\right)=0$ for all $s \in\left(T_{\text {left }}, T_{\text {right }}\right)$.

So, our plan is to first simulate enough information about $\mathbf{X}(\cdot)$ (that is, the driving Brownian motion) so that conditional on such information we have the representation

$$
\begin{align*}
\mathbf{Y}(T) & =\mathbf{Y}\left(T_{l e f t}\right)+\left(\mathbf{X}(T)-\mathbf{X}\left(T_{l e f t}\right)\right) \\
& =: \mathbf{Y}\left(T_{l e f t}\right)+\boldsymbol{\Delta} \tag{6}
\end{align*}
$$

for a suitable $T_{l e f t}$ identified from the simulated information. Naturally, we identify $\mathbf{Y}(T)$ and $\mathbf{Y}\left(T_{\text {left }}\right)$, respectively, with the variables $W$ and $Y$ introduced in the abstract setting discussed previously in Section 2.1. Therefore, our objective is to simulate just enough information so that conditioned on the simulated information,

R1') the probability density of the Brownian increment $\boldsymbol{\Delta}:=\mathbf{X}(T)-\mathbf{X}\left(T_{l e f t}\right)$, denoted by $f_{\boldsymbol{\Delta}}(\cdot)$, has bounded support, and is lipschitz continuous, and
R2') the probability density of $\mathbf{Y}(T)$, evaluated at $w$, is simply $f_{\boldsymbol{\Delta}}\left(w-\mathbf{Y}\left(T_{\text {left }}\right)\right)$.

The requirements R1') and R2') mirror the earlier requirements R1) and R2) in the abstract setting in Section 2.2. Once these requirements are met, we can follow the logic in Section 2.1 to develop a refine until accept / reject sampler for obtaining samples from the distribution of $\mathbf{Y}(T)$. Thus, the proposed algorithm for exact sampling of $\mathbf{Y}(T)$ can be roughly divided into two steps:

1) a preconditioning step where we simulate enough information to arrive at the representation (6), and
2) exploiting the representation (6) arrived in the preconditioning step, we perform 'refine until accept / reject' to obtain samples of $\mathbf{Y}(T)$.
2.2.1. Overview of the preconditioning step. In order to sample enough information which will enable us to obtain the representation (6) along with satisfying the above two requirements, we use another important property of the Skorokhod map, $S$, namely, $S$ is Lipschitz continuous as a function of the driving process in the uniform norm over the time interval $[0,1]$. Consequently, to identify $T_{\text {left }}$ we use so-called $\varepsilon$-strong simulation techniques, also known as Tolerance-Enforced Simulation (TES), which allows us to simulate $\mathbf{X}^{\varepsilon}(\cdot)$ piecewise linear and guaranteed to be within $\varepsilon$-close in uniform norm to $\mathbf{X}(\cdot)$. This construction is, conceptually at least, not complicated. There are several methods that can be applied for the same: based on wavelets as in [5], localization using stopping times as in [7], or tracking jointly the maxima and minima on dyadic intervals as in [3]. We have chosen to use the latter construction, thereby ultimately obtaining ( $T_{\text {left }}, T_{\text {right }}$ ] as a dyadic interval (i.e. $T_{\text {left }}=i 2^{-N}$ and $T_{\text {right }}=j 2^{-N}$ for some $0 \leq i<j \leq 2^{N}$ and $N>0$ ). The reason for choosing the construction in [3] is because it allows us to recursively develop more refined approximations $\mathbf{X}^{\varepsilon^{\prime}}$ for desired $\varepsilon^{\prime}<\varepsilon$ while preserving the conditional independence of $\boldsymbol{\Delta}$ and $\mathbf{Y}\left(T_{\text {left }}\right)$ given all the information required to conclude that $\mathbf{L}\left(T_{\text {right }}\right)-\mathbf{L}\left(T_{\text {left }}\right)=0$. Refer Section 3.1.1 for an overview of the algorithm in [3] that allows us to obtain the desired piecewise linear approximation $\mathbf{X}^{\varepsilon}(\cdot)$ for the driving Brownian motion.

The Skorokhod problem is easy to solve for piecewise linear input $\mathbf{X}^{\varepsilon}$, because in such case the solution to Skorokhod problem, denoted by $\left(\mathbf{Y}^{\varepsilon}, \mathbf{L}^{\varepsilon}\right)$ is piecewise linear as well, and the gradients can be obtained by solving linear systems based on (1) (see Section 3.1.2 for an explicit algorithm). Since the piecewise linear approximation $\mathbf{Y}^{\varepsilon_{n}}(\cdot)$ to $\mathrm{RBM} \mathbf{Y}(\cdot)$ can be identified explicitly for a computable $\left\{\varepsilon_{n}\right\}_{n \geq 1}$ such that $\varepsilon_{n} \rightarrow 0$ as $n \rightarrow \infty$, the Lipschitz continuity of $\mathbf{Y}=S(\mathbf{X})$ as a function of $\mathbf{X}$, combined with the approximation $\mathbf{X}^{\varepsilon_{n}}$, and the fact that $\mathbf{Y}(T)$ must be strictly positive coordinate-wise, eventually can be used to identify $T_{\text {left }}$ used in the additive representation (6). See Section 3.1 for details.
2.2.2. Requirements for refine until accept / reject step. Once we arrive at representation (6), we can use the refine until accept / reject algorithm introduced in Section 2.1 to obtain samples of $\mathbf{Y}(T)$. In order to be able to do this, we need to guarantee that the requirements R1') and R2') mentioned earlier are met. Our construction of $\mathbf{X}^{\varepsilon}(\cdot)$, as indicated earlier, based on [3] will give rise to a conditional density for $\boldsymbol{\Delta}$, denoted by $f_{\boldsymbol{\Delta}}(\cdot)$, which is expressed as an infinite series. So, the Lipschitz continuity of $f_{\boldsymbol{\Delta}}(\cdot)$ used in (4) and (5) is obtained by means of some careful estimates. Consequently, as we shall see in Section 3.2, we will be able to implement the basic refine until accept / reject strategy underlying (3), (4), and (5).

In Section 3 below, we provide more specific details behind our sampling methodology and point to future relevant sections where details are fully fleshed out.

## 3. The exact sampling scheme

We first describe essential components of the pre-conditioning step (such as $\varepsilon$-strong approximation techniques) before presenting the refine until accept / reject sampler for $\mathbf{Y}(T)$.
3.1. The preconditioning step. As mentioned in Section 2.2, the objective of the preconditioning step is to simulate just enough information in order to obtain the representation (6) while satisfying requirements R1') and R2').
3.1.1. Generating $\varepsilon$-strong approximation for the driving Brownian motion. Here, we first provide a brief description of the $\varepsilon$-strong algorithm of [3] that simulates a piecewise linear approximation to 1-dimensional standard Brownian motion $B(\cdot)$. The algorithm iteratively generates a sequence of pairs of piecewise constant dominating processes, $\left\{B_{n}^{\uparrow}(t): t \in[0,1]\right\}$ and $\left\{B_{n}^{\downarrow}(t): t \in[0,1]\right\}$, that satisfy the following properties: For all $t \in[0,1]$,

$$
\begin{gathered}
B_{n}^{\downarrow}(t) \leq B_{n+1}^{\downarrow}(t) \leq B(u) \leq B_{n+1}^{\uparrow}(t) \leq B_{n}^{\uparrow}(t), \text { and } \\
\sup _{t \in[0,1]}\left|B_{n}^{\uparrow}(t)-B_{n}^{\downarrow}(t)\right| \searrow 0, \text { a.s. as } n \nearrow \infty .
\end{gathered}
$$

At every step $n \geq 1$, the algorithm generates information about the Brownian motion $B(\cdot)$ in dyadic intervals $\left\{\left((j-1) 2^{-n}, j 2^{-n}\right]: j=1, \ldots, 2^{n}\right\}$ conditional on the information available on dyadic intervals from the $(n-1)^{\text {th }}$ step. Let $m_{j, n}$ and $M_{j, n}$ denote the extrema of $B(\cdot)$ :

$$
m_{j, n}:=\inf \left\{B(t): t \in\left((j-1) 2^{-n}, j 2^{-n}\right]\right\} \text { and } M_{j, n}=\sup \left\{B(t): t \in\left((j-1) 2^{-n}, j 2^{-n}\right]\right\}
$$

During $n^{\text {th }}$ iteration, the $\varepsilon$-strong algorithm simulates the following random quantities for each dyadic interval (indexed by $j=1, \ldots, 2^{n}$ ):

1) an interval that contains the minimum: $L_{j, n}^{\downarrow}$ and $L_{j, n}^{\uparrow}$ such that $m_{j, n} \in\left[L_{j, n}^{\downarrow}, L_{j, n}^{\uparrow}\right]$ and $L_{j, n}^{\uparrow}-L_{j, n}^{\downarrow}<2^{-(n+1) / 2}$,
2) an interval that contains the maximum: $U_{j, n}^{\downarrow}$ and $U_{j, n}^{\uparrow}$ such that $M_{j, n} \in\left[U_{j, n}^{\downarrow}, U_{j, n}^{\uparrow}\right]$ and $U_{j, n}^{\uparrow}-U_{j, n}^{\downarrow}<2^{-(n+1) / 2}$, and
3) the end-points of Brownian motion: $B\left((j-1) 2^{-n}\right)$ and $B\left(j 2^{-n}\right)$.

Let $\mathcal{I}_{j, n}$ denote the collective information,

$$
\mathcal{I}_{j, n}:=\left\{L_{j, n}^{\downarrow}, L_{j, n}^{\uparrow}, U_{j, n}^{\downarrow}, U_{j, n}^{\uparrow}, B\left((j-1) 2^{-n}\right), B\left(j 2^{-n}\right)\right\},
$$

which is referred to as intersection layer in [3]. Let $\mathcal{I}$ denote the collection of all the intersection layers; at the end of $n^{\text {th }}$ iteration, the collection $\mathcal{I}$ is updated as below:

$$
\mathcal{I}:=\left\{\mathcal{I}_{j, n}: j=1, \ldots, 2^{n}\right\} .
$$

The $(n+1)^{\text {th }}$ step makes use of $\mathcal{I}$ generated in the $n^{\text {th }}$ step to arrive at $\left\{\mathcal{I}_{j, n+1}: j=1, \ldots, 2^{n+1}\right\}$. Specific details of how these random quantities are simulated can be found in [3]. From the intersection layers $\mathcal{I}_{j, n}$ generated by the algorithm at $n^{\text {th }}$ step, the dominating piecewise constant
processes can be formed as below:

$$
\begin{aligned}
& B_{n}^{\uparrow}(t)=\sum_{j=1}^{2^{n}} U_{j, n}^{\uparrow} \mathbf{1}\left(t \in\left((j-1) 2^{-n}, j 2^{-n}\right]\right), \text { and } \\
& B_{n}^{\downarrow}(t)=\sum_{j=1}^{2^{n}} L_{j, n}^{\downarrow} \mathbf{1}\left(t \in\left((j-1) 2^{-n}, j 2^{-n}\right]\right) .
\end{aligned}
$$

Further define the following piecewise linear process which shall serve as our approximation for $B(\cdot)$ :

$$
\begin{equation*}
B_{n}(t)=\sum_{j=1}^{2^{n}}\left[B\left((j-1) 2^{-n}\right)+2^{n}\left(B\left(j 2^{-n}\right)-B\left((j-1) 2^{-n}\right)\right)\left(t-(j-1) 2^{-n}\right)\right] \tag{7}
\end{equation*}
$$

which is just a linear interpolation of the points $\left\{B\left((j-1) 2^{-n}\right): j=1, \ldots, 2^{n}\right\}$ over the dyadic intervals in $[0,1]$. Note that all the random variables used in the construction of $B_{n}^{\uparrow}(\cdot), B_{n}^{\downarrow}(\cdot)$ and $B_{n}(\cdot)$ are available in $\mathcal{I}$, and can be simulated on a personal computer without any discretisation error. It is proved in [3] that the dominating processes $B_{n}^{\uparrow}(\cdot)$ and $B_{n}^{\downarrow}(\cdot)$ have the following convergence behavior:

$$
\begin{align*}
& \varepsilon_{n}:=\sup _{t \in[0,1]}\left|B_{n}^{\uparrow}(t)-B_{n}^{\downarrow}(t)\right|=\max _{1 \leq j \leq 2^{n}}\left|U_{j, n}^{\uparrow}-L_{j, n}^{\downarrow}\right| \searrow 0, \text { and }  \tag{8}\\
& \mathbb{E}\left[\int_{0}^{1}\left|B_{n}^{\uparrow}(t)-B_{n}^{\downarrow}(t)\right| d t\right]=O\left(2^{-n / 2}\right), \quad \text { as } n \rightarrow \infty .
\end{align*}
$$

To generate a piecewise linear approximation of the $d$-dimensional Brownian motion $\mathbf{B}(\cdot)=$ $\left(B_{1}(\cdot), \ldots, B_{d}(\cdot)\right)$, we generate approximating processes $B_{n, i}(\cdot), B_{n, i}^{\uparrow}(\cdot)$, and $B_{n, i}^{\downarrow}(\cdot)$ independently for each 1-dimensional Brownian motion $B_{i}(\cdot)$ as explained above, and use

$$
\mathbf{B}_{n}(t)=\left(B_{n, 1}(t), \ldots, B_{n, d}(t)\right), \quad t \in[0,1]
$$

as piecewise linear approximation for $\mathbf{B}(\cdot)$. Similar to the 1-dimensional case, the simulated information is stored in the intersection layers $\mathcal{I}=\left\{\mathcal{I}_{j, n}^{i}: j=1, \ldots, 2^{n}, i=1, \ldots, d\right\}$; here, $\mathcal{I}_{j, n}^{i}$ simply denotes the intersection layer simulated to generate the approximation to $i^{\text {th }}$ component $B_{i}(t)$ of the driving Brownian motion $\mathbf{B}(\cdot)$. As in the 1-dimensional case, we use $\varepsilon_{n}$ to denote the error in approximation at the $n^{\text {th }}$ step:

$$
\begin{equation*}
\varepsilon_{n}:=\sup \left\{\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right|: t \in[0,1], i=1, \ldots, d\right\} \tag{9}
\end{equation*}
$$

3.1.2. Generating $\varepsilon$-strong approximation of $R B M$.. Given a linear path $\left(\mathbf{x}(t): t \in\left[t_{0}, t_{1}\right)\right)$ specified by initial condition $\mathbf{x}\left(t_{0}\right) \in \mathbb{R}^{d}$ and $\dot{\mathbf{x}}(t)=\mathbf{m} \in \mathbb{R}^{d}$ for $t \in\left(t_{0}, t_{1}\right)$, we explain in this section how to identify the reflected path $\mathbf{y}(\cdot)=S(\mathbf{x}(\cdot))$ that solves the Skorokhod problem in (1) with $\left(\mathbf{x}(t): t \in\left[t_{0}, t_{1}\right)\right)$ as the path of the driving (free) process. Once we know how to solve (1) for a linear path in the interval $\left(t_{0}, t_{1}\right)$, it is straightforward to iteratively solve for any piecewise linear path of the driving (free) process.

As the slope of each component of the driving path $\mathbf{x}(\cdot)$ is fixed in the interval $\left[t_{0}, t_{1}\right)$, the slope of $\mathbf{y}$ at time $t \in\left(t_{0}, t_{1}\right)$, denoted by $\dot{\mathbf{y}}(t)$, is obtained by,

$$
\dot{\mathbf{y}}(t)=\mathbf{m}+R \mathbf{z}
$$

where, as per conditions 1 )-3) in Skorokhod problem (1), $\mathbf{y}(t)$ and $\mathbf{z} \in \mathbf{R}^{d}$ should satisfy,

$$
\mathbf{y}(t) \geq \mathbf{0}, \quad \mathbf{z} \geq \mathbf{0}, \quad \text { and } \mathbf{y}(t) \mathbf{z}=\mathbf{0}
$$

As a component $z_{i}$ of $\mathbf{z}=\left(z_{1}, \ldots, z_{d}\right)$ is nonzero only when the respective component $y_{i}(t)$ of $\mathbf{y}(t)$ is zero, it is useful to dynamically keep track of which of the components of $\mathbf{y}(t)=\left(y_{1}(t), \ldots, y_{d}(t)\right)$ are zero. We accomplish this algorithmically by letting $C:=\left\{i: y_{i}(t)=0\right\}$. In addition, we use $R_{C}$ to denote the submatrix of $R$ formed by letting $R_{C}=\left[R_{i, j}\right]_{i, j \in C}$; similarly, let $\mathbf{z}_{C}=\left[z_{i}\right]_{i \in C}, \dot{\mathbf{y}}_{C}(t)=$ $\left[\dot{y}_{i}(t)\right]_{i \in C}$ denote the vectors formed by entries restricted to indices from the set $C$. As $\mathbf{z}_{C}$ is the minimal non-negative vector that maintains $\mathbf{y}_{C}(t)=0$ (see [11]), it is obtained by solving a linear program as in Algorithm 1 when any of the components of $\mathbf{y}$ hit zero.

```
Algorithm 1 Algorithm to solve Skorokhod problem (1) for a linear path ( \(\left.\mathbf{x}(t): t \in\left[t_{0}, t_{1}\right)\right)\) with component wise constant slopes \(\mathbf{m}\) in the interval \(\left[t_{0}, t_{1}\right)\) for given initial condition \(\mathbf{y}\left(t_{0}\right)\)
```

```
procedure ApproxRBM \(\left(t_{0}, t_{1}, \mathbf{y}\left(t_{0}\right), \mathbf{m}\right)\)
    Initialize the time data structure \(\vec{t}=\left[t_{0}\right]\) and path space data structure \(\overrightarrow{\mathbf{y}}=\left[\mathbf{y}\left(t_{0}\right)\right]\)
    Initialize \(\mathbf{y}=\mathbf{y}\left(t_{0}\right), C=\left\{i: y_{i}=0\right\}, t^{\prime}=t_{0}, \mathbf{z}=0\).
    while \(t^{\prime}<t_{1}\) do
        Solve the linear program \(\min _{\mathbf{z}_{C}, \dot{\mathbf{y}}_{C}} \mathbf{1}^{T} \mathbf{z}\),
            subject to \(\mathbf{z}_{C} \geq \mathbf{0}, \dot{\mathbf{y}}_{C}=\mathbf{m}_{C}+R_{C} \mathbf{z}_{C}, \dot{\mathbf{y}}_{C} \geq \mathbf{0}\).
        Update \(\quad \dot{\mathbf{y}}=\mathbf{m}+R \mathbf{z}\),
                        \(t^{\prime}=\min \left\{t_{1}, \min _{i: \dot{y}_{i}<0}\left(t_{0}-\frac{y_{i}}{\dot{y}_{i}}\right)\right\}\),
                \(\mathbf{y}=\mathbf{y}+\dot{\mathbf{y}}\left(t^{\prime}-t_{0}\right)\) and
                \(C=\left\{i: y_{i}=0\right\}\).
```

Append the time and path space data structures with the latest entries as in
end while

$$
\vec{t} \leftarrow\left[\begin{array}{ll}
\vec{t} & t^{\prime}
\end{array}\right] \quad \text { and } \quad \overrightarrow{\mathbf{y}} \leftarrow\left[\begin{array}{ll}
\overrightarrow{\mathbf{y}} & \mathbf{y}
\end{array}\right] .
$$

Return $\vec{t}, \overrightarrow{\mathbf{y}}$.

Given component-wise constant slopes $\mathbf{m}$ of the driving process $\mathbf{x}(\cdot)$ in the interval $\left(t_{0}, t_{1}\right)$ and the initial condition $\mathbf{y}\left(t_{0}\right)=\mathbf{y}$, Algorithm 1 returns a vector $\vec{t}$ and a matrix $\vec{y}$ of dimension $d \times \operatorname{size}(\vec{t})$, where $\operatorname{size}(\vec{t})$ is the dimension of $\vec{t}$. The data structures $\vec{t}$ and $\overrightarrow{\mathbf{y}}$ returned by Algorithm 1 can be used to construct the piecewise linear path $\left(\mathbf{y}(t): t \in\left[t_{0}, t_{1}\right)\right)$ as follows: if the $i^{t h}$ entry of $\vec{t}$ is $t_{i}$ and the $i^{\text {th }}$ column of the matrix $\vec{y}$ is $\mathbf{y}_{i}$, then linear interpolation of the points $\left(t_{i}, \mathbf{y}_{i}\right)$ yields us the reflected path $\left(\mathbf{y}(t): t \in\left[t_{0}, t_{1}\right)\right)$.
3.1.3. Detecting the interval $\left(T_{\text {left }}, T_{\text {right }}\right]$.. Recall that the objective of the preconditioning step is to simulate just enough information about the RBM so that we arrive at the representation (6). As explained in Section 2.2, we propose to achieve this by detecting an interval $\left(T_{\text {left }}, T_{\text {right }}\right]$ containing $T$ such that the RBM stays in the interior of positive orthant without hitting the reflecting boundary anywhere in that interval. In order to accomplish this, we first make the following observations:
A) It is well-known that the Skorokhod map $S$ is lipschitz continuous (with respect to the uniform metric on the path space $\left.C\left([0,1] ; \mathbb{R}^{d}\right)\right)$ with Lipschitz constant $K_{1}:=(1-\alpha)^{-1}$; here $\alpha \in(0,1)$ denotes the spectral radius of the matrix $Q=I-R$. (see [10]). Therefore, if we solve (1) with $\mathbf{X}(\cdot)=\mathbf{B}_{n}(\cdot)$ as the driving (free) process, the corresponding reflected process $\mathbf{Y}_{n}(\cdot):=S\left(\mathbf{B}_{n}\right)(\cdot)$ satisfies,

$$
\begin{equation*}
\left\|\mathbf{Y}_{n}-\mathbf{Y}\right\|:=\sup _{t \in[0,1], i=1, \ldots, d}\left|Y_{n, i}(t)-Y_{i}(t)\right|<K_{1} \varepsilon_{n} \tag{10}
\end{equation*}
$$

where $\varepsilon_{n}:=\sup \left\{B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t): t \in[0,1], i=1, \ldots, d\right\}$, and $B_{n, i}^{\uparrow}(\cdot), B_{n, i}^{\downarrow}(\cdot)$ are, respectively, the simulated upper and lower bounding processes of the $i^{\text {th }}$ component of the driving Brownian motion $\mathbf{B}(\cdot)$. Since

$$
\left\|\mathbf{B}_{n}(t)-\mathbf{B}(t)\right\|=\sup \left\{\left|B_{n, i}(t)-B_{i}(t)\right|: t \in[0,1], i=1, \ldots, d\right\} \leq \varepsilon_{n}
$$

(10) follows as a simple consequence of Lipschitz continuity of Skorokhod map.
B) Let us say $a:=\min _{i=1, \ldots, d} Y_{i}(T)>0$; that is, the $\mathrm{RBM} \mathbf{Y}(\cdot)$ lies in the interior of positive orthant at time $T$. Then, due to continuity of Brownian motion paths, there exists a random interval $\left(T-t^{\prime}, T+t^{\prime}\right)$ where $\left|B_{i}(t)-B_{i}(T)\right|<a$ for all $t \in\left(T-t^{\prime}, T+t^{\prime}\right)$, and consequently,

$$
\begin{aligned}
& \mathbf{Y}(t)-\mathbf{Y}(T)=\mathbf{B}(t)-\mathbf{B}(T) \text { and } \\
& \mathbf{Y}(t)=\mathbf{Y}(T)+(\mathbf{B}(t)-\mathbf{B}(T))>a \mathbf{1}-a \mathbf{1}=\mathbf{0}
\end{aligned}
$$

for all $t \in\left(T-t^{\prime}, T+t^{\prime}\right)$. In other words, the $\mathrm{RBM} \mathbf{Y}(t)$ lies in the interior of positive orthant for every $t \in\left(T-t^{\prime}, T+t^{\prime}\right)$ where $\left|B_{i}(t)-B_{i}(T)\right|<a$. In order to make use of this observation, recall that our constructed piecewise constant upper and lower bounding processes $B_{n, i}^{\uparrow}$ and $B_{n, i}^{\downarrow}$ satisfy $B_{n, i}^{\uparrow}(\cdot)-B_{n, i}^{\downarrow}(\cdot)<\varepsilon_{n}$ for each component $i$, and consequently,

$$
\left|B_{i}(t)-B_{i}(s)\right|<\varepsilon_{n}, \quad t, s \in\left[(j-1) 2^{-n}, j 2^{-n},\right]
$$

for all $i=1, \ldots, d$, and $j=1, \ldots, 2^{n}$. As a result, if $\min _{i=1, \ldots, d} Y_{i}(T)>\varepsilon_{n}$, then as $\mid B_{i}(t)-$ $B_{i}(T) \mid<\varepsilon_{n}$ for every $t$ in the dyadic interval containing $T$, the RBM $\mathbf{Y}(\cdot)$ does not hit the reflecting boundary anywhere in the specific dyadic interval $\left[(j-1) 2^{-n}, j 2^{-n}\right]$ containing $T$.

If we know that $\varepsilon_{n}$ is small enough such that every component of the RBM satisfies $Y_{i}(T) \geq \varepsilon_{n}$, then from Observation B) noted above, the $\mathrm{RBM} \mathbf{Y}(t)$ stays in the interior of positive orthant for every $t$ in the dyadic interval $\left((j-1) 2^{-n}, j 2^{-n}\right]$ containing $T$; consecutively, we can declare the corresponding interval $\left((j-1) 2^{-n}, j 2^{-n}\right]$ as $\left(T_{\text {left }}, T_{\text {right }}\right]$. However, since we do not know $\mathbf{Y}(T)$, the immediate objective is to figure out how to guarantee that $Y_{i}(T)$ is indeed larger than $\varepsilon_{n}$, for every $i=1, \ldots, d$. From the Lipschitz continuity in observation A), if $\varepsilon_{n}$ is small enough so that $Y_{n, i}(T)=S\left(\mathbf{B}_{n}\right)(T)>\left(K_{1}+1\right) \varepsilon_{n}$ for some $n$, then $Y_{i}(T)>\varepsilon_{n}$. Since $\mathbf{Y}(t)$ lies in the interior of the positive orthant almost everywhere, we will indeed have that $Y_{n, i}(T)>\left(K_{1}+1\right) \varepsilon_{n}$ for suitably small approximation error $\varepsilon_{n}$. Now define,

$$
N:=\inf \left\{n \geq 1: Y_{n, i}(T)>\left(K_{1}+1\right) \varepsilon_{n}, i=1, \ldots, d\right\}, \text { and } \delta:=\varepsilon_{N}
$$

Recall that $\varepsilon_{n}:=\sup \left\{\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right|: t \in[0,1], i=1, \ldots, d\right\}$. The preconditioning procedure for detecting the interval $\left(T_{\text {left }}, T_{\text {right }}\right]$ that simulates approximations to Brownian motion and RBM until the stopping time $N$ is summarized in Algorithm 2.

With this construction, since the RBM $\mathbf{Y}(\cdot)$ does not hit the reflecting boundary anywhere in the interval ( $\left.T_{\text {left }}, T_{\text {right }}\right]$, the dynamics of $\mathbf{B}(\cdot)$ and $\mathbf{Y}(\cdot)$ match in $\left(T_{\text {left }}, T_{\text {right }}\right]$; in particular,

$$
\begin{equation*}
\mathbf{Y}(t)-\mathbf{Y}\left(T_{l e f t}\right)=\mathbf{B}(t)-\mathbf{B}\left(T_{l e f t}\right), \text { for all } t \in\left(T_{l e f t}, T_{\text {right }}\right] \tag{11}
\end{equation*}
$$

thus resulting in the desired additive representation $\mathbf{Y}(T)=\mathbf{Y}\left(T_{l e f t}\right)+\boldsymbol{\Delta}$, in (6), where the increment $\boldsymbol{\Delta}:=\mathbf{B}(T)-\mathbf{B}\left(T_{l e f t}\right)$ is simply the Brownian increment.
3.2. The conditional probability density of $\Delta$ for the refine until accept / reject sampler. The requirements R1') and R2'), listed in Section 2.2 for the implementation of the refine until accept / reject exact sampler, necessitate us to know the law of $\boldsymbol{\Delta}$ conditional on all the simulated collection of random variables $\mathcal{I}$. For ease of exposition, we consider the 1-dimensional case; the

```
Algorithm 2 Preconditioning step to arrive at the representation (6).
(achieved mainly via \(\varepsilon\)-strong approximation of the underlying stochastic processes)
```

    procedure Preconditioning \((T)\)
        Initialize \(\mathcal{I}=\emptyset, n=0, \delta=1, \mathbf{Y}_{0}(t)=0, t \in[0,1]\)
        while \(n=0 \quad\) OR \(\quad Y_{n, i}(T)<\left(K_{1}+1\right) \delta\) for some \(i \in\{1, \ldots, d\}\) do
            Increment \(n \longleftarrow n+1\)
            Simulate the intersection layers \(\mathcal{I}_{j, n}^{i}\) for \(j=1, \ldots, 2^{n}, i=1, \ldots, d\) and the piecewise constant upper
    and lower bounding processes \(B_{n, i}^{\uparrow}(\cdot), B_{n, i}^{\downarrow}(\cdot)\), conditional on \(\mathcal{I}\)
            Form \(\mathbf{B}_{n}(\cdot)=\left(B_{n, 1}, \ldots, B_{n, d}\right)\) as in (7), which serves as a piecewise linear approximation to \(\mathbf{B}(\cdot)\)
            Update \(\quad \delta=\sup \left\{\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right|: t \in[0,1], i=1, \ldots, d\right\}\)
                    \(\mathcal{I}=\left\{\mathcal{I}_{j, n}: j=1, \ldots, 2^{n}\right\}\)
            Letting \(\mathbf{X}(\cdot)=\mathbf{B}_{n}(\cdot)\) in (1), solve for the reflected process \(\mathbf{Y}(\cdot)\) using Algorithm 1; call the solution
    \(\mathbf{Y}_{n}(\cdot)\)
    end while
    Find \(J=\left\{1 \leq j \leq 2^{n}:(j-1) 2^{-n}<T \leq j 2^{-n}\right\}\)
    Set \(N=n, T_{\text {left }}=(J-1) 2^{-n}\) and \(T_{\text {right }}=J 2^{-n}\)
    Return \(\mathcal{I}, N, T_{\text {left }}, T_{\text {right }}, \delta\) and \(\mathbf{Y}^{\delta}(\cdot):=\mathbf{Y}_{n}(\cdot)\)
    conditional probability density of the Brownian increment $\boldsymbol{\Delta}$ in $d$-dimensions, denoted by $f_{\boldsymbol{\Delta}}(\cdot)$, is given simply by the product form of 1-dimensional densities.

At any stage of algorithm, all the information simulated about the driving Brownian motion are available in the intersection layers $\mathcal{I}=\left\{\mathcal{I}_{j, n}: j=1, \ldots, 2^{n}\right\}$. From Algorithm 2, recall that $J$ is the index corresponding to the dyadic interval $\left(T_{\text {left }}, T_{\text {right }}\right]$ that contains $T$; that is $(J-1) 2^{-N}=$ $T_{\text {left }}<T \leq T_{\text {right }}=J 2^{-N}$. For ease of notation, let

$$
\begin{gathered}
L^{\downarrow}:=L_{J, N}^{\downarrow}-B\left(T_{l e f t}\right), L^{\uparrow}:=L_{J, N}^{\uparrow}-B\left(T_{l e f t}\right) \\
U^{\downarrow}:=U_{J, N}^{\downarrow}-B\left(T_{l e f t}\right), U^{\uparrow}:=U_{J, N}^{\uparrow}-B\left(T_{l e f t}\right) \\
l:=T_{\text {right }}-T_{l e f t}, s:=T-T_{l e f t} \text { and } v:=B\left(T_{\text {right }}\right)-B\left(T_{l e f t}\right) .
\end{gathered}
$$

Further let $W(\cdot)$ denote an independent standard Brownian motion on $C[0,1]$ under measure $\mathbb{P}(\cdot)$. Then due to Markov property of $B(\cdot)$, the increment $\Delta$ conditional on $\mathcal{I}$ has the following density:

$$
\begin{equation*}
f_{\Delta}(x) d x=\mathbb{P}\left\{W(s) \in d x \mid W(l)=v, \inf _{0 \leq t \leq l} W(t) \in\left(L^{\downarrow}, L^{\uparrow}\right), \sup _{0 \leq t \leq l} W(t) \in\left(U^{\downarrow}, U^{\uparrow}\right)\right\} \tag{12}
\end{equation*}
$$

Note that the support of $f_{\Delta}(\cdot)$ is $\left(L^{\downarrow}, U^{\uparrow}\right)$. A closed form expression for $f_{\Delta}(\cdot)$ follows from Proposition 5.1 of [3], and is given here:

$$
f_{\Delta}(x) \propto \rho(x) \times \pi(x)
$$

where for any fixed $L^{\downarrow}, L^{\uparrow}, U^{\downarrow}, U^{\uparrow}, v, s$ and $l$,

$$
\begin{align*}
\pi(x) & :=\exp \left(-\frac{1}{2}\left(x-\frac{s}{l} v\right)^{2} /\left(\frac{s(l-s)}{l}\right)\right), \text { and }  \tag{13}\\
\rho(x) & :=\mathbb{P}\left\{\inf _{0 \leq t \leq l} W(t) \in\left(L^{\downarrow}, L^{\uparrow}\right), \sup _{0 \leq t \leq l} W(t) \in\left(U^{\downarrow}, U^{\uparrow}\right) \mid W(s)=x, W(l)=v\right\} \\
& =\gamma_{1}(x) \gamma_{2}(x)-\gamma_{3}(x) \gamma_{4}(x)-\gamma_{5}(x) \gamma_{6}(x)+\gamma_{7}(x) \gamma_{8}(x) \tag{14}
\end{align*}
$$

To define $\gamma_{1}, \ldots, \gamma_{8}$, first consider the probability that the Brownian bridge from $a$ to $b$ in the time interval $[0, r]$ stays within $(L, U)$ :

$$
\begin{align*}
\gamma(L, U ; r, a, b) & :=\mathbb{P}\left\{L<\inf _{0 \leq t \leq l} W(t) \leq \sup _{0 \leq t \leq l} W(t)<U \mid W(0)=a, W(r)=b\right\} \\
& =\left(1-\sum_{j=1}^{\infty}\left(\sigma_{j}-\tau_{j}\right)\right) \mathbf{1}(a, b \in(L, U)) \tag{15}
\end{align*}
$$

where, $\sigma_{j}:=\exp \left(-\frac{2}{r}((U-L) j+L-a)((U-L) j+L-b)\right)$

$$
\begin{gathered}
+\exp \left(-\frac{2}{r}((U-L) j-U+a)((U-L) j-U+b)\right), \text { and } \\
\tau_{j}:=\exp \left(-\frac{2(U-L) j}{r}((U-L) j+a-b)\right)+\exp \left(-\frac{2(U-L) j}{r}((U-L) j+b-a)\right)
\end{gathered}
$$

The expression (15) for $\gamma(L, U ; l, a, b)$ is originally from [13]. Now we are ready to define $\gamma_{1}, \ldots, \gamma_{8}$ mentioned in (14):

$$
\begin{array}{lll}
\gamma_{1}(x)=\gamma\left(L^{\downarrow}, U^{\uparrow} ; s, 0, x\right), & \gamma_{2}(x)=\gamma\left(L^{\downarrow}, U^{\uparrow} ; l-s, x, v\right), & \gamma_{3}(x)=\gamma\left(L^{\uparrow}, U^{\uparrow} ; s, 0, x\right), \\
\gamma_{4}(x)=\gamma\left(L^{\uparrow}, U^{\uparrow} ; l-s, x, v\right), & \gamma_{5}(x)=\gamma\left(L^{\downarrow}, U^{\downarrow} ; s, 0, x\right), & \gamma_{6}(x)=\gamma\left(L^{\downarrow}, U^{\downarrow} ; l-s, x, v\right), \\
\gamma_{7}(x)=\gamma\left(L^{\uparrow}, U^{\downarrow} ; s, 0, x\right), & \gamma_{8}(x)=\gamma\left(L^{\uparrow}, U^{\downarrow} ; l-s, x, v\right) . &
\end{array}
$$

To perform acceptance / rejection type-sampling, we need that the conditional density $f_{\Delta}$ (of the Brownian increment) is Lipschitz continuous (see Requirement R1') in Section 2.2. Lemma 1 is a step towards establishing this fact.

Lemma 1. There exists positive constants $c_{\pi}, K_{\pi}, c_{\rho}$ and $K_{\rho}$ such that for any fixed $L^{\downarrow}, L^{\uparrow}, U^{\downarrow}, U^{\uparrow}$, $v, s$ and $l$,

$$
\begin{array}{ll}
\pi(x)<c_{\pi}, & |\pi(x)-\pi(y)|<K_{\pi}|x-y| \text { and } \\
\rho(x)<c_{\rho}, & |\rho(x)-\rho(y)|<K_{\rho}|x-y|
\end{array}
$$

for all $x, y \in\left(L^{\downarrow}, U^{\uparrow}\right)$.
Explicit closed-form expressions for the constants $K_{\pi}, K_{\rho}, c_{\pi}$ and $c_{\rho}$ are presented in the Appendix.

Following the representation that $Y(T)=Y\left(T_{l e f t}\right)+\Delta$, the conditional density of $Y(T)$ given $Y\left(T_{\text {left }}\right)$ is given by $f_{\Delta}\left(w-Y\left(T_{\text {left }}\right)\right)$, which is supported on $\left(Y\left(T_{\text {left }}\right)+L^{\downarrow}, Y\left(T_{\text {left }}\right)+U^{\uparrow}\right) \subset$ $\left(Y\left(T_{l e f t}\right)-\delta, Y\left(T_{l e f t}\right)+\delta\right)$. As the unknown $Y\left(T_{l e f t}\right)$ differs from $Y^{\delta}\left(T_{l e f t}\right)$ only by $\pm K_{1} \delta$, the support of $Y(T)$ conditional on the simulated information is, in turn, a subset of

$$
\left(Y^{\delta}\left(T_{l e f t}\right)-\left(K_{1}+1\right) \delta, Y^{\delta}\left(T_{l e f t}\right)+\left(K_{1}+1\right) \delta\right)
$$

Here, recall that $K_{1}:=(1-\alpha)^{-1}$ is the Lipschitz constant of the Skorokhod reflection map $S$. Consecutively, if we propose a sample $Z$ from the uniform distribution in the interval $\left(Y^{\delta}\left(T_{l e f t}\right)-\right.$ $\left.\left(K_{1}+1\right) \delta, Y^{\delta}\left(T_{l e f t}\right)+\left(K_{1}+1\right) \delta\right)$, then the likelihood ratio (or the Radon-Nikodym derivative) between the true conditional density and proposal density is proportional to $f_{\Delta}\left(w-Y\left(T_{l e f t}\right)\right)$;
consecutively, a traditional accept / reject algorithm would accept the proposed sample $Z$ if

$$
V<L\left(Z ; Y\left(T_{\text {left }}\right)\right):=\frac{\pi\left(Z-Y\left(T_{\text {left }}\right)\right) \rho\left(Z-Y\left(T_{\text {left }}\right)\right)}{c_{\pi} c_{\rho}},
$$

for an independent $V \sim \operatorname{Unif}(0,1)$; hereafter, we use $L(z ; y)$ to denote

$$
L(z ; y):=\frac{\pi(z-y) \rho(z-y)}{c_{\rho} c_{\phi}} .
$$

However, as we do not know $Y\left(T_{\text {left }}\right)$ exactly, if $L(Z ; \cdot)$ is Lipschitz continuous, the fact that $Y\left(T_{\text {left }}\right)$ differs from $Y^{\delta}\left(T_{\text {left }}\right)$ by $\pm K_{1} \delta$, can be used to implement a refine until accept / reject sampler discussed in Section 2.1.

The Lipschitz continuity of $L(z ; y)$ as a function of $y$, follows as a simple consequence of Lemma 1 established earlier. If $f$ and $g$ are Lipschitz continuous with Lipschitz constants $K_{f}$ and $K_{g}$, and respective absolute bounds $c_{f}$ and $c_{g}$, then $f g$ is a Lipschitz continuous function with Lipschitz constant at most $c_{f} K_{g}+c_{g} K_{f}$. As a result, the product $\pi(\cdot) \times \rho(\cdot)$ is Lipschitz continuous as well with Lipschitz constant $c_{\pi} K_{\rho}+c_{\rho} K_{\pi}$. Consequently, the function $L(z, y)$ is Lipschitz continous, as a function of $y$, with Lipschitz constant

$$
K_{2}:=\frac{K_{\pi}}{c_{\pi}}+\frac{K_{\rho}}{c_{\rho}} .
$$

Given this Lipschitz continuity of $L$, one can unambiguously accept the proposal $Z$ if,

$$
V<L\left(Z ; Y^{\delta}\left(T_{l e f t}\right)\right)-K_{1} K_{2} \delta
$$

or reject the proposal $Z$ conclusively if

$$
V>L\left(Z ; Y^{\delta}\left(T_{\text {left }}\right)\right)+K_{1} K_{2} \delta .
$$

However, if $V$ is within $L\left(Z ; Y^{\delta}\left(T_{\text {left }}\right)\right) \pm K_{1} K_{2} \delta$, then we obtain a more refined approximation $Y^{\varepsilon_{n}}\left(T_{l e f t}\right)$, for a suitable $\varepsilon_{n}<\delta$, that is good enough to decide whether to accept / reject. In particular, if $\varepsilon_{n}$ is smaller than $\left(K_{1} K_{2}\right)^{-1}\left|V-L\left(Z ; Y^{\varepsilon_{n}}\left(T_{\text {left }}\right)\right)\right|$, then

$$
\begin{equation*}
V<L\left(Z ; Y\left(T_{l e f t}\right)\right) \text { if and only if } V<L\left(Z ; Y^{\varepsilon_{n}}\left(T_{\text {left }}\right)\right)-K_{1} K_{2} \varepsilon_{n} . \tag{16}
\end{equation*}
$$

This equivalent comparison is at the heart of the refine until accept / reject sampler in Algorithm 3 below. It takes the intersection layers $\mathcal{I}$ returned by Algorithm 2 as input, and generates further refined approximations $\left(Y^{\varepsilon_{n}}(t): t \in\left[0, T_{l e f t}\right]\right)$ of the RBM, if necessary, in order to perform the equivalent comparisons in (16).

For $d$-dimensional processes, the probability density of the increment $\boldsymbol{\Delta}$ and proposal density $g(\cdot)$ are both given by product of 1-dimensional densities. This results in a likelihood ratio which is also of product form, leading to a straightforward generalization of the refine until accept / reject procedure given in Algorithm 3.

## 4. A note on computational complexity

Our objective in this section is to understand the computational effort required to execute the refine until accept / reject exact sampler described in Section 3. For ease of exposition, we do not keep track of multiplying constants, and instead adopt the following standard notation to describe the asymptotic behaviour of functions: For given functions $f: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$and $g: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$, we say $f(x)=O(g(x))$ if there exists $c_{1}>0$ and $x_{1}$ large enough such that $f(x) \leq c_{1} g(x)$ for all

```
Algorithm 3 To accept/reject the proposal \(Z \sim \operatorname{Unif}\left(Y^{\delta}\left(T_{l e f t}\right)-\left(K_{1}+1\right) \delta, Y^{\delta}\left(T_{l e f t}\right)+\left(K_{1}+1\right) \delta\right)\).
If \(V<L\left(Z ; Y\left(T_{\text {left }}\right)\right)\), the algorithm returns \(Z\); otherwise it rejects Z and returns nothing. Recall
that \(L(z ; y):=c_{\pi}^{-1} c_{\rho}^{-1} \pi(z-y) \rho(z-y), K_{1}:=1 /(1-\alpha)\) and \(K_{2}:=K_{\pi} / c_{\pi}+K_{\rho} / c_{\rho}\).
procedure RAR-SAMPLER \(\left(\mathcal{I}, T_{l e f t}, N, \delta, Y^{\delta}\left(T_{l e f t}\right), Z\right)\)
    Initialize \(n=N, \varepsilon_{n}=\delta\)
    Draw \(V\) uniformly from \([0,1]\)
    while \(\varepsilon_{n}>\left(K_{1} K_{2}\right)^{-1}\left|V-L\left(Z ; Y^{\varepsilon_{n}}\left(T_{l e f t}\right)\right)\right|\) do
        Increment \(n \longleftarrow n+1\)
        Simulate the intersection layers \(\mathcal{I}_{j, n}\) for \(j=1, \ldots, 2^{n}\) conditional on \(\mathcal{I}\)
        Form \(B_{n}(\cdot)\) as in (7), which serves as piecewise linear approximation to \(B(\cdot)\)
        Set \(\varepsilon_{n}=\max \left\{U_{j, n}^{\uparrow}-L_{j, n}^{\downarrow}: j=1, \ldots, 2^{n}\right\}\) and \(\mathcal{I}=\left\{\mathcal{I}_{j, n}: j=1, \ldots, 2^{n}\right\}\)
        Letting \(X(t)=B_{n}(t)\) for \(t \in\left[0, T_{l e f t}\right]\) in (1), solve for the reflected process \(Y(\cdot)\); call the solution
    \(Y_{n}(\cdot)\); set the required refined approximation \(Y^{\varepsilon_{n}}\left(T_{\text {left }}\right)=Y_{n}\left(T_{\text {left }}\right)\).
    end while
    if \(V<L\left(Z ; Y^{\varepsilon_{n}}\left(T_{\text {left }}\right)\right)-K_{1} K_{2} \varepsilon_{n}\) then
        Return \(Z\)
    else
        Return \(\emptyset\)
    end if
```

$x>x_{1}$; further, we say $f(x)=\Theta(g(x))$ if there also exists $c_{2}>0$ and $x_{2}$ large enough such that $c_{2} g(x) \leq f(x) \leq c_{1} g(x)$ for all $x>x_{2}$.

Recall the definition of error in approximation $\varepsilon_{n}$ in (9), and to achieve this accuracy we needed to simulate relevant information (such as maxima, minima and endpoints for $d$ independent 1dimensional Brownian motions $\left.\mathbf{B}_{i}(\cdot)\right)$ in $2^{n}$ dyadic intervals as described in Section 3.1.1. As this entails constant amount of expected computation for each dimension in each of the $2^{n}$ dyadic intervals, the computational cost at the end of $n$ steps of the iterative procedure described in Section 3.1.1 is $\Theta\left(d 2^{n}\right)$.

Next, observe that the preconditioning step requires us to iterate until the stopping time $N=$ $\inf \left\{n \geq 1: Y_{n, i}(t)>\left(K_{1}+1\right) \varepsilon_{n}, i=1, \ldots, d\right\}$. As the computational cost of solving the Skorokhod problem (as in Algorithm 1) with a piecewise linear input $\mathbf{B}_{k}(\cdot)$ is $\Theta\left(d^{3} 2^{k}\right)$ uniformly in $d$, the total cost of executing the entire preconditioning step is $\Theta\left(d^{3} \sum_{k=1}^{N} 2^{k}\right)=\Theta\left(d^{3} 2^{N}\right)$. Here, we have used that $O\left(d^{3}\right)$ computations are needed to solve the linear program in Algorithm 1 when the set $C$ contains $O(d)$ elements. Following the same line of reasoning, if we let

$$
N^{\prime}=\inf \left\{n \geq N: \varepsilon_{n}<\left(K_{1} K_{2}\right)^{-1}\left|V-L\left(\mathbf{Z} ; \mathbf{Y}^{\varepsilon_{n}}\left(T_{l e f t}\right)\right)\right|\right\}
$$

as required in the refine until accept / reject step in Algorithm 3, the corresponding computational cost is $\Theta\left(d^{3} 2^{N^{\prime}}\right)$. As $N^{\prime} \geq N$, the total computational cost of the sampling procedure is $\Theta\left(d^{3} 2^{N}+\right.$ $\left.d^{3} 2^{N^{\prime}}\right)=\Theta\left(d^{3} 2^{N^{\prime}}\right)$. In other words, there exists positive constants $c_{1}$ and $c_{2}$ such that

$$
c_{1} d^{3} 2^{N^{\prime}} \leq \text { computational cost } \leq c_{2} d^{3} 2^{N^{\prime}} .
$$

Therefore, the expected computational cost of the entire sampling procedure is $\Theta\left(d^{3} E\left[2^{N^{\prime}}\right]\right)$.
Next, to compute $E\left[2^{N^{\prime}}\right]$, we first use the definition of $\varepsilon_{n}$ in (9) to observe that

$$
P\left(N^{\prime}>n\right)=P\left(\varepsilon_{n}>\frac{D}{K_{1} K_{2}}\right)=P\left(\max _{i=1, \ldots, d} \sup _{t \in[0,1]}\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right|>\frac{D}{K_{1} K_{2}}\right),
$$

where we have let $D=\left|V-L\left(\mathbf{Z} ; \mathbf{Y}^{\varepsilon_{n}}\left(T_{l e f t}\right)\right)\right|$ for notational convenience. For each fixed $i$, it follows from the construction of 1-dimensional piecewise constant bounding processes $B_{n, i}^{\uparrow}(\cdot)$ and $B_{n, i}^{\downarrow}$ in Section 3.1.1 that

$$
\begin{aligned}
\sup _{t \in[0,1]}\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right| & =\max _{j=1, \ldots, 2^{n}}\left|U_{j, n}^{\uparrow}-L_{j, n}^{\downarrow}\right| \\
& \leq \max _{j=1, \ldots, 2^{n}}\left\{\left|U_{j, n}^{\uparrow}-M_{j, n}\right|+\left|M_{j, n}-m_{j, n}\right|+\left|m_{j, n}-L_{j, n}^{\downarrow}\right|\right\} \\
& \leq \max _{j=1, \ldots, 2^{n}}\left\{2^{-(n+1) / 2}+\left|M_{j, n}-m_{j, n}\right|+2^{-(n+1) / 2}\right\} \\
& \stackrel{D}{=} 2^{-n / 2}+\max _{j=1, \ldots, 2^{n}}\left(\sup _{t \in\left[0,2^{n}\right]} W_{j}(t)-\inf _{t \in\left[0,2^{n}\right]} W_{j}(t)\right)
\end{aligned}
$$

where $\left(W_{j}(t): j=1, \ldots, 2^{n}\right)$ are $2^{n}$ independent copies of standard Brownian motion. Here, the notation $\stackrel{D}{=}$ is used to denote equality in distribution. If we let

$$
\bar{Z}_{j}:=1+\sup _{t \in[0,1]} W_{j}(t)-\inf _{t \in[0,1]} W_{j}(t)
$$

then due to self-similarity of Brownian motion,

$$
\begin{aligned}
\sup _{t \in[0,1]}\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right| & \stackrel{D}{=} 2^{-n / 2}+2^{-n / 2}\left(\sup _{t \in[0,1]} W_{j}(t)-\inf _{t \in[0,1]} W_{j}(t)\right) \\
& =2^{-n / 2} \max _{j=1, \ldots, 2^{n}} \bar{Z}_{j}
\end{aligned}
$$

Since the approximations $B_{n, i}$ are independently obtained for each $i=1, \ldots, d$, we have

$$
\sup _{t \in[0,1]}\left|B_{n, i}^{\uparrow}(t)-B_{n, i}^{\downarrow}(t)\right|=\max _{k=1, \ldots, d 2^{n}} \bar{Z}_{k}
$$

where $\left(\bar{Z}_{k}: k=1, \ldots, d 2^{n}\right)$ are $d 2^{n}$ independent copies of $\bar{Z}:=1+\sup _{t \in[0,1]} W(t)-\inf _{t \in[0,1]} W(t)$. For notational convenience, let us denote $M_{n}=\max _{j=1, \ldots, d 2^{n}} \bar{Z}_{j}$. Then

$$
\begin{aligned}
P\left(N^{\prime}>n\right) & =P\left(2^{-n / 2} M_{n}>\frac{D}{K_{1} K_{2}}\right)=P\left(D<K_{1} K_{2} \frac{M_{n}}{2^{n / 2}}\right) \\
& =P\left(L\left(\mathbf{Z} ; \mathbf{Y}^{\varepsilon_{n}}\left(T_{l e f t}\right)\right)-K_{1} K_{2} \frac{M_{n}}{2^{n / 2}}<V<L\left(\mathbf{Z} ; \mathbf{Y}^{\varepsilon_{n}}\left(T_{l e f t}\right)\right)+K_{1} K_{2} \frac{M_{n}}{2^{n / 2}}\right)
\end{aligned}
$$

because of our earlier definition that $D=\left|V-L\left(\mathbf{Z} ; \mathbf{Y}^{\varepsilon_{n}}\left(T_{l e f t}\right)\right)\right|$. As $V \sim \operatorname{Unif}[0,1]$, it is immediate that

$$
P\left(N^{\prime}>n\right)=\frac{2 K_{1} K_{2}}{2^{n / 2}} E\left[M_{n}\right]=\Theta\left(\sqrt{\frac{n}{2^{n}}}\right)
$$

where the second equality follows from the observation that $E\left[M_{n}\right]=\Theta(\sqrt{n})$, which is proved in Lemma 3 in appendix. Therefore, $P\left(2^{N^{\prime}}>x\right)=\Theta\left(x^{-0.5} \sqrt{\log x}\right)$. As the random variable $2^{N^{\prime}}$ has regularly varying tails with index -0.5 ,

$$
E\left[2^{N^{\prime}}\right]=\infty
$$

and consequently, expected total computational cost is infinite.
An alternative, intuitive explanation for why the expected termination time is infinite is as follows: Note that conditional on $\mathbf{Z}$ and $\mathbf{Y}^{\varepsilon_{n}}\left(T_{\text {left }}\right)$, the distance $D=\mid V-L\left(\mathbf{Z} ; \mathbf{Y}^{\varepsilon_{n}}\left(T_{\text {left }}\right) \mid\right.$ is less than $\delta$ with probability $O(\delta)$ (because $V$ is uniformly distributed). Thus, if the cost of generating $\mathbf{Y}^{D}$
(required to decide whether to accept or reject) is $C(D)$, the running time of the algorithm would be finite if $\int_{0}^{1} C(u) d u<\infty$. Unfortunately, however, the cost of producing an $\varepsilon$-strong approximation to Brownian motion $\left(\mathbf{X}^{\varepsilon}\right)$ is roughly $O\left(1 / \varepsilon^{2}\right)$ (see, for example, $[3]$ ) and therefore $C(D) \geq c / D^{2}$, with positive probability, for some $c>0$, which results in an infinite expected running time.

## 5. Conclusions

We provide the first exact sampling algorithm to obtain samples from a multi-dimensional reflected Brownian motion. The algorithm relies on a novel conditional acceptance / rejection step, which is implemented by carefully refining $\varepsilon$-strong approximations of the reflected Brownian motion path $\mathbf{Y}(\cdot)$ until we can conclusively accept or reject a proposal $\mathbf{Z}$ from a suitable uniform distribution. Unfortunately, as shown in Section 4, the proposed algorithm has expected termination time because of the large amount of computational effort required to conclusively decide whether $V<L\left(\mathbf{Z} ; \mathbf{Y}\left(T_{\text {left }}\right)\right)$ when the proposal likelihood $L\left(\mathbf{Z} ; \mathbf{Y}\left(T_{\text {left }}\right)\right)$ and the uniform random variable $V$ are close. It may be of interest to the readers to know whether the entire exact sampling scheme can be executed with finite expected computational effort if we, somehow, are able to resolve the difficulty in deciding whether $V<L\left(\mathbf{Z} ; \mathbf{Y}\left(T_{\text {left }}\right)\right)$ with finite expected computational effort. We believe this is indeed the case because of the following reasoning.

Apart from the refine until accept / reject step in Algorithm 3, the only other step where we execute a 'while' loop performing a random comparison is in the pre-conditioning procedure in Algorithm 2. Recall that the preconditioning step must develop a piecewise approximation to Brownian motion that is accurate enough to satisfy $\varepsilon_{n}:=\left\|\mathbf{B}_{n}(\cdot)-\mathbf{B}(\cdot)\right\|<\left(K_{1}+1\right) Y_{i}(T)$, for $i=$ $1, \ldots, d$, in order to identify $\left(T_{\text {left }}, T_{\text {right }}\right)$. If the probability density of $Y_{i}(T)$ evaluated at 0 is positive (as in the 1-dimensional RBM case), then the probability that $Y_{i}(T)$ is smaller than $\delta$ is at least $c_{1} \delta$ (for every $\delta<\delta^{\prime}$ suitably small), and the computational effort required to generate a Brownian approximation that satisfies $\varepsilon_{n}<\left(K_{1}+1\right) \delta$ is larger than $c_{2} / \delta^{2}$ with positive probability; here, $c_{1}$ and $c_{2}$ are suitable positive constants. As the required computational effort $O\left(1 /\left(\min _{i} Y_{i}(T)\right)^{2}\right)$ is high when $Y_{i}(T)$ is close to 0 for some $i$, the expected computational effort required in the preconditioning step is at least

$$
\int_{c_{2} / \delta^{\prime 2}}^{\infty} P(\text { computational cost }>u) d u \geq \int_{c_{2} / \delta^{\prime 2}}^{\infty} c_{1} \sqrt{c_{2} / u} d u=\infty
$$

However, this difficulty can be easily resolved if we imagine, for a moment, that it is possible to resolve the earlier difficulty explained in Section 4 (on deciding whether $V<L\left(\mathbf{Z} ; Y\left(T_{l e f t}\right)\right.$ ) within finite expected time), and it is possible to obtain exact samples of $\mathbf{Y}(T)$ whenever $\mathbf{Y}(T)$ is bounded away from the reflecting barrier; specifically, let us assume we can obtain samples of $\mathbf{Y}(T)$ with finite expected computational effort $C_{\gamma}$ when $\min _{i} Y_{i}(T)>\gamma$ for some fixed constant $\gamma \in(0,1)$. In that case, we first obtain an exact sample of $\mathbf{Y}\left(t^{\prime}\right)$ for the latest $t^{\prime} \leq T$ during which $\min _{i} \mathbf{Y}_{i}\left(t^{\prime}\right)>\gamma$. To be specific, define $t^{\prime}:=\sup \left\{t \leq T: \min _{i} Y_{i}(t)>\gamma\right\}, \tilde{\mathbf{Y}}:=\mathbf{Y}_{i}\left(t^{\prime}\right)$ and $T_{\gamma}:=T-t^{\prime}$. Then due to the Markov property and self-similarity of RBM, the original objective of obtaining a sample of $\mathbf{Y}(T)$ can be equivalently written as follows: Obtain a sample of $\mathbf{Y}_{n e w}\left(T_{\gamma} / \gamma^{2}\right)$, where $\mathbf{Y}_{\text {new }}(\cdot)$ is also an RBM obtained by shifting and scaling the RBM $\mathbf{Y}(\cdot)$ as in $\mathbf{Y}_{n e w}(t):=\mathbf{Y}\left(t^{\prime}+\gamma^{2} t\right) / \gamma$ with initial condition $\mathbf{Y}_{\text {new }}(0):=\tilde{\mathbf{Y}} / \gamma$. Thus, even if the original problem of detecting $\left(T_{\text {left }}, T_{\text {right }}\right)$ is difficult when $\mathbf{Y}(T)<\gamma$, by a suitable translation and scaling (magnification) of the underlying Brownian and RBM paths, we have a new, but equivalent, objective of sampling from $\mathbf{Y}_{\text {new }}\left(T_{\gamma} / \gamma^{2}\right)=\mathbf{Y}(T) / \gamma$. In case if $\mathbf{Y}_{\text {new }}\left(T_{\gamma} / \gamma^{2}\right)$ is smaller than $\gamma$ as well, we perform a similar translation and scaling once again recursively. Since $E\left[T_{\gamma} / \gamma^{2}\right]$ is uniformly bounded as a function of $\gamma$, a simple recursive algorithm complexity analysis (see, for example, Chapter 4 of [8]) yields us that the total expected
computational effort of the described recursive procedure is $O\left(C_{\gamma} / P\left(\min _{i} Y_{i}(T)>\gamma\right)\right)$, which is finite, as per our assumption on $C_{\gamma}$.

As the described recursive construction for the preconditioning step is built on the assumption that we can conclusively decide whether $V<L\left(\mathbf{Z} ; \mathbf{Y}\left(T_{l e f t}\right)\right)$ within finite expected computational effort, we identify the difficulty explained carefully in Section 4 as the only fundamental bottleneck in obtaining exact samples of multi-dimensional RBM. Future research that addresses this bottleneck by means of new techniques will make the proposed algorithm, which is currently of theoretical importance, to be more suitable for practice as well.

## Appendix

Here we provide the proof of Lemma 1, and present explicit expressions for the constants $K_{\pi}, K_{\rho}, c_{\pi}$ and $c_{\rho}$. For proving Lemma 1, we need the following result.

Lemma 2. For any given $U>L, r>0$, the function $\gamma(L, U ; r, a, b)$ defined in (15) is Lipschitz continuous with respect to the variables $a$ and $b$; that is,

$$
\begin{aligned}
& \left|\gamma\left(L, U ; r, a_{1}, \cdot\right)-\gamma\left(L, U ; r, a_{2}, \cdot\right)\right|<K(L, U, r)\left|a_{1}-a_{2}\right| \\
& \left|\gamma\left(L, U ; r, \cdot, b_{1}\right)-\gamma\left(L, U ; r, \cdot, b_{2}\right)\right|<K(L, U, r)\left|b_{1}-b_{2}\right|
\end{aligned}
$$

for all $a_{1}, a_{2}, b_{1}, b_{2} \in(L, U)$. The Lipschitz constant $K(L, U, r)$ is given by

$$
K(L, U, r):=\sum_{j \geq 1} K_{j}=\frac{8(U-L)}{r} \sum_{j \geq 1} j \exp \left(-\frac{2}{r}(U-L)^{2}(j-1)^{2}\right)
$$

Proof. Let $\gamma_{n}(a, b)=1-\sum_{j=1}^{n}\left(\sigma_{j}-\tau_{j}\right)$. Since $a, b$ take values in $(L, U)$, it is easily checked that for all $j \geq 1$ both

$$
\left|\frac{d}{d a}\left(\sigma_{j}-\tau_{j}\right)\right|<K_{j}, \text { and }\left|\frac{d}{d b}\left(\sigma_{j}-\tau_{j}\right)\right|<K_{j},
$$

where

$$
K_{j}:=\frac{8(U-L) j}{r} \exp \left(-\frac{2}{r}(U-L)^{2}(j-1)^{2}\right)
$$

Then it is immediate that for all $n$,

$$
\left|\frac{d \gamma_{n}(a, b)}{d a}\right|<\sum_{j=1}^{\infty} K_{j} \text { and }\left|\frac{d \gamma_{n}(a, b)}{d b}\right|<\sum_{j=1}^{\infty} K_{j}
$$

As a consequence, we use the following elementary properties of Lipschitz continuity to establish the Lipschitz continuity of $\gamma(L, U ; r, a, b)$ with respect to variables $a$ and $b$ :

1) If a differentiable function $f(\cdot)$ on a convex domain is such that its first derivative $\left|f^{\prime}(x)\right|<K$ for some constant $K$, then the function $f(\cdot)$ is Lipschitz continuous with Lipschitz constant at most $K$.
2) If a sequence of Lipschitz functions $f_{n}(\cdot)$ all having Lipschitz constant bounded by $K$ converge uniformly to $f(\cdot)$, then $f(\cdot)$ is also Lipschitz continuous with Lipschitz constant at most $K$.

Since $\gamma_{n}(a, b)$ converge uniformly to $\gamma(a, b)$ for $a, b \in(L, U)$, it follows immediately from the above two facts that $\gamma(L, U ; r, a, b)$ is Lipschitz continuous with Lipschitz constant at most

$$
K(L, U, r):=\sum_{j \geq 1} K_{j}=\frac{8(U-L)}{r} \sum_{j \geq 1} j \exp \left(-\frac{2}{r}(U-L)^{2}(j-1)^{2}\right)
$$

Proof of Lemma 1. For all $x \in\left(L^{\downarrow}, U^{\uparrow}\right)$,

$$
\begin{equation*}
\pi(x)=\exp \left(-\frac{1}{2}\left(x-\frac{s}{l} v\right)^{2} /\left(\frac{s(l-s)}{l}\right)\right) \leq 1=: c_{\pi} \tag{17}
\end{equation*}
$$

The Lipschitz continuity of $\pi(\cdot)$ follows from the boundedness of its first derivative $\pi^{\prime}(\cdot)$ on the convex domain $(L, U)$ : for all $x \in(L, U)$,

$$
\begin{equation*}
\left|\frac{d \pi(x)}{d x}\right| \leq \frac{|x l-s v|}{s(l-s)}<\frac{\max \{|U l-s v|,|L l-s v|\}}{s(l-s)}=: K_{\pi} \tag{18}
\end{equation*}
$$

To prove the Lipschitz continuity of $\rho(\cdot)$, we first note the boundedness of $\gamma(L, U ; r, \cdot, \cdot)$ : Simple substitution will yield that $\gamma(L, U ; r, a, b)=0$ whenever either $a$ or $b$ equals one of $L, U$. Then due to the Lipschitz continuity of $\gamma(L, U ; r, \cdot, \cdot)$ from Lemma 2, we have that

$$
\begin{equation*}
|\gamma(L, U ; r, a, b)| \leq K(L, U, r)(U-L) \tag{19}
\end{equation*}
$$

Now consider the first term $\gamma_{1}(x) \gamma_{2}(x)$ in (14):

1) Because of (19), $\left|\gamma_{1}(\cdot)\right|$ and $\left|\gamma_{2}(\cdot)\right|$ are bounded by $K\left(L^{\downarrow}, U^{\uparrow}, s\right)\left(U^{\uparrow}-L^{\downarrow}\right)$ and $K\left(L^{\downarrow}, U^{\uparrow}, l-\right.$ $s)\left(U^{\uparrow}-L^{\downarrow}\right)$, respectively, in the interval $x \in\left(L^{\downarrow}, U^{\uparrow}\right)$.
2) From Lemma 2, we have that $\gamma_{1}(\cdot)$ and $\gamma_{2}(\cdot)$ are Lipschitz continuous (with respect to the variable $x$ ) with Lipschitz constants at most $K\left(L^{\downarrow}, U^{\uparrow}, s\right)$ and $K\left(L^{\downarrow}, U^{\uparrow}, l-s\right)$ respectively.

From the above two observations, we conclude that $\gamma_{1}(\cdot) \gamma_{2}(\cdot)$ is Lipschitz continuous with respect to $x$ with Lipschitz constant at most

$$
K_{1,2}:=2 K\left(L^{\downarrow}, U^{\uparrow}, s\right) K\left(L^{\downarrow}, U^{\uparrow}, l-s\right)\left(U^{\uparrow}-L^{\downarrow}\right)
$$

This is because if $f, g$ are Lipschitz continuous with respective Lipschitz constants $K_{f}$ and $K_{g}$ and absolute bounds $C_{f}$ and $C_{g}$, then $f g$ is Lipschitz continuous with Lipschitz constant at most $C_{f} K_{g}+C_{g} K_{f}$. Using the same reasoning, the Lipschitz constants of other terms in (14), namely $\gamma_{3}(\cdot) \gamma_{4}(\cdot), \gamma_{5}(\cdot) \gamma_{6}(\cdot)$ and $\gamma_{7}(\cdot) \gamma_{8}(\cdot)$ are at most

$$
\begin{aligned}
K_{3,4} & :=2 K\left(L^{\uparrow}, U^{\uparrow}, s\right) K\left(L^{\uparrow}, U^{\uparrow}, l-s\right)\left(U^{\uparrow}-L^{\uparrow}\right), \\
K_{5,6} & :=2 K\left(L^{\downarrow}, U^{\downarrow}, s\right) K\left(L^{\downarrow}, U^{\downarrow}, l-s\right)\left(U^{\downarrow}-L^{\downarrow}\right), \text { and } \\
<K_{7,8} & :=2 K\left(L^{\uparrow}, U^{\downarrow}, s\right) K\left(L^{\uparrow}, U^{\downarrow}, l-s\right)\left(U^{\downarrow}-L^{\uparrow}\right)
\end{aligned}
$$

respectively. Therefore, $\rho(x)$ is Lipschitz continuous with Lipschitz constant $K_{\rho}$ given by,

$$
K_{\rho}:=K_{1,2}+K_{3,4}+K_{5,6}+K_{7,8}
$$

Since $\rho(x)=0$ whenever $x$ takes either $L^{\downarrow}$ or $U^{\uparrow}$, using Lipschitz continuity of $\rho$ we reason that,

$$
|\rho(x)| \leq K_{\rho}\left(U^{\uparrow}-L^{\downarrow}\right)=: c_{\rho}
$$

This along with (17) and (18) proves the claim.

Lemma 3. Recall the definition $\bar{Z}:=1+\sup _{t \in[0,1]} W(t)-\inf _{t \in[0,1]} W(t)$, where $(W(t): t \in[0,1])$ is a standard Brownian motion. If $\left(\bar{Z}_{i}: i=1, \ldots, k\right)$ are $k$ independent copies of $\bar{Z}$, then

$$
E\left[\max _{i=1, \ldots, k} \bar{Z}_{i}\right]=\Theta(\sqrt{\log k}), \quad \text { as } k \rightarrow \infty
$$

Proof. We first observe that

$$
\begin{aligned}
\max _{i=1, \ldots, k} \bar{Z}_{i} & =\max _{i=1, \ldots, k}\left(1+\sup _{t \in[0,1]} W(t)-\inf _{t \in[0,1]} W(t)\right) \\
& \leq 1+\max _{i=1, \ldots, k} \sup _{t \in[0,1]} W(t)+\max _{i=1, \ldots, k}\left(-\inf _{t \in[0,1]} W(t)\right)
\end{aligned}
$$

As $\sup _{t \in[0,1]} W(t) \stackrel{D}{=}-\inf _{t \in[0,1]} W(t)$, we have,

$$
1+\max _{i=1, \ldots, k} \sup _{t \in[0,1]} W(t) \leq \max _{i=1, \ldots, k} \bar{Z}_{i}{ }^{D} \leq 1+2 \max _{i=1, \ldots, k} \sup _{t \in[0,1]} W(t)
$$

where the notation $X \stackrel{D}{\leq} Y$ denotes that $X$ is stochastically upper bounded by $Y$. Further, as $\sup _{t \in[0,1]} W(t) \stackrel{D}{=}|Z|$ when $Z$ follows standard normal distribution, we have

$$
\begin{equation*}
E\left[\max _{i=1, \ldots, k} \bar{Z}_{i}\right]=\Theta\left(E\left[\max _{i=1, \ldots, k} \sup _{t \in[0,1]} W(t)\right]\right)=\Theta\left(E\left[\max _{i=1, \ldots, k}\left|Z_{i}\right|\right]\right) \tag{20}
\end{equation*}
$$

as $k \rightarrow \infty$. Here, $\left(Z_{i}: i=1, \ldots, k\right)$ are simply $k$ independent copies of a standard normal variable. Next, if we denote the positive and negative parts of $Z_{i}$ as $Z_{i}^{+}=\max \left\{Z_{i}, 0\right\}$ and $Z_{i}^{-}=-\min \left\{Z_{i}, 0\right\}$, then $\left|Z_{i}\right|=Z_{i}^{+}+Z_{i}^{-}$. Further, as

$$
\max _{i=1, \ldots, k} Z_{i}^{+} \leq \max _{i=1, \ldots, k}\left|Z_{i}\right| \leq \max _{i=1, \ldots, k} Z_{i}^{+}+\max _{i=1, \ldots, k} Z_{i}^{-}
$$

it follows from (20) that

$$
E\left[\max _{i=1, \ldots, k} \bar{Z}_{i}\right]=\Theta\left(E\left[\max _{i=1, \ldots, k} Z_{i}^{+}\right]\right)=\Theta\left(E\left[M^{+}\right]\right),
$$

where $M^{+}$is the positive part of $M:=\max _{i=1, \ldots, k} Z_{i}$. Since $E[M]=\Theta(\sqrt{\log k})$ and $E\left[M^{-}\right]=$ $\int_{0}^{\infty}(P(Z<-u))^{k} d u \rightarrow 0$ as $k \rightarrow \infty$, we obtain $E\left[M^{+}\right]=E[M]+E\left[M^{-}\right]=\Theta(\sqrt{\log k})$, thus proving the claim.

## References

[1] S. Asmussen and P. W. Glynn. Stochastic Simulation: Algorithms and Analysis, volume 57. Springer-Verlag, 2007.
[2] A. Beskos, O. Papaspiliopoulos, and G. O. Roberts. Retrospective exact simulation of diffusion sample paths with applications. Bernoulli, 12(6):1077-1098, 2006.
[3] A. Beskos, S. Peluchetti, and G. O. Roberts. $\varepsilon$-strong simulation of the Brownian path. Bernoulli, 18(4):12231248, 2012.
[4] A. Beskos and G. O. Roberts. Exact simulation of diffusions. The Annals of Applied Probability, 15(4):2422-2444, 112005.
[5] J. Blanchet and X. Chen. Steady-state simulation of reflected Brownian motion and related stochastic networks. arXiv preprint arXiv:1202.2062, 2012.
[6] H. Chen and D. D. Yao. Fundamentals of Queuing Networks: Performance, Asymptotics, and Optimization. Applications of mathematics : stochastic modelling and applied probability. Springer, 2001.
[7] N. Chen and Z. Huang. Localization and exact simulation of Brownian motion-driven stochastic differential equations. Mathematics of Operations Research, 2013.
[8] T. H. Cormen, C. Stein, R. L. Rivest, and C. E. Leiserson. Introduction to Algorithms. McGraw-Hill Higher Education, 2nd edition, 2001.
[9] P. Étoré and M. Martinez. Exact simulation of one-dimensional stochastic differential equations involving the local time at zero of the unknown process. Monte Carlo Methods and Applications, 19(1):41-71, 2013.
[10] J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. Ann. Probab., 9(2):302-308, 1981.
[11] O. Kella and W. Whitt. Stability and structural properties of stochastic storage networks. Journal of Applied Probability, 33(4):1169-1180, 1996.
[12] M. Pollock, A. M. Johansen, and G. O. Roberts. On the exact and $\varepsilon$-strong simulation of (jump) diffusions. Bernoulli, 22(2):794-856, 052016.
[13] K. Pötzelberger and L. Wang. Boundary crossing probability for Brownian motion. J. Appl. Probab., 38(1):152164, 2001.
[14] M. I. Reiman. Open queueing networks in heavy traffic. Mathematics of Operations Research, 9(3):441-458, 1984.

Stanford University, Management Science \& Engineering Department, 475 Via Ortega, Stanford, CA 94305-4121, United States.

E-mail address: jblanche@stanford.edu

Columbia University, Department of Industrial Engineering \& Operations Research, 340 S. W. Mudd Building, 500 W. 120 Street, New York, NY 10027, United States.

E-mail address: karthyek.murthy@columbia.edu


[^0]:    Date: Dec 15, 2016.
    Key words and phrases. Unbiased Sampling; Refine until Accept / Reject; Tolerance Enforced Simulation; Acceptance / Rejection sampling; $\varepsilon$ - strong simulation; intersection layers; iterative algorithm.

    Author 1 gratefully acknowledges the support from NSF Award 1538217.

