

On first exit times and their means for Brownian bridges

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Abstract

For a Brownian bridge from 0 to y we prove that the mean of the first exit time from interval $(-h, h)$, $h > 0$, behaves as $O(h^2)$ when $h \downarrow 0$. Similar behavior is seen to hold also for the 3-dimensional Bessel bridge. For Brownian bridge and 3-dimensional Bessel bridge this mean of the first exit time has a puzzling representation in terms of the Kolmogorov distribution. The result regarding the Brownian bridge is applied to prove in detail an estimate needed by Walsh to determine the convergence of the binomial tree scheme for European options.

Keywords: Brownian motion, Brownian bridge, Bessel process, Bessel bridge, first exit time, last exit time, Kolmogorov distribution function, binomial tree scheme

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1 Introduction

We start with the description of our setting. Let $C[0, \infty)$ denote the space of continuous functions $\omega : [0, \infty) \mapsto \mathbb{R}$, and

$$\mathcal{C}_t := \sigma\{\omega(s) : s \leq t\}$$

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the smallest σ -algebra making the co-ordinate mappings up to time t measurable. Furthermore, let \mathcal{C} be the smallest σ -algebra containing all \mathcal{C}_t , $t \geq 0$. For an interval $I \subset \mathbb{R}$ let $(\mathbb{P}_x^X)_{x \in I}$ be a family of probability measures defined in the filtered canonical space $(C[0, \infty), \mathcal{C}, (\mathcal{C}_t)_{t \geq 0})$ such that under \mathbb{P}_x^X for a given $x \in I$ the co-ordinate process $X = (X_t)_{t \geq 0} := (\omega_t)_{t \geq 0}$ is a regular diffusion taking values in I and starting from x . Here, X is considered in the sense of Itô and McKean [8], see also [3]. A crucial property of X is that there exists a (speed) measure m^X such that the transition probability has a continuous strictly positive density $(t, x, y) \mapsto q_t(x, y)$, $t > 0, x, y \in I$ with respect to m^X i.e.,

$$\mathbb{P}_x^X(X_t \in dy) = q_t(x, y)m^X(dy), \quad (1)$$

see [8, page 149 and 157].

For $(X_t)_{t \geq 0}$, $X_0 = x$, as defined above and $T > 0$, one can construct a new non-homogeneous strong Markov process by conditioning X to be at a given point $y \in I$ at time T . Although the conditioning is, in general, with respect to a zero set $\{X_T = y\}$, it can be realized using the Bayes formula and the notion of regular conditional distributions. Another approach is to apply the theory of the Doob h -transforms. To explain this briefly, consider X in space-time i.e., the process $\bar{X} = ((X_t, t))_{t \geq 0}$. Introduce for $z \in I$ and $t < T$ the function

$$h(z, t) := h(z, t; y, T) := q_{T-t}(z, y).$$

By the Chapman-Kolmogorov equation it holds for $x \in I$ and $s < t$

$$\begin{aligned} \mathbb{E}_{(x,s)}^X[h(X_t, t)] &= \int_I q_{t-s}(x, z)h(z, t) m(dz) \\ &= \int_I q_{t-s}(x, z)q_{T-t}(z, y) m(dz) \\ &= q_{T-s}(x, y) \\ &= h(x, s), \end{aligned}$$

where $\mathbb{E}_{(x,s)}^X$ refers to the expectation associated with the space-time process \bar{X} initiated from x at time s . Consequently, we may define for $f \in \mathcal{B}_b(I)$ (= bounded measurable functions on I) and $s < t < T$ a non-homogeneous Markov semi-group $P_{t,s}^{X,h}$, $0 < s < t < T$, via

$$P_{t,s}^{X,h} f(x) := \mathbb{E}_x^X \left[f(X_{t-s}) \frac{h(X_{t-s}, t)}{h(x, s)} \right]. \quad (2)$$

The process governed by the probability measure induced by this semigroup is called the X -bridge to y of length T . The notations $X^{x,T,y}$, $\mathbb{P}_{x,T,y}^X$, and $\mathbb{E}_{x,T,y}^X$ are used for this process, its probability measure and the expectation, respectively, when the initial state is $x \in I$. From (2) one may deduce the following absolute continuity relation for $A_t \in \mathcal{C}_t$, $t < T$,

$$\mathbb{P}_{x,T,y}^X(A_t) = \mathbb{E}_x^X \left[\frac{h(X_t, t)}{h(x, 0)} ; A_t \right] = \mathbb{E}_x^X \left[\frac{q_{T-t}(X_t, y)}{q_T(x, y)} ; A_t \right].$$

We refer to Chung and Walsh [4] for a general discussion on h -transforms, to Fitzsimmons et al. [6], in particular, Proposition 1, for general Markovian bridges, and to Salminen [14] for some properties of diffusion bridges.

Our main interest in this paper is focused on the case where the underlying process is a Brownian motion. The notations $(W_t)_{t \geq 0}$ and $(B_t^{x,T,y})_{0 \leq t < T}$ are used for standard Brownian motion and Brownian bridge from x to y of length T , respectively, and, for notational simplicity, the corresponding probability measures are \mathbb{P}_x and $\mathbb{P}_{x,T,y}$ with the expectations \mathbb{E}_x and $\mathbb{E}_{x,T,y}$, respectively. Brownian bridge has in addition to the general h -transform approach a few equivalent specific characterizations. Indeed, Brownian bridge can be viewed as (i) a Gaussian process, (ii) a deterministic time change with an additional drift of standard Brownian motion or (iii) as a solution of a SDE, see e.g. [3] p. 66. In Section 3 we apply, in particular, (ii) and (iii). The second one states that

$$B_t^{x,T,y} \stackrel{d}{=} \begin{cases} (1 - \frac{t}{T})W(\frac{Tt}{T-t}) + x + \frac{(y-x)t}{T}, & t < T \\ y & t = T, \end{cases} \quad (3)$$

where $\stackrel{d}{=}$ means that the processes on the left and the right hand side are identical in law. The third one says that

$$B_t^{x,T,y} \stackrel{d}{=} \begin{cases} (T-t) \int_0^t \frac{dW_s}{T-s} + x + \frac{(y-x)t}{T}, & t < T \\ y & t = T. \end{cases} \quad (4)$$

Here it is assumed that the canonical filtration $(\mathcal{C}_t)_{t \geq 0}$ is augmented with the null sets of \mathcal{C} with respect to \mathbb{P}_0 in order to have the usual conditions satisfied.

For a general diffusion bridge, we present an integral representation of the mean of the first exit time from an interval the aim being to deduce the limiting behavior of the mean, when the length of the interval around the starting value of the bridge decreases to 0. The main body of the paper concerns Brownian bridge

and Bessel bridge (with dimension parameter 3). For Brownian bridge (a similar result holds for Bessel bridge), it is shown in Theorem 2.4 that

$$\lim_{h \downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2} = 1, \quad y \neq 0,$$

where

$$\mathcal{T}_{(-h,h)} := \inf\{t > 0 : B_t^{0,T,y} \notin (-h, h)\}$$

denotes the first exit time from the interval $(-h, h)$, $h > 0$. To avoid ambiguity, in some cases we indicate the process for which the first exit time is considered by writing for any continuous process $(X_t)_{t \geq 0}$

$$\mathcal{T}_{(a,b)}^X := \inf\{t > 0 : X_t \notin (a, b)\}, \quad a < x < b,$$

and put $\mathcal{T}_{(a,b)}^X := \infty$ if $\{t > 0 : X_t \notin (a, b)\} = \emptyset$. Recall that for Brownian motion (this can be deduced, e.g., from the Laplace transform of $\mathcal{T}_{(a,b)}$ given in [3, Part II, Section 1, 3.0.1]) it holds

$$\mathbb{E}_x [\mathcal{T}_{(a,b)}] = (b-x)(x-a), \quad a < x < b. \quad (5)$$

Consequently,

$$\lim_{h \downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{\mathbb{E}_0[\mathcal{T}_{(-h,h)}]} = 1, \quad y \neq 0.$$

For some other diffusion bridges a similar asymptotic behavior can be found. But we have not been able to prove such a result in generality.

The paper is organized as follows. Section 2 contains the main results. In Subsection 2.1 integral representations are given for the mean of $\mathcal{T}_{(a,b)}^X$ when X is, firstly, a general regular diffusion and, secondly, a corresponding diffusion bridge. We also calculate for a regular diffusion X , $X_0 = x$, the limiting behavior of the mean of $\mathcal{T}_{(x-h,x+h)}^X$ as $h \rightarrow 0$. In Subsection 2.2 we focus on Brownian bridge and 3-dimensional Bessel bridge starting from $x > 0$ and find the limiting behavior of the mean of $\mathcal{T}_{(x-h,x+h)}$ as $h \rightarrow 0$. For the means of the first exit times – considered in Subsection 2.2 – Subsection 2.3 provides puzzling representations in terms of the Kolmogorov distribution function. We discuss also other properties of the Kolmogorov distribution, in particular, its connection with the last exit time distribution of Brownian motion. As an application, in Section 3, we use our results concerning Brownian bridge to give a detailed proof of an estimate in Walsh [17] needed therein when deriving the convergence rate of an option price calculated from the binomial tree scheme to the Black-Scholes price. The estimate is used also in a forthcoming paper [12].

2 Main results

2.1 Preliminaries

Let $X = (X_t)_{t \geq 0}$ be a regular diffusion taking values on an interval I , and recall from (1) the notation $q_t(x, y)$ for its transition density with respect to the speed measure m^X . For $a < b$, $a, b \in I$, let \widehat{X} denote X killed at $\mathcal{T}_{(a,b)}$. Then \widehat{X} is a regular diffusion on (a, b) . The speed measure of \widehat{X} is m^X , and \widehat{X} has a continuous strictly positive transition density $\widehat{q}_t(x, y)$ such that for $x, y \in (a, b)$

$$\begin{aligned} \mathbb{P}_x^{\widehat{X}}(\widehat{X}_t \in dy) &= \widehat{q}_t(x, y)m^X(dy) \\ &= \mathbb{P}_x^X(X_t \in dy, \mathcal{T}_{(a,b)} > t) \\ &= \mathbb{P}_x^X(a < \inf_{0 \leq s \leq t} X_s, \sup_{0 \leq s \leq t} X_s < b, X_t \in dy). \end{aligned} \quad (6)$$

This yields immediately the following result.

Proposition 2.1. *In case $\mathbb{P}_x^X(\mathcal{T}_{(a,b)} < \infty) = 1$, i.e., X does not die inside (a, b) , it holds*

$$\mathbb{E}_x^X[\mathcal{T}_{(a,b)}] = \int_0^\infty dt \int_a^b m^X(dz) \widehat{q}_t(x, z). \quad (7)$$

The next proposition will serve as an important tool for the calculations below.

Proposition 2.2. *For $x \in (a, b) \subset I$ and $y \in I \setminus (a, b)$ it holds*

$$\mathbb{E}_{x,T,y}^X[\mathcal{T}_{(a,b)}] = \int_0^T dt \int_a^b m^X(dz) \widehat{q}_t(x, z) \frac{q_{T-t}(z, y)}{q_T(x, y)}. \quad (8)$$

Proof. Since $x \in (a, b)$ and $y \in I \setminus (a, b)$ we have $\mathbb{P}_{x,T,y}^X(\mathcal{T}_{(a,b)} < T) = 1$. Consider

$$\begin{aligned} \mathbb{E}_{x,T,y}^X[\mathcal{T}_{(a,b)}] &= \int_0^T dt \int_a^b \mathbb{P}_{x,T,y}^X(\mathcal{T}_{(a,b)} > t, X_t^{x,T,y} \in dz) \\ &= \int_0^T dt \int_a^b \mathbb{P}_x^X(\mathcal{T}_{(a,b)} > t, X_t \in dz) \frac{q_{T-t}(z, y)}{q_T(x, y)} \\ &= \int_0^T \int_a^b \mathbb{P}_x^X(a < \inf_{0 \leq s \leq t} X_s, \sup_{0 \leq s \leq t} X_s < b, X_t \in dz) \frac{q_{T-t}(z, y)}{q_T(x, y)} dt, \end{aligned}$$

where the Markov property and formula (2) are used. □

One of our main issues concerns the limiting behavior of the mean of $\mathcal{T}_{(x-h, x+h)}^X$ as $h \rightarrow 0$ for diffusion bridges. For regular diffusions we have the following fairly complete characterization.

Proposition 2.3. *Assume that the differential operator associated with the regular diffusion X is given by*

$$\mathcal{G}u(x) := \frac{1}{2}a^2(x)u''(x) + b(x)u'(x), \quad x \in I,$$

where $x \mapsto a^2(x) > 0$ and $x \mapsto b(x)$ are continuous in I . Let $x_o \in \text{Int}(I)$. Then

$$\lim_{h \downarrow 0} \frac{\mathbb{E}_{x_o}^X[\mathcal{T}_{(x_o-h, x_o+h)}]}{h^2} = a^{-2}(x_o).$$

Proof. Recall from [3, Part I, Chapter II, No 7, p.17] that the speed measure m^X and the scale function s^X can be taken to be

$$m^X(dx) = m^X(x)dx, \quad \frac{d}{dx}s^X(x) = e^{-B(x)}, \quad (9)$$

where

$$m^X(x) = 2a^{-2}(x)e^{B(x)}, \quad B(x) = \int^x 2a^{-2}(y)b(y)dy. \quad (10)$$

Consider now the process X initiated at x_o and killed when it leaves the interval $(x_o - h, x_o + h)$. We let $(\widehat{X}_t)_{t \geq 0}$ denote this diffusion:

$$\widehat{X}_t = \begin{cases} X_t, & t < \mathcal{T}_{(x_o-h, x_o+h)}, \\ \partial, & t \geq \mathcal{T}_{(x_o-h, x_o+h)}, \end{cases}$$

where ∂ is a cemetery point. It is well known (cf. [3, Part I, Chapter II, No 11]) that the 0-resolvent kernel of \widehat{X} is given by

$$\widehat{G}_0(x, y) := \int_0^\infty \widehat{q}_t(x, y)dt = \begin{cases} \frac{(s(x) - s(x_o - h))(s(x_o + h) - s(y))}{s(x_o + h) - s(x_o - h)}, & x < y, \\ \frac{(s(y) - s(x_o - h))(s(x_o + h) - s(x))}{s(x_o + h) - s(x_o - h)}, & x > y, \end{cases}$$

where $s := s^X$ and $\widehat{q}_t(x, y)$ is the transition density w.r.t. the speed measure m^X . Consequently, relation (7) implies that

$$\begin{aligned}
\mathbb{E}_{x_o}^X[\mathcal{T}_{(x_o-h, x_o+h)}] &= \int_0^\infty \left(\int_{x_o-h}^{x_o+h} \widehat{q}_t(x_o, y) m^X(dy) \right) dt \\
&= \int_{x_o-h}^{x_o+h} \widehat{G}_0(x_o, y) m^X(dy) \\
&= \frac{s(x_o+h) - s(x_o)}{s(x_o+h) - s(x_o-h)} \int_{x_o-h}^{x_o} (s(y) - s(x_o-h)) m^X(y) dy \\
&\quad + \frac{s(x_o) - s(x_o-h)}{s(x_o+h) - s(x_o-h)} \int_{x_o}^{x_o+h} (s(x_o+h) - s(y)) m^X(y) dy.
\end{aligned} \tag{11}$$

Since s is assumed to be continuously differentiable we have

$$\lim_{h \downarrow 0} \frac{s(x_o+h) - s(x_o)}{s(x_o+h) - s(x_o-h)} = \lim_{h \downarrow 0} \frac{s(x_o) - s(x_o-h)}{s(x_o+h) - s(x_o-h)} = \frac{1}{2},$$

and l'Hospital's rule yields

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{1}{h^2} \int_{x_o-h}^{x_o} (s(y) - s(x_o-h)) m^X(y) dy \\
&= \lim_{h \downarrow 0} \frac{1}{2h} \int_{x_o-h}^{x_o} \left(-\frac{d}{dh} s(x_o-h) \right) m^X(y) dy \\
&= \lim_{h \downarrow 0} \left(-\frac{d}{dh} s(x_o-h) \right) \frac{1}{2h} \int_{x_o-h}^{x_o} m^X(y) dy \\
&= \frac{1}{2} s'(x_o) m^X(x_o) \\
&= a^{-2}(x_o),
\end{aligned}$$

where the last equality follows from relations (9) and (10). Similarly,

$$\lim_{h \downarrow 0} \frac{1}{h^2} \int_{x_o}^{x_o+h} (s(x_o+h) - s(y)) m^X(y) dy = a^{-2}(x_o).$$

Hence, by (11),

$$\lim_{h \downarrow 0} \frac{\mathbb{E}_{x_o}^X[\mathcal{T}_{(x_o-h, x_o+h)}]}{h^2} = a^{-2}(x_o).$$

□

2.2 The mean of the first exit time for Brownian bridge and 3-dimensional Bessel bridge

We introduce the following function

$$\Delta(z, h, t) := \frac{1}{\sqrt{2\pi t}} \sum_{m=-\infty}^{\infty} \left(e^{-\frac{(z+4mh)^2}{2t}} - e^{-\frac{(z+2h(2m+1))^2}{2t}} \right), \quad |z| < h, t > 0. \quad (12)$$

Theorem 2.4.

(i) For the Brownian bridge with $|y| \geq h$,

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = \int_0^T \int_{-h}^h \frac{p_{T-t}(z, y)}{p_T(0, y)} \Delta(z, h, t) dz dt, \quad (13)$$

where $p_t(x, y)$ denotes the transition density of the standard Brownian motion, and, moreover,

$$\lim_{h \downarrow 0} h^{-2} \mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = 1, \quad y \neq 0. \quad (14)$$

(ii) For the 3-dimensional Bessel bridge with $x > h$ and $y \notin (x-h, x+h)$,

$$\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = \int_0^T \int_{-h}^h \frac{z+x}{x} \Delta(z, h, t) \frac{r_{T-t}^{(3)}(z+x, y)}{r_T^{(3)}(x, y)} dz dt,$$

where

$$r_t^{(3)}(x, y) dy = \frac{y}{x} (p_t(x, y) - p_t(x, -y)) dy, \quad x > 0, y > 0 \quad (15)$$

describes the transition density of the 3-dimensional Bessel process, and, moreover,

$$\lim_{h \downarrow 0} h^{-2} \mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = 1. \quad (16)$$

(iii) If $y \in (-h, h)$ (or $y \in (x-h, x+h)$ with $x > h, y > 0$) the mean of the first exit time is infinite for both bridges, i.e.

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = \mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = \infty. \quad (17)$$

Proof. (i) According to (6) and (8),

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = \int_0^T \int_{-h}^h \frac{p_{T-t}(z,y)}{p_T(0,y)} \mathbb{P}_0(\inf_{0 \leq s \leq t} W_s > -h, \sup_{0 \leq s \leq t} W_s < h, W_t \in dz) dt.$$

By [3, Part II, Section 1, 1.15.8 (p. 180)] and (12),

$$\mathbb{P}_0(\inf_{0 \leq s \leq t} W_s > -h, \sup_{0 \leq s \leq t} W_s < h, W_t \in dz) = \Delta(z, h, t) dz.$$

To show (14), substitute $z = hu$ and $t = h^2s$, and notice that $\Delta(hu, h, h^2s)h = \Delta(u, 1, s)$, so that

$$\begin{aligned} \mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] &= \int_0^T \int_{-h}^h \frac{p_{T-t}(z,y)}{p_T(0,y)} \Delta(z, h, t) dz dt \\ &= h^2 \int_0^{T/h^2} \int_{-1}^1 \frac{p_{T-h^2s}(hu,y)}{p_T(0,y)} \Delta(u, 1, s) dud s. \end{aligned}$$

To apply dominated convergence for $h \downarrow 0$ let y be fixed and assume that $h < \frac{|y|}{2}$. Notice that for $y \neq 0$ there exists a constant $C(T, y) > 0$ such that

$$\sup_{0 < t < T} p_t(0, y) \leq C(T, y). \quad (18)$$

This implies that

$$\sup_{u \in (-1, 1), s \in (0, T/h^2)} \frac{p_{T-h^2s}(hu, y)}{p_T(0, y)} \leq \frac{C(T, y/2)}{p_T(0, y)}.$$

Therefore, since $(u, s) \mapsto \Delta(u, 1, s)$ is integrable and $(u, s, h) \mapsto \frac{p_{T-h^2s}(hu, y)}{p_T(0, y)}$ is bounded on $(-1, 1) \times (0, T/h^2) \times (0, |y|/2)$, dominated convergence yields

$$\begin{aligned} \lim_{h \downarrow 0} \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2} &= \int_0^\infty \int_{-1}^1 \Delta(u, 1, s) dud s \\ &= \int_0^\infty \int_{-1}^1 \mathbb{P}_0(\mathcal{T}_{(-1,1)} > s, W_s \in du) ds \\ &= \mathbb{E}_0[\mathcal{T}_{(-1,1)}] \\ &= 1, \end{aligned} \quad (19)$$

where (5) is used for the last equality.

(ii) By (6) and (8), the expectation $\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}]$ has the representation

$$\int_0^T \int_{x-h}^{x+h} \mathbb{P}_x^{(3)}(\sup_{0 \leq s \leq t} |X_s - x| < h, X_t \in dz) \frac{r_{T-t}^{(3)}(z, y)}{r_T^{(3)}(x, y)} dt.$$

According to [3, Part II, Section 5, 1.15.8] we have for $z \in (x - h, x + h)$ that

$$\begin{aligned} & \mathbb{P}_x^{(3)}(\sup_{0 \leq s \leq t} |X_s - x| < h, X_t \in dz) \\ &= \frac{z}{x\sqrt{2\pi t}} \sum_{m=-\infty}^{\infty} \left(\exp\left(-\frac{((z-x)+2h(2m))^2}{2t}\right) - \exp\left(-\frac{((z-x)+2h(2m+1))^2}{2t}\right) \right) dz \\ &= \frac{z}{x} \Delta(z - x, h, t) dz, \end{aligned}$$

with Δ defined in (12). We substitute $z - x = \alpha$ and get

$$\begin{aligned} \mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] &= \int_0^T \int_{x-h}^{x+h} \frac{z}{x} \Delta(z - x, h, t) \frac{r_{T-t}^{(3)}(z, y)}{r_T^{(3)}(x, y)} dz dt \\ &= \int_0^T \int_{-h}^h \frac{\alpha + x}{x} \Delta(\alpha, h, t) \frac{r_{T-t}^{(3)}(\alpha + x, y)}{r_T^{(3)}(x, y)} d\alpha dt. \end{aligned} \quad (20)$$

For the proof of (16) we substitute $t = h^2 s$ and $\alpha = h\beta$ so that

$$\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = h^3 \int_0^{T/h^2} \int_{-1}^1 \frac{h\beta + x}{x} \Delta(h\beta, h, h^2 s) \frac{r_{T-h^2 s}^{(3)}(h\beta + x, y)}{r_T^{(3)}(x, y)} d\beta ds.$$

Using relation (15) yields

$$\left| \frac{h\beta + x}{x} \frac{r_{T-h^2 s}^{(3)}(h\beta + x, y)}{r_T^{(3)}(x, y)} \right| = \left| \frac{p_{T-h^2 s}(h\beta + x, y) - p_{T-h^2 s}(h\beta + x, -y)}{p_T(x, y) - p_T(x, -y)} \right|.$$

To see that this expression is bounded in $s \in (0, T/h^2)$ and $\beta \in (-1, 1)$ notice that $y \notin (h - x, h + x)$, with $x > h$ and $y > 0$ implies

$$0 < (|y - x| - h)^2 \leq (h\beta + x - y)^2 \leq (h\beta + x + y)^2,$$

so that

$$|p_{T-h^2 s}(h\beta + x, y) - p_{T-h^2 s}(h\beta + x, -y)| \leq p_{T-h^2 s}(|y - x|, h).$$

For $h \leq |y - x|/2$ we infer from (18) that

$$\sup_{s \in (0, T/h^2)} p_{T-h^2s}(|y - x|, h) \leq C(T, |y - x|/2).$$

Recall that $\Delta(\beta, 1, s) = \Delta(h\beta, h, h^2s)h$, so that dominated convergence gives

$$\lim_{h \downarrow 0} \frac{\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}]}{h^2} = \int_0^\infty \int_{-1}^1 \Delta(\beta, 1, s) d\beta ds = 1,$$

where the last equality was shown in (19).

(iii) For a regular diffusion X , (17) follows from $\mathbb{P}_{x,T,y}^X(\mathcal{T}_{(x-h,x+h)} \geq T) > 0$. It holds

$$\mathbb{P}_{x,T,y}^X(\mathcal{T}_{(x-h,x+h)} \geq T) = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{P}_x^X(\mathcal{T}_{(x-h,x+h)} \geq T, y \leq X_T \leq y + \varepsilon)}{\mathbb{P}_x^X(y \leq X_T \leq y + \varepsilon)} \quad (21)$$

and

$$\mathbb{P}_x^X(\mathcal{T}_{(x-h,x+h)} \geq T, y \leq X_T \leq y + \varepsilon) = \int_y^{y+\varepsilon} \hat{q}_T(x, z) dz,$$

where \hat{q} is the transition density of X killed at $\mathcal{T}_{(x-h,x+h)}$. By [8, p.157] the transition density of any regular diffusion is strictly positive. In our case this means

$$\hat{q}_T(x, z) > 0 \quad \text{for all } |z| < h.$$

Using l'Hospital's rule in (21) yields

$$\mathbb{P}_{x,T,y}^X(\mathcal{T}_{(x-h,x+h)} \geq T) = \frac{\hat{q}_T(x, y)}{q_T(x, y)} > 0. \quad \square$$

Remark 2.5. For the 3-dimensional Bessel bridge starting in 0 with $y > h$ it holds

$$\mathbb{E}_{0,T,y}^{(3)}[\mathcal{T}_h] = \int_0^T \int_0^h \mathbb{P}_0^{(3)}(\sup_{0 \leq s \leq t} X_s < h, X_t \in dz) \frac{r_{T-t}^{(3)}(z, y)}{r_T^{(3)}(0, y)} dt,$$

where $\mathcal{T}_h^X := \inf\{t > 0 : X_t = h\}$ denotes the first hitting time. The representation follows from (6) and (8), which can be extended to $x = 0$ since 0 is an entrance boundary point. Similarly as above it can be shown that

$$\lim_{h \downarrow 0} \frac{\mathbb{E}_{0,T,y}^{(3)}[\mathcal{T}_h]}{h^2} = \lim_{h \downarrow 0} \frac{\mathbb{E}_0^{(3)}[\mathcal{T}_h]}{h^2} = \frac{1}{3}.$$

We leave the proof to the reader.

2.3 The Kolmogorov distribution function and the mean of the first exit time

A classical result due to Doob [5] is that the distribution of the supremum of the absolute value of the standard Brownian bridge is given by

$$\mathbb{P}_{0,1,0} \left(\sup_{0 \leq s \leq 1} |B_s^{0,1,0}| \leq h \right) = \sum_{m=-\infty}^{\infty} (-1)^m e^{-2m^2 h^2} =: F(h).$$

The function $h \mapsto F(h)$, $h > 0$, is called the Kolmogorov distribution function due to Kolmogorov's fundamental work [10] (and also Smirnov [16]) on empirical distributions. We refer also to [13] and [11, Section 5.7]. The main result of this section – Theorem 2.6 – provides representations for the mean of the first exit time of the Brownian bridge and of the 3-dimensional Bessel bridge involving the Kolmogorov distribution function. We present now some formulas related to the Kolmogorov distribution which we need later. The following Jacobi's theta function identity, an instance of the Poisson summation formula, is stated in [1, equation (2.1)]:

$$\sum_{m=-\infty}^{\infty} \cos(2m\pi v) e^{-m^2 \pi^2 u} = \frac{1}{\sqrt{\pi u}} \sum_{m=-\infty}^{\infty} e^{-\frac{(m+v)^2}{u}}, \quad u > 0, v \in \mathbb{R}.$$

Putting here $u = 2x^2/\pi^2$ and $v = 1/2$ yields

$$F(x) = \sum_{m=-\infty}^{\infty} (-1)^m e^{-2m^2 x^2} = \frac{\sqrt{2\pi}}{x} \sum_{k=1}^{\infty} \exp\left(-\frac{(2k-1)^2 \pi^2}{8x^2}\right). \quad (22)$$

Notice also that

$$\begin{aligned} F(h/\sqrt{t}) &= \mathbb{P}_{0,1,0} \left(\sup_{0 \leq s \leq 1} |B_s^{0,1,0}| \leq h/\sqrt{t} \right) \\ &= \mathbb{P}_{0,t,0} \left(\sup_{0 \leq s \leq 1} |B_{ts}^{0,t,0}| \leq h \right) \\ &= \mathbb{P}_{0,t,0} \left(\sup_{0 \leq s \leq t} |B_s^{0,t,0}| \leq h \right), \end{aligned}$$

where it is used that

$$(\sqrt{t} B_s^{0,1,0})_{0 \leq s \leq 1} \stackrel{d}{=} (B_{st}^{0,t,0})_{0 \leq s \leq 1},$$

which can be seen by applying the scaling property of Brownian motion to the representation (3). Consequently,

$$F(h/\sqrt{t}) = \mathbb{P}_{0,t,0}(\mathcal{T}_{(-h,h)} > t) = \mathbb{P}_{0,t,0}(\mathcal{T}_{(-h,h)} = \infty). \quad (23)$$

Theorem 2.6.

(i) For the Brownian bridge with $|y| \geq h$,

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = h \int_0^T \frac{p_{T-t}(0,y)}{p_T(0,y)} \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt. \quad (24)$$

(ii) For the 3-dimensional Bessel bridge with positive $y \notin (x-h, x+h)$ and $x > 0$,

$$\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = h \int_0^T \frac{r_{T-t}^{(3)}(x,y)}{r_T^{(3)}(x,y)} \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt. \quad (25)$$

Proof. (i) Since $\mathbb{E}_{0,T,-y}[\mathcal{T}_{(-h,h)}] = \mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]$, we may assume that $y \geq h$. We derive (24) from (6) and (8) by showing that the Laplace transforms of the functions

$$T \mapsto \int_0^T \int_{-h}^h p_{T-t}(x,y) \mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t) p_t(0,x) dx dt$$

and

$$T \mapsto h \int_0^T p_{T-t}(0,y) \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt \quad (26)$$

coincide. In the following, we will denote the Laplace transform of a function f by

$$\mathcal{L}_{t,\gamma}(f(t)) = \int_0^\infty e^{-\gamma t} f(t) dt, \quad \gamma > 0.$$

Notice that we also indicate the integration variable t . Using Fubini's theorem and the convolution formula, we get

$$\mathcal{L}_{T,\gamma} \left(\int_0^T \int_{-h}^h p_{T-t}(x,y) \mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t) p_t(0,x) dx dt \right)$$

$$= \int_{-h}^h \mathcal{L}_{t,\gamma}(p_t(x, y)) \mathcal{L}_{t,\gamma}(\mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t) p_t(0, x)) dx. \quad (27)$$

To compute the second Laplace transform expression of (27) we use the series representation

$$\mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t) = \sum_{m=-\infty}^{\infty} (-1)^m e^{-\frac{2mh(mh-x)}{t}}, \quad |x| < h, \quad (28)$$

(see [2, formula (4.12)] or [15, formula (17)]). For $x \in (-h, h)$ it holds that

$$\left| \sum_{|m| \geq 2} (-1)^m e^{-\frac{2mh(mh-x)}{t}} \right| \leq \sum_{|m| \geq 2} e^{-\frac{2mh(mh-x)}{t}} \leq 2 \sum_{m \geq 2} \left(e^{-\frac{2h^2}{t}} \right)^m.$$

Hence one may interchange the summation and the Laplace transform, and since

$$\mathcal{L}_{t,\gamma}(p_t(x, z)) = \frac{1}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}|z-x|},$$

one gets

$$\begin{aligned} & \mathcal{L}_{t,\gamma}(\mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t) p_t(0, x)) \\ &= \mathcal{L}_{t,\gamma} \left(\sum_{m=-\infty}^{\infty} (-1)^m e^{-\frac{2mh(mh-x)}{t}} \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \right) \\ &= \sum_{m=-\infty}^{\infty} (-1)^m \mathcal{L}_{t,\gamma}(p_t(x, 2mh)) \\ &= \frac{1}{\sqrt{2\gamma}} \left(e^{-\sqrt{2\gamma}|x|} + (e^{\sqrt{2\gamma}x} + e^{-\sqrt{2\gamma}x}) \sum_{m=1}^{\infty} (-1)^m e^{-\sqrt{2\gamma}2mh} \right). \end{aligned}$$

By a straightforward calculation, this yields

$$\begin{aligned} & \int_{-h}^h \mathcal{L}_{t,\gamma}(p_t(x, y)) \mathcal{L}_{t,\gamma}(\mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t) p_t(0, x)) dx \\ &= \frac{h}{2\gamma} e^{-\sqrt{2\gamma}y} \frac{1 - e^{-\sqrt{2\gamma}2h}}{1 + e^{-\sqrt{2\gamma}2h}} \\ &= \frac{h}{2\gamma} e^{-\sqrt{2\gamma}y} \tanh(h\sqrt{2\gamma}). \end{aligned}$$

We continue with the Laplace transform of (26). From (22) we get

$$\begin{aligned}\mathcal{L}_{t,\gamma}\left(\frac{F(h/\sqrt{t})}{h\sqrt{2\pi t}}\right) &= \frac{1}{h^2}\sum_{k=1}^{\infty}\int_0^{\infty}\exp\left(-\frac{(2k-1)^2\pi^2 t}{8h^2}\right)\exp(-\gamma t)dt \\ &= \sum_{k=1}^{\infty}\frac{8}{(2k-1)^2\pi^2 t + 8h^2\gamma} \\ &= \frac{\tanh(h\sqrt{2\gamma})}{h\sqrt{2\gamma}},\end{aligned}$$

where we use that for any $x \in \mathbb{R}$ it holds

$$\tanh\left(\frac{\pi x}{2}\right) = \frac{4x}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2 + x^2}$$

(see [7, Subsection 1.421]). From the convolution formula we conclude that for $y \geq h$ it holds

$$\begin{aligned}\mathcal{L}_{T,\gamma}\left(h\int_0^T p_{T-t}(0,y)\frac{F(h/\sqrt{t})}{\sqrt{2\pi t}}dt\right) &= h\mathcal{L}_{t,\gamma}(p_t(0,y))\mathcal{L}_{t,\gamma}\left(\frac{F(h/\sqrt{t})}{\sqrt{2\pi t}}\right) \\ &= \frac{h}{\sqrt{2\gamma}}e^{-\sqrt{2\gamma}y}\frac{\tanh(h\sqrt{2\gamma})}{\sqrt{2\gamma}}.\end{aligned}$$

This implies (24), since we have shown that

$$\begin{aligned}\int_0^T\int_{-h}^h p_{T-t}(x,y)\mathbb{P}_{0,t,x}(\mathcal{T}_{(-h,h)} > t)p_t(0,x)dxdt \\ = h\int_0^T p_{T-t}(0,y)\frac{F(h/\sqrt{t})}{\sqrt{2\pi t}}dt.\end{aligned}\tag{29}$$

(ii) By (20) it holds

$$\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = \int_0^T\int_{x-h}^{x+h}\frac{z}{x}\Delta(z-x,h,t)\frac{r_{T-t}^{(3)}(z,y)}{r_T^{(3)}(x,y)}dzdt.$$

We notice that $\Delta(z-x,h,t)$ given in (12) can be written as

$$\Delta(z-x,h,t) = \frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}\left(e^{-\frac{(z-x+2h(2m))^2}{2t}} - e^{-\frac{(z-x+2h(2m+1))^2}{2t}}\right)$$

$$\begin{aligned}
&= p_t(x, z) \sum_{m=-\infty}^{\infty} (-1)^m e^{-\frac{2mh(mh-(x-z))}{t}} \\
&= p_t(x, z) \mathbb{P}_{0,t,z-x}(\mathcal{T}_{(-h,h)} > t),
\end{aligned}$$

where (28) is used for the last line. Since by (15) it holds

$$\frac{r_{T-t}^{(3)}(z, y)}{r_T^{(3)}(x, y)} = \frac{x p_{T-t}(z, y) - p_{T-t}(z, -y)}{z p_T(x, y) - p_T(x, -y)}, \quad (30)$$

we have

$$\begin{aligned}
&(p_T(x, y) - p_T(x, -y))^{-1} \mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] \\
&= \int_0^T \int_{x-h}^{x+h} \mathbb{P}_{0,t,z-x}(\mathcal{T}_{(-h,h)} > t) p_t(x, z) (p_{T-t}(z, y) - p_{T-t}(z, -y)) dz dt \\
&= \int_0^T \int_{-h}^h \mathbb{P}_{0,t,u}(\mathcal{T}_{(-h,h)} > t) p_t(0, u) p_{T-t}(u, y-x) du dt \\
&\quad - \int_0^T \int_{-h}^h \mathbb{P}_{0,t,u}(\mathcal{T}_{(-h,h)} > t) p_t(0, u) p_{T-t}(u, -(y+x)) du dt \\
&= h \int_0^T p_{T-t}(0, y-x) \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt - h \int_0^T p_{T-t}(0, y+x) \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt,
\end{aligned}$$

where the last equality is implied by (29). Then (30) yields

$$\mathbb{E}_{x,T,y}^{(3)}[\mathcal{T}_{(x-h,x+h)}] = h \int_0^T \frac{r_{T-t}^{(3)}(x, y)}{r_T^{(3)}(x, y)} \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt.$$

□

Remark 2.7. (i) We have not been able to find a probabilistic explanation for the appearance of the Kolmogorov distribution function in the representations (24) and (25).

(ii) Notice that the integrands w.r.t. t of the expressions in [Theorem 2.4, equation (13)] and in [Theorem 2.6, equation (24)] do not coincide. To see this, one can compare the double Laplace transform of

$$h p_{T-t}(0, y) \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} \quad \text{and} \quad \int_{-h}^h p_{T-t}(z, y) \Delta(z, h, t) dz.$$

Recall that $\frac{1}{\sqrt{2\pi t}} = p_t(0, 0)$, and $F(h/\sqrt{t}) = \mathbb{P}_{0,t,0}(\mathcal{T}_{(-h,h)} > t)$ by (23), so that, by a similar computation as in the proof of Theorem 2.6,

$$\begin{aligned} \mathcal{L}_{T,\gamma} & \left(h \int_0^T p_{T-t}(0, y) \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} e^{-\lambda t} dt \right) \\ &= \mathcal{L}_{T,\gamma} \left(h \int_0^T p_{T-t}(0, y) \mathbb{P}_{0,t,0}(\mathcal{T}_{(-h,h)} > t) p_t(0, 0) e^{-\lambda t} dt \right) \\ &= \frac{h}{\sqrt{2\gamma}} e^{-\sqrt{2\gamma}y} \frac{\tanh(h\sqrt{2(\gamma+\lambda)})}{\sqrt{2(\gamma+\lambda)}}. \end{aligned}$$

On the other hand, we have $\Delta(z, h, t) = \mathbb{P}_{0,t,z}(\mathcal{T}_{(-h,h)} > t) p_t(0, z)$, which yields after some calculations that

$$\begin{aligned} \mathcal{L}_{T,\gamma} & \left(\int_0^T \int_{-h}^h p_{T-t}(z, y) \mathbb{P}_{0,t,z}(\mathcal{T}_{(-h,h)} > t) p_t(0, z) dz e^{-\lambda t} dt \right) \\ &= \frac{1}{\lambda\sqrt{2\gamma}} e^{-\sqrt{2\gamma}|y|} \left[1 - \frac{e^{-\sqrt{2\gamma}h} + e^{\sqrt{2\gamma}h}}{e^{-\sqrt{2(\gamma+\lambda)}h} + e^{\sqrt{2(\gamma+\lambda)}h}} \right]. \end{aligned}$$

We conclude this section by pointing out a connection between the Kolmogorov distribution function and the density of the last visit of 0 by a Brownian motion before $\mathcal{T}_{(-h,h)}$. This connection has been noticed by Knight in [9, Corollary 2.1].

We provide here a different proof for this fact.

Proposition 2.8. *When \widehat{W} is a Brownian motion killed at $\mathcal{T}_{(-h,h)}$, then $t \mapsto \frac{F(h/\sqrt{t})}{h\sqrt{2\pi t}}$ is the density of the last passage time $\lambda_0 := \sup\{t > 0 : \widehat{W}_t = 0\}$.*

Proof. It is well known that (cf. [3, Part I, Chapter II, No 20, p. 26])

$$\mathbb{P}_0^{\widehat{W}}(\lambda_0 \in dt) = \frac{\widehat{q}_t(0, 0)}{\widehat{G}_0(0, 0)} dt, \quad (31)$$

where

$$\widehat{q}_t(x, y) = \frac{1}{\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \left(e^{-\frac{(x-y+2k\cdot 2h)^2}{2t}} - e^{-\frac{(x+y+(2k+1)2h)^2}{2t}} \right)$$

is the transition density (w.r.t. the Lebesgue measure) of \widehat{W} (cf. [3, Part I, Appendix I, No 6, p. 126]), and

$$\widehat{G}_0(x, y) = \begin{cases} \frac{(x+h)(h-y)}{h}, & -h \leq x \leq y \leq h, \\ \frac{(y+h)(h-x)}{h}, & -h \leq y \leq x \leq h, \end{cases}$$

denotes the 0-resolvent kernel (see [3, Part I, Appendix I, No 6, p. 126])). From (31) we get

$$\begin{aligned} \mathbb{P}_0^{\widehat{W}}(\lambda_0 \in dt)/dt &= \frac{1}{h\sqrt{2\pi t}} \sum_{k=-\infty}^{\infty} \left(e^{-\frac{4(2k)^2 h^2}{2t}} - e^{-\frac{4(2k+1)^2 h^2}{2t}} \right) \\ &= \frac{1}{h\sqrt{2\pi t}} \left(1 + 2 \sum_{k=1}^{\infty} (-1)^k e^{-\frac{4k^2 h^2}{2t}} \right). \end{aligned}$$

□

3 Application

In this section we apply our previous results to prove in Corollary 3.3 an estimate which was used by Walsh in [17]. The convergence analysis there succeeds to identify the leading constants appearing in the error expansion of

$$\mathbb{E}[g(X_n^{(n)}) - g(X_T)]$$

(cf. [17, equation (14)]) in terms of expressions depending on the function g (which is assumed to be exponentially bounded and piecewise twice continuously differentiable). Here $X_t = \sigma W_t$ for $t \geq 0$ and $(X_k^{(n)})_{k=0}^n$ denotes a symmetric simple random walk with step size $\sigma\sqrt{T/n}$. For simplicity, we will put $\sigma = 1$ and hence consider

$$\mathbb{E}[g(W_n^{(n)}) - g(W_T)].$$

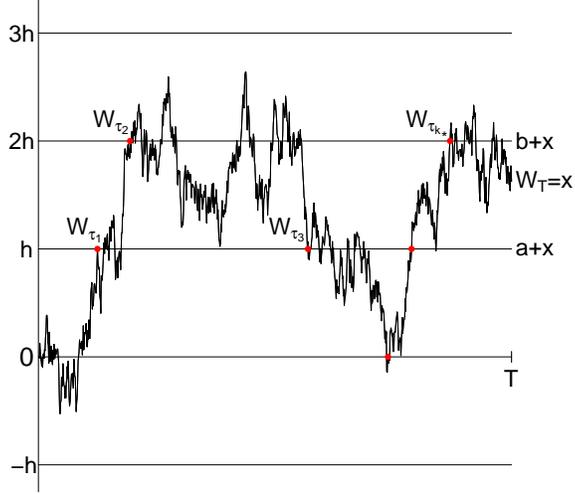
The central idea for this error analysis is to build this random walk from a given Brownian motion $(W_t)_{t \geq 0}$ so that both processes are on the same probability space: Fix $T > 0$ and $n \in \mathbb{N}$. For $h := \sqrt{T/n}$ define $\tau_0 := 0$ and

$$\tau_k := \inf\{t > \tau_{k-1} : |W_t - W_{\tau_{k-1}}| = h\}, \quad k \geq 1.$$

Then $(W_{\tau_k} - W_{\tau_{k-1}})_{k=1}^{\infty}$ is a sequence of i.i.d. random variables with $\mathbb{P}(W_{\tau_k} - W_{\tau_{k-1}} = h) = \mathbb{P}(W_{\tau_k} - W_{\tau_{k-1}} = -h) = \frac{1}{2}$. Let k_* be such that $\tau_{k_*} \leq T < \tau_{k_*+1}$. In [17, Section 9] the conditional probability

$$q(x) := \mathbb{P}(k_* \text{ is even} | W_T = x), \quad x \in \mathbb{R}$$

is introduced (there k_* is denoted by L). We want to study and estimate q . Notice that k_* is an even number if and only if $W_{\tau_{k_*}}$ is an even multiple of h . The process $(W_t)_{0 \leq t \leq T}$ given $W_0 = 0, W_T = x$ is identical in law with a Brownian bridge from 0 to x and of length T . We denote this bridge by $(B_t^{0,T,x})_{0 \leq t \leq T}$. By time reversion, we get the Brownian bridge $(B_t^{x,T,0})_{0 \leq t \leq T}$.



We fix $k \in \mathbb{Z}$ and assume $x \in ((2k-1)h, (2k+1)h)$. For simplicity put

$$\underline{h} := (2k-1)h, \quad \bar{h} := (2k+1)h, \quad h_e := 2kh \quad (32)$$

for the lower and upper value, and for the 'even' midpoint of the interval. Then, given $W_T = x$, we have that ' k_* is even' is the same as ' $W_{\tau_{k_*}} = h_e$ '. For the time-reversed bridge associated to $\mathbb{P}_{x,T,0}$ we get

$$q(x) = \mathbb{P}(k_* \text{ is even} | W_T = x) = \begin{cases} \mathbb{P}_{x,T,0}(\mathcal{T}_{h_e} < \mathcal{T}_{\underline{h}}), & \underline{h} < x < h_e, \\ \mathbb{P}_{x,T,0}(\mathcal{T}_{h_e} < \mathcal{T}_{\bar{h}}), & h_e < x < \bar{h}, \end{cases}$$

where $\mathcal{T}_y(\omega) := \inf\{t > 0 : \omega(t) = y\}$ for $\omega \in C[0, T]$. For the time-reversed and by $-x$ shifted bridge this means it hits the 'shifted even line' $h_e - x$ before the shifted odd one:

$$\mathbb{P}(k_* \text{ is even} | W_T = x) = \begin{cases} \mathbb{P}_{0,T,-x}(\mathcal{T}_{h_e-x} < \mathcal{T}_{\underline{h}-x}), & \underline{h} < x < h_e, \\ \mathbb{P}_{0,T,-x}(\mathcal{T}_{h_e-x} < \mathcal{T}_{\bar{h}-x}), & h_e < x < \bar{h}. \end{cases}$$

For any $a < 0 < b$ and $y \notin (a, b)$ it holds

$$\mathbb{E}[B_{\mathcal{T}_{(a,b)}}^{0,T,y}] = a(1 - \mathbb{P}_{0,T,y}(\mathcal{T}_b < \mathcal{T}_a)) + b\mathbb{P}_{0,T,y}(\mathcal{T}_b < \mathcal{T}_a).$$

Consequently,

$$\mathbb{P}_{0,T,y}(\mathcal{T}_b < \mathcal{T}_a) = \frac{-a}{b-a} + \frac{\mathbb{E}[B_{\mathcal{T}_{(a,b)}}^{0,T,y}]}{b-a}.$$

Hence

$$q(x) = \begin{cases} \frac{x - \underline{h}}{h} + \frac{\mathbb{E}[B_{\mathcal{T}_{(\underline{h}-x, h_e-x)}}^{0,T,-x}]}{h}, & \underline{h} < x < h_e, \\ \frac{\bar{h} - x}{h} - \frac{\mathbb{E}[B_{\mathcal{T}_{(h_e-x, \bar{h}-x)}}^{0,T,-x}]}{h}, & h_e < x < \bar{h}. \end{cases} \quad (33)$$

Arguing that Brownian bridge and Brownian motion have a similar exit behavior for small h , in [17, equation (20)] it is stated that

$$q(x) = \frac{\text{dist}(x, \mathbb{N}_o^h)}{h} + O(h), \quad (34)$$

where $x \in \mathbb{R}$, $\mathbb{N}_o^h = \{(2k+1)h : k \in \mathbb{Z}\}$ and $\text{dist}(x, \mathbb{N}_o^h) := \inf\{|x-y| : y \in \mathbb{N}_o^h\}$. If one compares (34) with (33) one notices that

$$\text{dist}(x, \mathbb{N}_o^h) \quad \text{is equal to} \quad x - \underline{h} \text{ or } \bar{h} - x,$$

so that we should have

$$\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}] = O(h^2),$$

where (32) was used to rewrite $\mathcal{T}_{(\underline{h}-x, h_e-x)}$ and $\mathcal{T}_{(h_e-x, \bar{h}-x)}$ as

$$\mathcal{T}(x) := \mathcal{T}_{(kh-x, (k+1)h-x)} \text{ if } x \in (kh, (k+1)h). \quad (35)$$

In view of this we prove the following lemma.

Lemma 3.1. *Suppose that $a < 0 < b$, $y \notin (a, b)$, $T > 0$ and $h > 0$. Then*

$$|\mathbb{E}[B_{\mathcal{T}_{(a,b)}}^{0,T,y}]| \leq \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}]}{T} \left(2(|a| \vee b) + |y| + 3\sqrt{2T} \right). \quad (36)$$

Proof. By Markov's inequality we have

$$\left| \mathbb{E}[B_{\mathcal{T}_{(a,b)}}^{0,T,y} \mathbf{1}_{\{\mathcal{T}_{(a,b)} > T/2\}}] \right| \leq (|a| \vee b) \mathbb{P}_{0,T,y}(\mathcal{T}_{(a,b)} > T/2) \leq \frac{2(|a| \vee b)}{T} \mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}].$$

To estimate $\left| \mathbb{E} \left[B_{\mathcal{T}_{(a,b)}}^{0,T,y} \mathbf{1}_{\{\mathcal{T}_{(a,b)} \leq T/2\}} \right] \right|$ we let

$$\tilde{B}_t^{0,T,y} := (T-t) \int_0^t \frac{dW_s}{T-s} + \frac{t}{T}y, \quad t \in [0, T],$$

and $\tilde{B}_T^{0,T,y} := y$. Then $(\tilde{B}_t^{0,T,y})_{0 \leq t \leq T} \stackrel{d}{=} (B_t^{0,T,y})_{0 \leq t \leq T}$ (cf. (4)). Setting

$$\tilde{\mathcal{T}} := \inf \{ t \in [0, T] : \tilde{B}_t^{0,T,y} \notin (a, b) \}$$

yields $\mathbb{E} \left[\tilde{B}_{\tilde{\mathcal{T}}}^{0,T,y} \mathbf{1}_{\{\tilde{\mathcal{T}} \leq T/2\}} \right] = \mathbb{E} \left[B_{\mathcal{T}_{(a,b)}}^{0,T,y} \mathbf{1}_{\{\mathcal{T}_{(a,b)} \leq T/2\}} \right]$ and

$$\begin{aligned} \left| \mathbb{E} \left[\tilde{B}_{\tilde{\mathcal{T}}}^{0,T,y} \mathbf{1}_{\{\tilde{\mathcal{T}} \leq T/2\}} \right] \right| &\leq T \left| \mathbb{E} \left[\mathbf{1}_{\{\tilde{\mathcal{T}} \leq T/2\}} \int_0^{\tilde{\mathcal{T}} \wedge (T/2)} (T-s)^{-1} dW_s \right] \right| \\ &\quad + \mathbb{E} \left| (\tilde{\mathcal{T}} \wedge (T/2)) \int_0^{\tilde{\mathcal{T}} \wedge (T/2)} (T-s)^{-1} dW_s \right| + \frac{|y|}{T} \mathbb{E}[\tilde{\mathcal{T}}]. \end{aligned}$$

By the optional stopping theorem, Hölder's and Markov's inequality it holds that

$$\begin{aligned} \left| \mathbb{E} \left[\mathbf{1}_{\{\tilde{\mathcal{T}} \leq T/2\}} \int_0^{\tilde{\mathcal{T}} \wedge \frac{T}{2}} (T-s)^{-1} dW_s \right] \right| &= \left| \mathbb{E} \left[\mathbf{1}_{\{\tilde{\mathcal{T}} > T/2\}} \int_0^{\tilde{\mathcal{T}} \wedge \frac{T}{2}} (T-s)^{-1} dW_s \right] \right| \\ &\leq \left(\frac{2}{T} \mathbb{E}[\tilde{\mathcal{T}}] \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_0^{\tilde{\mathcal{T}} \wedge \frac{T}{2}} \frac{ds}{(T/2)^2} \right] \right)^{\frac{1}{2}} \\ &\leq \left(\frac{2}{T} \mathbb{E}[\tilde{\mathcal{T}}] \right)^{\frac{1}{2}} \left(\left(\frac{2}{T} \right)^{\frac{1}{2}} \mathbb{E} \left(\tilde{\mathcal{T}} \wedge \frac{T}{2} \right) \right)^{\frac{1}{2}} \\ &\leq (\sqrt{2/T})^3 \mathbb{E}[\tilde{\mathcal{T}}]. \end{aligned}$$

Again by Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left| \left(\tilde{\mathcal{T}} \wedge \frac{T}{2} \right) \int_0^{\tilde{\mathcal{T}} \wedge \frac{T}{2}} \frac{dW_s}{T-s} \right| &\leq \left(\mathbb{E} \left(\tilde{\mathcal{T}} \wedge \frac{T}{2} \right)^2 \right)^{\frac{1}{2}} \left(\mathbb{E} \left[\int_0^{\tilde{\mathcal{T}} \wedge \frac{T}{2}} \frac{ds}{(T-s)^2} \right] \right)^{\frac{1}{2}} \\ &\leq \left(\frac{T}{2} \mathbb{E}[\tilde{\mathcal{T}}] \right)^{\frac{1}{2}} \left(\left(\frac{2}{T} \right)^{\frac{1}{2}} \mathbb{E} \left(\tilde{\mathcal{T}} \wedge \frac{T}{2} \right) \right)^{\frac{1}{2}} \\ &\leq \sqrt{2/T} \mathbb{E}[\tilde{\mathcal{T}}]. \end{aligned}$$

From the above estimates and $\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] = \mathbb{E}[\tilde{\mathcal{T}}]$ we get (36). \square

Now we derive estimates for $\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}]$.

Lemma 3.2. *Let $T > 0$ be fixed. Suppose that $a < 0 < b$ and $y \notin (a, b)$.*

(i) *It holds*

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \leq \begin{cases} 4b(|a| + y/2) \wedge T & \text{if } y \geq b, \\ 4|a|(b + |y|/2) \wedge T & \text{if } y \leq a. \end{cases} \quad (37)$$

(ii) *Define for $|y| \geq h$*

$$C(T, h, y) := \frac{\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}]}{h^2}. \quad (38)$$

Then

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \leq C(b-a)^2 \quad \text{if } |y| \geq b-a,$$

where $C = C(T, b-a, y)$, and it holds

$$\lim_{b-a \rightarrow 0, a < 0 < b} C(T, b-a, y) = 1.$$

Proof. (i) We first assume that $y \geq b$. Then

$$\begin{aligned} \mathcal{T}_{(a,b)}(B^{0,T,y}) &\stackrel{d}{=} \inf \left\{ t \in [0, T) : (1 - \frac{t}{T})W_{\frac{Tt}{T-t}} + y\frac{t}{T} \notin (a, b) \right\} \\ &= \inf \left\{ t \in [0, T) : (1 - \frac{t}{T})W_{\frac{Tt}{T-t}} \notin (a - y\frac{t}{T}, b - y\frac{t}{T}) \right\} \\ &\leq \inf \left\{ t \in [0, T) : (1 - \frac{t}{T})W_{\frac{Tt}{T-t}} \notin (a - y\frac{t}{T}, b) \right\} \\ &\leq \inf \left\{ t \in [0, T/2) : W_{\frac{Tt}{T-t}} \notin (2a - y, 2b) \right\}. \end{aligned}$$

Since $u : [0, T/2) \rightarrow [0, T)$ given by $t \mapsto \frac{Tt}{T-t}$ is one-to-one, increasing and $t = \frac{Tu(t)}{T+u(t)}$, we get

$$\inf \left\{ t \in [0, T/2) : W_{\frac{Tt}{T-t}} \notin (2a - y, 2b) \right\} \leq \inf \{ u \in [0, T) : W_u \notin (2a - y, 2b) \}.$$

Since by definition, $\mathcal{T}_{(a,b)}(B^{0,T,y}) \leq T$ if $y \notin (a, b)$, we have

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \leq \mathbb{E}[\inf \{ u \in [0, T) : W_u \notin (2a - y, 2b) \}] \wedge T$$

$$\begin{aligned}
&\leq \mathbb{E}_0[\mathcal{T}_{(2a-y, 2b)}] \wedge T \\
&\leq |2a - y| 2b \wedge T = 4b(|a| + y/2) \wedge T.
\end{aligned}$$

The case $y \leq a$ follows similarly.

(ii) For $|y| \geq b - a$ we have $\mathbb{E}_{0,T,y}[\mathcal{T}_{(a,b)}] \leq \mathbb{E}_{0,T,y}[\mathcal{T}_{(-(b-a), b-a)}]$. Using (38) with $h = b - a$ gives for $|y| \geq h$ that

$$\mathbb{E}_{0,T,y}[\mathcal{T}_{(-h,h)}] = h^2 C(T, h, y),$$

and from Theorem 2.4 (i) we have that $C(T, h, y)$ converges to 1 as $h \rightarrow 0$. \square

Hence for $x \in (kh, (k+1)h)$ and $k \in \{-1, 0\}$ we get by Lemma 3.2 (i) that $\mathbb{E}_{0,T,-x}[\mathcal{T}_{(kh-x, (k+1)h-x)}] \leq ch^2$, and for $k \notin \{-1, 0\}$ we use (ii). Then Lemma 3.1 implies

$$\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}] = O(h^2).$$

However, this equality does not hold uniformly in x with the consequence that we can not use (34) in integrals like (40) below. For example, for the sequence $x_k := (k + 0.5)h$ it holds

$$\mathbb{E}[B_{\mathcal{T}(x_k)}^{0,T,-x_k}] = \mathbb{E}[B_{\mathcal{T}(-h/2, h/2)}^{0,T,-x_k}] \rightarrow -h/2, \quad k \rightarrow \infty, \quad (39)$$

which contradicts that $\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}] = O(h^2)$ holds uniformly in x . The limit in (39) can be easily seen from the representation $(B_t^{0,T,-x_k})_{0 \leq t \leq T} \stackrel{d}{=} (B_t^{0,T,0} - \frac{x_k t}{T})_{0 \leq t \leq T}$. For any path $t \mapsto B_t^{0,T,0}(\omega)$ one can find a sufficiently large x_k , such that the transformed path $t \mapsto B_t^{0,T,0}(\omega) - \frac{x_k t}{T}$ exits $(-h/2, h/2)$ first at $-h/2$.

Nevertheless, one can prove the estimates needed in [17], where q was used inside an integral over the real line. We only discuss here [17, equation (38)], because the calculations for the other cases where the function q appears are similar. For $\sigma = 1$ and denoting $\mathbb{N}_e^h := \{2kh : k \in \mathbb{Z}\}$ the last term in [17, equation (38)] can be written as

$$\begin{aligned}
&\int_{-\infty}^{\infty} (2h^2 - \text{dist}^2(x, \mathbb{N}_e^h)) q(x) p_T(0, x) dx \\
&= \int_{-\infty}^{\infty} (2h^2 - \text{dist}^2(x, \mathbb{N}_e^h)) \text{dist}(x, \mathbb{N}_e^h) h^{-1} p_T(0, x) dx \\
&\quad + \int_{-\infty}^{\infty} (2h^2 - \text{dist}^2(x, \mathbb{N}_e^h)) (q(x) - \text{dist}(x, \mathbb{N}_e^h) h^{-1}) p_T(0, x) dx. \quad (40)
\end{aligned}$$

The calculation for the integral containing $\text{dist}(x, \mathbb{N}_o^h)h^{-1}$ is carried out in [17]. It remains to show that the other integral behaves like $O(h^3)$. Since it holds that $(2h^2 - \text{dist}^2(x, \mathbb{N}_e^h)) \leq 2h^2$ and

$$|q(x) - \text{dist}(x, \mathbb{N}_o^h)h^{-1}| = |\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}]| h^{-1}$$

by (33) and (35), we get the desired estimate from the next corollary.

Corollary 3.3. *For $T > 0$ and $h = \sqrt{T/n}$, there exists a $C = C(T) > 0$ such that*

$$\int_{-\infty}^{\infty} |\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}]| p_T(0, x) dx \leq Ch^2,$$

where $\mathcal{T}(x)$ is given in (35).

Proof. Since $B_{\mathcal{T}(x)}^{0,T,-x} \stackrel{d}{=} -B_{\mathcal{T}(-x)}^{0,T,x}$, it suffices to estimate the integral over $[0, \infty)$. By (36),

$$|\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}]| \leq \frac{\mathbb{E}_{0,T,-x}[\mathcal{T}(x)]}{T} (2h + |x| + 3\sqrt{2T}).$$

If $x \in (0, h)$, then $\mathcal{T}(x) = \mathcal{T}_{(-x, h-x)}$, and estimate (37) gives

$$\mathbb{E}_{0,T,-x}[\mathcal{T}(x)] \leq 4x \left(h - x + \frac{x}{2} \right) \leq 2h^2.$$

For $x \geq h$ it holds

$$\mathbb{E}_{0,T,-x}[\mathcal{T}(x)] \leq C(T, h, -x)h^2$$

by Lemma 3.2. From the above estimates we get

$$\int_0^h |\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}]| p_T(0, x) dx \leq 2h^2 \int_0^h \frac{2h + x + 3\sqrt{2T}}{T} p_T(0, x) dx \leq C(T)h^2, \quad (41)$$

and

$$\begin{aligned} \int_h^\infty |\mathbb{E}[B_{\mathcal{T}(x)}^{0,T,-x}]| p_T(0, x) dx &\leq h^2 \int_h^\infty C(T, h, -x) \frac{2h + x + 3\sqrt{2T}}{T} p_T(0, x) dx \\ &\leq C(T)h^2 \int_h^\infty (1 + x) C(T, h, -x) p_T(0, x) dx \\ &\leq C(T)h^2, \end{aligned} \quad (42)$$

where the constant $C(T)$ varies from line to line. The last inequality in (42) can be seen as follows. We use the representation (24) for (38) and substitute $t = h^2u$, so that

$$\begin{aligned} C(T, h, -x) &= \mathbb{E}_{0,T,-x}[\mathcal{T}_{(-h,h)}]h^{-2} = h^{-1} \int_0^T \frac{p_{T-t}(0, x)}{p_T(0, x)} \frac{F(h/\sqrt{t})}{\sqrt{2\pi t}} dt \\ &= \int_0^{T/h^2} \frac{p_{T-h^2u}(0, x)}{p_T(0, x)} \frac{F(1/\sqrt{u})}{\sqrt{2\pi u}} du. \end{aligned}$$

By Fubini's theorem it holds

$$\int_0^\infty \frac{F(1/\sqrt{u})}{\sqrt{2\pi u}} \int_h^\infty (1+x)p_{T-h^2u}(0, x)\mathbb{1}_{[0,T/h^2)}(u) dx du \leq C(T),$$

where we used for the last line that $u \mapsto \frac{F(1/\sqrt{u})}{\sqrt{2\pi u}} \mathbb{1}_{(0,\infty)}(u)$ is a density. The claim then follows by (41) and (42). \square

References

- [1] Ph. Biane, J. Pitman and M. Yor, *Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions*, Bull. Amer. Math. Soc. (N.S.), 38 (2001), 435–465.
- [2] L. Beghin and E. Orsingher, *On the maximum of the generalized Brownian bridge*, Lith. Math. J., 39 (1999), 157–167.
- [3] A. N. Borodin and P. Salminen, *Handbook of Brownian Motion - Facts and Formulae*, Probability and its Applications. Birkhäuser Verlag, Basel, 2nd edition, 2nd printing, 2015.
- [4] K. L. Chung and J. B. Walsh, *Markov Processes, Brownian Motion, and Time Symmetry*, Grundlehren der Mathematischen Wissenschaften, 249, 2nd edition, Springer, New York, 2005.
- [5] J. L. Doob, *Heuristic approach to the Kolmogorov-Smirnov theorems*, Ann. Math. Stat., 20 (1949), 393–403.

- [6] P. Fitzsimmons, J. Pitman and M. Yor, *Markovian bridges: construction, Palm interpretation, and splicing*. In *Seminar on Stochastic Processes*, Progress in Probability 32 (1992), 101–134.
- [7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 8th edition, Elsevier/Academic Press, Amsterdam, 2014.
- [8] K. Itô and H.P. McKean, *Diffusion Processes and their Sample Paths*, Die Grundlehren der Mathematischen Wissenschaften, 125, 2nd printing, corrected, Berlin-New York, 1974.
- [9] F. B. Knight, *Brownian local times and taboo processes*, Trans. Amer. Math. Soc. 143 (1969), 173–185.
- [10] A. N. Kolmogorov, *Sulla determinazione empirica delle leggi di probabilita*, Giorn. Ist. Ital. Attuari, 4 (1933), 1–11.
- [11] D. Kroese, T. Taimre and Z. Botev, *Handbook of Monte Carlo Methods*, Wiley Series in Probability and Statistics, John Wiley and Sons, New York, 2011.
- [12] A. Luoto, *Time-dependent weak rate of convergence for functions of generalized bounded variation*, preprint (2017), arXiv:1609.05768v3 [math.PR].
- [13] J. Pitman and M. Yor, *The law of the maximum of a Bessel bridge*, Electr. J. Probab., 4 (1999), 1–15.
- [14] P. Salminen, *On last exit decomposition of linear diffusions*, Studia Sci. Math. Hungar., 33 (1997), 251–262.
- [15] P. Salminen and M. Yor, *On hitting times of affine boundaries by reflecting Brownian motion and Bessel processes*, Period. Math. Hungar., 62 (2011), 75–101.
- [16] N. V. Smirnov, *On the estimation of the discrepancy between empirical curves of distribution for two independent samples*, Bul. Math. de l’Univ. de Moscou, 2 (1939), 3–14 (in Russian).
- [17] J. B. Walsh, *The rate of convergence of the binomial tree scheme*, Finance Stoch., 7 (2003), 337–361.