

CONDITIONAL TAIL INDEPENDENCE IN ARCHIMEDEAN COPULA MODELS

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ABSTRACT. Consider a random vector \mathbf{U} , whose distribution function coincides in its upper tail with that of an Archimedean copula. We report the fact that the conditional distribution of \mathbf{U} , conditional on one of its components, has under a mild condition on the generator function independent upper tails, no matter what the unconditional tail behavior is. This finding is extended to Archimax copulas.

1. INTRODUCTION

Let $\mathbf{U} = (U_1, \dots, U_d)$ be a random vector (rv), whose distribution function (df) F is in the domain of attraction of a multivariate extreme value df G , denoted by $F \in \mathcal{D}(G)$, i.e., there are constants $\mathbf{a}_n = (a_{n1}, \dots, a_{nd}) > \mathbf{0} \in \mathbb{R}^d$, $\mathbf{b}_n = (b_{n1}, \dots, b_{nd}) \in \mathbb{R}^d$, $n \in \mathbb{N}$, such that for each $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$F^n(\mathbf{a}_n \mathbf{x} + \mathbf{b}_n) \rightarrow_{n \rightarrow \infty} G(\mathbf{x}).$$

Note that all operations on vectors such as $\mathbf{x} + \mathbf{y}$, $\mathbf{x}\mathbf{y}$ etc. are always meant componentwise.

The rv \mathbf{U} , or, equivalently, the df F , is said to have asymptotically independent (upper) tails, if

$$G(\mathbf{x}) = \prod_{i=1}^d G_i(x_i),$$

where G_i , $1 \leq i \leq d$, denote the univariate margins of G .

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We require in this paper that the df F of \mathbf{U} coincides in its upper tail with a copula, say C , i.e., there exists $\mathbf{u}_0 = (u_{01}, \dots, u_{0d}) \in (0, 1)^d$ such that

$$F(\mathbf{u}) = C(\mathbf{u}), \quad \mathbf{u} \in [\mathbf{u}_0, \mathbf{1}] \subset \mathbb{R}^d.$$

Each univariate margin of a copula is the uniform distribution $H(u) = u$, $0 \leq u \leq 1$, and, thus, each univariate margin of F equals $H(u)$ for $u \in [v_0, 1]$, where $v_0 := \max_{1 \leq i \leq d} u_{0i}$.

The significance of copulas is due to Sklar's theorem ([Sklar \(1959, 1996\)](#)), by which an arbitrary multivariate df can be represented as a copula together with its univariate margins. The dependence structure among the margins of an arbitrary rv is, therefore, determined by the copula. For an introduction to copulas we refer to [Nelsen \(2006\)](#).

We require in this paper that the upper tail of C is that of an Archimedean copula C_φ , i.e., there exists a convex and strictly decreasing function $\varphi : (0, 1] \rightarrow [0, \infty)$ with $\varphi(1) = 0$, such that

$$C_\varphi(\mathbf{u}) = \varphi^{-1}(\varphi(u_1) + \dots + \varphi(u_d))$$

for $\mathbf{u} \in [\mathbf{u}_0, \mathbf{1}] \subset \mathbb{R}^d$, where $\mathbf{u}_0 = (u_{01}, \dots, u_{0d}) \in (0, 1)^d$.

A prominent example is $\varphi_p(s) := (1 - s)^p$, $s \in [0, 1]$, where $p \geq 1$. In this case we obtain

$$(1) \quad C_{\varphi_p}(\mathbf{u}) = 1 - \left(\sum_{i=1}^d (1 - u_i)^p \right)^{1/p}, \quad \mathbf{u} \in [\mathbf{u}_0, \mathbf{1}].$$

Note that

$$C_{\varphi_p}(\mathbf{u}) := \max \left(0, 1 - \left(\sum_{i=1}^d (1 - u_i)^p \right)^{1/p} \right), \quad \mathbf{u} \in [0, 1]^d,$$

defines a multivariate df only in dimension $d = 2$, see, e.g., [McNeil and Nešlehová \(2009, Examples 2.1, 2.2\)](#). But one can find for arbitrary dimension $d \geq 2$ a rv, whose df satisfies equation (1), see, e.g., [Falk \(2019, \(2.15\)\)](#). This is the reason, why

we require the Archimedean structure of C_φ only on some upper interval $[\mathbf{u}_0, \mathbf{1}]$ and we do not speak of C_φ as a copula, but rather of a distribution function.

The behavior of $C_\varphi(\mathbf{u})$ for \mathbf{u} close to $\mathbf{1} \in \mathbb{R}^d$ determines the upper tail behavior of the components of \mathbf{U} . Precisely, suppose that $C_\varphi \in \mathcal{D}(G)$, i.e.,

$$C_\varphi\left(\mathbf{1} + \frac{\mathbf{x}}{n}\right)^n \rightarrow_{n \rightarrow \infty} G(\mathbf{x}), \quad \mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d,$$

where the norming constants are prescribed by the univariate margins of C_φ , which is the df $H(u) = u$, $u \in [v_0, 1]$. We obviously have for arbitrary $x \leq 0$ and n large enough

$$H\left(1 + \frac{x}{n}\right)^n = \left(1 + \frac{x}{n}\right)^n \rightarrow \exp(x).$$

The multivariate max-stable df G , consequently, has standard negative exponential margins $G_i(x) = \exp(x)$, $x \leq 0$.

Moreover, there exists a norm $\|\cdot\|_D$ on \mathbb{R}^d , such that $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_D)$, $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$; see, e.g., [Falk \(2019\)](#). This norm $\|\cdot\|_D$ describes the asymptotic tail dependence of the margins of C_φ ; the index D , therefore, means dependence. In particular $\|\cdot\|_D = \|\cdot\|_1$ is the case of (asymptotic) independence of the margins, whereas $\|\cdot\|_D = \|\cdot\|_\infty$ yields their total dependence. For the df C_{φ_p} in (1) we obtain, for example, for n large,

$$\begin{aligned} C_{\varphi_p}\left(\mathbf{1} + \frac{\mathbf{x}}{n}\right)^n &= \left(1 - \frac{1}{n} \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}\right)^n \\ &\rightarrow_{n \rightarrow \infty} \exp\left(-\|\mathbf{x}\|_p\right), \quad \mathbf{x} = (x_1, \dots, x_d) \leq \mathbf{0} \in \mathbb{R}^d, \end{aligned}$$

where $\|\mathbf{x}\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$, $p \geq 1$, is the logistic norm on \mathbb{R}^d . In this case we have tail independence only for $p = 1$.

In this paper we investigate the problem, if conditioning on a margin $U_j = u$ has an influence on the tail dependence of the left margins $U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_d$. Actually, we will show that the rv $(U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_d)$, conditional on $U_j = u$, has in general independent tails, for each choice of j , no matter what

the unconditional tail behavior is; see Section 3. This is achieved under a mild condition on the generator function φ , which is introduced in Section 2.

2. CONDITION ON THE GENERATOR FUNCTION

Our results are achieved under the following condition on the generator function φ . There exists a number $p \geq 1$ such that

$$(C0) \quad \lim_{s \downarrow 0} \frac{\varphi(1 - sx)}{\varphi(1 - s)} = x^p, \quad x > 0.$$

REMARK 2.1. The exponent p in condition (C0) is necessarily greater than one by the convexity of φ , which can easily be seen as follows. We have for arbitrary $\lambda, x, y \in (0, 1]$

$$\varphi(\lambda x + (1 - \lambda)y) \leq \lambda \varphi(x) + (1 - \lambda)\varphi(y).$$

Setting $x = 1 - s$ and $y = 1$, we obtain

$$\varphi(\lambda(1 - s) + 1 - \lambda) = \varphi(1 - \lambda s) \leq \lambda \varphi(1 - s)$$

and, thus,

$$\lim_{s \downarrow 0} \frac{\varphi(1 - \lambda s)}{\varphi(1 - s)} = \lambda^p \leq \lambda.$$

But this requires $p \geq 1$.

A df C_φ , whose generator satisfies condition (C0), is in the domain of attraction of a multivariate extreme value distribution. Precisely, we have the following result.

Proposition 2.2. *Suppose that the generator φ satisfies condition (C0). Then we have $C_\varphi \in \mathcal{D}(G)$, where $G(\mathbf{x}) = \exp\left(-\|\mathbf{x}\|_p\right)$, $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$.*

Proof. First we show that condition (C0) implies for $x > 0$

$$(2) \quad \lim_{s \downarrow 0} \frac{1 - \varphi^{-1}(sx)}{1 - \varphi^{-1}(s)} = x^{1/p}.$$

Choose $\delta_{sx}, \delta_s \in (0, 1)$ such that

$$\varphi(1 - \delta_{sx}) = sx, \quad \varphi(1 - \delta_s) = s,$$

i.e.,

$$\varphi^{-1}(sx) = 1 - \delta_{sx}, \quad \varphi^{-1}(s) = 1 - \delta_s.$$

Condition (C0) implies for $s \downarrow 0$

$$x = \frac{\varphi(1 - \delta_{sx})}{\varphi(1 - \delta_s)} = \frac{\varphi\left(1 - \delta_s \frac{\delta_{sx}}{\delta_s}\right)}{\varphi(1 - \delta_s)} \sim \left(\frac{\delta_{sx}}{\delta_s}\right)^p,$$

where \sim means that the ratio of the left hand side and the right hand side converges to one as s converges to zero. But this is

$$\lim_{s \downarrow 0} \frac{1 - \varphi^{-1}(sx)}{1 - \varphi^{-1}(s)} = x^{1/p}.$$

Next we show that for $\mathbf{x} = (x_1, \dots, x_d) \leq \mathbf{0} \in \mathbb{R}^d$

$$\lim_{n \rightarrow \infty} C_{\vartheta}^n\left(\mathbf{1} + \frac{\mathbf{x}}{n}\right) = \lim_{n \rightarrow \infty} \left[\varphi^{-1} \left(\sum_{i=1}^d \varphi \left(1 + \frac{x_i}{n} \right) \right) \right]^n = \exp(-\|\mathbf{x}\|_p).$$

Taking logarithms on both sides, this is equivalent with

$$\lim_{n \rightarrow \infty} n \left[1 - \varphi^{-1} \left(\sum_{i=1}^d \varphi \left(1 + \frac{x_i}{n} \right) \right) \right] = \|\mathbf{x}\|_p.$$

Write

$$\frac{1}{n} = 1 - \varphi^{-1} \left(\varphi \left(1 - \frac{1}{n} \right) \right).$$

Then

$$\begin{aligned} n \left[1 - \varphi^{-1} \left(\sum_{i=1}^d \varphi \left(1 + \frac{x_i}{n} \right) \right) \right] &= \frac{1 - \varphi^{-1} \left(\sum_{i=1}^d \varphi \left(1 + \frac{x_i}{n} \right) \right)}{1 - \varphi^{-1} \left(\varphi \left(1 - \frac{1}{n} \right) \right)} \\ &= \frac{1 - \varphi^{-1} \left(\varphi \left(1 - \frac{1}{n} \right) \sum_{i=1}^d \frac{\varphi \left(1 + \frac{x_i}{n} \right)}{\varphi \left(1 - \frac{1}{n} \right)} \right)}{1 - \varphi^{-1} \left(\varphi \left(1 - \frac{1}{n} \right) \right)} \\ &\rightarrow_{n \rightarrow \infty} \left(\sum_{i=1}^d (-x_i)^p \right)^{1/p} \end{aligned}$$

by condition (C0) and equation (2), which is the assertion. \square

Condition (C0) on φ is, for example, implied by the condition

$$(C1) \quad \lim_{s \downarrow 0} \frac{\varphi(1-s)}{s^p} = A$$

for some constant $A > 0$ and $p \geq 1$, which is obviously satisfied by the generator $\varphi_p(s) = (1-s)^p$.

Condition (C1) is by l'Hospital's rule implied by

$$(C2) \quad -\lim_{s \downarrow 0} \frac{\varphi'(1-s)}{s^{p-1}} = pA.$$

As a consequence, (C2) implies the condition

$$(C3) \quad -\lim_{s \downarrow 0} \frac{s\varphi'(1-s)}{\varphi(1-s)} = p.$$

Charpentier and Segers (2009, Theorem 4.1) showed, among others, that a copula C_φ , whose generator satisfies (C3), is in the domain of attraction of $G(\mathbf{x}) = \exp(-\|\mathbf{x}\|_p)$, $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$; see also Falk (2019, Corollary 3.1.15). In this case we have tail independence only if $p = 1$.

The Clayton family with generator $\varphi_\vartheta(t) := (t^{-\vartheta} - 1)/\vartheta$ and $\vartheta > 0$, satisfies condition (C2) with $p = 1$ and $A = 1$. As a consequence, we have independent tails for each $\vartheta > 0$.

The Frank family has the generator

$$\varphi_\vartheta(t) := -\log \left(\frac{e^{-\vartheta t} - 1}{e^{-\vartheta} - 1} \right), \quad \vartheta > 0.$$

It satisfies condition (C0) with $p = 1$, i.e., we have again independent tails for each $\vartheta > 0$.

Consider, on the other hand, the generator $\varphi_\vartheta(t) := (-\log(t))^\vartheta$, $\vartheta \geq 1$, of the Gumbel-Hougaard family of Archimedean copulas. This generator satisfies condition (C0) with $p = \vartheta$ and, thus, we have tail independence only for $\vartheta = 1$.

3. MAIN THEOREM

In this section we establish conditional tail independence of the margins of C_φ , if the generator φ satisfies condition (C0). First we compute the conditional df of $(U_1, \dots, U_{j-1}, U_{j+1}, \dots, U_d)$, given that $U_j = u$.

Lemma 3.1. *We have for $j \in \{1, \dots, d\}$ and $\mathbf{u} = (u_1, \dots, u_{j-1}, u, u_{j+1}, \dots, u_d) \in [\mathbf{u}_0, \mathbf{1})$*

$$\begin{aligned} H_{j,u}(u_1, \dots, u_{j-1}, u_{j+1}, \dots, u_d) &:= P(U_i \leq u_i, 1 \leq i \leq d, i \neq j \mid U_j = u) \\ &= \frac{\varphi'(u)}{\varphi'(C(\mathbf{u}))} \\ &= \frac{\varphi'(u)}{\varphi' \left(\varphi^{-1} \left(\varphi(u) + \sum_{1 \leq i \leq d, i \neq j} \varphi(u_i) \right) \right)}, \end{aligned}$$

provided the derivative $\varphi'(v)$ exists in a neighborhood of u , that φ' is continuous at u with $\varphi'(u) \neq 0$, and that $C(\mathbf{u}) \neq 0$ as well.

Proof. For notational simplicity we establish the result for the choice $j = d$. We have for $\mathbf{u} = (u_1, \dots, u_d) \in [\mathbf{u}_0, \mathbf{1})$

$$\begin{aligned} &P(U_i \leq u_i, 1 \leq i \leq d-1 \mid U_d = u_d) \\ &= \lim_{\varepsilon \downarrow 0} \frac{P(U_i \leq u_i, 1 \leq i \leq d-1, U_d \in [u_d, u_d + \varepsilon])}{P(U_d \in [u_d, u_d + \varepsilon])} \\ &= \lim_{\varepsilon \downarrow 0} \frac{P(U_i \leq u_i, 1 \leq i \leq d-1, U_d \leq u_d + \varepsilon) - P(U_i \leq u_i, 1 \leq i \leq d-1, U_d \leq u_d)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \frac{\varphi^{-1} \left(\sum_{i=1}^{d-1} \varphi(u_i) + \varphi(u_d + \varepsilon) \right) - \varphi^{-1} \left(\sum_{i=1}^d \varphi(u_i) \right)}{\varepsilon} \\ &= (\varphi^{-1})' \left(\sum_{i=1}^d \varphi(u_i) \right) \varphi'(u_d) \\ &= \frac{\varphi'(u_d)}{\varphi' \left(\varphi^{-1} \left(\sum_{i=1}^d \varphi(u_i) \right) \right)} \\ &= \frac{\varphi'(u_d)}{\varphi'(C_\varphi(\mathbf{u}))}, \end{aligned}$$

which is the assertion. \square

Note that the univariate margins of the df $H_{j,u}$, $1 \leq j \leq d$, coincide in their upper tails, where they are equal to

$$H_u(v) := \frac{\varphi'(u)}{\varphi'(\varphi^{-1}(\varphi(u) + \varphi(v)))}, \quad v_0 \leq v \leq 1,$$

with $v_0 = \max_{1 \leq i \leq d} u_{0i}$.

The upper endpoint of H_u is one, and, therefore, if the df H_u is in the domain of attraction of a univariate extreme value df G , then the family of negative Weibull distributions $G_\alpha(x) := \exp(-|x|^\alpha)$, $x \leq 0$, with $\alpha > 0$, is the first choice. Note that $\alpha = 1$ yields the standard negative exponential distribution.

The univariate df H_u is in the domain of attraction of G_α for some $\alpha > 0$ if and only if (iff)

$$\lim_{s \downarrow 0} \frac{1 - H_u(1 - sx)}{1 - H_u(1 - s)} = x^\alpha, \quad x > 0,$$

see, e.g., [Galambos \(1987, Theorem 2.1.2\)](#).

Lemma 3.2. *Suppose that the second derivative of φ exists in a neighborhood of $u > v_0$, and that it is continuous in u with $\varphi''(u) \neq 0 \neq \varphi'(u)$. The univariate df H_u satisfies $H_u \in \mathcal{D}(G_p)$ for some $p \geq 1$ iff φ satisfies condition (C0).*

Proof. Applying Taylor's formula twice shows that

$$\begin{aligned} 1 - H_u(1 - s) &= \frac{\varphi'(\varphi^{-1}(\varphi(u) + \varphi(1 - s))) - \varphi'(u)}{\varphi'(\varphi^{-1}(\varphi(u) + \varphi(1 - s)))} \\ &\sim \frac{\varphi''(u)}{\varphi'(u)^2} \varphi(1 - s) \end{aligned}$$

as $s \downarrow 0$, which is the assertion. \square

The next result is our main theorem.

Theorem 3.3. *Suppose the generator φ of C_φ satisfies condition (C0). Then, if $u > u_{0j}$, and φ satisfies the differentiability conditions in Lemma 3.2, we obtain*

for $\mathbf{x} = (x_1, \dots, x_d) \leq \mathbf{0} \in \mathbb{R}^{d-1}$

$$H_{j,u}(\mathbf{1} + ca_n \mathbf{x})^n \rightarrow_{n \rightarrow \infty} \exp \left(- \sum_{i=1}^{d-1} (-x_i)^p \right),$$

with $c := (\varphi'(u)^2 / \varphi''(u))^{1/\alpha}$ and $a_n := 1 - \varphi^{-1}(1/n)$, $n \geq n_0$.

Note that the convexity of φ implies that $\varphi''(u) \geq 0$.

REMARK 3.4. The preceding result shows tail independence of $H_{j,u}$, as the limiting df is the product of its margins.

Lemma 3.2 implies, moreover, that also the reverse implication in the previous result holds, i.e., if $H_{j,u}$ is in the domain of attraction of a multivariate max-stable df G with negative Weibull margins having parameter at least one, then condition (C0) is satisfied by Lemma 3.2, and G has by the preceding result identical independent margins.

Finally, by the preceding arguments, we have $H_{j,u} \in \mathcal{D}(G)$, where G has negative Weibull margins, iff just one univariate margin of $H_{j,u}$ is in the domain of attraction of a univariate extreme value distribution, and in this case G has identical and independent margins.

Proof. For notational simplicity we establish this result for $j = d$. It is sufficient to establish for $\mathbf{x} = (x_1, \dots, x_d) \leq \mathbf{0} \in \mathbb{R}^{d-1}$

$$(3) \quad n(1 - H_{d,u}(\mathbf{1} + ca_n \mathbf{x})) \rightarrow_{n \rightarrow \infty} \sum_{i=1}^{d-1} (-x_i)^p.$$

We know from Lemma 3.1 that for $(u_1, \dots, u_{d-1}, u) \in [\mathbf{u}_0, \mathbf{1}]$,

$$(4) \quad H_{d,u}(u_1, \dots, u_{d-1}) = \frac{\varphi'(u)}{\varphi' \left(\varphi^{-1} \left(\varphi(u) + \sum_{i=1}^{d-1} \varphi(u_i) \right) \right)}.$$

As a consequence we obtain

$$n(1 - H_{d,u}(\mathbf{1} + ca_n \mathbf{x}))$$

$$\begin{aligned}
&= n \left(1 - \frac{\varphi'(u)}{\varphi' \left(\varphi^{-1} \left(\varphi(u) + \sum_{i=1}^{d-1} \varphi(1 + ca_n x_i) \right) \right)} \right) \\
&= n \frac{\varphi' \left(\varphi^{-1} \left(\varphi(u) + \sum_{i=1}^{d-1} \varphi(1 + ca_n x_i) \right) \right) - \varphi'(u)}{\varphi' \left(\varphi^{-1} \left(\varphi(u) + \sum_{i=1}^{d-1} \varphi(1 + ca_n x_i) \right) \right)},
\end{aligned}$$

where the denominator converges to $\varphi'(u)$ as n increases.

Taylor's formula yields that the nominator equals

$$\varphi''(\vartheta_n) \left(\varphi^{-1} \left(\varphi(u) + \sum_{i=1}^{d-1} \varphi(1 + ca_n x_i) \right) - u \right),$$

where $\varphi''(\vartheta_n)$ converges to $\varphi''(u)$ as n increases. Applying Taylor's formula again yields

$$\varphi^{-1} \left(\varphi(u) + \sum_{i=1}^{d-1} \varphi(1 + ca_n x_i) \right) - u = \frac{1}{\varphi'(\varphi^{-1}(\xi_n))} \sum_{i=1}^{d-1} \varphi(1 + ca_n x_i),$$

where ξ_n converges to $\varphi(u)$ as n increases. But

$$n \sum_{i=1}^d \varphi(1 + ca_n x_i) = \sum_{i=1}^d \frac{\varphi(1 + ca_n x_i)}{\varphi(1 - a_n)} \rightarrow_{n \rightarrow \infty} \sum_{i=1}^{d-1} (-cx_i)^p.$$

by condition (C0). This yields the assertion. \square

4. ARCHIMAX COPULAS

Let $\varphi : (0, 1] \rightarrow [0, \infty)$ be the generator of an Archimedean copula $C_\varphi(\mathbf{u}) = \varphi^{-1} \left(\sum_{i=1}^d \varphi(u_i) \right)$, $\mathbf{u} = (u_1, \dots, u_d) \in (0, 1]^d$, and let $\|\cdot\|_D$ be an arbitrary D -norm. Put

$$(5) \quad C(\mathbf{u}) := \varphi^{-1} (\|(\varphi(u_1), \dots, \varphi(u_d))\|_D), \quad \mathbf{u} \in (0, 1]^d.$$

It was established by [Charpentier et al. \(2014\)](#) that C actually defines a copula on \mathbb{R}^d , called Archimax copula. Choosing $\|\cdot\|_D = \|\cdot\|_1$ yields $C(\mathbf{u}) = C_\varphi(\mathbf{u})$ and, thus, the concept of Archimax copulas generalizes that of Archimedean copulas considerably.

To include also the generator family $\varphi_p(s) = (1-s)^p$, $s \in [0, 1]$, $p \geq 1$, we require the representation of C in equation (5) only for $\mathbf{u} \in [\mathbf{u}_0, \mathbf{1}] \subset (0, 1]^d$. There actually exists a rv, whose copula satisfies

$$C(\mathbf{u}) = \varphi^{-1} \left(\left\| (\varphi(u_1), \dots, \varphi(u_d)) \right\|_p \right), \quad \mathbf{u} \in [\mathbf{u}_0, \mathbf{1}]$$

with some $\mathbf{u}_0 \in (0, 1)^d$. This follows from the fact that $\|(|x_1|^p, \dots, |x_d|^p)\|_D^{1/p}$ is again a D -norm, with an arbitrary D -norm $\|\cdot\|_D$ and $p \geq 1$, see Proposition 2.6.1 and equations (2.14), (2.15) in Falk (2019).

An Archimax copula is in the domain of attraction of a multivariate extreme value distribution, if the generator satisfies condition (C0). Precisely, we have the following result.

Proposition 4.1. *Suppose the generator φ satisfies condition (C0). Then the corresponding Archimax copula C , with arbitrary D -norm $\|\cdot\|_D$, satisfies $C \in \mathcal{D}(G)$, where $G(\mathbf{x}) = \exp \left(-\|(|x_1|^p, \dots, |x_d|^p)\|_D^{1/p} \right)$, $\mathbf{x} \leq \mathbf{0} \in \mathbb{R}^d$.*

Proof. We have for $\mathbf{x} = (x_1, \dots, x_d) \leq \mathbf{0} \in \mathbb{R}^d$

$$\begin{aligned} & n \left[1 - \varphi^{-1} \left(\left\| \left(\varphi \left(1 + \frac{x_1}{n} \right), \dots, \varphi \left(1 + \frac{x_d}{n} \right) \right) \right\|_D \right) \right] \\ &= \frac{1 - \varphi^{-1} \left(\varphi \left(1 - \frac{1}{n} \right) \left\| \left(\frac{\varphi(1 + \frac{x_1}{n})}{\varphi(1 - \frac{1}{n})}, \dots, \frac{\varphi(1 + \frac{x_d}{n})}{\varphi(1 - \frac{1}{n})} \right) \right\|_D \right)}{1 - \varphi^{-1} \left(\varphi \left(1 - \frac{1}{n} \right) \right)} \\ & \rightarrow_{n \rightarrow \infty} \|(|x_1|^p, \dots, |x_d|^p)\|_D^{1/p} \end{aligned}$$

by condition (C0) and equation (2). Repeating the arguments in the proof of Proposition 2.2 yields the assertion. \square

Let the rv $\mathbf{U} = (U_1, \dots, U_d)$ follow an Archimax copula with generator function φ and D -norm $\|\cdot\|_D$. Does it also have independent tails, conditional on one of its components? We give a partial answer to this question.

Suppose the underlying $\|\cdot\|_D$ is a logistic one $\|\cdot\|_q$, with $q \geq 1$. Then

$$\begin{aligned} \varphi^{-1} \left(\|(\varphi(u_1), \dots, \varphi(u_d))\|_q \right) &= \varphi^{-1} \left(\left(\sum_{i=1}^d \varphi(u_i)^q \right)^{1/q} \right) \\ &= \psi^{-1} \left(\sum_{i=1}^d \psi(u_i) \right), \end{aligned}$$

where

$$\psi(s) := \varphi(s)^q, \quad s \in [0, 1].$$

If the generator φ satisfies condition (C0), then the generator ψ clearly satisfies condition (C0) as well:

$$\lim_{s \downarrow 0} \frac{\psi(1 - sx)}{\psi(1 - s)} = x^{pq}, \quad x > 0.$$

If φ satisfies the differentiability conditions in Lemma 3.2, then the conclusion of Theorem 3.3 applies, i.e., with the choice $\|\cdot\|_D = \|\cdot\|_q$, $q \geq 1$, the rv \mathbf{U} has again independent tails, conditional on one of its components.

Set, on the other hand $\mathbf{U} = (U, \dots, U)$, where U is a rv that follows the uniform distribution on $(0, 1)$. Choose $\|\cdot\|_D = \|\cdot\|_\infty$ with $\|\mathbf{x}\|_\infty = \max_{1 \leq i \leq d} (|x_i|)$. Then we have for every function $\varphi : (0, 1] \rightarrow [0, \infty)$, which is continuous and strictly decreasing,

$$\begin{aligned} C(\mathbf{u}) &= P(U \leq u_1, \dots, U \leq u_d) \\ &= \min_{1 \leq i \leq d} u_i \\ &= \varphi^{-1} (\|(\varphi(u_1), \dots, \varphi(u_d))\|_\infty), \quad \mathbf{u} \in (0, 1]^d. \end{aligned}$$

The copula C is, therefore, an Archimax copula, but it has completely dependent conditional margins.

5. SIMULATION STUDY

We conducted a simulation study to illustrate our findings on the conditional tail independence of the Archimedean Gumbel-Hougaard copula family with dimension

$d > 2$ and dependence parameter $\vartheta > 1$. The condition on ϑ implies that copula's tails are asymptotically dependent. There are several statistical tests to verify whether the tails of a multivariate distribution are asymptotically independent, provided that the latter is in the domain of attraction of a multivariate extreme value df. In the bivariate case, some tests have been suggested by [Draisma et al. \(2004\)](#), [Hüsler and Li \(2009\)](#), Chapter 6.5 in [Falk et al. \(2011\)](#). However, to extend them in higher dimensions than two is not straightforward. Therefore, we rely on the hypothesis testing proposed by [Guillou et al. \(2018\)](#), which is based on the componentwise maximum approach and is meant for an arbitrary dimension $d \geq 2$. Such a test is based on a system of hypotheses where under the null hypothesis it is assumed that $A(\mathbf{t}) = 1$ for all $\mathbf{t} \in \mathcal{S}_d$, i.e. the tails are asymptotically independent, while under the alternative hypothesis it is assumed that $A(\mathbf{t}) < 1$ for at least one $\mathbf{t} \in \mathcal{S}_d$, i.e. some tails are asymptotically dependent. Here, A is the Pickands dependence function and \mathcal{S}_d is d -dimensional unit simplex (e.g., [Falk et al., 2011](#), Ch. 4). In [Guillou et al. \(2018\)](#) the authors proposed to use the test statistic $\hat{S}_n = \sup_{\mathbf{t} \in \mathcal{S}_d} \sqrt{n} |\hat{A}_n(\mathbf{t}) - 1|$ to decide whether or not to reject null hypothesis, where \hat{A}_n is an appropriated estimator of the the Pickands dependence function and n is the sample size of the componentwise maxima. Under the null hypothesis, the test statistic converges to a suitable random variable S , for large samples. Large values of the observed test statistic provide evidence against the null hypothesis and in particular the quantiles of the distribution of S to use for rejection of the null hypothesis are reported in Table 1 of [Guillou et al. \(2018\)](#).

We performed the following simulation experiment. In the first step we simulated a sample of size $n = 110K$ of independent observations from a Gumbel-Hougaard copula with $d = 3$ and $\vartheta = 3$. Then, we computed the vector of normalized componentwise maxima $m_{n,j} = \max_{i=1,\dots,n} (u_{i,j} - b_{n,j}) / a_{n,j}$ with $a_{n,j} = n$, $b_{n,j} = 1$ and $j = 1, \dots, d$. In the second step, for $u = 0.99$ and $\varepsilon = 0.0005$ we selected the observations $(u_{i,1}, \dots, u_{i,j-1}, u_{i,j+1}, \dots, u_{i,d})$ such that $u_{i,j} \in [u - \varepsilon, u + \varepsilon]$, $i = 1, \dots, n$. To work with a sample with fixed size we considered only $k = 1000$

of such observations. Then, we computed the vector of normalized componentwise maxima $m_{k,s}^* = \max_{i=1,\dots,k} u_{i,s}/(ca_{k,s})$, where $c = (\varphi'(u)^2/\varphi''(u))^{1/\vartheta}$ and $a_{k,s} := 1 - \varphi^{-1}(1/k)$ with $\varphi(t) := (-\log(t))^\vartheta$ and $s = 1, \dots, j-1, j+1, \dots, d$. We repeated the first and second steps $N = 100$ times obtaining two samples of componentwise maxima, one from the d -dimensional copula and one from the corresponding $d - 1$ conditional distribution. The top-left and top-right panel of Figure 1 display an example of maxima obtained from the Gumbel-Hougaard and the associated estimate of the Pickands dependence function, respectively. A strong dependence among the variables is evident. To see this better in the middle panels the maxima of a pair of variables and the relative estimate of the Pickands dependence function are reported. Indeed, the latter is close to lower bound $\max(1-t, t)$, i.e. the case of complete dependence. The bottom panels of Figure 1 display the maxima obtained with the second step of the simulation experiment and the associated estimate of the Pickands dependence function. These maxima, in contrast to the previous ones, seem to be independent and indeed the estimated Pickands dependence function is close to the upper bound (i.e. the case of independence). Then, we applied the hypothesis test with the sample of maxima obtained in the first and second step of the simulation experiment, leading to the observed values of test statistic of 3.843 and 0.348, respectively. Since the 0.95-quantiles of the distribution of S are 1.300 and 0.960 for $d = 3$ and $d = 2$, respectively (Guillou et al., 2018), we conclude that we reject the hypothesis of tails independence with the first sample of maxima whereas we do not reject it with the second sample. These results are consistent with our theoretical finding.

We repeated this simulation experiment $M = 1000$ times and with the maxima obtained with the second step of the simulation experiment we computed the rejection rate of the null hypothesis. Since we simulated data under the null hypothesis we expect that the rejection rate is close the nominal value of the first type error, i.e. 5%. We did this for different dimension d and values of the parameter ϑ .

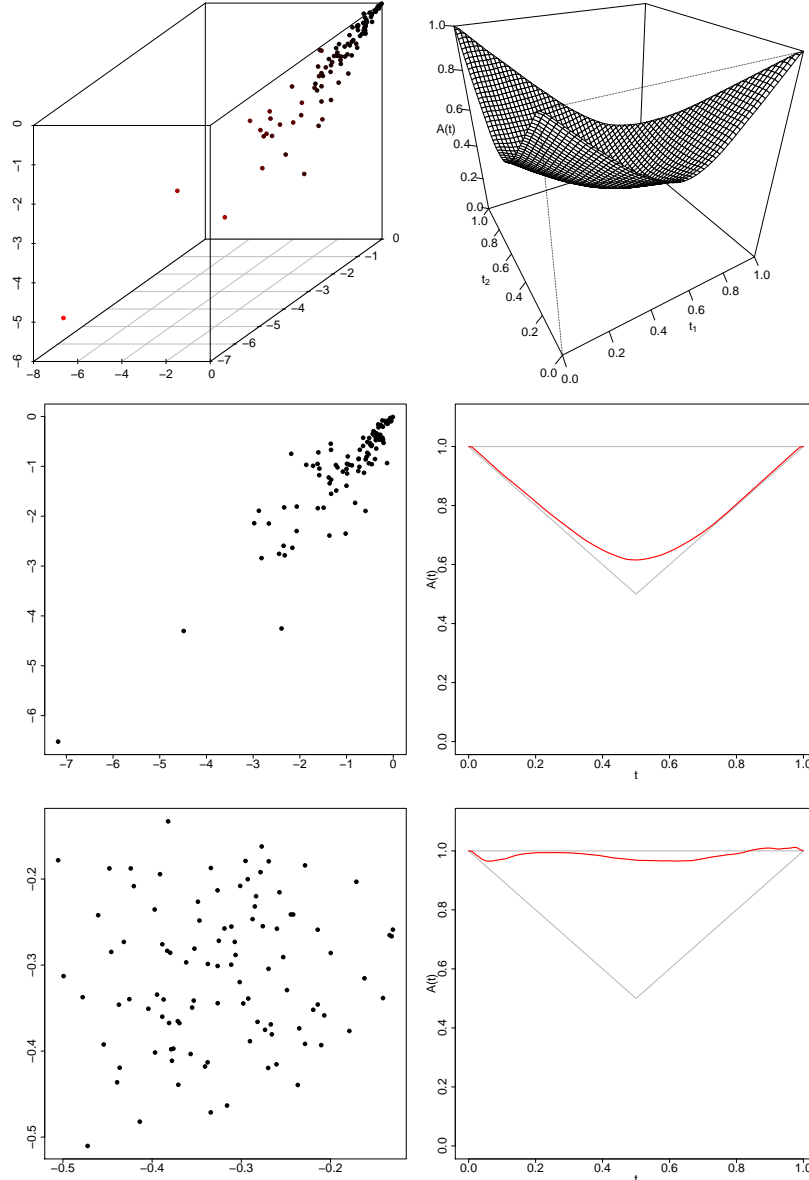


FIGURE 1. Top-left panel displays the maxima obtain with the data simulated from a trivariate Gumbel-Hougaard copula with $\vartheta = 4$. The middle one shows the maxima corresponding to two components. Finally, the one below shows the maxima obtain with the simulated data where one component is set to be a high value. The right-column report the relative estimated Pickands dependence function.

The results are collected in Table 1. Again the simulation results show that our theoretical findings are correct.

TABLE 1. Rejection rate (in percentage) of the null hypothesis (asymptotic independent tails) based on $M = 1000$ simulations.

Dimension		Dependence parameter				
d	$\vartheta :$	2	3	4	5	6
3		5.414	4.877	5.438	5.352	5.725
4		5.216	5.783	5.491	4.841	4.591
5		5.353	4.396	5.791	4.685	4.454

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