Branching processes in random environment with immigration stopped at zero*

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Abstract

A critical branching process with immigration which evolve in a random environment is considered. Assuming that immigration is not allowed when there are no individuals in the aboriginal population we investigate the tail distribution of the so-called life period of the process, i.e., the length of the time interval between the moment when the process is initiated by a positive number of particles and the moment when there are no individuals in the population for the first time.

1 Introduction and statement of main results

We consider branching processes allowing immigration and evolving in a random environment. In such a process individuals reproduce independently of each other according to random offspring distributions which vary from one generation to the other. In addition, immigrants arrive to each generation independently on the development of the population and according to the laws varying at random from generation to generation. To give a formal definition let $\Delta = (\Delta_1, \Delta_2)$ be the space of all pairs of probability measures on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Equipped with the component-wise metric of total variation Δ becomes a Polish space. Let $\mathbf{Q} = \{F, G\}$ be a random vector with independent components taking values in Δ, and let $\mathbf{Q}_n = \{F_n, G_n\}, n = 1, 2, \ldots,$ be a sequence of independent copies of $\mathbf{Q}$. The infinite sequence $\mathcal{E} = \{\mathbf{Q}_1, \mathbf{Q}_2, ...\}$ is called a random environment.

A sequence of $\mathbb{N}_0$-valued random variables $\mathbf{Y} = \{Y_n,\ n \in \mathbb{N}_0\}$ specified on the respective probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a branching process with

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immigration in the random environment (BPIRE), if Y_0 is independent of $\mathcal{E}$ and, given $\mathcal{E}$ the process $\mathbf{Y}$ is a Markov chain with

$$\mathcal{L}\left(Y_n|Y_{n-1}=y_{n-1},\mathcal{E}=(\mathbf{q}_1,\mathbf{q}_2,...)\right)=\mathcal{L}(\xi_{n1}+\ldots+\xi_{ny_{n-1}}+\eta_n) \tag{1}$$

for every $n\in\mathbb{N}:=\mathbb{N}_0\backslash\{0\}$, $y_{n-1}\in\mathbb{N}_0$ and $\mathbf{q}_1=(f_1,g_1)\,,\mathbf{q}_2=(f_2,g_2)\,,...\in\mathbf{Q}$, where $\xi_{n1},\xi_{n2},\ldots$ are i.i.d. random variables with distribution f_n and independent of the random variable η_n with distribution g_n. In the language of branching processes Y_{n-1} is the $(n-1)$th generation size of the population, f_n is the distribution of the number of children of an individual at generation $n-1$ and g_n is the reproduction law of immigrants at generation n.

Along with the process $\mathbf{Y}$ we consider a branching process $\mathbf{Z}=\{Z_n,\ n\in\mathbb{N}_0\}$ in the random environment $\mathcal{E}_1=\{F_1,F_2,...\}$ which, given $\mathcal{E}_1$ is a Markov chain with $Z_0=1$ and, for $n\in\mathbb{N}$

$$\mathcal{L}\left(Z_n|Z_{n-1}=z_{n-1},\mathcal{E}_1=(f_1,f_2,...)\right)=\mathcal{L}(\xi_{n1}+\ldots+\xi_{nz_{n-1}}). \tag{2}$$

It will be convenient to assume that if $Y_{n-1}=y_{n-1}>0$ is the population size of the $(n-1)$th generation of $\mathbf{Y}$ then first $\xi_{n1}+\ldots+\xi_{ny_{n-1}}$ individuals of the nth generation are born and than η_n immigrants enter the population.

This agreement allows us to consider a modified version $\mathbf{W}=\{W_n,\ n\in\mathbb{N}_0\}$ of the process $\mathbf{Y}$ specified as follows. Assume, without loss of generality that $Y_0>0$. Let $W_0=Y_0$ and for $n\geq 1$,

$$W_n:=\begin{cases} 0, & \text{if } T_n:=\xi_{n1}+\ldots+\xi_{nW_{n-1}}=0, \\ T_n+\eta_n, & \text{if } T_n>0. \end{cases} \tag{3}$$

We call $\mathbf{W}$ as a branching process with immigration stopped at zero and evolving in the random environment.

The aim of the present paper is to study the tail distribution of the random variable

$$\zeta:=\min\{n\geq 1: W_n=0\}$$

under the annealed approach. To formulate our main result we consider the so-called associated random walk $\mathbf{S}=(S_0,S_1,...)$. This random walk has initial state S_0 and increments $X_n=S_n-S_{n-1}$, $n\geq 1$, defined as

$$X_n:=\log\mathfrak{m}\left(F_n\right)$$

which are i.i.d. copies of the logarithmic mean offspring number $X:=\log\mathfrak{m}(F)$ with

$$\mathfrak{m}(F):=\sum_{j=0}^{\infty}jF\left(\{j\}\right).$$

We suppose that X is a.s. finite.

With each pair of measures (F,G) we associate the respective probability generation functions

$$F(s):=\sum_{j=0}^{\infty}F\left(\{j\}\right)s^j,\qquad G(s):=\sum_{j=0}^{\infty}G\left(\{j\}\right)s^j.$$

We impose the following restrictions on the distributions of F and G.

Hypothesis A1. The probability generating function $F(s)$ is geometric with probability 1, that is

$$F(s) = \frac{q}{1-ps} = \frac{1}{1+\mathfrak{m}(F)(1-s)} \tag{4}$$

with random $p, q \in (0,1)$ satisfying $p+q=1$ and

$$\mathfrak{m}(F) = \frac{p}{q} = e^{\log(p/q)} = e^X.$$

Hypothesis A2. There exist real numbers $\kappa \in [0,1)$ and $\gamma, \sigma \in (0,1]$ such that, with probability 1

1) the inequality $F(0) \geq \kappa$ is valid;

2) the estimate

$$G(s) \leq s^\gamma \tag{5}$$

holds for all $s \in [\kappa^\sigma, 1]$.

To formulate one more assumption we set

$$M_n := \max(S_1, ..., S_n), \quad L_n := \min(S_0, S_1, ..., S_n),$$

and, given $S_0 = 0$, introduce the right-continuous function $U : \mathbb{R} \to [0, \infty)$ specified by the relation

$$U(x) := I\{x \geq 0\} + \sum_{n=1}^{\infty} \mathbf{P}(S_n \geq -x, M_n < 0), \tag{6}$$

where $I(A)$ is the indicator of the event A.

One may check (see, for instance, [2] and [3]) that for any oscillating random walk

$$\mathbf{E}[U(x+X); X+x \geq 0] = U(x), \quad x \geq 0. \tag{7}$$

Hypothesis A3. The distribution of X is nonlattice, the sequence $\{S_n, n \geq 0\}$ satisfies the Doney-Spitzer condition

$$\lim_{n\to\infty} \mathbf{P}(S_n > 0) =: \rho \in (0,1), \tag{8}$$

and there exists $\varepsilon > 0$ such that

$$\mathbf{E}\left(\log^+ G'(1)\right)^{\rho^{-1}+\varepsilon} < \infty \quad \text{and} \quad \mathbf{E}\left(U(X)\log^+ G'(1)\right)^{1+\varepsilon} < \infty, \tag{9}$$

where $\log^+ x = \max(0, \log x)$.

We now formulate our main result.

Theorem 1 *Let Hypotheses A1 - A3 be satisfied. Then there exists a function $l(n)$ slowly varying at infinity such that*

$$\mathbf{P}(\zeta > n) \sim \frac{l(n)}{n^{1-\rho}}$$

as $n \to \infty$.

It is convenient to describe the range of possible values of the parameter κ by examples.

Let

$$\mathcal{A} := \{0 < \alpha < 1; |\beta| < 1\} \cup \{1 < \alpha < 2; |\beta| \leq 1\} \cup \{\alpha = 1, \beta = 0\} \cup \{\alpha = 2, \beta = 0\}$$

be a subset in $\mathbb{R}^2$. For $(\alpha, \beta) \in \mathcal{A}$ and a random variable X we write $X \in \mathcal{D}(\alpha, \beta)$ if the distribution of X belongs to the domain of attraction of a stable law with characteristic function

$$\mathcal{G}_{\alpha,\beta}(t) := \exp\left\{-c|t|^{\alpha}\left(1 - i\beta\frac{t}{|t|}\tan\frac{\pi\alpha}{2}\right)\right\}, \ c > 0, \tag{10}$$

and, in addition, $\mathbf{E}[X] = 0$ if this moment exists. If $X_n \stackrel{d}{=} X \in \mathcal{D}(\alpha, \beta)$ then the parameter ρ in (8) is given by the formula (see, for instance, [17])

$$\rho = \begin{cases} \frac{1}{2}, \text{ if } \ \alpha = 1, \\ \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right), \text{ otherwise.} \end{cases} \tag{11}$$

Note that if $\mathbf{E}[X] = 0$ and $\mathbf{Var}X \in (0, \infty)$ then the central limit theorem implies $\rho = 1/2$.

Example 1 *If Hypothesis A1 is valid and*

$$X = \log\mathfrak{m}(F) = \log(p/q) \in \mathcal{D}(\alpha, \beta)$$

with $\alpha \in (0, 2)$ *then*

$$\mathbf{P}(\log(p/q) > x) \sim \frac{1}{x^{\alpha}l_1(x)} \quad \text{as } x \to \infty, \tag{12}$$

where $l_1(x)$ *is a function slowly varying at infinity. Therefore,*

$$\mathbf{P}\left(\log\frac{q}{1-q} < -x\right) \sim \frac{1}{x^{\alpha}l_1(x)}$$

as $x \to \infty$ *implying*

$$\mathbf{P}\left(F(0) = q < \frac{e^{-x}}{1 + e^{-x}}\right) \sim \frac{1}{x^{\alpha}l_1(x)}.$$

As a result, $\mathbf{P}(F(0) < y) > 0$ *for any* $y > 0$.

Thus, if $\alpha \in (0, 2)$ *then point 1) of Hypothesis A2 reduces to the trivial inequality* $F(0) \geq \kappa = 0$. *Moreover, given* $\kappa = 0$ *point 2) of Hypothesis A2 implies* $G(0) = 0$ *which, in turn, leads to the inequality*

$$G(s) = \sum_{j=1}^{\infty} G(\{j\})\, s^j \leq s$$

for all $s \in [0, 1]$. *The last means that at least one immigrant enters* $\mathbf{W}$ *each time when it is allowed by (3).*

The case $\mathbf{E}\left[X^2\right] < \infty$ is less restrictive and allows for $\kappa > 0$, i.e., for the absence of immigrants in some generations of $\mathbf{W}$ (even they are allowed).

Example 2 *Let*

$$F(s) = \begin{cases} \frac{1}{1+63(1-s)} & \textit{with probability } \frac{1}{2}, \\ \\ \frac{63}{64-s} & \textit{with probability } \frac{1}{2} \end{cases}$$

and the probability generating function of immigrants be deterministic:

$$G(s) = \frac{2}{3}s^2 + \frac{1}{3} \textit{ with probability 1.}$$

Clearly, $\mathbf{E}\left[\log \mathfrak{m}(F)\right] = 0$*,* $\mathbf{Var}\left[\log \mathfrak{m}(F)\right] \in (0, \infty)$*. It is not difficult to see that*

$$F(0) \geq 1/64 \textit{ and } G(s) \leq s^{1/3} \textit{ for all } s \in \left[8^{-1}, 1\right] = \left[64^{-1/2}, 1\right].$$

Thus, the conditions of Theorem 1 fulfill with $\kappa = 1/64$*,* $\gamma = 1/3$ *and* $\sigma = 1/2$*.*

We note that Zubkov [18] considered a problem similar to ours for a branching process with immigration $\{Y_c(n), n \geq 0\}$ evolving in a constant environment. He assumed that $G(0) > 0$ and investigated the distribution of the so-called life period ζ_c of such a process initiated at time N and defined as

$$Y_c(N-1) = 0, \min_{N \leq k < N+\zeta_c} Y_c(k) > 0, Y_c(N+\zeta_c) = 0.$$

The same problem for other models of branching processes with immigration evolving in a constant environment was analysed, for instance, in [4], [11], [14] and [16].

Various properties of BPIRE were investigated by several authors (see, for instance, [1], [7], [9],[10],[13], [15]). However, asymptotic properties of the life periods of BPIRE were not considered up to now.

2 Auxiliary statements

Given the environment $\mathcal{E} = \{(F_n, G_n), n \in \mathbb{N}\}$, we construct the i.i.d. sequence of pairs of generating functions

$$F_n(s) := \sum_{j=0}^{\infty} F_n(\{j\})\, s^j, \qquad G_n(s) := \sum_{j=0}^{\infty} G_n(\{j\})\, s^j \quad s \in [0,1],$$

and use below the convolutions of the generating functions $F_1, ..., F_n$ specified for $0 \leq i \leq n-1$ by the equalities

$$\begin{aligned} F_{i,n}(s) &:= \quad F_{i+1}(F_{i+2}(\ldots(F_n(s))\ldots)), \\ F_{n,i}(s) &:= \quad F_n(F_{n-1}(\ldots(F_{i+1}(s))\ldots)) \text{ and } F_{n,n}(s) := s. \end{aligned}$$

The evolution of the BPIRE defined by (3) may be now described for $n \geq 1$ by the relation

$$\begin{aligned} \mathbf{E}[s^{W_n}|\mathcal{E}, W_{n-1}] &= (F_n(0))^{W_{n-1}} + \left((F_n(s))^{W_{n-1}} - (F_n(0))^{W_{n-1}}\right) G_n(s) \\ &= (F_n(0))^{W_{n-1}}(1 - G_n(s)) + (F_n(s))^{W_{n-1}} G_n(s)\ . \end{aligned} \tag{13}$$

We assume for convenience that $W_0 = Y_0 > 0$ has the (random) probability generating function

$$N(0; s) := \frac{G_0(s) - G_0(0)}{1 - G_0(0)}$$

where $G_0(s) \stackrel{d}{=} G(s)$. Other classes of the initial distribution may be considered in a similar way.

Setting

$$N(n; s) := \mathbf{E}[s^{W_n}|\mathcal{E}],\ n \geq 1$$

we have by (13)

$$\begin{aligned} N(n; s) &= \mathbf{E}\left[(F_n(0))^{W_{n-1}}(1 - G_n(s)) + (F_n(s))^{W_{n-1}} G_n(s)|\mathcal{E}\right] \\ &= N(n-1; F_n(0))(1 - G_n(s)) + N(n-1; F_n(s)) G_n(s) \qquad (14) \\ &= N(n-1; F_n(0))(1 - G_n(s)) + N(n-2; F_{n-1}(0))(1 - G_{n-1}(F_n(s))) G_n(s) \\ &\quad + N(n-2; F_{n-1}(F_n(s))) G_{n-1}(F_n(s)) G_n(s), \end{aligned}$$

where for $n = 1$ one should take into account only the first two equalities. Assuming $\prod_{j=n+1}^{n} = 1$ we obtain by induction

$$\begin{aligned} N(n; s) &= \sum_{k=0}^{n-1} N(n-k-1; F_{n-k}(0))(1 - G_{n-k}(F_{n-k,n}(s))) \prod_{j=n-k+1}^{n} G_j(F_{j,n}(s)) \\ &\quad + N(0; F_{0,n}(s)) \prod_{j=1}^{n} G_j(F_{j,n}(s)). \end{aligned}$$

Note that according to (14)

$$N(n; 0) = N(n-1; F_n(0)),\ n \geq 1.$$

Besides,

$$\mathbf{E} N(n; 0) = \mathbf{P}(W_n = 0) = \mathbf{P}(\zeta \leq n)\,.$$

Hence, setting $s = F_{n+1}(0)$, taking the expectation with respect to the environment and using the independency of the elements of the environment we get

$$\begin{aligned} \mathbf{E}\left[N(n+1; 0)\right] &= \sum_{k=0}^{n-1} \mathbf{E}\left[N(n-k; 0)\right] \mathbf{E}\left[(1 - G_{n-k}(F_{n-k,n+1}(0))) \prod_{j=n-k+1}^{n} G_j(F_{j,n+1}(0))\right] \\ &\quad + \mathbf{E}\left[N(0; F_{0,n+1}(0)) \prod_{j=1}^{n} G_j(F_{j,n+1}(0))\right]. \end{aligned} \tag{15}$$

Denoting for $n \geq 0$

$$
\begin{aligned}
R_n :=& \quad 1-\mathbf{E}\left[N(n;0)\right]=\mathbf{E}\left[1-N(n;0)\right]=\mathbf{P}\left(\zeta>n\right), \\
H_n^* :=& \quad \mathbf{E}\left[\frac{1-G_0(F_{0,n+1}(0))}{1-G_0(0)}\prod_{i=1}^{n}G_i(F_{i,n+1}(0))\right], \\
d_n :=& \quad \mathbf{E}\left[\prod_{i=1}^{n}G_i\left(F_{i,n+1}(0)\right)\right]=\mathbf{E}\left[\prod_{i=1}^{n}G_i\left(F_{i,0}(0)\right)\right],
\end{aligned}
$$

observing that

$$
\begin{aligned}
H_n \quad &: \quad =\mathbf{E}\left[\left(1-G_0(F_{0,n+1}(0))\right)\prod_{i=1}^{n}G_i\left(F_{i,n+1}(0)\right)\right] \\
&= \quad \mathbf{E}\left[\prod_{i=1}^{n}G_i\left(F_{i,n+1}(0)\right)\right]-\mathbf{E}\left[\prod_{i=1}^{n+1}G_i\left(F_{i,n+2}(0)\right)\right]=d_n-d_{n+1},
\end{aligned}
$$

and using the equality

$$
\mathbf{E}\left[\left(1-G_{n-k}(F_{n-k,n+1}(0))\right)\prod_{j=n-k+1}^{n}G_j(F_{j,n+1}(0))\right]=\mathbf{E}\left[\left(1-G_0(F_{0,k+1}(0))\right)\prod_{j=1}^{k}G_j(F_{j,k+1}(0))\right]
$$

we rewrite (15) as a renewal type equation

$$
R_{n+1}=\sum_{k=0}^{n-1}H_kR_{n-k}+H_n^*,\ n\geq 0. \tag{16}
$$

Let

$$
\mathcal{R}(s):=\sum_{n=1}^{\infty}R_ns^n.
$$

Lemma 1

$$
\mathcal{R}(s)=\frac{s\mathcal{H}^*(s)+sR_1}{(1-s)\,D\,(s)} \tag{17}
$$

where

$$
D\left(s\right):=\sum_{n=0}^{\infty}d_ns^n \ \text{ and } \mathcal{H}^*(s):=\sum_{n=1}^{\infty}H_n^*s^n.
$$

Proof. Set

$$
\mathcal{H}(s):=\sum_{n=0}^{\infty}H_ns^n.
$$

Clearly,

$$
s\mathcal{H}(s)=\sum_{n=0}^{\infty}(d_n-d_{n+1})s^{n+1}=sD\left(s\right)-D(s)+1.
$$

Multiplying (16) by s^{n+1} and summing over n from 1 to ∞ we get

$$\mathcal{R}(s) - sR_1 = s\mathcal{H}(s)\mathcal{R}(s) + s\mathcal{H}^*(s)$$

or

$$\mathcal{R}(s) = \frac{s\mathcal{H}^*(s) + sR_1}{1 - s\mathcal{H}(s)} = \frac{s\left(\mathcal{H}^*(s) + R_1\right)}{(1-s)\,D\,(s)}.$$

The lemma is proved.

Denote for $0 \le i \le n$

$$A_n := e^{S_n}, \quad B_{i,n} := \sum_{k=i}^{n} e^{S_k}, \quad B_n := B_{0,n},$$

and introduce the function

$$C_n(s) := \prod_{i=1}^{n} F_{i,0}(s).$$

Lemma 2 *Under Hypothesis A*1

$$C_n := C_n(0) = \frac{1}{B_n}.$$

Proof. Hypothesis A1 implies

$$F_i(s) = \frac{q_i}{1 - p_i s} = \frac{1}{1 + e^{X_i}\,(1-s)} \tag{18}$$

for all $i = 1, 2, \ldots$. Using these equalities it is not difficult to check by induction that, for $n \ge 1$

$$F_{n,0}(s) = 1 - \frac{A_n}{(1-s)^{-1} + B_{1,n}} = \frac{(1-s)^{-1} + B_{1,n-1}}{(1-s)^{-1} + B_{1,n}},$$

where $B_{1,0} = 0$ by definition. Therefore,

$$C_n(s) = \prod_{i=1}^{n} \frac{(1-s)^{-1} + B_{1,i-1}}{(1-s)^{-1} + B_{1,i}} = \frac{(1-s)^{-1}}{(1-s)^{-1} + B_{1,n}}. \tag{19}$$

Setting $s = 0$ in (19) we prove the lemma.

To go further we need more notation. Let $\mathcal{E} = \{\mathbf{Q}_1, \mathbf{Q}_2, \ldots\}$ be a random environment and let $\mathcal{F}_n, n \ge 1$, be the σ-field of events generated by the random pairs $\mathbf{Q}_1 = \{F_1, G_1\}, \mathbf{Q}_2 = \{F_2, G_2\}, \ldots, \mathbf{Q}_n = \{F_n, G_n\}$ and the sequence $W_0, W_1, \ldots, W_n$. These σ-fields form a filtration $\mathfrak{F}$. Now the increments $\{X_n, n \ge 1\}$ of the random walk S are measurable with respect to the σ-field $\mathcal{F}_n$. Using the martingale property (7) of U we introduce a sequence of probability measures $\left\{\mathbf{P}^+_{(n)}, n \ge 1\right\}$ on the σ-field $\mathcal{F}_n$ by means of the density

$$d\mathbf{P}^+_{(n)} := U(S_n) I\left\{L_n \ge 0\right\} d\mathbf{P}.$$

This and Kolmogorov's extension theorem show that, on a suitable probability space there exists a probability measure $\mathbf{P}^+$ on the σ-field $\mathfrak{F}$ such that (see [2] and [3] for more detail)

$$\mathbf{P}^+|\mathcal{F}_n = \mathbf{P}^+_{(n)}, \ n \geq 1.$$

We now formulate two known statements dealing with conditioning $\{L_n \geq 0\}$.

Lemma 3 *(see Lemma 2.5 in [2] or Lemma 5.2 in [8]) Let the condition (8) hold and let $\xi_1, \xi_2, \ldots$ be a sequence of uniformly bounded random variables adapted to the filtration $\mathfrak{F}$ such that the limit*

$$\xi_\infty := \lim_{n\to\infty} \xi_n \tag{20}$$

exists $\mathbf{P}^+$ - a.s. Then

$$\lim_{n\to\infty} \mathbf{E}[\xi_n \mid L_n \geq 0] = \mathbf{E}^+ [\xi_\infty] . \tag{21}$$

Let

$$\tau(n) := \min\{i \geq 0 : S_i = L_n\} .$$

Lemma 4 *(see Lemma 2.2 in [2]) Let $u(x), x \geq 0$, be a nonnegative, nonincreasing function with $\int_0^\infty u(x)dx < \infty$. If the condition (8) holds then, for every $\varepsilon > 0$, there exists a positive number $m = m(\varepsilon)$ such that for all $n \geq m$*

$$\sum_{k=m}^{n} \mathbf{E}\left[u(-S_k); \tau(k) = k\right] \mathbf{P}\left(L_{n-k} \geq 0\right) \leq \varepsilon \mathbf{P}\left(L_n \geq 0\right).$$

3 Proof of the main result

It is known (see, for instance, [12] or [5], Theorem 8.9.12) that if Hypothesis A3 is valid then there exists a slowly varying function $l_2(n)$ such that

$$\mathbf{P}\left(L_n \geq 0\right) \sim \frac{l_2(n)}{n^{1-\rho}}, \quad n \to \infty. \tag{22}$$

We now prove an important statement describing the asymptotic behavior of d_n as $n \to \infty$. To this aim we introduce the reflected random walk

$$\tilde{S}_0 = 0, \ \tilde{S}_k = \tilde{X}_1 + ... + \tilde{X}_k, \ k \geq 1,$$

where $\tilde{X}_k = -X_k$ and supply in the sequel the relevant variables and measures by the upper symbol $\tilde{}$.

Note that $\tilde{X}_k \in \mathcal{D}(\alpha, -\beta)$ and

$$\lim_{n\to\infty} \mathbf{P}\left(\tilde{S}_n > 0\right) = \lim_{n\to\infty} \mathbf{P}\left(S_n < 0\right) = 1 - \rho.$$

Hence it follows that

$$\mathbf{P}\left(\tilde{L}_n \geq 0\right) \sim \frac{l_3(n)}{n^{\rho}}, \quad n \to \infty, \tag{23}$$

for a slowly varying function $l_3(n)$.

Lemma 5 *If Hypotheses A1-A3 are satisfied then there exists a constant $\theta > 0$ such that*

$$d_n \sim \theta \mathbf{P}\left(\tilde{L}_n \geq 0\right) \sim \theta \frac{l_3(n)}{n^\rho}, \quad n \to \infty. \tag{24}$$

Proof. According to Lemma 2

$$C_n = \frac{1}{B_n} = \frac{1}{1 + e^{-\tilde{S}_1} + \ldots + e^{-\tilde{S}_n}} =: \frac{1}{\tilde{B}_n}.$$

We set

$$\tilde{\tau}(n) := \min\left\{i \geq 0 : \tilde{S}_i = \tilde{L}_n\right\}$$

and write

$$d_n = \sum_{k=0}^{n} \mathbf{E}\left[\prod_{i=1}^{n} G_i\left(F_{i,0}(0)\right); \tilde{\tau}(n) = k\right].$$

Recalling point 1) of Hypothesis A2 we conclude that, for any $i \geq 1$

$$F_{i,0}^{\sigma}(0) = F_{i,i-1}^{\sigma}(F_{i-1,0}(0)) \geq F_{i,i-1}^{\sigma}(0) \geq \kappa^{\sigma}.$$

This estimate, point 2) of Hypothesis A2 and Lemma 2 imply

$$\begin{aligned}
\mathbf{E}\left[\prod_{i=1}^{n} G_i\left(F_{i,0}(0)\right); \tilde{\tau}(n) = k\right] &\leq \mathbf{E}\left[\prod_{i=1}^{n} G_i\left(F_{i,0}^{\sigma}(0)\right); \tilde{\tau}(n) = k\right] \\
&\leq \mathbf{E}\left[\left(\prod_{i=1}^{n} F_{i,0}^{\sigma}(0)\right)^{\gamma}; \tilde{\tau}(n) = k\right] = \mathbf{E}\left[\frac{1}{\left(\tilde{B}_n\right)^{\sigma\gamma}}; \tilde{\tau}(n) = k\right].
\end{aligned}$$

Further,

$$\mathbf{E}\left[\frac{1}{\left(\tilde{B}_n\right)^{\sigma\gamma}}; \tilde{\tau}(n) = k\right] \leq \mathbf{E}\left[e^{\sigma\gamma\tilde{S}_k}; \tilde{\tau}(n) = k\right] = \mathbf{E}\left[e^{\sigma\gamma\tilde{S}_k}; \tilde{\tau}(k) = k\right] \mathbf{P}\left(\tilde{L}_{n-k} \geq 0\right).$$

Using Lemma 4 with $u(x) = e^{-\sigma\gamma x}$ we conclude that, for any $\varepsilon > 0$ there exists $m = m(\varepsilon)$ such that

$$\begin{aligned}
&\sum_{k=m}^{n} \mathbf{E}\left[\prod_{i=1}^{n} G_i\left(F_{i,0}(0)\right); \tilde{\tau}(n) = k\right] \\
&\quad \leq \sum_{k=m}^{n} \mathbf{E}\left[e^{\sigma\gamma\tilde{S}_k}; \tilde{\tau}(k) = k\right] \mathbf{P}\left(\tilde{L}_{n-k} \geq 0\right) \leq \varepsilon \mathbf{P}\left(\tilde{L}_n \geq 0\right).
\end{aligned} \tag{25}$$

We now consider fixed $k \leq m$ and write

$$\begin{aligned}
&\mathbf{E}\left[\prod_{i=1}^{n} G_i\left(F_{i,0}(0)\right); \tilde{\tau}(n)=k\right] \\
&\quad = \mathbf{E}\left[\prod_{i=1}^{k} G_i\left(F_{i,0}(0)\right) \prod_{j=k+1}^{n} G_j\left(F_{j,k}(F_{k,0}(0))\right); \tilde{\tau}(n)=k\right] \\
&\quad = \mathbf{E}\left[\prod_{i=1}^{k} G_i\left(F_{i,0}(0)\right) \Theta\left(n-k; F_{k,0}(0)\right); \tilde{\tau}(k)=k\right],
\end{aligned}$$

where

$$\Theta\left(n; s\right) := \mathbf{E}\left[\prod_{j=1}^{n} G_j\left(F_{j,0}(s)\right); \tilde{L}_n \geq 0\right].$$

Using the arguments applied to establish Lemma 2.7 in [2], one may check that, under the conditions of Theorem 1

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(1-G_j\left(F_{j,0}(s)\right)\right) \leq \sum_{j=1}^{\infty} G_j'(1)\left(1-F_{j,0}(s)\right) \\
&\quad \leq \sum_{j=1}^{\infty} G_j'(1)\left(1-F_{j,0}(0)\right) \leq \sum_{j=1}^{\infty} G_j'(1) e^{-\tilde{S}_j} < \infty \quad \tilde{\mathbf{P}}^{+} - a.s.
\end{aligned}$$

Hence it follows that,

$$\xi_n(s) := \prod_{j=1}^{n} G_j\left(F_{j,0}(s)\right) \to \xi_\infty(s) := \prod_{j=1}^{\infty} G_j\left(F_{j,0}(s)\right) > 0$$

$\tilde{\mathbf{P}}^{+}-$a.s. Since $\xi_n(s) \to \xi_\infty(s)$ $\tilde{\mathbf{P}}^{+}-$a.s. as $n \to \infty$, it follows from Lemma 3 that, for each $s \in [0,1)$

$$\Theta\left(n; s\right) \sim \tilde{\mathbf{E}}^{+}\left[\xi_\infty(s)\right] \mathbf{P}\left(\tilde{L}_n \geq 0\right),\ n \to \infty.$$

Applying the dominated convergence theorem gives on account of (23) and properties of slowly varying functions

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\left[\prod_{i=1}^{k} G_i\left(F_{i,0}(0)\right) \frac{\Theta\left(n-k; F_{k,0}(0)\right)}{\mathbf{P}\left(\tilde{L}_n \geq 0\right)}; \tilde{\tau}(k)=k\right] \\
&\quad = \mathbf{E}\left[\prod_{i=1}^{k} G_i\left(F_{i,0}(0)\right) \tilde{\mathbf{E}}^{+}\left[\prod_{j=0}^{\infty} \hat{G}_j\left(\hat{F}_{j,0}(F_{k,0}(0))\right)\right]; \tilde{\tau}(k)=k\right], \quad (26)
\end{aligned}$$

where $\hat{G}_j, \hat{F}_{j,0}$ are independent copies of $G_j, F_{j,0}$.

Combining (26) with (25) we get

$$\lim_{n\to\infty}\frac{1}{\mathbf{P}\left(\tilde{L}_n\geq 0\right)}\mathbf{E}\left[\prod_{i=0}^{n-1}G_i\left(F_{i,n}(0)\right)\right]=\theta,$$

where

$$\theta:=\sum_{k=0}^{\infty}\mathbf{E}\left[\prod_{i=1}^{k}G_i\left(F_{i,0}(0)\right)\tilde{\mathbf{E}}^{+}\left[\prod_{j=0}^{\infty}\hat{G}_j\left(\hat{F}_{j,0}(F_{k,0}(0))\right)\right];\tilde{\tau}(k)=k\right].$$

This proves Lemma 5.

Proof of Theorem 1. We know that

$$d_n\sim\theta\frac{l_3(n)}{n^{\rho}}$$

as $n\to\infty$. This and a Tauberian theorem (see [6], Chapter XIII.5, Theorem 5) imply

$$D(s)=\sum_{n=1}^{\infty}d_ns^n\sim\theta\Gamma\left(1-\rho\right)\frac{l_3\left(1/(1-s)\right)}{(1-s)^{1-\rho}}.$$

Thus,

$$\mathcal{R}(s)=\frac{s\left(\mathcal{H}^*(s)+R_1\right)}{(1-s)\,D\left(s\right)}\sim\frac{\mathcal{H}^*(1)+R_1}{\theta\Gamma\left(1-\rho\right)l_3\left(1/(1-s)\right)(1-s)^{\rho}}$$

as $s\uparrow 1$. Since the sequence $\{R_n,n\geq 1\}$ is monotone decreasing, it follows that (see [6], Chapter XIII.5, Theorem 5)

$$R_n\sim\frac{\mathcal{H}^*(1)+R_1}{\theta\Gamma\left(\rho\right)\Gamma\left(1-\rho\right)}\frac{n^{\rho-1}}{l_3\left(n\right)}\qquad\text{as } n\to\infty.$$

Theorem 1 is proved.

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