# General drawdown of general tax model in a time-homogeneous Markov framework

Florin Avram<sup>\*</sup> Bin Li<sup>†</sup> Shu Li<sup>‡</sup>

October 5, 2018

#### Abstract

Drawdown/regret times feature prominently in optimal stopping problems, in statistics (CUSUM procedure) and in mathematical finance (Russian options). Recently it was discovered that a first passage theory with general drawdown times, which generalize classic ruin times, may be explicitly developed for spectrally negative Lévy processes – see Avram, Vu, Zhou(2017), Li, Vu, Zhou(2017). In this paper, we further examine general drawdown related quantities for taxed time-homogeneous Markov processes, using the pathwise connection between general drawdown and tax process.

## 1 Introduction

Our paper is part of a larger program to improve the control of a reserves/risk process X. The rough idea is that when below low levels a, the reserves should be replenished at some cost, and when above high levels b, the reserves should be invested to yield dividends – see for example [1]. The low levels first considered historically have been those of X, but one may equally consider low levels of the drawdown/regret/process reflected at the maximum, defined by

$$D_t = \overline{X}_t - X_t, \quad \overline{X}_t := \sup_{0 \le s \le t} X_s,$$

which turn out to be of interest in several problems in statistics, mathematical finance and risk theory [7–9, 12, 16–18, 20–23, 25]. The book [26] summarizes most of the recent developments on drawdown.

Assume from now on that our underlying process X is time-homogeneous and Markovian. The first passage times of X across a level  $x \in \mathbb{R}$  are denoted by

$$\tau_x^+ = \inf \{ t \ge 0 : X_t > x \}$$
 and  $\tau_x^- = \inf \{ t \ge 0 : X_t < x \}$ .

For simplicity, we assume X is upward skip-free. Moreover, we assume X is regular in the sense that  $\mathbb{P}_y(\tau_x^{+(-)} < \infty) > 0$ , for all  $x, y \in \mathbb{R}$ .

Instrumental in achieving the control of one dimensional risk processes are the distributions of the two-sided smooth and non-smooth first passage times from a bounded interval [u, v]. For

<sup>\*</sup>Laboratoire de Mathématiques Appliquées, Université de Pau, France

<sup>&</sup>lt;sup>†</sup>Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, ON, N2L 3G1, Canada

<sup>&</sup>lt;sup>‡</sup>Department of Statistical and Actuarial Sciences, Western University, London, ON, N6A 5B7, Canada

upward skip-free processes, it turns out easier to study the corresponding Laplace transforms:

$$B^{(q)}(x; u, v) := \mathbb{E}_x \left[ e^{-q\tau_v^+} \mathbf{1}_{\{\tau_v^+ < \tau_u^-\}} \right],$$
(1.1)

$$C^{(q,s)}(x;u,v) := \mathbb{E}_x \left[ e^{-q\tau_u^- - s(u - X_{\tau_u^-})} \mathbf{1}_{\{\tau_u^- < \tau_v^+\}} \right],$$
(1.2)

where  $q, s \ge 0$ , and  $u \le x \le v$ . Indeed, for Lévy processes for example it holds that:

$$B^{(q)}(x; u, v) = rac{W_q(x-u)}{W_q(v-u)},$$

where  $W_q(x)$  is called the scale function [11, 14, 24], and for some non-homogeneous spectrally negative Markov processes [13] a similar formula holds

$$B^{(q)}(x; u, v) = \frac{W_q(x; u)}{W_q(v; u)},$$

where now the newly defined scale function naturally depends on the two variables x, u.

Several control problems for (X, D) are known to reduce to the study of the process  $X_t$  with all its negative excursions excised, which turns out to be a deterministic process, killed at a random time [2,3]–see Figure 1 below. This supports the parallel fundamental idea of [18] to base the study of (X, D) on the existence of two differential parameters.

**Assumption 1.1** For all  $q, s \ge 0$  and  $u \le x$  fixed, assume that  $B^{(q)}(x; u, v)$  and  $C^{(q,s)}(x; u, v)$  are differentiable in v at v = x and denote

$$\frac{\partial B^{(q)}(x;u,v)}{\partial v}\bigg|_{v=x} = -b_u^{(q)}(x) \text{ and } \left. \frac{\partial C^{(q,s)}(x;u,v)}{\partial v} \right|_{v=x} = c_u^{(q,s)}(x).$$

A necessary condition for Assumption 1.1 to hold is that

$$\tau_x^+ = 0$$
 and  $X_{\tau_x^+} = x, \mathbb{P}_x - a.s.$  for all  $x \in \mathbb{R}$ .

To understand the joint dynamics of two dimensional process  $t \mapsto (X_t, D_t)$ , it is useful to look at Figure 1, reproduced from [8], which depicts a sample path of (X, D), where X is chosen to be the standard Brownian motion and the exit region is  $R = [-6, 7] \times [0, 10]$ . As is clear from the figure and from its definition, the process (X, D) has very particular dynamics on R: away from the boundary  $\partial_1 := \{x \in \mathbb{R} \times \mathbb{R}_+ : x_2 = 0\}$  it oscillates on the line segment  $L_{\overline{X}_t}$  where, for  $c \in \mathbb{R}, L_c := \{x \in \mathbb{R} \times \mathbb{R}_+ : x_1 + x_2 = c\}$ . These oblique lines represent each a negative excursion. On  $\partial_1$ , we observe the evolution of the process  $\overline{X}_t$  with all its negative excursions excised; as  $\overline{X}_t$ increases, the line segment  $L_{\overline{X}_t}$  on which (X, D) oscillates during a negative excursion advances continuously to the right.

To fully specify the process  $\overline{X}_t$  with its negative excursions excised, we must give a rule for killing a negative excursion; two classic choices are  $X_t < a$  (ruin stopping) and  $D_t > d$  (drawdown stopping), which are the left and upper boundaries in Figure 1, respectively. A linear combination of these, translating into an oblique upper boundary, has been studied in [9].

In our paper we consider more general upper boundaries, which include the previous works as particular cases. Following [20], we consider stopping times

$$\tau_f = \inf \left\{ t \ge 0 : X_t < f(\overline{X}_t) \right\} = \inf \{ t \ge 0 : Y_t > 0 \},\$$



Figure 1: A sample path of (X, D) (sampled at time step  $\Delta t = 0.1$ ) when X is a standard Brownian motion with  $X_0 = a + d = 4$ , and the region R with d = 10, a = -6 and b = 7; the dark shaded region shows the possible points of exit of (X, D) from  $R = [-6, 7] \times [0, 10]$ 

where

$$Y_t = f(\overline{X}_t) - X_t = D_t - \overline{f}(\overline{X}_t), \ t \ge 0$$

will be called a general drawdown process. Here  $\overline{f}(m) := m - f(m)$ , and f must be nondecreasing such that

$$f(x) < x \Leftrightarrow \overline{f}(x) > 0, \quad x \in \mathbb{R}.$$

Note that we have  $Y_0 = f(X_0) - X_0 < 0$ .

General drawdown times include many important particular subcases which have been extensively studied in the literature:

- 1. If f(x) = 0,  $\tau_f = \tau_{0,0}$  is the run time.
- 2. If f(x) = x d,  $\tau_f = \tau_{1,d}$  is the classic drawdown time.
- 3. If  $f(x) = \xi x, \xi < 1$ , when  $\tau_f = \tau_{\xi,0}$  is the proportional drawdown time.
- 4. If

$$f(x) = \xi x - d \Leftrightarrow \bar{f}(m) = m - f(m) = (1 - \xi)m + d, \ \xi \in (-\infty, 1], \ d \ge 0, \ (1 - \xi) > 0, \ (1.3)$$

the corresponding drawdown time is

$$\tau_f = \tau_{\xi,d} = \inf\left\{t \ge 0 : X_t \le \xi \overline{X}_t - d\right\} = \inf\left\{t \ge 0 : \overline{X}_t - X_t > (1 - \xi)\overline{X}_t + d\right\}.$$
 (1.4)

This is called the affine drawdown studied in [9]. It turns out that this extension complicates only slightly the classic drawdown results, while allowing treating simultaneously times cases 2 and 3.

5. Nonlinear drawdown times emerged in [16] and were used by Azéma and Yor [10] to provide a solution of the Skorokhod problem of stopping a Brownian motion to obtain a given desired centered marginal measure.

**Contents.** Below, we extend first the general drawdown results of [20] from spectrally negative Lévy processes to spectrally negative time-homogeneous Markov processes – see Section 2. Then, in Section 3 we allow also for the possibility of general taxation. The method of proof involves a nontrivial use of the "differential exit problems" of [18]. The results in Section 2 are applied in the three particular cases in which the "differential exit parameters" of [18] are analytically computable: spectrally negative Lévy processes and diffusions. A third example, which is illustrated in [18], is of Ornstein-Uhlenbeck-type processes with exponential jumps.

## 2 Main results of general drawdown in the time-homogeneous Markov process

The following pathwise inequalities are central to the construction of tight bounds for the joint law of the triplet  $(\tau_f, \overline{X}_{\tau_f}, Y_{\tau_f})$ .

**Proposition 2.1** For  $q, s \ge 0$ ,  $x \in \mathbb{R}$  and  $\varepsilon > 0$ , we have  $\mathbb{P}_x$ -a.s.

$$1_{\{\tau_{x+\varepsilon}^+ < \tau_{\overline{f}(x+\varepsilon)}^-\}} \le 1_{\{\tau_{x+\varepsilon}^+ < \tau_f\}} \le 1_{\{\tau_{x+\varepsilon}^+ < \tau_{\overline{f}(x)}^-\}},\tag{2.1}$$

and

$$e^{-q\tau_f - sY_{\tau_f}} 1_{\{\tau_f < \tau_{x+\varepsilon}^+\}} \ge e^{-q\tau_{f(x)}^- - s(f(x+\varepsilon) - X_{\tau_{f(x)}^-})} 1_{\{\tau_{f(x)}^- < \tau_{x+\varepsilon}^+\}},$$
(2.2)

$$e^{-q\tau_f - sY_{\tau_f}} 1_{\{\tau_f < \tau_{x+\varepsilon}^+\}} \le e^{-q\tau_{f(x+\varepsilon)}^- - s(f(x) - X_{\tau_{f(x+\varepsilon)}^-})} 1_{\{\tau_{f(x+\varepsilon)}^- < \tau_{x+\varepsilon}^+\}}.$$
(2.3)

**Proof.** By analyzing the sample paths of X, it is easy to see that  $\tau_f \leq \tau_{f(x)}^- \mathbb{P}_x$ -a.s. Thus,  $\mathbb{P}_x$ -a.s. we have

$$(\tau_{x+\varepsilon}^+ < \tau_f) = (\tau_{x+\varepsilon}^+ < \tau_f \le \tau_{f(x)}^-) \subset (\tau_{x+\varepsilon}^+ < \tau_{f(x)}^-)$$

and

$$(\tau_{x+\varepsilon}^+ < \tau_{f(x+\varepsilon)}^-) = (\tau_{x+\varepsilon}^+ < \tau_{f(x+\varepsilon)}^-, \tau_{x+\varepsilon}^+ < \tau_f) \subset (\tau_{x+\varepsilon}^+ < \tau_f)$$

which immediately implies (2.1).

On the other hand, by using the same argument, we have,  $\mathbb{P}_x$ -a.s.,

$$(\tau_{f(x)}^{-} < \tau_{x+\varepsilon}^{+}) = (\tau_f \le \tau_{f(x)}^{-} < \tau_{x+\varepsilon}^{+}) \subset (\tau_f < \tau_{x+\varepsilon}^{+}),$$
(2.4)

and

$$(\tau_f < \tau_{x+\varepsilon}^+) = (\tau_{f(x+\varepsilon)}^- \le \tau_f < \tau_{x+\varepsilon}^+) \subset (\tau_{f(x+\varepsilon)}^- < \tau_{x+\varepsilon}^+).$$
(2.5)

For any path  $\omega \in (\tau_{f(x)}^- < \tau_{x+\varepsilon}^+)$ , we know from (2.4) that  $\omega \in (\tau_f \le \tau_{f(x)}^- < \tau_{x+\varepsilon}^+)$ . This implies  $\overline{X}_{\tau_f}(\omega) \le x + \varepsilon$  and  $X_{\tau_f}(\omega) \ge X_{\tau_{f(x)}^-}(\omega)$ , which further entails that  $Y_{\tau_f}(\omega) = f(\overline{X}_{\tau_f}(\omega)) - X_{\tau_f}(\omega) \le X_{\tau_f}(\omega)$ 

 $f(x + \varepsilon) - X_{\tau_{f(x)}^{-}}(\omega)$ . Therefore, by the above analysis and (2.4),  $\mathbb{P}_{x}$ -a.s.,

$$e^{-q\tau_{f(x)}^{-}-s(f(x+\varepsilon)-X_{\tau_{f(x)}^{-}})}1_{\{\tau_{f(x)}^{-}<\tau_{x+\varepsilon}^{+}\}} \le e^{-q\tau_{f}-sY_{\tau_{f}}}1_{\{\tau_{f}<\tau_{x+\varepsilon}^{+}\}}$$

which naturally leads to (2.2).

Similarly, for any sample path  $\omega \in (\tau_f < \tau_{x+\varepsilon}^+)$ , we know from (2.5) that  $\omega \in (\tau_{\overline{f}(x+\varepsilon)} \leq \tau_f < \tau_{x+\varepsilon}^+)$ , which implies that  $f(x) - X_{\tau_{\overline{f}(x+\varepsilon)}^-}(\omega) \leq Y_{\tau_{\overline{f}(x+\varepsilon)}}(\omega) \leq Y_{\tau_f}(\omega)$ . Here the last inequality is because  $Y_{\tau_{\overline{f}(x+\varepsilon)}^-}(\omega) \leq 0 \leq Y_{\tau_f}(\omega)$  if  $\tau_{\overline{f}(x+\varepsilon)}^- < \tau_f$ , and  $Y_{\tau_{\overline{f}(x+\varepsilon)}^-}(\omega) = Y_{\tau_f}(\omega)$  if  $\tau_{\overline{f}(x+\varepsilon)}^- = \tau_f$ . Therefore, we obtain,  $\mathbb{P}_x$ -a.s.,

$$e^{-q\tau_{f}-sY_{\tau_{f}}}1_{\{\tau_{f}<\tau_{x+\varepsilon}^{+}\}} \le e^{-q\tau_{f(x+\varepsilon)}^{-}-s(f(x)-X_{\tau_{f(x+\varepsilon)}^{-}})}1_{\{\tau_{f(x+\varepsilon)}^{-}<\tau_{x+\varepsilon}^{+}\}},$$

which proves (2.3).

By Proposition 2.1, we easily obtain the following useful estimates.

**Corollary 2.1** For  $q, s \ge 0, x \in \mathbb{R}$  and  $\varepsilon > 0$ ,

$$B^{(q)}(x; f(x+\varepsilon), x+\varepsilon) \le \mathbb{E}_x \left[ e^{-q\tau_{x+\varepsilon}^+} \mathbb{1}_{\{\tau_{x+\varepsilon}^+ < \tau_f\}} \right] \le B^{(q)}(x; f(x), x+\varepsilon),$$

and

$$\mathbb{E}_{x}\left[e^{-q\tau_{f}-sY_{\tau_{f}}}1_{\left\{\tau_{f}<\tau_{x+\varepsilon}^{+}\right\}}\right] \leq e^{s(f(x+\varepsilon)-f(x))}C^{(q,s)}(x;f(x+\varepsilon),x+\varepsilon)$$
$$\mathbb{E}_{x}\left[e^{-q\tau_{f}-sY_{\tau_{f}}}1_{\left\{\tau_{f}<\tau_{x+\varepsilon}^{+}\right\}}\right] \geq e^{-s(f(x+\varepsilon)-f(x))}C^{(q,s)}(x;f(x),x+\varepsilon)$$

Next we present our main results of the general drawdown.

**Theorem 2.1** Consider an upward skip-free time-homogeneous Markov process X such that Assumption 1.1 holds. For  $q, s \ge 0$  and  $x < K \in \mathbb{R}$ , we have

$$\mathbb{E}_{x}\left[e^{-q\tau_{K}^{+}}1_{\{\tau_{K}^{+}<\tau_{f}\}}\right] = e^{-\int_{x}^{K}b_{f}^{(q)}(z)\mathrm{d}z},\tag{2.6}$$

$$\mathbb{E}_{x}\left[e^{-q\tau_{f}-sY_{\tau_{f}}}1_{\{\overline{X}_{\tau_{f}}\leq K\}}\right] = \int_{x}^{K} e^{-\int_{x}^{y} b_{f}^{(q)}(z) \mathrm{d}z} c_{f}^{(q,s)}(y) \mathrm{d}y.$$
(2.7)

**Proof.** Let

$$g(x) = \mathbb{E}_x \left[ e^{-q\tau_K^+} \mathbf{1}_{\{\tau_K^+ < \tau_f\}} \right], \quad x < K.$$

By the strong Markov property of X at maxima, for any  $X_0 = x \le y < K$  and  $0 < \varepsilon < K - y$ , we have

$$g(y) = \mathbb{E}_y \left[ e^{-q\tau_{y+\varepsilon}^+} \mathbb{1}_{\{\tau_{y+\varepsilon}^+ < \tau_f\}} \right] g(y+\varepsilon).$$

By Corollary 2.1, it follows that

$$B^{(q)}(y; f(y+\varepsilon), y+\varepsilon)g(y+\varepsilon) \le g(y) \le B^{(q)}(y; f(y), y+\varepsilon)g(y+\varepsilon).$$

It follows that

$$\begin{cases} g(y+\varepsilon) - g(y) \leq \left[1 - B^{(q)}(y; f(y+\varepsilon), y+\varepsilon)\right] g(y+\varepsilon) \\ g(y+\varepsilon) - g(y) \geq \left[1 - B^{(q)}(y; f(y), y+\varepsilon)\right] g(y+\varepsilon) \end{cases}$$

By Assumption 1.1, it follows that

$$g'(y) = b_f^{(q)}(y)g(y), \quad y < K,$$

with boundary condition g(K) = 1. Thus,

$$g(x) = e^{-\int_x^K b_f^{(q)}(z) dz}, \quad x < K.$$

Similarly, let

$$h(x) = \mathbb{E}_x \left[ e^{-q\tau_f - sY_{\tau_f}} \mathbb{1}_{\{\overline{X}_{\tau_f} \le K\}} \right], \quad x < K.$$

By the strong Markov property of X at maxima, for any  $X_0 = x \le y < K$  and  $0 < \varepsilon < K - y$ , we have

$$h(y) = \mathbb{E}_y \left[ e^{-q\tau_f - sY_{\tau_f}} \mathbb{1}_{\left\{\tau_f < \tau_{y+\varepsilon}^+\right\}} \right] + \mathbb{E}_y \left[ e^{-q\tau_{y+\varepsilon}^+} \mathbb{1}_{\left\{\tau_{y+\varepsilon}^+ < \tau_f\right\}} \right] h(y+\varepsilon).$$

By Corollary 2.1, it follows that

$$\begin{cases} h(y) \leq e^{s(f(y+\varepsilon)-f(y))}C^{(q,s)}(y;f(y+\varepsilon),y+\varepsilon) + B^{(q)}(y;f(y),y+\varepsilon)h(y+\varepsilon),\\ h(y) \geq e^{-s(f(y+\varepsilon)-f(y))}C^{(q,s)}(y;f(y),y+\varepsilon) + B^{(q)}(y;f(y+\varepsilon),y+\varepsilon)h(y+\varepsilon). \end{cases}$$

It follows that

$$\begin{cases} h(y+\varepsilon) - h(y) \ge -e^{s(f(y+\varepsilon) - f(y))}C^{(q,s)}(y; f(y+\varepsilon), y+\varepsilon) + \left[1 - B^{(q)}(y; f(y), y+\varepsilon)\right]h(y+\varepsilon), \\ h(y+\varepsilon) - h(y) \le -e^{-s(f(y+\varepsilon) - f(y))}C^{(q,s)}(y; f(y), y+\varepsilon) + \left[1 - B^{(q)}(y; f(y+\varepsilon), y+\varepsilon)\right]h(y+\varepsilon). \end{cases}$$

By Assumption 1.1, we deduce that

$$h'(y) = -c_f^{(q,s)}(y) + b_f^{(q)}(y)h(y), \quad y < K,$$

with boundary condition h(K) = 0. Therefore,

$$h(x) = \int_{x}^{K} e^{-\int_{x}^{y} b_{f}^{(q)}(z) dz} c_{f}^{(q,s)}(y) dy, \quad x < K.$$

This ends the proof.  $\blacksquare$ 

### 3 Extension to the general loss-carry-forward taxation model

The loss-carry-forward taxation model is first proposed by Albrecher and Hipp [4] under the compound Poisson model. It has been extended to the spectrally negative Lévy model by Albrecher et al. [6], the time-homogeneous diffusion model by Li et al. [19], and the Markov additive model by Albrecher [2].

In this section, we will further incorporate the general taxation proposed by Kyprianou and Zhou [15]. As our underlying model is upward skip-free Markov processes, our results will generalize [6], [15], and [19]. It is worth to mention that the methodologies adopt in these previous works are quite different, while this paper utilizes a unified and also more direct approach.

Consider a loss-carry-forward type tax strategy, where the tax payment is made whenever the surplus process reaches a new running maximum, (e.g., Kyprianou and Zhou [15])

$$\mathrm{d}U_t = \mathrm{d}X_t - \gamma(\overline{X}_t)\mathrm{d}\overline{X}_t,\tag{3.1}$$

where  $\gamma : [0, \infty) \to [0, \infty)$  is a measurable function. Note that for  $\gamma : [0, \infty) \to (1, \infty)$ , it is heavy-perturbation regime; while for  $\gamma : [0, \infty) \to [0, 1)$ , it is light-perturbation regime;  $\gamma(x) = 1_{\{x \ge a\}}$  corresponds to a reflection strategy, which sits between the previous two regimes, see, e.g., Kyprianou [14]. In what follows, we only consider the light-perturbation case with a non-decreasing function  $\gamma(\cdot)$ , and in addition, we assume the following condition holds:

$$\int_x^\infty (1 - \gamma(s)) ds = \infty.$$

For  $X_0 = x$ , define

$$\overline{\gamma}_x(y) := y - \int_x^y \gamma(z) \mathrm{d}z = x + \int_x^y (1 - \gamma(z)) \mathrm{d}z, \quad y \ge x,$$

which is strictly increasing and continuous with  $\overline{\gamma}(x) = x$ , and let  $\gamma_x(y) := y - \overline{\gamma}_x(y)$ .

The first passage times of U are defined in the same manner, i.e.,

$$\tau_x^{U,+} = \inf \left\{ t \ge 0 : U_t > x \right\} \text{ and } \tau_x^{U,-} = \inf \left\{ t \ge 0 : U_t < x \right\}.$$

Note that, conditional on  $X_0 = U_0 = x$ , for any  $y \ge x$ , we have

$$\overline{U}_t = \overline{\gamma}_x(\overline{X}_t) \text{ and } \tau_y^{U,+} = \tau_{\overline{\gamma}^{-1}(y)}^+.$$

The general drawdown process of the tax model U is denoted by  $Y^U = (Y^U_t)_{t \ge 0}$  with

$$Y_t^U = f(\overline{U}_t) - U_t,$$

where  $\overline{U}_t = \sup_{0 \le s \le t} U_t$  and f is an increasing function such that

$$f(x) < x$$
, for all  $x \in \mathbb{R}$ 

Hence,  $Y_0^U = f(\overline{U}_0) - U_0 < 0$ . The time of general drawdown is defined by

$$\sigma_f = \inf \left\{ t \ge 0 : Y_t^U > 0 \right\} = \inf \left\{ t \ge 0 : U_t < f(\overline{U}_t) \right\}.$$

Actually, from the general drawdown results for a general model X in Theorem 2.1, by noting the pathwise connection between X and U, one can easily find the general drawdown results for a general tax model U associated with the time-homogeneous Markov process X.

In the following, we first provide some time correspondences between processes U and X. Given  $X_0 = U_0 = x$ , in the light-perturbation case,

$$\tau_b^{U,+} = \tau_{\overline{\gamma}_x^{-1}(b)}^+, \quad a.s.,$$
(3.2)

since  $\overline{U}_t = \overline{X}_t - \int_{(0,t]} \gamma(\overline{X}_u) d\overline{X}_u = \overline{\gamma}_x(\overline{X}_t)$ ; see Equation (10.44) in Kyprianou [14].

$$U_0^{U,-} = \tau_{\gamma_x}, \quad a.s., \tag{3.3}$$

since  $U_t = X_t - \gamma_x(\overline{X}_t)$  and  $\{U_t < 0\} = \{X_t < \gamma_x(\overline{X}_t)\}.$ 

(iii)

$$\sigma_f = \tau_{f^*}, \quad a.s., \tag{3.4}$$

since 
$$\{U_t < f(\overline{U}_t)\} = \{X_t < f(\overline{\gamma}_x(\overline{X}_t)) + \gamma_x(\overline{X}_t)\} = \{X_t < f^*(\overline{X}_t)\}, \text{ where}$$
  
$$f^*(z) := f(\overline{\gamma}_x(z)) + \gamma_x(z).$$
(3.5)

**Theorem 3.1** Consider an upward skip-free time-homogeneous Markov process X such that Assumption 1.1 holds, and its general tax process U is defined in (3.1). For  $q, s \ge 0$  and  $x < K \in \mathbb{R}$ , we have

$$\mathbb{E}_{x}\left[e^{-q\tau_{K}^{U,+}}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}\right] = \exp\left\{-\int_{x}^{\overline{\gamma}_{x}^{-1}(K)}b_{f^{*}}^{(q)}(z)dz\right\},$$
(3.6)

$$\mathbb{E}_{x}\left[e^{-q\sigma_{f}-sY_{\sigma_{f}}^{U}}1_{\{\overline{U}_{\sigma_{f}}\leq K\}}\right] = \int_{x}^{\overline{\gamma}_{x}^{-1}(K)} e^{-\int_{x}^{y} b_{f^{*}}^{(q)}(z)\mathrm{d}z} c_{f^{*}}^{(q,s)}(y)\mathrm{d}y,$$
(3.7)

with  $f^*(\cdot)$  given in (3.5).

**Proof.** Using time correspondences (3.2) and (3.4), as well as Equation (2.6), one finds

$$\mathbb{E}_{x}[e^{-q\tau_{K}^{U,+}}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}] = \mathbb{E}_{x}[e^{-q\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+}}1_{\{\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+}<\tau_{f^{*}}\}}] = \exp\left\{-\int_{x}^{\overline{\gamma}_{x}^{-1}(K)}b_{f^{*}}^{(q)}(z)dz\right\},$$

which proves (3.6).

Similarly, noting  $\left\{\sigma_f, Y^U_{\sigma_f}, \overline{U}_{\sigma_f}\right\} \stackrel{d}{=} \left\{\tau_{f^*}, Y_{\tau_{f^*}}, \overline{\gamma}_x(\overline{X}_{f^*})\right\}$ , we have

$$\mathbb{E}_x \left[ e^{-q\sigma_f - sY^U_{\sigma_f}} \mathbf{1}_{\{\overline{U}_{\sigma_f} \le K\}} \right] = \mathbb{E}_x \left[ e^{-q\tau_{f^*} - sY_{\tau_{f^*}}} \mathbf{1}_{\{\overline{\gamma}_x(\overline{X}_{f^*}) \le K\}} \right]$$
$$= \mathbb{E}_x \left[ e^{-q\tau_{f^*} - sY_{\tau_{f^*}}} \mathbf{1}_{\{\overline{X}_{f^*} \le \overline{\gamma}_x^{-1}(K)\}} \right]$$
$$= \int_x^{\overline{\gamma}_x^{-1}(K)} e^{-\int_x^y b_{f^*}^{(q)}(z) \mathrm{d}z} c_{f^*}^{(q,s)}(y) \mathrm{d}y.$$

In the following proposition, we provide the results relating to the expected present value of tax up to some certain stopping times. We denote  $\eta(\cdot)$  as a general tax payment function, which depends on the surplus level at the moment of paying tax.

**Proposition 3.1** For x < K and any function  $\eta(\cdot) > 0$ , the expected present value of tax until general drawdown or exiting above is

$$\mathbb{E}_x\left[\int_0^{\tau_K^{U,+}\wedge\sigma_f} e^{-qu}\eta(\overline{X}_u)d\overline{X}_u\right] = \int_x^{\overline{\gamma}_x^{-1}(K)}\eta(y)\exp\left\{-\int_x^y b_{f^*}^{(q)}(z)dz\right\}dy,$$

and the expected present value of tax until reaching level K before general drawdown is

$$\mathbb{E}_{x}\left[\int_{0}^{\tau_{K}^{U,+}} e^{-qu}\eta(\overline{X}_{u})d\overline{X}_{u}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}\right] = \int_{x}^{\overline{\gamma}_{x}^{-1}(K)}\eta(y)\exp\left\{-\int_{x}^{y}b_{f^{*}}^{(q)}(z)dz - \int_{y}^{\overline{\gamma}_{x}^{-1}(K)}b_{f^{*}}^{(0)}(z)dz\right\}dy.$$

**Proof.** Thanks to the path/time correspondences in (3.2)-(3.4), we have

$$\begin{split} \mathbb{E}_{x}\left[\int_{0}^{\tau_{K}^{U,+}\wedge\sigma_{f}}e^{-qu}\eta(\overline{X}_{u})d\overline{X}_{u}\right] = \mathbb{E}_{x}\left[\int_{0}^{\tau_{T}^{+}(K)}^{\tau_{T}^{+}(K)}\wedge\tau_{f^{*}}\wedge e_{q}}\eta(\overline{X}_{u})d\overline{X}_{u}\right] \\ &= \int_{x}^{\infty}\int_{x}^{z}\eta(y)dy\mathbb{P}_{x}(\overline{X}(\tau_{T}^{+}_{\overline{\gamma}_{x}^{-1}(K)}\wedge\tau_{f^{*}}\wedge e_{q})\in dz) \\ &= \int_{x}^{\overline{\gamma}_{x}^{-1}(K)}\eta(y)\mathbb{P}_{x}(\overline{X}(\tau_{f^{*}}\wedge e_{q})>y)dy \\ &= \int_{x}^{\overline{\gamma}_{x}^{-1}(K)}\eta(y)\mathbb{P}_{x}(\tau_{y}^{+}<\tau_{f^{*}}\wedge e_{q})dy \\ &= \int_{x}^{\overline{\gamma}_{x}^{-1}(K)}\eta(y)\mathbb{E}_{x}(e^{-q\tau_{y}^{+}}\mathbf{1}_{\{\tau_{y}^{+}<\tau_{f^{*}}\}})dy \\ &= \int_{x}^{\overline{\gamma}_{x}^{-1}(K)}\eta(y)\exp\left\{-\int_{x}^{y}b_{f^{*}}^{(q)}(z)dz\right\}dy, \end{split}$$

and

$$\begin{split} \mathbb{E}_{x} \left[ \int_{0}^{\tau_{K}^{U,+}} e^{-qu} \eta(\overline{X}_{u}) d\overline{X}_{u} \mathbf{1}_{\{\tau_{K}^{U,+} < \sigma_{f}\}} \right] = \mathbb{E}_{x} \left[ \int_{0}^{\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} \wedge e_{q}} \eta(\overline{X}_{u}) d\overline{X}_{u} \mathbf{1}_{\{\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} < \tau_{f^{*}}\}} \right] \\ = \int_{x}^{\infty} \int_{x}^{z} \eta(y) dy \mathbb{P}_{x} [\overline{X}(\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} \wedge e_{q}) \in dz, \tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} < \tau_{f^{*}}] \\ = \int_{x}^{\infty} \eta(y) \mathbb{P}_{x} [\overline{X}(\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} \wedge e_{q}) > y, \tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} < \tau_{f^{*}}] dy \\ = \int_{x}^{\infty} \eta(y) \mathbb{P}_{x}(\tau_{y}^{+} < \tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} \wedge \tau_{f^{*}} \wedge e_{q}) \mathbb{P}_{y}(\tau_{\overline{\gamma}_{x}^{-1}(K)}^{+} < \tau_{f^{*}}) dy \\ = \int_{x}^{\overline{\gamma}_{x}^{-1}(K)} \eta(y) \exp\left\{ -\int_{x}^{y} b_{f^{*}}^{(q)}(z) dz \right\} \exp\left\{ -\int_{y}^{\overline{\gamma}_{x}^{-1}(K)} b_{f^{*}}^{(0)}(z) dz \right\} dy \end{split}$$

which completes the proof.  $\blacksquare$ 

**Remark 3.1** In a special case with  $\gamma$  being a constant, we have

$$\overline{\gamma}_x(y) = y - \gamma(y - x), \text{ and } \overline{\gamma}_x^{-1}(y) = \frac{y - \gamma x}{1 - \gamma},$$

and

$$f^*(z) = f(\overline{\gamma}_x(z)) + z - \overline{\gamma}_x(z) = f(z - \gamma(z - x)) + \gamma(z - x).$$

Rewriting (3.6) using a change of variable,

$$\mathbb{E}_{x}\left[e^{-q\tau_{K}^{U,+}}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}\right] = \exp\left\{-\int_{x}^{\overline{\gamma}_{x}^{-1}(K)}b_{f^{*}}^{(q)}(z)\mathrm{d}z\right\} = \exp\left\{-\int_{x}^{K}\frac{1}{1-\gamma}b_{f^{*}}^{(q)}(\overline{\gamma}_{x}^{-1}(y))\mathrm{d}y\right\},$$

Introducing

$$W_{f^*}^{(q)}(z) = e^{\int_{z_0}^z b_{f^*}^{(q)}(\overline{\gamma}_x^{-1}(y)) \mathrm{d}y} \Leftrightarrow b_{f^*}^{(q)}(\overline{\gamma}_x^{-1}(z)) = \frac{W_{f^*}^{(q)\prime}(z)}{W_{f^*}^{(q)}(z)},$$

for some fixed  $x_0 \leq K$ , and we may rewrite (3.6) as

$$\mathbb{E}_{x}\left[e^{-q\tau_{K}^{U,+}}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}\right] = \left(\frac{W_{f^{*}}^{(q)}(x)}{W_{f^{*}}^{(q)}(K)}\right)^{\frac{1}{1-\gamma}}$$

Thus, the multiplicative structure is still present with generalized drawdown times, and tax introduces an extra power, see, e.g., [4] and [5].

## 4 Examples

In this section, we consider the Spectrally Negative Lévy process, time-homogeneous diffusion process and Ornstein-Uhlenbeck process with exponential jumps for specific examples. These processes are of particular interests thanks to their various applications in insurance and finance.

#### 4.1 Spectrally negative Lévy process

Consider a spectrally negative Lévy process X. Let  $\psi(s) := \frac{1}{t} \log \mathbb{E}[e^{sX_t}]$ ,  $s \ge 0$ , be the Laplace exponent of X. Further, let  $W^{(q)} : \mathbb{R} \to [0, \infty)$  be the well-known q-scale function of X. The second scale function is defined as  $Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy$ . We assume the scale functions are continuously differentiable. For  $p = q - \psi(s)$ , let  $W_s^{(p)}(Z_s^{(p)})$  be the (second) scale function of X under a new probability measure  $\mathbb{P}^s$  defined by the Radon-Nikodym derivative process  $\frac{d\mathbb{P}^s}{d\mathbb{P}}|_{\mathcal{F}_t} = e^{sX_t - \psi(s)t}$  for  $t \ge 0$ . Recall that

$$B^{(q)}(x;u,v) = \mathbb{E}_x \left[ e^{-q\tau_v^+} \mathbf{1}_{\left\{\tau_v^+ < \infty, \tau_v^+ < \tau_u^-\right\}} \right] = \frac{W^{(q)}(x-u)}{W^{(q)}(v-u)},$$

and

$$C^{(q,s)}(x;u,v) = \mathbb{E}_x \left[ e^{-q\tau_u^- - s(u - X_{\tau_u^-})} \mathbb{1}_{\left\{\tau_u^- < \infty, \tau_u^- < \tau_v^+\right\}} \right] = Z_s^{(p)}(x-u) - Z_s^{(p)}(v-u) \frac{W_s^{(p)}(x-u)}{W_s^{(p)}(v-u)}$$

It is direct to check that Assumption 1.1 is satisfied. More specifically,

$$b_f^{(q)}(x) = -\left. \frac{\partial B^{(q)}(x; f(x), v)}{\partial v} \right|_{v=x} = \frac{W^{(q)\prime}(x - f(x))}{W^{(q)}(x - f(x))}$$

and

$$c_f^{(q,s)}(x) = \left. \frac{\partial C^{(q,s)}(x;f(x),v)}{\partial v} \right|_{v=x} = Z_s^{(p)}(x-f(x)) \frac{W_s^{(p)\prime}(x-f(x))}{W_s^{(p)}(x-f(x))} - Z_s^{(p)\prime}(x-f(x))$$

Then Theorem 2.1 implies, for  $x \leq K$ ,

$$\mathbb{E}_{x}\left[e^{-q\tau_{f}-sY_{\tau_{f}}}1_{\{\tau_{f}<\infty,\overline{X}_{\tau_{f}}\leq K\}}\right] = \int_{x}^{K}e^{-\int_{x}^{y}\frac{W^{(q)'}(z-f(z))}{W^{(q)}(z-f(z))}\mathrm{d}z}\left(Z_{s}^{(p)}(y-f(y))\frac{W_{s}^{(p)'}(y-f(y))}{W_{s}^{(p)}(y-f(y))}-Z_{s}^{(p)'}(y-f(y))\right)\mathrm{d}y$$

which is consistent with Proposition 3.1 in [20].

In particular, suppose that

$$f(x) = \xi x - d$$

where  $\xi \leq 1$  and d>0 are two fixed constants. One has a simplified formula because

$$e^{-\int_{x}^{y} \frac{W^{(q)'(z-f(z))}}{W^{(q)}(z-f(z))} \mathrm{d}z} = e^{-\int_{x}^{y} \frac{W^{(q)'((1-\xi)z+d)}}{W^{(q)}((1-\xi)z+d)} \mathrm{d}z} = e^{-\frac{1}{1-\xi} \ln W^{(q)}((1-\xi)z+d)|_{z=x}^{z=y}} = \left(\frac{W^{(q)}((1-\xi)x+d)}{W^{(q)}((1-\xi)y+d)}\right)^{\frac{1}{1-\xi}}.$$

Below is a direct corollary from Theorem 3.1 and Proposition 3.1. Corollary 4.1 For x < K and any function  $\eta(\cdot)$ ,

$$\mathbb{E}_x\left[e^{-q\tau_K^{U,+}}\mathbf{1}_{\{\tau_K^{U,+}<\sigma_f\}}\right] = \exp\left\{-\int_x^{\overline{\gamma}_x^{-1}(K)} \frac{W^{(q)\prime}(\overline{f}(\overline{\gamma}_x(t)))}{W^{(q)}(\overline{f}(\overline{\gamma}_x(t)))}dt\right\},\,$$

$$\mathbb{E}_x \left[ \int_0^{\tau_K^{U,+} \wedge \sigma_f} e^{-qu} \eta(\overline{X}_u) d\overline{X}_u \right] = \int_x^{\overline{\gamma}_x^{-1}(K)} \eta(y) \exp\left( -\int_x^y \frac{W^{(q)'}(\overline{f}(\overline{\gamma}_x(t)))}{W^{(q)}(\overline{f}(\overline{\gamma}_x(t)))} dt \right) dy,$$

and

$$\mathbb{E}_{x}\left[\int_{0}^{\overline{\gamma}_{K}^{U,+}} e^{-qu}\eta(\overline{X}_{u})d\overline{X}_{u}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}\right]$$

$$=\int_{x}^{\overline{\gamma}_{x}^{-1}(K)}\eta(y)\exp\left(-\int_{x}^{y}\frac{W^{(q)'}(\overline{f}(\overline{\gamma}_{x}(t)))}{W^{(q)}(\overline{f}(\overline{\gamma}_{x}(t)))}dt-\int_{y}^{\overline{\gamma}_{x}^{-1}(K)}\frac{W'(\overline{f}(\overline{\gamma}_{x}(t)))}{W(\overline{f}(\overline{\gamma}_{x}(t)))}dt\right)dy,$$

where  $\overline{f}(x) = x - f(x)$ .

**Remark 4.1** In the special case, where  $\gamma(\cdot) = \gamma$  and  $f(s) = \xi s - d$ , we have

$$\overline{\gamma}_x(s) = s - \gamma s + \gamma x, \qquad \overline{f}(s) = (1 - \xi)s + d,$$
$$\overline{f}(\overline{\gamma}_x(t)) = \overline{\gamma_x + f(\overline{\gamma}_x)} = (1 - \xi)(s - \gamma s + \gamma x) + d.$$

Hence, Corollary 4.1 reduces to

$$\mathbb{E}_{x}\left[e^{-q\tau_{K}^{U,+}}1_{\{\tau_{K}^{U,+}<\sigma_{f}\}}\right] = \exp\left(-\frac{1}{(1-\xi)(1-\gamma)}\int_{(1-\xi)x+d}^{(1-\xi)K+d}\frac{W^{(q)\prime}(y)}{W^{(q)}(y)}dy\right) = \left(\frac{W^{(q)}((1-\xi)x+d)}{W^{(q)}((1-\xi)K+d)}\right)^{\frac{1}{(1-\xi)(1-\gamma)}}$$

and furthermore, by letting  $\eta(\cdot) = 1$ , we have

$$\mathbb{E}_{x}\left[\int_{0}^{\tau_{K}^{U,+}\wedge\sigma_{f}}e^{-qu}d\overline{X}_{u}\right] = \frac{1}{1-\gamma}\int_{x}^{K}\left(\frac{W^{(q)}((1-\xi)x+d)}{W^{(q)}((1-\xi)z+d)}\right)^{\frac{1}{(1-\xi)(1-\gamma)}}dz,$$

and

$$\mathbb{E}_{x}\left[\int_{0}^{\tau_{K}^{U,+}} e^{-qu}\gamma(\overline{X}_{u})d\overline{X}_{u}; \tau_{K}^{U,+} < \sigma_{f}\right] = \frac{1}{1-\gamma}\int_{x}^{K} \left(\frac{W^{(q)}((1-\xi)x+d)}{W^{(q)}((1-\xi)z+d)}\frac{W((1-\xi)z+d)}{W((1-\xi)a+d)}\right)^{\frac{1}{(1-\xi)(1-\gamma)}}dz.$$

which are consistent with Theorems 1.1 and 1.2 in [9] respectively.

#### 4.2 Time-homogeneous diffusion process

Consider a linear diffusion process X of the form

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t,$$

where  $(B_t)_{t\geq 0}$  is a standard Brownian motion, and the drift term  $\mu(\cdot)$  and local volatility  $\sigma(\cdot) > 0$ satisfy the usual Lipschitz continuity and linear growth conditions. The infinitesimal generator of X is given by

$$\mathcal{L}_X = \frac{1}{2}\sigma^2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \mu(x)\frac{\mathrm{d}}{\mathrm{d}x}$$

It is well-known that, for any q > 0, there exist two independent and positive solutions, denoted as  $\phi_q^{\pm}(y)$ , to the Sturm-Liouville equation

$$\mathcal{L}_X \phi_q^{\pm}(y) = q \phi_q^{\pm}(y), \tag{4.1}$$

where  $\phi_q^+(\cdot)$  is strictly increasing and  $\phi_q^-(\cdot)$  is strictly decreasing.

Thanks to  $\phi_q^{\pm}(y)$ , it is known that

$$B^{(q)}(x; u, v) = \mathbb{E}_x \left[ e^{-q\tau_v^+} \mathbf{1}_{\{\tau_v^+ < \tau_u^-\}} \right] = \frac{\Phi_q(u, x)}{\Phi_q(u, v)}$$

and

$$C^{(q,s)}(x;u,v) = \mathbb{E}_x \left[ e^{-q\tau_u^-} \mathbf{1}_{\{\tau_u^- < \tau_v^+\}} \right] = \frac{\Phi_q(x,v)}{\Phi_q(u,v)},$$

where  $\Phi_q(x,y) := \phi_q^+(x)\phi_q^-(y) - \phi_q^+(y)\phi_q^-(x)$ . Note that  $C^{(q,s)}(x;u,v)$  does not depend on the argument s since the diffusion process has  $X_{\tau_u^-} = u$  a.s.

Then Assumption 1.1 is satisfied, and we have

$$b_f^{(q)}(x) = -\left. \frac{\partial B^{(q)}(x; f(x), v)}{\partial v} \right|_{v=x} = \frac{\Phi_{q,2}(f(x), x)}{\Phi_q(f(x), x)},$$

and

$$c_{f}^{(q,s)}(x) = \left. \frac{\partial C^{(q,s)}(x;f(x),v)}{\partial v} \right|_{v=x} = \frac{\Phi_{q,2}(x,x)}{\Phi_{q}(f(x),x)} = -\frac{\Phi_{q,1}(x,x)}{\Phi_{q}(f(x),x)}$$

where  $\Phi_{q,1}(x,y) := \frac{\partial}{\partial x} \Phi_q(x,y)$  and  $\Phi_{q,2}(x,y) := \frac{\partial}{\partial y} \Phi_q(x,y)$ . Notice the fact that  $\Phi_{q,2}(x,x) = -\Phi_{q,1}(x,x)$ .

**Corollary 4.2** For  $q, s \ge 0$  and  $x < K \in \mathbb{R}$ , we have

$$\mathbb{E}_{x} \left[ e^{-q\tau_{K}^{+}} 1_{\{\tau_{K}^{+} < \tau_{f}\}} \right] = e^{-\int_{x}^{K} \frac{\Phi_{q,2}(f(z),z)}{\Phi_{q}(f(z),z)} \mathrm{d}z},$$
  
$$\mathbb{E}_{x} \left[ e^{-q\tau_{f}} 1_{\{\overline{X}_{\tau_{f}} \le K\}} \right] = \int_{x}^{K} e^{-\int_{x}^{y} \frac{\Phi_{q,2}(f(z),z)}{\Phi_{q}(f(z),z)} \mathrm{d}z} \frac{\Phi_{q,2}(x,x)}{\Phi_{q}(f(x),x)} \mathrm{d}y.$$

**Remark 4.2** In the special case, with proper choices of q and x = 0, it is easy to check that the results in Corollary 4.2 are consistent with Equations (20) and (21) in [16].

#### 4.3 Ornstein-Uhlenbeck process with exponential jumps

Consider a generalized Ornstein-Uhlenbeck process X with negative jumps, where

$$dX_t = \theta(\mu - X_t)dt + \sigma dB_t - d\left(\sum_{i=1}^{N_t} P_i\right),$$

where  $\theta > 0$ ,  $\mu \in \mathbb{R}$  and  $X_0 = x$ . Also,  $(B_t)_{t\geq 0}$  is a standard Brownian motion, and  $\sum_{i=1}^{N_t} P_i$  is an independent compound Poisson process. In particular, we assume the Poisson process  $(N_t)_{t\geq 0}$ has intensity  $\lambda$ , and the jumps follow the exponential distribution with mean  $1/\eta$ . Note that one could rewrite the process X as

$$X_t = X_0 - \theta \int_0^t X_s ds + K_t,$$
  
$$X_t = X_0 e^{-\theta t} + e^{-\theta t} \int_0^t e^{\theta s} dK_s,$$

with  $K_t = (\theta \mu)t + \sigma B_t - \sum_{i=1}^{N_t} P_i$  being a Brownian perturbed Cramér-Lundberg process, whose Laplace exponent is

$$\psi(s) := \frac{1}{t} \log \mathbb{E}[e^{sK_t}] = \theta \mu s + \frac{\sigma^2}{2}s^2 + \lambda(\frac{\eta}{\eta+s} - 1).$$

From Lemmas 2.1 and 2.2 in [27], where the authors examined the occupation times of Ornstein-Uhlenbeck process with two-sided exponential jumps, we have the following results:

$$\mathbb{E}_{x}\left[e^{-q\tau_{u}^{-}}1_{\{X_{\tau_{u}^{-}}=u\}}\right] = \frac{C_{2}^{q}(u)F_{1}^{q}(x) - C_{1}^{q}(u)F_{2}^{q}(x)}{C_{2}^{q}(u)F_{1}^{q}(u) - C_{1}^{q}(u)F_{2}^{q}(u)} =: I_{1}(x, u), \tag{4.2}$$

$$\mathbb{E}_{x}\left[e^{-q\tau_{u}^{-}-s(u-X_{\tau_{u}^{-}})}1_{\{X_{\tau_{u}^{-}}< u\}}\right] = \frac{F_{1}^{q}(u)F_{2}^{q}(x) - F_{2}^{q}(u)F_{1}^{q}(x)}{C_{2}^{q}(u)F_{1}^{q}(u) - C_{1}^{q}(u)F_{2}^{q}(u)}\frac{\eta}{\eta+s} =: I_{2}(x,u)\frac{\eta}{\eta+s}, \qquad (4.3)$$

$$\mathbb{E}_x\left[e^{-q\tau_v^+}\right] = \frac{F_3^q(x)}{F_3^q(v)},\tag{4.4}$$

where

$$\phi_q(x) := |x|^{\frac{q}{\theta} - 1} e^{-\frac{\sigma^2}{4\theta} x^2 + \mu x} |x - \eta|^{\frac{\lambda}{\theta}}, \quad F_i^q(x) := \int_{\Gamma_i} \phi_q(z) e^{-xz} dz, \quad C_i^q := -\int_{\Gamma_i} \frac{\eta}{z - \eta} \phi_q(z) e^{-xz} dz,$$

with  $\Gamma_1 = (0, \eta)$ ,  $\Gamma_2 = (\eta, \infty)$  and  $\Gamma_3 = (-\infty, 0)$ . Hence, using the strong Markov property, one has

$$B^{(q)}(x;u,v) = \mathbb{E}_{x} \left[ e^{-q\tau_{v}^{+}} 1_{\left\{\tau_{v}^{+} < \tau_{u}^{-}\right\}} \right]$$
$$= \mathbb{E}_{x} \left[ e^{-q\tau_{v}^{+}} \right] - \mathbb{E}_{x} \left[ e^{-q\tau_{u}^{-}} 1_{\left\{X_{\tau_{u}^{-}} = u, \tau_{u}^{-} < \tau_{v}^{+}\right\}} \right] \mathbb{E}_{u} \left[ e^{-q\tau_{v}^{+}} \right] - \int_{0}^{\infty} \mathbb{E}_{x} \left[ e^{-q\tau_{u}^{-}} 1_{\left\{u - X_{\tau_{u}^{-}} \in dy, \tau_{u}^{-} < \tau_{v}^{+}\right\}} \right] \mathbb{E}_{u-y} \left[ e^{-q\tau_{v}^{+}} \right]$$

and

$$C^{(q,s)}(x;u,v) = \mathbb{E}_{x} \left[ e^{-q\tau_{u}^{-} - s(u - X_{\tau_{u}^{-}})} 1_{\{\tau_{u}^{-} < \tau_{v}^{+}\}} \right]$$
$$= \mathbb{E}_{x} \left[ e^{-q\tau_{u}^{-} - s(u - X_{\tau_{u}^{-}})} \right] - \mathbb{E}_{x} \left[ e^{-q\tau_{v}^{+}} 1_{\{\tau_{v}^{+} < \tau_{u}^{-}\}} \right] \mathbb{E}_{v} \left[ e^{-q\tau_{u}^{-} - s(u - X_{\tau_{u}^{-}})} \right].$$

It is easy to solve  $B^{(q)}(x; u, v)$  and  $C^{(q,s)}(x; u, v)$  using Equations (4.2)-(4.4) (noticing that the 'deficit' in (4.3) has an exponential density)

$$B^{(q)}(x;u,v) = \frac{F_3^q(x) - I_1(x,u)F_3^q(u) - I_2(x,u)\int_0^\infty \eta e^{-\eta y}F_3^q(u-y)dy}{F_3^q(v) - I_1(v,u)F_3^q(u) - I_2(v,u)\int_0^\infty \eta e^{-\eta y}F_3^q(u-y)dy},$$
  

$$C^{(q,s)}(x;u,v) = \left[I_1(x,u) + I_2(x,u)\frac{\eta}{\eta+s}\right] - B^{(q)}(x;u,v)\left[I_1(v,u) + I_2(v,u)\frac{\eta}{\eta+s}\right].$$

Then Assumption 1.1 is satisfied, and we could obtain the differential exit parameters  $b_f^{(q)}(x) = -\frac{\partial B^{(q)}(x;f(x),v)}{\partial v}\Big|_{v=x}$  and  $c_f^{(q,s)}(x) = \frac{\partial C^{(q,s)}(x;f(x),v)}{\partial v}\Big|_{v=x}$ . The differential calculations are omitted for conciseness and left for interested readers.

## References

- [1] Ablrecher, H.; Asmussen, S. Ruin probabilities. Volume 14, World Scientific, 2010.
- [2] Albrecher, H.; Avram, F.; Constantinescu, C.; Ivanovs, J. The tax identity for Markov additive risk processes. Methodology and Computing in Applied Probability, 16(1), 245–258, 2014.
- [3] Albrecher, H.; Borst, S.; Boxma, O.; Resing, J. The tax identity in risk theory a simple proof and an extension. Insurance: Mathematics and Economics, 44(2), 304–306, 2009.
- [4] Albrecher, H.; Hipp, C. Lundberg's risk process with tax. Blätter der DGVFM, 28(1), 13–28, 2007.
- [5] Albrecher, H.; Ivanovs, J. Power identities for Lévy risk models under taxation and capital injections. Stochastic Systems, 4(1), 157–172, 2014.
- [6] Albrecher, H.; Renaud, J. F.; Zhou, X. A Lévy insurance risk process with tax. Journal of Applied Probability, 45(2), 363-375, 2008.
- [7] Avram, F.; Kyprianou, A.; Pistorius, M. Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. The Annals of Applied Probability, 14(1), 215–238, 2004.
- [8] Avram, F.; Vardar-Acar, C. Exit time of a strong Markov process with negative jumps and its draw-down from a rectangular region. Submitted, 2018.
- [9] Avram, F.; Vu, N.; Zhou, X. On taxed spectrally negative Lévy processes with draw-down stopping. Insurance: Mathematics and Economics, 76, 69–74, 2017.
- [10] Azéma, J.; Yor, M. Une solution simple au problème de Skorokhod. In Séminaire de probabilités XIII, 90–115, 1979.
- [11] Bertoin, J. Lévy processes. Volume 121, Cambridge university press, 1998.
- [12] Carr, P. First-order calculus and option pricing. Journal of Financial Engineering, 1(01), 1450009, 2014.
- [13] Czarna, I.; Pérez, J.; Rolski, T.; Yamazaki, K. Fluctuation theory for level-dependent Lévy risk process. arXiv: 1712.00050, 2017.
- [14] Kyprianou, A. Fluctuations of Lévy Processes with Applications. Introductory Lectures. Springer, 2014.
- [15] Kyprianou, A.; Zhou, X. General tax structures and the Lévy insurance risk model. Journal of Applied Probability, 46(4), 1146–1156, 2009.
- [16] Lehoczky, J. Formulas for stopped diffusion processes with stopping times based on the maximum. The Annals of Probability, 5(4), 601–607, 1977.

- [17] Landriault, D.; Li, B.; Li, S. Analysis of a drawdown-based regime-switching Lévy insurance model. Insurance: Mathematics and Economics, 60, 98–107, 2015.
- [18] Landriault, D.; Li, B.; Zhang, H. A unified approach for drawdown (drawup) of time-homogeneous Markov processes. Journal of Applied Probability, 54(2), 603–626, 2017.
- [19] Li, B.; Tang, Q.; Zhou, X. A time-homogeneous diffusion model with tax. Journal of Applied Probability, 50(1), 195-207, 2013.
- [20] Li, B.; Vu, L.; Zhou, X. Exit problems for general draw-down times of spectrally negative Lévy processes. arXiv:1702.07259, 2017.
- [21] Mijatovic, A.; Pistorius, M. On the drawdown of completely asymmetric Lévy processes. Stochastic Processes and their Applications, 122(11), 3812–3836, 2012.
- [22] Page, E. Continuous inspection schemes. Biometrika, 41(1/2), 100–115, 1954.
- [23] Shepp, L.; Shiryaev, A. The Russian option: reduced regret. The Annals of Applied Probability, 3(3), 631–640, 1993.
- [24] Suprum, V. Problem of destruction and resolvent of a terminating process with independent increments. Ukrainian Mathematical Journal, 28(1), 39–51, 1976.
- [25] Taylor, H. A stopped Brownian motion formula. The Annals of Probability, 3(2), 234–246, 1975.
- [26] Zhang, H. Stochastic Drawdowns. Volume 2, World Scientific, 2018.
- [27] Zhou, J.; Wu, L.; Bai, Y. Occupation times of Lévy-driven Ornstein-Uhlenbeck processes with two-sided exponential jumps and applications. Statistics and Probability Letters, 125, 80–90, 2017.