# RATE OF STRONG CONVERGENCE TO MARKOV-MODULATED BROWNIAN MOTION 

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#### Abstract

In [13], the authors constructed a sequence of stochastic fluid processes and showed that it converges weakly to a Markov-modulated Brownian motion (MMBM). Here, we construct a different sequence of stochastic fluid processes and show that it converges strongly to an MMBM. To the best of our knowledge, this is the first result on strong convergence to a Markov-modulated Brownian motion.

We also prove that the rate of this almost sure convergence is $o\left(n^{-1 / 2} \log n\right)$. When reduced to the special case of standard Brownian motion, our convergence rate is an improvement over that obtained by a different approximation in [9], which is $o\left(n^{-1 / 2}(\log n)^{5 / 2}\right)$.


1. Introduction. The family of fip-flop processes corresponds to a class of piecewiselinear Markov processes that converges, in some sense, to a standard Brownian motion. Specifically, for $\lambda>0$, let $=\left\{\varphi^{\lambda}(t)\right\}_{t \geq 0}$ be a Markov jump process with state space $\{+,-\}$, initial distribution ( $1 / 2,1 / 2$ ) and intensity matrix

$$
\left[\begin{array}{rr}
-\lambda & \lambda \\
\lambda & -\lambda
\end{array}\right] .
$$

Let $r(+)=\sqrt{\lambda}, r(-)=-\sqrt{\lambda}$ and define

$$
\begin{equation*}
F^{\lambda}(t)=\int_{0}^{t} r\left(\varphi^{\lambda}(s)\right) \mathrm{d} s, \quad t \geq 0 . \tag{1.1}
\end{equation*}
$$

We call $\left\{\left(F^{\lambda}(t), \varphi^{\lambda}(t)\right)\right\}_{t \geq 0}$ a flip-flop process. It can be shown (see, e.g., [18]) that $\mathcal{F}^{\lambda}=$ $\left\{F^{\lambda}(t)\right\}_{t \geq 0}$ converges weakly to a standard Brownian motion $\mathcal{B}=\{B(t)\}_{t \geq 0}$ as $\lambda \rightarrow \infty$. In other words,

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty} \mathbb{E}\left[h\left(\mathcal{F}^{\lambda}\right)\right]=\mathbb{E}[h(\mathcal{B})] \tag{1.2}
\end{equation*}
$$

whenever $h: \mathcal{C}([0, \infty)) \mapsto \mathbb{R}$ is a bounded Borel-measurable functional continuous with respect to the topology of uniform convergence on compact intervals. Weak convergence implies that the family of probability laws induced by $\left\{\mathcal{F}^{\lambda}\right\}_{\lambda>0}$ is tight, and that, for any $0 \leq t_{1}<t_{2}<$ $\cdots<t_{n}<\infty$,

$$
\lim _{\lambda \rightarrow \infty}\left(F^{\lambda}\left(t_{1}\right), F^{\lambda}\left(t_{2}\right), \ldots, F^{\lambda}\left(t_{n}\right)\right) \stackrel{d}{=}\left(B\left(t_{1}\right), B\left(t_{2}\right), \ldots, B\left(t_{n}\right)\right) .
$$

[^0]These two properties are also sufficient conditions for (1.2) to hold [4]. As weak convergence is a statement regarding probability laws, the stochastic processes involved do not need to be defined on a common probability space.

An alternative definition of the flip-flop process $\mathcal{F}^{\lambda}$ is as follows. For $t>0$, let

$$
N^{\lambda}(t)=\#\left\{s \in(0, t]: \varphi^{\lambda}\left(s^{-}\right) \neq \varphi^{\lambda}(s)\right\}
$$

and $N^{\lambda}(0)=0$. Then, $\left\{N^{\lambda}(t)\right\}_{t \geq 0}$ is the Poisson process of intensity $\lambda$ which counts the jumps of $\left\{\varphi^{\lambda}(t)\right\}_{t \geq 0}$, and we can rewrite (1.1) as

$$
\begin{equation*}
F^{\lambda}(t)=\sqrt{\lambda} \int_{0}^{t}(-1)^{N^{\lambda}(s)} \mathrm{d} s, \quad t \geq 0 \tag{1.3}
\end{equation*}
$$

The process $\mathcal{F}^{\lambda}$ defined as in (1.3) was first considered in $[6,11]$, where a link between its transition probabilities and the telegraph equation was developed. In this context, $\mathcal{F}^{\lambda}$ became known as a telegraph process or uniform transport process, of which the weak convergence to $\mathcal{B}$ was proved in [17] and [19].

Later on, it was proved in [10] that such a convergence also holds in a pathwise sense. More precisely, the authors showed that there exists a common probability space in which the family of flip-flop (or uniform transport) processes $\left\{\mathcal{F}^{\lambda}\right\}_{\lambda>0}$ and a standard Brownian motion $\mathcal{B}$ are defined such that for any $T>0$

$$
\begin{equation*}
\lim _{\lambda \rightarrow \infty} \sup _{0 \leq t \leq T}\left|F^{\lambda}(t)-B(t)\right|=0 \quad \text { almost surely. } \tag{1.4}
\end{equation*}
$$

Whenever (1.4) holds, we say that $\mathcal{F}^{\lambda}$ converges strongly to $\mathcal{B}$ as $\lambda \rightarrow \infty$. By applying the Bounded Convergence Theorem to (1.2), we trivially get that strong convergence implies weak convergence. Strong convergence results also lead to stronger approximations for diffusions and for solutions to stochastic differential equations (e.g. in [8] and [7], respectively). In [9], the rate of strong convergence of $\mathcal{F}^{\lambda}$ to $\mathcal{B}$ was computed. The key step in [10, 9] consisted in embedding certain values of $\mathcal{F}^{\lambda}$ into $\mathcal{B}$ using the Skorokhod embedding theorem.

In recent years, the study of flip-flop processes was generalised into different directions, most of which are based on the following. Consider a process $(\mathcal{R}, \mathcal{J})=\{(R(t), J(t))\}_{t \geq 0}$ where the phase process $\mathcal{J}$ is a Markov jump process on a finite state space $\mathcal{S}$, initial distribution $\boldsymbol{p}$, and intensity matrix $Q$, and the level process $\mathcal{R}$ is defined by

$$
\begin{equation*}
R(t)=\int_{0}^{\infty} \mu_{J(s)} \mathrm{d} s+\int_{0}^{\infty} \sigma_{J(s)} \mathrm{d} B(s), \quad t \geq 0 \tag{1.5}
\end{equation*}
$$

with $\mu_{i} \in \mathbb{R}$ and $\sigma_{i} \geq 0$ for $i \in \mathcal{S}$. If $\sigma_{i}=0$ for all $i \in \mathcal{S}$, the process $(\mathcal{R}, \mathcal{J})$ is known as a stochastic fluid process (SFP). If $\sigma_{i}>0$ for all $i \in \mathcal{S}$, then $(\mathcal{R}, \mathcal{J})$ is called a Markov modulated Brownian motion (MMBM). In [13], it is shown that there exists a family of SFPs that converges weakly to any given MMBM. This result was later used to study MMBM with two boundaries in [12], [14] and [1], Markov-modulated sticky Brownian motion in [15], and MMBM with temporary change of regime at zero in [16].

In this paper, we construct a sequence of stochastic fluid processes which converges strongly to an MMBM of any given parameters. More specifically, we prove the following result.

Theorem 1.1. For any given $\boldsymbol{p}, Q,\left\{\mu_{i}\right\}_{i \in \mathcal{S}}$ and $\left\{\sigma_{i}>0\right\}_{i \in \mathcal{S}}$, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ on which live an $\operatorname{MMBM}(\mathcal{R}, \mathcal{J})=\{(R(t), J(t))\}_{t \geq 0}$ defined as in (1.5) and a sequence of stochastic fluid models $\left\{\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)\right\}_{n \geq 0}=\left\{\left(R^{n}(t), J^{n}(t)\right)\right\}_{t \geq 0}$, where $\mathcal{J}^{n}$ has the state space $\{+,-\} \times \mathcal{S}$, such that for all $T \geq 0$

$$
\begin{align*}
\lim _{n \rightarrow \infty} \sup _{0 \leq s \leq T}\left|R(s)-R^{n}(s)\right| & =0 \quad \text { a.s. }  \tag{1.6}\\
\lim _{n \rightarrow \infty} \pi_{2}\left(J^{n}(T)\right) & =J(T) \quad \text { a.s. } \tag{1.7}
\end{align*}
$$

where $\pi_{2}:\{+,-\} \times \mathcal{S} \mapsto \mathcal{S}$ denotes the second-coordinate projection.
In fact, Theorem 1.1 is a consequence of the following result which concerns the rate of the strong convergence of $\left\{\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)\right\}_{n \geq 0}$ to $(\mathcal{R}, \mathcal{J})$.

Theorem 1.2. Fix $T \in[0,1)$. In the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ of Theorem 1.1,
(i) for each $q>0$ there exists a constant $\alpha=\alpha(q)>0$ such that

$$
\begin{equation*}
\mathbb{P}\left(\sup _{0 \leq s \leq T}\left|R(s)-R^{n}(s)\right|>\alpha \varepsilon_{n}\right)=o\left(n^{-q}\right) \quad \text { as } n \rightarrow \infty \tag{1.8}
\end{equation*}
$$

with $\varepsilon_{n}:=n^{-1 / 2} \log (n)$, where $o(g(n))$ for $g: \mathbb{N} \mapsto \mathbb{R}_{+}$denotes a function $f: \mathbb{N} \mapsto \mathbb{R}$ such that $\lim _{n \rightarrow \infty} f(n) / g(n)=0$;
(ii) furthermore, the process $\left\{\pi_{2}\left(J^{n}(t)\right)\right\}_{t \geq 0}$ converges in an a.s. local uniform sense to $\{J(t)\}_{t \geq 0}$; that is,

$$
\begin{equation*}
\lim _{\rho \downarrow 0}\left[\limsup _{n \rightarrow \infty}\left(\sup _{s \in(T-\rho, T+\rho)} d\left(\pi_{2}\left(J^{n}(s)\right), J(s)\right)\right)\right]=0 \quad \text { a.s., } \tag{1.9}
\end{equation*}
$$

where $d(\cdot, \cdot)$ denotes the discrete metric in $\mathcal{S}$.
The case $T \in[0,1)$ of Theorem 1.1 is a consequence of Theorem 1.2 and the Borel-Cantelli lemma, with the case $T \geq 1$ following by elementary time-scaling arguments.

REmark 1.3. The proof of Theorem 1.2 is inspired by the work of [9], where we replace the use of the Skorokhod embedding theorem with a Poissionian observations argument. Our approach yields tighter and simpler bounds, which ultimately enables us to obtain a faster rate of convergence than the one of [9] (which was proportional to $n^{-1 / 2}(\log (n))^{5 / 2}$ ) when reduced to the case of the standard Brownian motion.

This paper is structured as follows. In Section 2 we construct $(\Omega, \mathscr{F}, \mathbb{P})$ and describe the distributional characteristics of each stochastic fluid process $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$, for $n \geq 0$. We compute in Section 3 the rate of convergence of $\mathcal{R}^{n}$ to $\mathcal{R}$, from which the proof of Theorem 1.2, and thus that of Theorem 1.1, follows. Finally, in Section 4 we develop some implications of Theorem 1.1 regarding the downcrossing probabilities of $\mathcal{R}$ and $\mathcal{R}^{n}$; in particular, we exhibit a new link between the solutions of certain Riccati and quadratic matrix equations.
2. Construction of $\left\{\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)\right\}_{n \geq 0}$. First, we construct the probability space suitable to prove Theorems 1.1 and 1.2. Fix $\boldsymbol{p}, Q,\left\{\mu_{i}\right\}_{i \in \mathcal{S}}$ and $\left\{\sigma_{i}>0\right\}_{i \in \mathcal{S}}$ of Theorem 1.1. Let $\lambda_{0}=2 \max _{i \in \mathcal{S}}\left|Q_{i i}\right|$, and consider a sequence $\left\{\lambda_{n}\right\}_{n \geq 1}$ such that $\lambda_{n} \geq \lambda_{n-1}$ for $n \geq 1$ and $\lim _{n \rightarrow \infty} \lambda_{n}=\infty$. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space that supports:

- a standard Brownian motion $\mathcal{B}=\{B(t)\}_{t \geq 0}$,
- a Poisson process $\mathcal{M}^{0}=\left\{M^{0}(t)\right\}_{t \geq 0}$ of rate $\lambda_{0} / 2$,
- a sequence of Poisson processes $\left\{\tilde{\mathcal{M}}^{n}\right\}_{n \geq 1}$, where $\widetilde{\mathcal{M}}^{n}$ has rate $\left(\lambda_{n}-\lambda_{n-1}\right) / 2$,
- a discrete-time Markov chain $\mathcal{X}^{0}=\left\{X^{0}(k)\right\}_{k \geq 0}$ with state space $\mathcal{S}$, initial distribution $\boldsymbol{p}$, and transition probability matrix $P_{0}:=I+\left(\lambda_{0} / 2\right)^{-1} Q$,
with $\mathcal{B}, \mathcal{M}^{0},\left\{\widetilde{\mathcal{M}}^{n}\right\}_{n \geq 1}$, and $\mathcal{X}^{0}$ being independent of each other. All the elements stated in Theorem 1.1 and of the whole manuscript will be constructed in $(\Omega, \mathscr{F}, \mathbb{P})$. To construct $(\mathcal{R}, \mathcal{J})$ on $(\Omega, \mathscr{F}, \mathbb{P})$, let

$$
\begin{equation*}
J(t)=X^{0}\left(M^{0}(t)\right), \quad t \geq 0 \tag{2.1}
\end{equation*}
$$

The uniformization method implies that $\mathcal{J}=\{J(t)\}_{t \geq 0}$ is a Markov jump process with initial distribution $\boldsymbol{p}$ and intensity matrix $Q$. Let $\mathcal{R}=\{R(t)\}_{t \geq 0}$ be defined as on (1.5), so that $(\mathcal{R}, \mathcal{J})$ corresponds to a Markov-modulated Brownian motion.

Next, for each $n \geq 0$, we construct the process $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$ as follows. Define the arrival process $\mathcal{M}^{n}=\left\{M^{n}(t)\right\}_{t \geq 0}$ to be the superposition of $\left\{\mathcal{M}^{0}, \widetilde{\mathcal{M}}^{1}, \widetilde{\mathcal{M}}^{2}, \ldots, \widetilde{\mathcal{M}}^{n}\right\}$. Then, $\mathcal{M}^{n}$ is itself a Poisson process of intensity

$$
\lambda_{0} / 2+\sum_{\ell=1}^{n}\left(\lambda_{\ell}-\lambda_{\ell-1}\right) / 2=\lambda_{n} / 2
$$

and its arrival epochs form a subset of the arrival epochs of $\mathcal{M}^{n+m}$ for any $m \geq 0$. In other words, $\left\{\mathcal{M}^{n}\right\}_{n \geq 0}$ is a sequence of Poisson process with nested time epochs whose new arrivals, as $n$ increases, are created independently of the existing ones. Let us emphasize that choosing to have Poissonan observations with rates $\lambda_{n} / 2$ allows a direct comparison of our construction with the models of [9] and of [18] in the special case of flip-flop approximations to a standard Brownian motion.

Intuitively, our aim is to construct $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$ in such a way that $\mathcal{R}^{n}$ visits the levels of $\mathcal{R}$ inspected at the arrival epochs of the Poisson process $\mathcal{M}^{n}$. To that end, we employ the wellknown Wiener-Hopf factorisation for the Brownian motion with drift; see [5, Corollary 2.4.10] for a proof.

Theorem 2.1 (Wiener-Hopf factorisation for BM). Let $\left\{W_{t}\right\}_{t \geq 0}$ be a Brownian motion with variance $\sigma^{2}>0$, drift $\mu$, and initial point $W_{0}=0$. Let $S$ be a stopping time and let $T \sim \exp (\beta)$, independent of $\left\{W_{t}\right\}_{t \geq 0}$. Then, $W_{S}-\min _{0 \leq t \leq T} W_{S+t}$ and $W_{S+T}-\min _{0 \leq t \leq T} W_{S+t}$ are independent and exponentially distributed with rates

$$
\omega=\sqrt{\frac{\mu^{2}}{\sigma^{4}}+\frac{2 \beta}{\sigma^{2}}}+\frac{\mu}{\sigma^{2}} \quad \text { and } \quad \eta=\sqrt{\frac{\mu^{2}}{\sigma^{4}}+\frac{2 \beta}{\sigma^{2}}}-\frac{\mu}{\sigma^{2}}, \quad \text { respectively. }
$$

Theorem 2.1 implies that, restricted to an exponentially distributed time interval, we can track both the value of the minimum over this period and that at the right endpoint of a

Brownian motion with drift. Let $\left\{T_{k}^{n}\right\}_{k \geq 1}$ be the interarrival times of the process $\mathcal{M}^{n}$, and define $\theta_{0}^{n}:=0$,

$$
\begin{equation*}
\theta_{k}^{n}:=\sum_{j=1}^{k} T_{j}^{n}, \quad k \geq 0 \tag{2.2}
\end{equation*}
$$

thus $\left\{\theta_{k}^{n}\right\}_{n \geq 0}$ are the arrival epochs of $\mathcal{M}^{n}$. See Figure 1 for an illustration.


Fig 1: Blue dots correspond to arrivals $\left\{\theta_{k}^{0}\right\}_{k \geq 0}$ of $\mathcal{M}^{0}$, red diamonds the arrivals $\left\{\theta_{k}^{n}\right\}_{k \geq 0}$ of $\mathcal{M}^{n}$, black squares the jump epochs of $\mathcal{J}$. As $\mathcal{J}$ is given by (2.1), its jump epochs form a subset of the arrival times of $\mathcal{M}^{0}$.

As $\left\{\theta_{k}^{n}\right\}_{n \geq 0}$ contain all the arrival epochs of $\mathcal{M}^{0}$, Equation (2.1) implies that $\{J(t)\}_{t \geq 0}$ remains constant on each interval $\left[\theta_{k}^{n}, \theta_{k+1}^{n}\right), k \geq 0$. Consequently, given $J\left(\theta_{k}^{n}\right)=i$ on $\left[\theta_{k}^{n}, \theta_{k+1}^{n}\right)$, $\{R(t)\}_{t \geq 0}$ behaves like a Brownian motion with drift $\mu_{i}$ and variance $\sigma_{i}^{2}$. Thus, by sequentially using the Wiener-Hopf factorisation between arrival epochs of $\mathcal{M}^{n}$, we can keep track of $\left\{R\left(\theta_{k}^{n}\right)\right\}_{k \geq 0}$ and of $\left\{\min _{\theta_{k}^{n} \leq t \leq \theta_{k+1}^{n}} R(t)\right\}_{k \geq 0}=\left\{\min _{0 \leq t \leq T_{k+1}^{n}} R\left(\theta_{k}^{n}+t\right)\right\}_{k \geq 0}$ in a simple manner, which we explain in detail next.

For each $k \geq 0$, define the random variables

$$
\begin{aligned}
X^{n}(k) & :=J\left(\theta_{k}^{n}\right) \\
L_{k+1}^{n} & :=R\left(\theta_{k}^{n}\right)-\min _{0 \leq t \leq T_{k+1}^{n}} R\left(\theta_{k}^{n}+t\right) \\
H_{k+1}^{n} & :=R\left(\theta_{k+1}^{n}\right)-\min _{0 \leq t \leq T_{k+1}^{n}} R\left(\theta_{k}^{n}+t\right)
\end{aligned}
$$

By Theorem 4.4 in the Appendix, $\mathcal{X}^{n}=\left\{X^{n}(k)\right\}_{k \geq 0}$ is a discrete-time Markov chain with transition probability matrix $P_{n}:=I+\left(\lambda_{n} / 2\right)^{-1} Q$. The strong Markov property of $\{(R(t), J(t))\}_{t \geq 0}$ and Theorem 2.1 imply that, conditioned on $\mathcal{X}^{n},\left\{L_{k+1}^{n}\right\}_{k \geq 0}$ is a collection of independent random variables. More specifically, given $X_{k}^{n}=i, L_{k+1}^{n}$ is exponentially distributed with rate

$$
\omega_{i}^{n}:=\sqrt{\frac{\mu_{i}^{2}}{\sigma_{i}^{4}}+\frac{\lambda_{n}}{\sigma_{i}^{2}}}+\frac{\mu_{i}}{\sigma_{i}^{2}} .
$$

Similarly, $\left\{H_{k+1}^{n}\right\}_{k \geq 0}$ is a collection of conditionally independent random variables for which, given $X_{k}^{n}=i, H_{k+1}^{n}$ is exponentially distributed with rate

$$
\eta_{i}^{n}:=\sqrt{\frac{\mu_{i}^{2}}{\sigma_{i}^{4}}+\frac{\lambda_{n}}{\sigma_{i}^{2}}}-\frac{\mu_{i}}{\sigma_{i}^{2}} .
$$

Moreover, $\left\{L_{k+1}^{n}\right\}_{k \geq 0}$ is conditionally independent of $\left\{H_{k+1}^{n}\right\}_{k \geq 0}$. Note that $\left\{L_{k+1}^{n}\right\}_{k \geq 0}$ and $\left\{H_{k+1}^{n}\right\}_{k \geq 0}$ completely describe $\left\{R\left(\theta_{k}^{n}\right)\right\}_{k \geq 0}$ and $\left\{\min _{\theta_{k}^{n} \leq t \leq \theta_{k+1}^{n}} R(t)\right\}_{k \geq 0}$, in the sense that for all $k \geq 0$,

$$
\begin{align*}
R\left(\theta_{k}^{n}\right) & =\sum_{j=1}^{k}\left(-L_{j}^{n}+H_{j}^{n}\right), \quad \text { and }  \tag{2.3}\\
\min _{\theta_{k}^{n} \leq t \leq \theta_{k+1}^{n}} R(t) & =\sum_{j=1}^{k}\left(-L_{j}^{n}+H_{j}^{n}\right)-L_{k+1}^{n} . \tag{2.4}
\end{align*}
$$

For all $k \geq 1$, if $X^{n}(k-1)=i$, define

$$
\widehat{L}_{k}^{n}:=\lambda_{n}^{-1} \omega_{i}^{n} L_{k}^{n} \quad \text { and } \quad \widehat{H}_{k}^{n}:=\lambda_{n}^{-1} \eta_{i}^{n} H_{k}^{n} .
$$

Then, the collections $\left\{\widehat{L}_{k}^{n}\right\}_{k \geq 1}$ and $\left\{\widehat{H}_{k}^{n}\right\}_{k \geq 1}$ are i.i.d. random variables exponentially distributed with parameter $\lambda_{n}$. Let $\chi_{n}^{0}:=0$, and define for all $k \geq 1$

$$
\chi_{k}^{n}:=\sum_{j=1}^{k}\left(\widehat{L}_{j}^{n}+\widehat{H}_{j}^{n}\right) .
$$

Let $\mathcal{J}^{n}=\left\{J^{n}(t)\right\}_{t \geq 0}$ be the process with state space $\{+,-\} \times \mathcal{S}$ defined by

$$
J^{n}(t)= \begin{cases}(-, i) & \text { if } t \in\left[\chi_{k}^{n}, \chi_{k}^{n}+\widehat{L}_{k+1}^{n}\right) \text { for some } k \geq 0 \text { and } X_{k}^{n}=i, \\ (+, i) & \text { if } t \in\left[\chi_{k}^{n}+\widehat{L}_{k+1}^{n}, \chi_{k+1}^{n}\right) \text { for some } k \geq 0 \text { and } X_{k}^{n}=i\end{cases}
$$

Figure 2 shows a sample path of $\mathcal{J}^{n}$ with $\mathcal{S}=\{1,2,3\}$.


Fig 2: A sample path of $\left\{J^{n}(t)\right\}$ with $\mathcal{S}=\{1,2,3\}$.
The process $\mathcal{J}^{n}$ jumps alternately between $\{-\} \times \mathcal{S}$ and $\{+\} \times \mathcal{S}$ with intensity given by $\lambda_{n}$; furthermore, changes in its second coordinate, which occur according to $P_{n}$, are only possible at jumps instants from $\{+\} \times \mathcal{S}$ to $\{-\} \times \mathcal{S}$. Thus, $\mathcal{J}^{n}$ is a Markov jump process with statespace $\{+,-\} \times \mathcal{S}$ (ordered lexicographically), initial distribution ( $\mathbf{0}, \boldsymbol{p}$ ) and intensity matrix given by

$$
\left[\begin{array}{rr}
-\lambda_{n} I & \lambda_{n} P_{n} \\
\lambda_{n} I & -\lambda_{n} I
\end{array}\right]=\left[\begin{array}{rr}
-\lambda_{n} I & 2 Q+\lambda_{n} I \\
\lambda_{n} I & -\lambda_{n} I
\end{array}\right] .
$$

Note that the sequence of states in $\mathcal{S}$ visited by $\pi_{2}\left(\mathcal{J}^{n}\right)$ coincides with that of $\mathcal{J}$, or more precisely,

$$
\begin{equation*}
\pi_{2}\left(J^{n}\left(\chi_{k}^{n}\right)\right)=J\left(\theta_{k}^{n}\right) \quad \text { for all } \quad k \geq 0 \tag{2.5}
\end{equation*}
$$

Also notice the jumps of $\pi_{2}\left(\mathcal{J}^{n}\right)$ can occur only at $\left\{\chi_{k}^{n}\right\}_{k \geq 0}$ while the jumps of $\mathcal{J}$ can occur only at $\left\{\theta_{k}^{n}\right\}_{k \geq 0}$. In general, $\left\{\chi_{k}^{n}\right\}_{k \geq 0} \neq\left\{\theta_{k}^{n}\right\}_{k \geq 0} ;$ nevertheless,

$$
\begin{aligned}
& \mathbb{E}\left[\theta_{k}^{n}\right]=\mathbb{E}\left[\sum_{j=1}^{k} T_{j}^{n}\right]=k \mathbb{E}\left[T_{1}^{n}\right]=\frac{2 k}{\lambda_{n}} \\
& \mathbb{E}\left[\chi_{k}^{n}\right]=\mathbb{E}\left[\sum_{j=1}^{k}\left(\widehat{L}_{j}^{n}+\widehat{H}_{j}^{n}\right)\right]=k \mathbb{E}\left[\widehat{L}_{1}^{n}+\widehat{H}_{1}^{n}\right]=\frac{2 k}{\lambda_{n}}
\end{aligned}
$$

In words, the average jump times of $\pi_{2}\left(\mathcal{J}^{n}\right)$ coincide with the average jump times of $\mathcal{J}$, so that the process $\pi_{2}\left(\mathcal{J}^{n}\right)$ is indeed similar to $\mathcal{J}$. A more precise and stronger version of this statement is proven in Section 3.

In order to construct $\mathcal{R}^{n}$, define $r^{n}:\{+,-\} \times \mathcal{S} \mapsto \mathbb{R}$ by

$$
r^{n}(+, i):=\lambda_{n} / \omega_{i}^{n}, \quad r^{n}(-, i):=-\lambda_{n} / \eta_{i}^{n}
$$

Let

$$
R^{n}(t):=\int_{0}^{t} r^{n}\left(J^{n}(s)\right) \mathrm{d} s, \quad t \geq 0
$$

The pair $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$ is indeed a stochastic fluid process. Moreover, from the construction of $\mathcal{J}^{n}$, (2.3) and (2.4), it follows that for all $k \geq 0$

$$
\begin{align*}
R^{n}\left(\chi_{k}^{n}\right) & =\sum_{j=1}^{k}\left(-L_{j}^{n}+H_{j}^{n}\right)=R\left(\theta_{k}^{n}\right)  \tag{2.6}\\
R^{n}\left(\chi_{k}^{n}+\widehat{L}_{k+1}^{n}\right) & =\sum_{j=1}^{k}\left(-L_{j}^{n}+H_{j}^{n}\right)-L_{k+1}^{n}=\min _{\theta_{k}^{n} \leq t \leq \theta_{k+1}^{n}} R(t) \tag{2.7}
\end{align*}
$$

This implies that the values at the inflection points of the level process $\mathcal{R}^{n}$ coincide with the values of $\left\{R\left(\theta_{k}^{n}\right)\right\}_{k \geq 0}$ and $\left\{\min _{\theta_{k}^{n} \leq t \leq \theta_{k+1}^{n}} R(t)\right\}_{k \geq 0}$. In conclusion, the values of $\mathcal{R}$ at the arrival epochs of $\mathcal{M}^{n}$, and the minimum level attained between them, are embedded in $\mathcal{R}^{n}$. Figure 3 illustrates the construction of the stochastic fluid process $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$ corresponding to the Markov-modulated Brownian motion ( $\mathcal{R}, \mathcal{J})$.


Fig 3: (Left) A sample path of an $\operatorname{MMBM}(\mathcal{R}, \mathcal{J})$ with $\mathcal{J}$ being on $\mathcal{S}=\{1,2\}$ : arrivals corresponding to $\mathcal{M}^{n}$ occur at $\left\{\theta_{i}^{n}\right\}_{i>0}$, and the minima of $\mathcal{R}$ attained between these arrivals are highlighted with blue crosses. When $J(\bar{t})=1$ (red), $\mu_{1}=5$ and $\sigma_{1}^{2}=4$. When $J(t)=2$ (blue), $\mu_{2}=-2$ and $\sigma_{2}^{2}=1$. (Right) An associated sample path of a stochastic fluid process $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$ : jumps from $\{+\} \times \mathcal{S}$ to $\{-\} \times \mathcal{S}$ occur at $\left\{\chi_{i}^{n}\right\}_{i \geq 1}$. The values of $R^{n}\left(\chi_{i}^{n}\right)$ match with those of $R\left(\theta_{i}^{n}\right)$ for $i=1, \ldots 4$, respectively.
3. Proof of Theorem 1.2. As $\lambda_{n} \rightarrow \infty$, the partitions induced by $\left\{\theta_{k}^{n}\right\}_{k \geq 0}$ and $\left\{\chi_{k}^{n}\right\}_{k \geq 0}$ become finer. Intuitively, this and (2.6) indicate that $\mathcal{R}^{n}$ approximates $\mathcal{R}$ as $n \rightarrow \infty$, which is stated more precisely in Theorems 1.1 and 1.2. We devote this section to rigorously prove Theorem 1.2, from which Theorem 1.1 follows as a corollary.

Let $\lambda_{n}=2 n^{2}$; this makes our results and rates comparable to those of [9] and related papers. Fix $T \in[0,1), q>0$ and w.l.o.g. consider $n \geq 2$ throughout.

Proof of Part (i). In order to prove (1.8), notice that

$$
\begin{equation*}
\mathbb{P}\left(\sup _{0 \leq s \leq T}\left|R(s)-R^{n}(s)\right|>\alpha \varepsilon_{n}\right) \leq \mathbb{P}\left(A^{n}\right)+\mathbb{P}\left(\chi_{n^{2}}^{n}<T\right), \tag{3.1}
\end{equation*}
$$

where

$$
A^{n}:=\left\{\sup _{0 \leq s \leq \chi_{n^{2}}^{n}}\left|R(s)-R^{n}(s)\right|>\alpha \varepsilon_{n}\right\}=\left\{\max _{1 \leq k \leq n^{2} \chi_{k-1}^{n} \leq s \leq \chi_{k}^{n}} \sup \left|R(s)-R^{n}(s)\right|>\alpha \varepsilon_{n}\right\},
$$

where $\alpha$ is a constant to be determined later. We now show that each of the quantities $P\left(A^{n}\right)$ and $P\left(\chi_{n^{2}}^{n}<T\right)$ are $o\left(n^{-q}\right)$.

The triangle inequality implies that

$$
A^{n} \subseteq B_{1}^{n} \cup B_{2}^{n} \cup B_{3}^{n} \cup B_{4}^{n},
$$

where

$$
\begin{aligned}
& B_{1}^{n}:=\left\{\max _{1 \leq k \leq n^{2} \chi_{k-1}^{n} \leq s \leq \chi_{k}^{n}} \sup \left|R(s)-R\left(\chi_{k}^{n}\right)\right|>\alpha \varepsilon_{n} / 4\right\}, \\
& B_{2}^{n}:=\left\{\max _{1 \leq k \leq n^{2}}\left|R\left(\chi_{k}^{n}\right)-R\left(k / n^{2}\right)\right|>\alpha \varepsilon_{n} / 4\right\},
\end{aligned}
$$

$$
\begin{aligned}
B_{3}^{n} & :=\left\{\max _{1 \leq k \leq n^{2}}\left|R\left(k / n^{2}\right)-R^{n}\left(\chi_{k}^{n}\right)\right|>\alpha \varepsilon_{n} / 4\right\}, \\
B_{4}^{n} & :=\left\{\max _{1 \leq k \leq n^{2} \chi_{k-1}^{n} \leq s \leq \chi_{k}^{n}} \sup ^{n}\left(\chi_{k}^{n}\right)-R^{n}(s) \mid>\alpha \varepsilon_{n} / 4\right\} .
\end{aligned}
$$

By (2.6), $B_{3}^{n}$ can be rewritten as

$$
B_{3}^{n}=\left\{\max _{1 \leq k \leq n^{2}}\left|R\left(\theta_{k}^{n}\right)-R\left(k / n^{2}\right)\right|>\alpha \varepsilon_{n} / 4\right\},
$$

so that the events $B_{1}^{n}, B_{2}^{n}$ and $B_{3}^{n}$ concern only the process $\mathcal{R}$, not $\mathcal{R}^{n}$.
Let $\left\{\delta_{n}\right\}_{n \geq 0}$ be any positive and decreasing sequence. Making a further partition, we obtain

$$
\begin{equation*}
A^{n} \subseteq\left(\left(B_{1}^{n} \cup B_{2}^{n} \cup B_{3}^{n}\right) \cap\left(C_{\chi}^{n} \cup C_{\theta}^{n}\right)^{c}\right) \cup\left(C_{\chi}^{n} \cup C_{\theta}^{n}\right) \cup B_{4}^{n}, \tag{3.2}
\end{equation*}
$$

where

$$
C_{\chi}^{n}:=\left\{\max _{1 \leq k \leq n^{2}}\left|\chi_{k}^{n}-k / n^{2}\right|>\delta_{n}\right\}, \quad C_{\theta}^{n}:=\left\{\max _{1 \leq k \leq n^{2}}\left|\theta_{k}^{n}-k / n^{2}\right|>\delta_{n}\right\}
$$

On $\left(C_{\chi}^{n} \cup C_{\theta}^{n}\right)^{c}$, for all $k=1, \ldots, n^{2}$ we have that

$$
\chi_{k}^{n}, \theta_{k}^{n}, \chi_{k+1}^{n} \in\left[k / n^{2}-\delta_{n},(k+1) / n^{2}+\delta_{n}\right] .
$$

Let $x_{+}:=\max \{x, 0\}$ for $x \in \mathbb{R}$. If there exists $a, b \in\left[k / n^{2}-\delta_{n},(k+1) / n^{2}+\delta_{n}\right]$ such that $\left|R\left(a_{+}\right)-R\left(b_{+}\right)\right|>\alpha \varepsilon_{n} / 4$, then by the triangle inequality either $\left|R\left(\left[k / n^{2}-\delta_{n}\right]_{+}\right)-R\left(a_{+}\right)\right|>$ $\alpha \varepsilon_{n} / 8$, or $\left|R\left(\left[k / n^{2}-\delta_{n}\right]_{+}\right)-R\left(b_{+}\right)\right|>\alpha \varepsilon_{n} / 8$. Therefore,

$$
\begin{equation*}
\left(B_{1}^{n} \cup B_{2}^{n} \cup B_{3}^{n}\right) \cap\left(C_{\chi}^{n} \cup C_{\theta}^{n}\right)^{c} \subseteq D^{n} \tag{3.3}
\end{equation*}
$$

where

$$
\begin{aligned}
D^{n} & :=\left\{\max _{1 \leq k \leq n^{2}} \sup _{a \in\left[k / n^{2}-\delta_{n},(k+1) / n^{2}+\delta_{n}\right]}\left|R\left(\left(k / n^{2}-\delta_{n}\right)_{+}\right)-R\left(a_{+}\right)\right|>\alpha \varepsilon_{n} / 8\right\} \\
& =\left\{\max _{1 \leq k \leq n^{2}} \sup _{s \in\left[0, n^{-2}+2 \delta_{n}\right]}\left|R\left(\left(k / n^{2}-\delta_{n}+s\right)_{+}\right)-R\left(\left(k / n^{2}-\delta_{n}\right)_{+}\right)\right|>\alpha \varepsilon_{n} / 8\right\} .
\end{aligned}
$$

Thus, by (3.2) and (3.3),

$$
\begin{equation*}
\mathbb{P}\left(A^{n}\right) \leq \mathbb{P}\left(D^{n}\right)+\mathbb{P}\left(C_{\chi}^{n}\right)+\mathbb{P}\left(C_{\theta}^{n}\right)+\mathbb{P}\left(B_{4}^{n}\right) \tag{3.4}
\end{equation*}
$$

In the following, we show that with an appropiate choice of $\alpha$ and $\left\{\delta_{n}\right\}_{n \geq 1}$, each summand in the RHS of (3.4) is an $o\left(n^{-q}\right)$ function. For the remainder of the section, $K_{j}$, for $j \in \mathbb{N}$, denote generic constants that are used to simplify bounds and are not dependent on $q$ or $n$.

Bounding $P\left(B_{4}^{n}\right)$. Let $\omega_{\min }^{n}:=\min _{i \in \mathcal{S}} \omega_{i}^{n}, \eta_{\min }^{n}:=\min _{i \in \mathcal{S}} \eta_{i}^{n}, \kappa_{n}:=\omega_{\min }^{n} \wedge \eta_{\min }^{n}$. Then,

$$
\begin{aligned}
\mathbb{P}\left(B_{4}^{n}\right) & \leq \sum_{1 \leq k \leq n^{2}} \mathbb{P}\left(\sup _{\chi_{k-1}^{n} \leq s \leq \chi_{k}^{n}}\left|R^{n}\left(\chi_{k}^{n}\right)-R^{n}(s)\right|>\alpha \varepsilon_{n} / 4\right) \\
& \leq \sum_{1 \leq k \leq n^{2}} \mathbb{P}\left(H_{k}^{n}>\alpha \varepsilon_{n} / 4\right)+\mathbb{P}\left(L_{k}^{n}>\alpha \varepsilon_{n} / 4\right) \\
& \leq n^{2}\left(e^{-\omega_{\min }^{n} \alpha \varepsilon_{n} / 4}+e^{-\eta_{\min }^{n} \alpha \varepsilon_{n} / 4}\right) \\
& \leq n^{2}\left(2 e^{-\kappa_{n}\left(\alpha \varepsilon_{n} / 4\right)}\right) .
\end{aligned}
$$

By definition,

$$
\kappa_{n}:=\min _{i \in \mathcal{S}}\left\{\left(\sqrt{\frac{\mu_{i}^{2}}{\sigma_{i}^{4}}+\frac{2 n^{2}}{\sigma_{i}^{2}}}+\frac{\mu_{i}}{\sigma_{i}^{2}}\right) \wedge\left(\sqrt{\frac{\mu_{i}^{2}}{\sigma_{i}^{4}}+\frac{2 n^{2}}{\sigma_{i}^{2}}}-\frac{\mu_{i}}{\sigma_{i}^{2}}\right)\right\}=O(n),
$$

where the notation $O(g(n))$, for $g: \mathbb{N} \mapsto \mathbb{R}_{+}$, denotes a function $f: \mathbb{N} \mapsto \mathbb{R}$ such that $\limsup _{n \rightarrow \infty}|f(n)| / g(n) \leq M$ for some $M \in \mathbb{R}$. Then, there exists $n_{1}$ such that $\kappa_{n}>n^{1 / 2}$ for all $n \geq n_{1}$, and so for $n \geq n_{1}$

$$
\mathbb{P}\left(B_{4}^{n}\right) \leq n^{2}\left(2 e^{-\kappa_{n}(\alpha / 4) n^{-1 / 2} \log (n)}\right) \leq n^{2}\left(2 e^{-(\alpha / 4) \log (n)}\right)=K_{1} n^{2-\alpha / 4}
$$

Choose $\alpha$ to be larger than $\alpha_{1}:=8 q+8$. Then, $\mathbb{P}\left(B_{4}^{n}\right)$ is an $O\left(n^{-2 q}\right)$ function and thus, it is an $o\left(n^{-q}\right)$ function.

Bounding $P\left(C_{\chi}^{n}\right)$ and $P\left(C_{\theta}^{n}\right)$. Let $\left\{p_{n}\right\}_{n}$ be a sequence taking values in $\mathbb{N}$. By Doob's $L_{p}$-maximal inequality, we have

$$
\begin{equation*}
\mathbb{P}\left(C_{\chi}^{n}\right) \leq \frac{\mathbb{E}\left[\left(\chi_{n^{2}}^{n}-1\right)^{2 p_{n}}\right]}{\left(\delta_{n}\right)^{2 p_{n}}} . \tag{3.5}
\end{equation*}
$$

Since $\chi_{n^{2}}^{n}$ is a convolution of $2 n^{2}$ exponential r.v.s of rate $2 n^{2}, \chi_{n^{2}}^{n} \sim \operatorname{Erlang}\left(2 n^{2}, 2 n^{2}\right)$, so that (3.5) and Lemma 4.5 (in the Appendix) imply that

$$
\begin{align*}
\mathbb{P}\left(C_{\chi}^{n}\right) & \leq\left(\delta_{n}\right)^{-2 p_{n}} \frac{\left(2 p_{n}\right)!\sqrt{2 n^{2}}}{\left(2 n^{2}\right)^{2 p_{n}}} \frac{\sqrt{2 n^{2}}{ }^{2 p_{n}+1}-1}{\sqrt{2 n^{2}}-1} \\
& \leq K_{2} \frac{\left(\delta_{n}\right)^{-2 p_{n}}\left(2 p_{n}\right)!}{n^{2 p_{n}-1}} \\
& \leq K_{2} n\left(\frac{2 p_{n}}{\delta_{n} n}\right)^{2 p_{n}} . \tag{3.6}
\end{align*}
$$

Similarly, since $\theta_{n^{2}}^{n} \sim \operatorname{Erlang}\left(n^{2}, n^{2}\right)$, we have for $n \geq 2$

$$
\begin{equation*}
\mathbb{P}\left(C_{\theta}^{n}\right) \leq K_{3} n\left(\frac{2 p_{n}}{\delta_{n} n}\right)^{2 p_{n}} \tag{3.7}
\end{equation*}
$$

Set

$$
\begin{equation*}
\delta_{n}:=2 p_{n} n^{(q+1 / 2) / p_{n}-1}, \quad n \geq 1 . \tag{3.8}
\end{equation*}
$$

With this choice of $\left\{\delta_{n}\right\}$, both (3.6) and the RHS of (3.7) are proportional to $n^{-2 q}$, so that $\mathbb{P}\left(C_{\chi}^{n}\right)$ and $\mathbb{P}\left(C_{\theta}^{n}\right)$ are $o\left(n^{-q}\right)$ functions.

Bounding $P\left(D^{n}\right)$. Set $p_{n}:=\lfloor\log (n)\rfloor$; one can verify that with this choice of $\left\{p_{n}\right\}$, the sequence $\left\{\delta_{n}\right\}$ is a $O\left(n^{-1} \log (n)\right)$ function. Define $\delta_{n}^{\prime}:=2 \delta_{n}+n^{-2}$ and let $n_{2} \geq n_{1}$ be such that $\alpha_{1} \varepsilon_{n} / 8-\mu_{\max } \delta_{n}^{\prime}>0$ for all $n \geq n_{2}$. Then, for any $\alpha \geq \alpha_{1}$, we have $\alpha \varepsilon_{n} / 8-\mu_{\max } \delta_{n}^{\prime}>0$ for all $n \geq n_{2}$. Thus,

$$
\begin{aligned}
\mathbb{P}\left(D^{n}\right) & \leq \sum_{k=1}^{n^{2}} \mathbb{P}\left(\sup _{0 \leq s \leq \delta_{n}^{\prime}}\left|R\left(\left(k / n^{2}-\delta_{n}\right)_{+}\right)-R\left(\left(k / n^{2}-\delta_{n}+s\right)_{+}\right)\right|>\alpha \varepsilon_{n} / 8\right) \\
& \leq \sum_{k=1}^{n^{2}} \sum_{i \in \mathcal{S}} \mathbb{P}\left(\sup _{0 \leq s \leq \delta_{n}^{\prime}}\left|R\left(\left(k / n^{2}-\delta_{n}\right)_{+}\right)-R\left(\left(k / n^{2}-\delta_{n}+s\right)_{+}\right)\right|>\alpha \varepsilon_{n} / 8 \mid G_{i, k, n}\right),
\end{aligned}
$$

where $G_{i, k, n}:=\left\{J\left(k / n^{2}-\delta_{n}\right)=i\right\}$. By strong Markov property, we can rewrite the above RHS to obtain

$$
\begin{align*}
\mathbb{P}\left(D^{n}\right) & =n^{2} \sum_{i \in \mathcal{S}} \mathbb{P}\left(\sup _{0 \leq s \leq \delta_{n}^{\prime}}|R(s)|>\alpha \varepsilon_{n} / 8 \mid J(0)=i\right) \\
& \leq n^{2} m\left[\frac{2}{\sqrt{2 \pi}} \frac{\sqrt{\sigma_{\max } \delta_{n}^{\prime}}}{\alpha \varepsilon_{n} / 8-\mu_{\max } \delta_{n}^{\prime}} \exp \left(-\frac{\left(\alpha \varepsilon_{n} / 8-\mu_{\max } \delta_{n}^{\prime}\right)^{2}}{2 \sigma_{\max } \delta_{n}^{\prime}}\right)\right] \\
(3.9) \quad & \leq K_{4} n^{2} \exp \left(-\frac{\left(\alpha \varepsilon_{n} / 8-\mu_{\max } \delta_{n}^{\prime}\right)^{2}}{2 \sigma_{\max } \delta_{n}^{\prime}}\right), \quad n \geq n_{2}, \tag{3.9}
\end{align*}
$$

where the first inequality follows from Lemma 4.6 (in the Appendix).
Let $n_{3} \geq n_{2}$ be such that $\varepsilon_{n} \leq 1$ and $\delta_{n}^{\prime} \leq 3 \delta_{n}$ for all $n \geq n_{3}$. Then

$$
\begin{aligned}
\mathbb{P}\left(D^{n}\right) & \leq K_{4} n^{2} \exp \left(-\frac{\left(\alpha \varepsilon_{n} / 8\right)^{2}+\left(\mu_{\max } \delta_{n}^{\prime}\right)^{2}-2\left(\alpha \varepsilon_{n} / 8\right)\left(\mu_{\max } \delta_{n}^{\prime}\right)}{2 \sigma_{\max }^{\prime} \delta_{n}^{\prime}}\right) \\
& \leq K_{4} n^{2} \exp \left(-\frac{\left(\alpha \varepsilon_{n} / 8\right)^{2}-2\left(\alpha \varepsilon_{n} / 8\right)\left(\mu_{\max } \delta_{n}^{\prime}\right)}{2 \sigma_{\max } \delta_{n}^{\prime}}\right) \\
& \leq K_{4} n^{2} \exp \left(-\frac{\left(\alpha \varepsilon_{n} / 8\right)^{2}}{2 \sigma_{\max } \delta_{n}^{\prime}}+\frac{\alpha \mu_{\max }}{8 \sigma_{\max }}\right) \\
& \leq K_{5} n^{2} \exp \left(-\frac{\left(\alpha \varepsilon_{n} / 8\right)^{2}}{6 \sigma_{\max } \delta_{n}}\right) \\
& =K_{5} n^{2} \exp \left(-\frac{\alpha^{2} n^{-1}(\log (n))^{2}}{K_{6}\lfloor\log (n)\rfloor n^{(q+1 / 2) /\lfloor\log (n)\rfloor-1}}\right) \\
& \leq K_{5} n^{2} \exp \left(-\frac{\alpha^{2} \log (n)}{K_{6} n^{(q+1 / 2) /\lfloor\log (n)\rfloor}}\right), \quad n \geq n_{3} .
\end{aligned}
$$

Let $\gamma(q)=\sup _{n>n_{3}} K_{6} n^{(q+1 / 2) /\lfloor\log (n)\rfloor}$, which is finite since $n^{(q+1 / 2) /\lfloor\log (n)\rfloor}$ converges to $e^{q+1 / 2}$. If $\alpha>\alpha_{2}:=\sqrt{(2 q+2) \gamma(q)}$, by (3.10) we have

$$
\begin{aligned}
\mathbb{P}\left(D^{n}\right) & \leq K_{5} n^{2} \exp \left(-\frac{(2 q+2) \gamma(q) \log (n)}{K_{6} n^{(q+1 / 2) / \log (n)\rfloor}}\right) \\
& \leq K_{5} n^{2} \exp (-(2 q+2) \log (n)) \\
& =K_{5} n^{-2 q}, \quad n \geq n_{3},
\end{aligned}
$$

which implies $\mathbb{P}\left(D^{n}\right)$ is an $o\left(n^{-q}\right)$ function. Thus, all four terms in the LHS of (3.4) are $o\left(n^{-q}\right)$ functions, and so is $\mathbb{P}\left(A^{n}\right)$.

Finally, let $n_{4} \geq n_{3}$ be such that $\delta_{n}<1-T$ for all $n \geq n_{4}$. Then,

$$
\mathbb{P}\left(\chi_{n^{2}}^{n}<T\right) \leq \mathbb{P}\left(C_{\chi}^{n}\right) \text { for all } n \geq n_{4},
$$

meaning that (3.1) is an $o\left(n^{-q}\right)$ function. The proof of (1.8) is now complete.
Proof of Part (ii). Now, let $\left\{\rho_{\ell}\right\}_{\ell \geq 1}$ be a sequence with $\rho_{\ell} \downarrow 0$, and define

$$
E^{\ell}:=\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\pi_{2}\left(J^{n}(s)\right) \neq J(s) \text { for some } s \in\left(T-\rho_{\ell}, T+\rho_{\ell}\right)\right\}
$$

Proving (1.9) is equivalent to showing that $\mathbb{P}\left(\cap_{\ell=1}^{\infty} E^{\ell}\right)=0$, which in turn is equivalent to proving that $\lim _{\ell \rightarrow \infty} \mathbb{P}\left(E^{\ell}\right)=0$.

Define $\beta_{0}:=0, \beta_{0}^{n}:=0$ for $n \geq 0$. For $k \geq 0$, let

$$
\begin{aligned}
& \beta_{k+1}:=\inf \left\{s>\beta_{k}: J\left(s^{-}\right) \neq J(s)\right\} \\
& \beta_{k+1}^{n}:=\inf \left\{s>\beta_{k}^{n}: \pi_{2}\left(J\left(s^{-}\right)\right) \neq \pi_{2}(J(s))\right\}, \quad n \geq 0 .
\end{aligned}
$$

For any $a, b \in \mathbb{R}$, define $M^{0}[a, b]:=M\left(b_{+}\right)-M\left(a_{+}\right)$; recall that $\mathcal{M}^{0}$ is a Poisson process of rate $\lambda_{0} / 2$ defined in Section 2. Then,

$$
\begin{equation*}
E^{\ell} \subseteq\left\{M^{0}\left[T-2 \rho_{\ell}, T+2 \rho_{\ell}\right]>0\right\} \cup\left(\left\{M^{0}\left[T-2 \rho_{\ell}, T+2 \rho_{\ell}\right]=0\right\} \cap E^{\ell}\right) \tag{3.11}
\end{equation*}
$$

Note that $\mathbb{P}\left(M^{0}\left[T-2 \rho_{\ell}, T+2 \rho_{\ell}\right]>0\right) \leq 1-e^{-\left(\lambda_{0} / 2\right) 4 \rho_{\ell}} \rightarrow 0$ as $\ell \rightarrow \infty$. Thus, in order to prove that $\lim _{\ell \rightarrow \infty} \mathbb{P}\left(E^{\ell}\right)=0$, it is sufficient to show that

$$
\begin{equation*}
\lim _{\ell \rightarrow \infty} \mathbb{P}\left(\left\{M^{0}\left[T-2 \rho_{\ell}, T+2 \rho_{\ell}\right]=0\right\} \cap E^{\ell}\right)=0, \tag{3.12}
\end{equation*}
$$

which we do next. A path inspection reveals that

$$
\begin{aligned}
& \left\{M^{0}\left[T-2 \rho_{\ell}, T+2 \rho_{\ell}\right]=0\right\} \cap E^{\ell} \\
& \subseteq\left\{\bigcup_{k \geq 0}\left\{\beta_{k}<T-2 \rho_{\ell}<T+2 \rho_{\ell}<\beta_{k+1}\right\}\right\} \cap E^{\ell} \\
& =\bigcup_{k \geq 0} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\begin{array}{l}
\left\{\beta_{k}<T-2 \rho_{\ell}<T+2 \rho_{\ell}<\beta_{k+1}, T-\rho_{\ell}<\beta_{k}^{n}\right\} \cup \\
\left\{\beta_{k}<T-2 \rho_{\ell}<T+2 \rho_{\ell}<\beta_{k+1}, \beta_{k+1}^{n}<T+\rho_{\ell}\right\}
\end{array}\right\} \\
& \subseteq \bigcup_{k \geq 0}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left(\left\{\left|\beta_{k}-\beta_{k}^{n}\right|>\rho_{\ell}, \beta_{k}<T-2 \rho_{\ell}\right\} \cup\left\{\left|\beta_{k+1}-\beta_{k+1}^{n}\right|>\rho_{\ell}, \beta_{k+1}^{n}<T+\rho_{\ell}\right\}\right) \\
& \subseteq\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\max _{k: \theta_{k}^{n}<T-2 \rho_{\ell}}\left|\chi_{k}^{n}-\theta_{k}^{n}\right|>\rho_{\ell}\right\}\right) \cup\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\max _{k: \chi_{k}^{n}<T+\rho_{\ell}}\left|\chi_{k}^{n}-\theta_{k}^{n}\right|>\rho_{\ell}\right\}\right)
\end{aligned}
$$

$$
\begin{equation*}
\subseteq\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\max _{1 \leq k \leq n^{2}}\left|\chi_{k}^{n}-\theta_{k}^{n}\right|>\rho_{\ell}\right\}\right) \cup\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\theta_{n^{2}}^{n}<T-2 \rho_{\ell}\right\}\right) \cup\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\chi_{n^{2}}^{n}<T+\rho_{\ell}\right\}\right) . \tag{3.13}
\end{equation*}
$$

Since $\left\{\delta_{n}\right\}_{n \geq 1}$ is a sequence such that $\delta_{n} \downarrow 0$, then for each $\ell \geq 1$,

$$
\mathbb{P}\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\max _{1 \leq k \leq n^{2}}\left|\chi_{k}^{n}-\theta_{k}^{n}\right|>\rho_{\ell}\right\}\right) \leq \mathbb{P}\left(\bigcap_{j=1}^{\infty} \bigcup_{n=j}^{\infty}\left\{\max _{1 \leq k \leq n^{2}}\left|\chi_{k}^{n}-\theta_{k}^{n}\right|>2 \delta_{n}\right\}\right)=0
$$

where the last equality follows from the fact that

$$
\mathbb{P}\left(\max _{1 \leq k \leq n^{2}}\left|\chi_{k}^{n}-\theta_{k}^{n}\right|>2 \delta_{n}\right) \leq \mathbb{P}\left(C_{\chi}^{n} \cup C_{\theta}^{n}\right)=o\left(n^{-q}\right),
$$

and applying Borel-Cantelli (choosing, say, $q=2$ ). Similar arguments follow for the two other events in (3.13). Thus, $\lim _{\ell \rightarrow \infty} \mathbb{P}\left(E^{\ell}\right)=0$ and so (1.9) follows.
4. An application: First passage probabilities. Theorem 1.1 implies that some first passage properties of $(\mathcal{R}, \mathcal{J})$ can be analysed as the limiting first passage properties of $\left(\mathcal{R}^{n}, \mathcal{J}^{n}\right)$ as $n \rightarrow \infty$. In particular, for any Borel set $A \subset \mathbb{R}$ define

$$
\begin{aligned}
\tau_{A} & :=\inf \{s \geq 0: R(s) \in A\}, \\
\tau_{A}^{n}: & :=\inf \left\{s \geq 0: R^{n}(s) \in A\right\}, \quad n \geq 0 .
\end{aligned}
$$

Then, Theorem 1.1 implies that for any open set $A$ and $j \in \mathcal{S}$,

$$
\tau_{A}=\lim _{n \rightarrow \infty} \tau_{A}^{n}
$$

and on the event $\left\{\tau_{A}<\infty\right\}$,

$$
\left\{J\left(\tau_{A}\right)=j\right\}=\bigcup_{i=0}^{\infty} \bigcap_{n=i}^{\infty}\left\{\pi_{2}\left(J^{n}\left(\tau_{A}\right)\right)=j\right\} \quad \text { a.s.. }
$$

In the case $A$ takes the form $(-\infty,-x)$, for $x \geq 0$, we have the following.
Proposition 4.1. For $x \geq 0$ and $n \geq 0$ define

$$
\tau_{x}:=\tau_{(-\infty,-x)} \quad \text { and } \quad \tau_{x}^{n}:=\tau_{(-\infty,-x)}^{n} .
$$

Then, for all $j \in \mathcal{S}$,

$$
\begin{equation*}
\left\{\tau_{x}<\infty, J\left(\tau_{x}\right)=j\right\}=\left\{\tau_{x}^{n}<\infty, \pi_{2}\left(J^{n}\left(\tau_{x}^{n}\right)\right)=j\right\} . \tag{4.1}
\end{equation*}
$$

Proof. Fix $x \geq 0$ and $n \geq 0$. Let

$$
N:=\sup \left\{k \geq 0: \tau_{x} \geq \theta_{k}^{n}\right\} .
$$

This implies that $\tau_{x} \in\left[\theta_{N}^{n}, \theta_{N+1}^{0}\right)$ on $\{N<\infty\}$, and since $\mathcal{J}$ is constant between the epochs $\left\{\theta_{k}^{n}\right\}_{k \geq 0}$, then $J\left(\theta_{N}^{n}\right)=J\left(\tau_{x}\right)$. Similarly, if we define

$$
N^{n}:=\sup \left\{k \geq 0: \tau_{x}^{n} \geq \chi_{k}^{n}\right\}
$$

then $\pi_{2}\left(J^{n}\left(\chi_{N^{n}}^{n}\right)\right)=\pi_{2}\left(J^{n}\left(\tau_{x}^{n}\right)\right)$ on $\left\{N^{n}<\infty\right\}$. Equations (2.6) and (2.7) imply that $N_{n}=N$, and since $J\left(\theta_{k}^{n}\right)=\pi_{2}\left(J^{n}\left(\chi_{k}^{n}\right)\right)$ for all $k \geq 0$ (see (2.5)), then

$$
J\left(\tau_{x}\right)=J\left(\theta_{N}^{n}\right)=\pi_{2}\left(J^{n}\left(\chi_{N^{n}}^{n}\right)\right)=\pi_{2}\left(J^{n}\left(\tau_{x}^{n}\right)\right) \quad \text { on } \quad\{N<\infty\},
$$

and (4.1) follows.
The following result describes one central first passage distributional property of our construction.

Theorem 4.2. For $n \geq 0$, let $U_{n}$ denote the infinitesimal generator associated to the process $\left\{\pi_{2}\left(J^{n}\left(\tau_{x}^{n}\right)\right)\right\}_{x \geq 0}$. Then, $U_{n}$ is a solution to the quadratic matrix equation

$$
\begin{equation*}
X^{2}+2 \Delta_{\mu} \Delta_{\sigma}^{-2} X+2 \Delta_{\sigma}^{-2} Q=0 \tag{4.2}
\end{equation*}
$$

where $\Delta_{\mu}=\operatorname{diag}\left\{\mu_{i}: i \in \mathcal{S}\right\}, \Delta_{\sigma}=\operatorname{diag}\left\{\sigma_{i}: i \in \mathcal{S}\right\}$. Furthermore $U_{n}$ corresponds to the infinitesimal generator associated to $\left\{J\left(\tau_{x}\right)\right\}_{x \geq 0}$.

Proof. Let $\Psi_{n}$ be the $p \times p$-dimensional matrix defined by

$$
\left(\Psi_{n}\right)_{i j}=\mathbb{P}\left(\tau_{0}^{n}<\infty, J^{n}\left(\tau_{0}\right)=(-, j) \mid R^{n}(0)=0, J^{n}(0)=(+, i)\right), \quad i, j \in \mathcal{S} .
$$

Define $\Delta_{r_{+}^{n}}=\operatorname{diag}\left\{r^{n}(+, i): i \in \mathcal{S}\right\}, \Delta_{r_{-}^{n}}=\operatorname{diag}\left\{\left|r^{n}(-, i)\right|: i \in \mathcal{S}\right\}$, and

$$
\left[\begin{array}{cc}
T_{++} & T_{+-} \\
T_{-+} & T_{--}
\end{array}\right]:=\left[\begin{array}{cc}
-\lambda_{n} I & 2 Q+\lambda_{n} I \\
\lambda_{n} I & -\lambda_{n} I
\end{array}\right] .
$$

It is known [3] that $\Psi_{n}$ is the minimal nonnegative solution to the Riccati matrix equation

$$
\begin{equation*}
\Delta_{r_{+}^{n}}^{-1} T_{++} \Psi_{n}+\Psi_{n} \Delta_{r_{-}^{n}}^{-1} T_{--}+\Psi_{n} \Delta_{r_{-}^{n}}^{-1} T_{-+} \Psi_{n}+\Delta_{r_{+}^{n}}^{-1} T_{+-}=0, \tag{4.3}
\end{equation*}
$$

and that

$$
\mathbb{P}\left(\tau_{0}^{n}<\infty, J^{n}\left(\tau_{x}\right)=(-, j) \mid R^{n}(0)=0, J^{n}(0)=(-, i)\right)=\boldsymbol{e}_{i}^{\top} e^{U_{n} x} \boldsymbol{e}_{j}
$$

where

$$
\begin{equation*}
U_{n}=\Delta_{r_{-}^{n}}^{-1}\left(T_{--}+T_{-+} \Psi_{n}\right)=\lambda_{n} \Delta_{r_{-}^{n}}^{-1}\left(\Psi_{n}-I\right), \tag{4.4}
\end{equation*}
$$

with $\boldsymbol{e}_{i}$ being the $i$ th unit column vector.
Premultiplying (4.3) by $\Delta_{r_{-}^{n}}^{-1}$ and commuting $\Delta_{r_{-}^{n}}^{-1}$ with $\Delta_{r_{+}^{n}}^{-1}$ give

$$
-\lambda_{n} \Delta_{r_{+}^{n}}^{-1} \Delta_{r_{-}^{n}}^{-1} \Psi_{n}-\lambda_{n} \Delta_{r_{-}^{n}}^{-1} \Psi_{n} \Delta_{r_{-}^{n}}^{-1}+\lambda_{n} \Delta_{r_{\underline{n}}}^{-1} \Psi_{n} \Delta_{r_{-}^{n}}^{-1} \Psi_{n}+\Delta_{r_{+}^{n}}^{-1} \Delta_{r_{-}^{n}}^{-1}\left(2 Q+\lambda_{n} I\right)=0,
$$

which leads to $\left(\Delta_{r_{-}^{n}}^{-1}-\Delta_{r_{+}^{n}}^{-1}\right) U_{n}+\lambda_{n}^{-1} U_{n}^{2}+2 \Delta_{r_{+}^{n}}^{-1} \Delta_{r_{-}^{n}}^{-1} Q=0$. As $\Delta_{r_{-}^{n}}^{-1}-\Delta_{r_{+}^{n}}^{-1}=2 \lambda_{n}^{-1} \Delta_{\mu} \Delta_{\sigma}^{-2}$ and $\Delta_{r_{+}^{n}}^{-1} \Delta_{r_{-}^{n}}^{-1}=\lambda_{n}^{-1} \Delta_{\sigma}^{-2}$, we obtain

$$
\begin{equation*}
U_{n}^{2}+2 \Delta_{\mu} \Delta_{\sigma}^{-2} U_{n}+2 \Delta_{\sigma}^{-2} Q=0 \tag{4.5}
\end{equation*}
$$

That $U_{n}$ is also the infinitesimal generator of $\left\{J\left(\tau_{x}\right)\right\}_{x \geq 0}$ follows from Proposition 4.1.

REmARK 4.3. Theorem 4.2 provides a novel understanding of the classic quadratic matrix equation associated to the down-crossing records of an MMBM (see [2]). Indeed, to compute the infinitesimal generator solution of (4.2) (which is unique by [13]), we can instead compute the minimal nonnegative solution to the Riccati matrix equation (4.3), say $\Psi_{n}$. The solution of (4.2) is then given by $U_{n}$ as defined in (4.4). A comparable result is that of [13], where the authors construct a sequence of matrices $\left\{U_{n}^{*}\right\}_{n \geq 0}$ that is shown to converge to $U$. One advantage of our construction is that each element of the sequence $\left\{U_{n}\right\}$ obtained through Theorem 4.2 is identical to $U$.

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Appendix. The following are some standalone results used in Sections 2 and 3.
Theorem 4.4. Let $\mathcal{A}=\{A(t)\}_{t \geq 0}$ be a Poisson process of parameter $\lambda_{a}>0$, and $\mathcal{X}=$ $\{X(n)\}_{n \geq 0}$ an independent discrete-time Markov chain with state space $\mathcal{S}$ and transition probability matrix $P$. Define the Markov jump process $\mathcal{J}=\{J(t)\}_{t \geq 0}$ be

$$
J(t)=X(A(t)), \quad t \geq 0
$$

Let $\mathcal{B}=\{B(t)\}_{t \geq 0}$ be an independent Poisson process of parameter $\lambda_{b}>0$. Define $\mathcal{C}$ to be the superposition of the Poisson processes $\mathcal{A}$ and $\mathcal{B}$, and denote by $\left\{\tau_{k}\right\}_{k \geq 0}$ the arrival times of $\mathcal{C}$. If we let

$$
Y(n)=J\left(\tau_{n}\right), \quad n \geq 0,
$$

then the process $\mathcal{Y}=\{Y(n)\}_{n \geq 0}$ is a Markov chain with transition probability matrix given by

$$
\begin{equation*}
\frac{\lambda_{a}}{\lambda_{a}+\lambda_{b}} P+\frac{\lambda_{b}}{\lambda_{a}+\lambda_{b}} I . \tag{4.6}
\end{equation*}
$$

Proof. First, we show that $\mathcal{Y}$ is a Markov process. Let $i \in \mathcal{S}$ and $k \geq 1$. Then,

$$
\begin{aligned}
& \mathbb{P}(Y(k)=i \mid Y(0), Y(1), \ldots, Y(k-1)) \\
& \quad=\mathbb{P}\left(J\left(\tau_{k}\right)=i \mid J\left(\tau_{0}\right), J\left(\tau_{1}\right), \ldots, J\left(\tau_{k-1}\right)\right) \\
& \quad=\mathbb{P}\left(J\left(\tau_{k}\right)=i \mid J\left(\tau_{k-1}\right)\right) \quad(\text { Strong Markov property of } \mathcal{J}) \\
& \quad=\mathbb{P}(Y(k)=i \mid Y(k-1)),
\end{aligned}
$$

so that the Markov property holds.
Next, let $\mathcal{C}^{*}$ be the marked Poisson process with arrivals corresponding to the superposition of $\mathcal{A}$, arrivals which we mark with an $a$, and $\mathcal{B}$, arrivals which we mark with a $b$. The $k$ th arrival of $\mathcal{C}^{*}$ occurs at $\tau_{k}$ carrying a mark, say $m_{k} \in\{a, b\}$. Then,

$$
\begin{aligned}
& \mathbb{P}(Y(k)=j \mid Y(k-1)=i) \\
& =\mathbb{P}\left(Y(k)=j, m_{k}=a \mid Y(k-1)=i\right)+\mathbb{P}\left(Y(k)=j, m_{k}=b \mid Y(k-1)=i\right) \\
& =\mathbb{P}\left(Y(k)=j \mid Y(k-1)=i, m_{k}=a\right) \mathbb{P}\left(m_{k}=a \mid Y(k-1)=i\right) \\
& \quad+\mathbb{P}\left(Y(k)=j \mid Y(k-1)=i, m_{k}=b\right) \mathbb{P}\left(m_{k}=b \mid Y(k-1)=i\right)
\end{aligned}
$$

The event $\left\{m_{k}=a\right\}$ is clearly independent from $\{Y(k-1)=i\}$ : the mark of a given Poisson arrival is independent of the history of the previous arrivals. Thus,

$$
\mathbb{P}\left(m_{k}=a \mid Y(k-1)=i\right)=\mathbb{P}\left(m_{k}=a\right)=\frac{\lambda_{a}}{\lambda_{a}+\lambda_{b}} .
$$

Similarly,

$$
\mathbb{P}\left(m_{k}=b \mid Y(k-1)=i\right)=\mathbb{P}\left(m_{k}=b\right)=\frac{\lambda_{b}}{\lambda_{a}+\lambda_{b}}
$$

Next, since $\mathcal{J}$ only (possibly) jumps at arrival times marked with $a$, then

$$
\mathbb{P}\left(Y(k)=j \mid Y(k-1)=i, m_{k}=b\right)=\delta_{i j},
$$

where $\delta_{i j}$ denotes the Kronecker delta. Finally, since $\mathcal{J}$ is piecewise constant between the arrival times $\left\{\tau_{k}\right\}_{k}$, then

$$
\{Y(k-1)=i\}=\left\{J\left(\tau_{k-1}\right)=i\right\}=\left\{J\left(\tau_{k}^{-}\right)=i\right\}
$$

This implies that

$$
\mathbb{P}\left(Y(k)=j \mid Y(k-1)=i, m_{k}=a\right)=\mathbb{P}\left(J\left(\tau_{k}\right)=j \mid J\left(\tau_{k}^{-}\right)=i, m_{k}=a\right)=p_{i j}
$$

Consequently,

$$
\mathbb{P}(Y(k)=j \mid Y(k-1)=i)=p_{i j} \frac{\lambda_{a}}{\lambda_{a}+\lambda_{b}}+\delta_{i j} \frac{\lambda_{a}}{\lambda_{a}+\lambda_{b}}
$$

and the proof is complete.
Lemma 4.5. For $a \in \mathbb{N}_{+} \backslash\{1\}$ and $b>0$, let $Y \sim \operatorname{Erlang}(a, b)$. Then

$$
\begin{equation*}
\mathbb{E}\left[(Y-\mathbb{E}[Y])^{k}\right] \leq \frac{k!\sqrt{a}}{b^{j}} \frac{\sqrt{a}^{k+1}-1}{\sqrt{a}-1} \quad \text { for } k \in \mathbb{N}_{+} \tag{4.7}
\end{equation*}
$$

Proof. W.l.o.g. suppose that $b=1$. Equation (4.7) can be rewritten as

$$
\begin{equation*}
\mathbb{E}\left[(Y-\mathbb{E}[Y])^{k}\right] \leq k!\sum_{j=1}^{k} \sqrt{a}^{j} \tag{4.8}
\end{equation*}
$$

We use induction to prove that (4.8) holds. First, since $\mathbb{E}[Y-\mathbb{E}[Y]]=0<1$ ! $\sqrt{a}$, the case $k=1$ holds trivially. Now, suppose (4.7) holds for all $k \in\left\{1,2, \ldots, k_{0}\right\}$ for some $k_{0} \geq 1$. By [20, third formula on p.704],

$$
\begin{align*}
\mathbb{E}\left[(Y-\mathbb{E}[Y])^{k_{0}+1}\right] & =k_{0}!a \sum_{i=0}^{k_{0}-1} \frac{\mathbb{E}\left[(Y-\mathbb{E}[Y])^{i}\right]}{i!} \\
& =k_{0}!a\left[1+\sum_{i=2}^{k_{0}-1} \frac{\mathbb{E}\left[(Y-\mathbb{E}[Y])^{i}\right]}{i!}\right] . \tag{4.9}
\end{align*}
$$

Using the induction hypothesis on the RHS of (4.9), we get

$$
\begin{aligned}
\mathbb{E}\left[(Y-\mathbb{E}[Y])^{k_{0}+1}\right] & \leq k_{0}!a\left[1+\sum_{i=2}^{k_{0}-1} \sum_{j=1}^{i} \sqrt{a}^{j}\right] \leq k_{0}!a \sum_{i=1}^{k_{0}-1} \sum_{j=1}^{i} \sqrt{a}^{j} \\
& =k_{0}!a \sum_{j=1}^{k_{0}-1} \sum_{i=j}^{k_{0}-1} \sqrt{a}^{j}=k_{0}!a \sum_{j=1}^{k_{0}-1}\left(k_{0}-j\right) \sqrt{a}^{j} \\
& \leq\left(k_{0}+1\right)!a \sum_{j=1}^{k_{0}-1} \sqrt{a}^{j} \leq\left(k_{0}+1\right)!\sum_{j=1}^{k_{0}+1} \sqrt{a}^{j}
\end{aligned}
$$

which proves (4.8) and thus (4.7).
Lemma 4.6. Let $(\mathcal{R}, \mathcal{J})=\{(R(t), J(t))\}_{t \geq 0}$ be a Markov-modulated Brownian motion defined as in (1.5). Then, for any $i \in \mathcal{S}, t>0$ and $a>\mu_{\max } t$,

$$
\begin{equation*}
\mathbb{P}\left(\sup _{0 \leq s \leq t}|R(s)|>a \mid J(0)=i\right) \leq \frac{2}{\sqrt{2 \pi}} \frac{\sqrt{\sigma_{\max } t}}{a-\mu_{\max } t} \exp \left(-\frac{\left(a-\mu_{\max } t\right)^{2}}{2 \sigma_{\max } t}\right) \tag{4.10}
\end{equation*}
$$

where $\mu_{\max }:=\max _{i \in \mathcal{S}}\left|\mu_{i}\right|$ and $\sigma_{\max }:=\max _{i \in \mathcal{S}} \sigma_{i}$.
Proof. Let $\{W(t)\}_{t \geq 0}$ be a standard Brownian motion, independent from $(\mathcal{R}, \mathcal{J})$. A standard bound for the Brownian motion gives us for $b>0$

$$
\begin{aligned}
\mathbb{P}\left(\sup _{0 \leq s \leq t}|W(s)|>b\right) & =2 \int_{b}^{\infty} \frac{1}{\sqrt{2 \pi t}} e^{-x^{2} / 2 t} \mathrm{~d} x \\
& \leq 2 \int_{b}^{\infty} \frac{x / t}{\sqrt{2 \pi t}} e^{-x^{2} / 2 t} \mathrm{~d} x \\
& \leq \frac{2}{\sqrt{2 \pi}}\left(\frac{\sqrt{t}}{b}\right) e^{-b^{2} / 2 t}
\end{aligned}
$$

Note that $\mathcal{R}$ is identically distributed to $\left\{W\left(I_{t}^{\sigma}\right)+I_{t}^{\mu}\right\}_{t \geq 0}$, where $I_{t}^{\sigma}:=\int_{0}^{t} \sigma_{J(s)} \mathrm{d} s$ and $I_{t}^{\mu}:=$ $\int_{0}^{t} \mu_{J(s)} \mathrm{d} s$. This implies that

$$
\begin{aligned}
& \mathbb{P}\left(\sup _{0 \leq s \leq t}|R(s)|>a \mid J(0)=i\right)=\mathbb{P}\left(\sup _{0 \leq s \leq t}\left|W\left(I_{s}^{\sigma}\right)+I_{s}^{\mu}\right|>a \mid J(0)=i\right) \\
& \quad \leq \mathbb{P}\left(\sup _{0 \leq s \leq t}\left|W\left(I_{s}^{\sigma}\right)\right|>a-\sup _{0 \leq s \leq t}\left|I_{s}^{\mu}\right| \mid J(0)=i\right) \\
& \quad=\mathbb{E}\left(\mathbb{P}\left(\sup _{0 \leq s \leq t}\left|W\left(I_{s}^{\sigma}\right)\right|>a-\sup _{0 \leq s \leq t}\left|I_{s}^{\mu}\right| \mid J(0)=i,\left\{I_{s}^{\sigma}\right\}_{0 \leq s \leq t},\left\{I_{s}^{\mu}\right\}_{0 \leq s \leq t}\right) \mid J(0)=i\right) \\
& \quad \leq \mathbb{E}\left(\left.\frac{2}{\sqrt{2 \pi}}\left(\frac{\sqrt{\sup _{0 \leq s \leq t}\left|I_{s}^{\sigma}\right|}}{a-\sup _{0 \leq s \leq t}\left|I_{s}^{\mu}\right|}\right) \exp \left(-\frac{\left(a-\sup _{0 \leq s \leq t}\left|I_{s}^{\mu}\right|\right)^{2}}{2 \sup _{0 \leq s \leq t}\left|I_{s}^{\sigma}\right|}\right) \right\rvert\, J(0)=i\right) \\
& \quad \leq \mathbb{E}\left(\left.\frac{2}{\sqrt{2 \pi}} \frac{\sqrt{\sigma_{\max } t}}{a-\mu_{\max } t} \exp \left(-\frac{\left(a-\mu_{\max } t\right)^{2}}{2 \sigma_{\max } t}\right) \right\rvert\, J(0)=i\right) \\
& \quad=\frac{2}{\sqrt{2 \pi}} \frac{\sqrt{\sigma_{\max } t}}{a-\mu_{\max } t} \exp \left(-\frac{\left(a-\mu_{\max } t\right)^{2}}{2 \sigma_{\max } t}\right),
\end{aligned}
$$

which completes the proof.

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