# Self-normalized Cramér moderate deviations for a supercritical Galton-Watson process

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#### Abstract

Let  $(Z_n)_{n\geq 0}$  be a supercritical Galton-Watson process. Consider the Lotka-Nagaev estimator for the offspring mean. In this paper, we establish self-normalized Cramér type moderate deviations and Berry-Esseen's bounds for the Lotka-Nagaev estimator. The results are believed to be optimal or near optimal.

*Keywords:* Lotka-Nagaev estimator; offspring mean; Self-normalized processes; Cramér moderate deviations; Berry-Esseen's bounds

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## 1. Introduction

A Galton-Watson process can be described as follows

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad \text{for } n \ge 0,$$
 (1.1)

where  $X_{n,i}$  is the offspring number of the *i*-th individual of the generation *n*. Moreover, the random variables  $(X_{n,i})_{i>1}$  are independent of each other with common distribution law

$$\mathbb{P}(X_{n,i}=k) = p_k, \quad k \in \mathbb{N}, \tag{1.2}$$

and are also independent of  $Z_n$ .

An important task in statistical inference of Galton-Watson processes is to estimate the average offspring number of an individual m, usually termed the offspring mean. Clearly, it holds

$$m = \mathbb{E}Z_1 = \mathbb{E}X_{n,i} = \sum_{k=0}^{\infty} kp_k.$$

Denote v the standard variance of  $Z_1$ , that is

$$v^2 = \mathbb{E}(Z_1 - m)^2. \tag{1.3}$$

To avoid triviality, assume that v > 0. For estimation of the offspring mean m, the Lotka-Nagaev [11, 12] estimator  $Z_{n+1}/Z_n$  plays an important role. For the Galton-Watson processes, Athreya [1]

has established large deviations for the normalized Lotka-Nagaev estimator (see also Chu [4] for selfnormalized large deviations); Ney and Vidyashankar [14, 15] obtained sharp rate estimates for the large deviation behavior of the Lotka-Nagaev estimator; Bercu and Touati [2] proved an exponential inequalities for the Lotka-Nagaev estimator via self-normalized martingale method. The main purpose of this paper is to establish self-normalized Cramér moderate deviations for the Lotka-Nagaev estimator  $Z_{n+1}/Z_n$  for the Galton-Watson processes.

The paper is organized as follows. In Section 2, we present Cramér moderate deviations for the self-normalized Lotka-Nagaev estimator, provided that  $(Z_n)_{n\geq 0}$  or  $(X_{n,i})_{1\leq i\leq Z_n}$  can be observed. In Section 3, we present some applications of our results in statistics. The rest sections devote to the proofs of theorems.

#### 2. Main results

#### 2.1. $(Z_k)_{k>0}$ can be observed

Assume that the total populations  $(Z_k)_{k\geq 0}$  of all generations can be observed. For any  $n_0 \geq 0$ , we define

$$M_{n_0,n} = \frac{\sum_{k=n_0}^{n_0+n-1} \sqrt{Z_k} (\frac{Z_{k+1}}{Z_k} - m)}{\sqrt{\sum_{k=n_0}^{n_0+n-1} Z_k (\frac{Z_{k+1}}{Z_k} - m)^2}}.$$
(2.1)

We assume that the set of extinction of the process  $(Z_k)_{k\geq 0}$  is negligible with respect to the annealed law  $\mathbb{P}$ . Then  $M_{n_0,n}$  is well defined  $\mathbb{P}$ -a.s. As  $(Z_k)_{k=n_0,\ldots,n_0+n}$  can be observed,  $M_{n_0,n}$  can be regarded as a time type self-normalized process for the Lotka-Nagaev estimator  $Z_{k+1}/Z_k$ . The following theorem gives a self-normalized Cramér moderate deviation result for the Galton-Watson processes.

**Theorem 2.1.** Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ .

[1] If 
$$\rho \in (0, 1)$$
, then for all  $x \in [0, o(\sqrt{n}))$ ,  

$$\left| \ln \frac{\mathbb{P}(M_{n_0, n} \ge x)}{1 - \Phi(x)} \right| \le C_{\rho} \left( \frac{x^{2+\rho}}{n^{\rho/2}} + \frac{(1+x)^{1-\rho(2+\rho)/4}}{n^{\rho(2-\rho)/8}} \right), \tag{2.2}$$

where  $C_{\rho}$  depends only on the constants  $\rho, v$  and  $\mathbb{E}Z_1^{2+\rho}$ .

**[ii]** If  $\rho = 1$ , then for all  $x \in [0, o(\sqrt{n}))$ ,

$$\left|\ln\frac{\mathbb{P}(M_{n_0,n} \ge x)}{1 - \Phi(x)}\right| \le C\left(\frac{x^3}{\sqrt{n}} + \frac{\ln n}{\sqrt{n}} + \frac{(1+x)^{1/4}}{n^{1/8}}\right),\tag{2.3}$$

where C depends only on the constants v and  $\mathbb{E}Z_1^3$ .

In particular, the inequalities (2.2) and (2.3) together implies that

$$\frac{\mathbb{P}(M_{n_0,n} \ge x)}{1 - \Phi(x)} = 1 + o(1) \tag{2.4}$$

uniformly for  $n_0 \in \mathbb{N}$  and for  $x \in [0, o(n^{\rho/(4+2\rho)}))$  as  $n \to \infty$ . Moreover, the same inequalities remain valid when  $\frac{\mathbb{P}(M_{n_0,n} \ge x)}{1-\Phi(x)}$  is replaced by  $\frac{\mathbb{P}(M_{n_0,n} \le -x)}{\Phi(-x)}$ .

Notice that  $C_{\rho}$  and C do not depend on  $n_0$ . Thus (2.4) holds uniformity in  $n_0$ , which is of particular interesting in applications. For instance, due to the uniformity, in (2.4) we can take  $n_0$  as a function of n.

Equality (2.4) implies that  $\mathbb{P}(M_{n_0,n} \leq x) \to \Phi(x)$  as n tends to  $\infty$ . Thus Theorem 2.1 implies the central limit theory for  $M_{n_0,n}$ . Moreover, equality (2.4) states that the relative error of normal approximation for  $M_{n_0,n}$  tends to zero uniformly for  $x \in [0, o(n^{\rho/(4+2\rho)}))$  as  $n \to \infty$ .

Theorem 2.1 implies the following moderate deviation principle (MDP) result for the time type self-normalized Lotka-Nagaev estimator.

**Corollary 2.1.** Assume the conditions of Theorem 2.1. Let  $(a_n)_{n\geq 1}$  be any sequence of real numbers satisfying  $a_n \to \infty$  and  $a_n/\sqrt{n} \to 0$  as  $n \to \infty$ . Then for each Borel set B,

$$-\inf_{x\in B^o}\frac{x^2}{2} \le \liminf_{n\to\infty}\frac{1}{a_n^2}\ln\mathbb{P}\left(\frac{M_{n_0,n}}{a_n}\in B\right) \le \limsup_{n\to\infty}\frac{1}{a_n^2}\ln\mathbb{P}\left(\frac{M_{n_0,n}}{a_n}\in B\right) \le -\inf_{x\in\overline{B}}\frac{x^2}{2},\qquad(2.5)$$

where  $B^{o}$  and  $\overline{B}$  denote the interior and the closure of B, respectively.

**Remark 2.1.** From (2.2) and (2.3), it is easy to derive the following Berry-Esseen bound for the self-normalized Lotka-Nagaev estimator:

$$\left| \mathbb{P}(M_{n_0,n} \le x) - \Phi(x) \right| \le \frac{C_{\rho}}{n^{\rho(2-\rho)/8}},$$
(2.6)

where  $C_{\rho}$  depends only on the constants  $\rho, v$  and  $\mathbb{E}Z_1^{2+\rho}$ . When  $\rho > 1$ , by the self-normalized Berry-Esseen bound for martingales in Fan and Shao [7], we can get a Berry-Esseen bound of order  $n^{-\frac{\rho}{6+2\rho}}$ .

The last remark gives a self-normalized Berry-Esseen bound for the Lotka-Nagaev estimator, while the next theorem presents a normalized Berry-Esseen bound for the Lotka-Nagaev estimator. Denote

$$H_{n_0,n} = \frac{1}{\sqrt{n}v} \sum_{k=n_0}^{n_0+n-1} \sqrt{Z_k} \left(\frac{Z_{k+1}}{Z_k} - m\right).$$

Notice that the random variables  $(X_{k,i})_{1 \le i \le Z_k}$  have the same distribution as  $Z_1$ , and that  $(X_{k,i})_{1 \le i \le Z_k}$  are independent of  $Z_k$ . Then for the Galton-Watson processes, it holds

$$\mathbb{E}[(Z_{k+1} - mZ_k)^2 | Z_k] = \mathbb{E}[(\sum_{i=1}^{Z_k} (X_{k,i} - m))^2 | Z_k] = Z_k v^2.$$

It is easy to see that  $H_{n_0,n} = \sum_{k=n_0}^{n_0+n-1} \frac{1}{\sqrt{nv^2/Z_k}} \left(\frac{Z_{k+1}}{Z_k} - m\right)$ . Thus  $H_{n_0,n}$  can be regarded as a normalized process for the Lotka-Nagaev estimator  $Z_{k+1}/Z_k$ . We have the following normalized Berry-Esseen bounds for the Galton-Watson processes.

**Theorem 2.2.** Assume the conditions of Theorem 2.1 are satisfied.

[i] If  $\rho \in (0, 1)$ , then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(H_{n_0, n} \le x) - \Phi(x) \right| \le \frac{C_{\rho}}{n^{\rho/2}},\tag{2.7}$$

where  $C_{\rho}$  depends only on  $\rho, v$  and  $\mathbb{E}Z_1^{2+\rho}$ .

**[ii]** If  $\rho = 1$ , then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(H_{n_0, n} \le x) - \Phi(x) \right| \le C \frac{\ln n}{\sqrt{n}},\tag{2.8}$$

where C depends only on v and  $\mathbb{E}Z_1^3$ .

Moreover, the same inequalities remain valid when  $H_{n_0,n}$  is replaced by  $-H_{n_0,n}$ .

The convergence rates of (2.7) and (2.8) are same to the best possible convergence rates of the Berry-Esseen bounds for martingales, see Theorem 2.1 of Fan [8] and its comment. Notice that  $H_{n_0,n}$  is a martingale with respect to the natural filtration.

2.2.  $(X_{n,i})_{1 \leq i \leq Z_n}$  can be observed for some n

Assume that the offspring numbers  $(X_{n,i})_{1 \le i \le Z_n}$  of each individual in some generation n can be observed. Denote

$$T_n = \frac{Z_n \left(\frac{Z_{n+1}}{Z_n} - m\right)}{\sqrt{\sum_{i=1}^{Z_n} (X_{n,i} - \frac{Z_{n+1}}{Z_n})^2}}$$

the space type self-normalized process for the Lotka-Nagaev estimator  $Z_{n+1}/Z_n$ . The following theorem gives a Cramér moderate deviation result for the space type self-normalized Lotka-Nagaev estimator  $T_n$ .

**Theorem 2.3.** Assume that  $p_0 = 0$  and  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Then

$$\left|\ln\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)}\right| = O\left(\frac{1 + x^{2+\rho}}{n^{\rho/2}}\right)$$
(2.9)

uniformly for  $x \in [0, o(\sqrt{n}))$  as  $n \to \infty$ . Moreover, the same equality remains valid when  $\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)}$  is replaced by  $\frac{\mathbb{P}(T_n \le -x)}{\Phi(-x)}$ .

The condition  $p_0 = 0$  means that each individual has at least one offspring. Moreover, it also implies that  $Z_n \to \infty$  a.s. as  $n \to \infty$ . Then by law of large numbers, we have  $\frac{Z_{n+1}}{Z_n}$  tends to m a.s. as  $n \to \infty$ .

For the Galton-Watson processes, we refer to [1] for closely related results of Theorem 2.3, where Athreya has established a precise large deviation rate for the Lotka-Nagaev estimator  $Z_{n+1}/Z_n$ .

Using the inequality  $|e^x - 1| \le e^C |x|$  valid for  $|x| \le C$ , from Theorem 2.3, we obtain the following estimation for the relative error of normal approximation.

Corollary 2.2. Assume the conditions of Theorem 2.3. Then

$$\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)} = 1 + O\left(\frac{1 + x^{2+\rho}}{n^{\rho/2}}\right)$$
(2.10)

uniformly for  $x \in [0, O(n^{\rho/(4+2\rho)}))$  as  $n \to \infty$ . In particular, it implies that

$$\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)} = 1 + o(1) \tag{2.11}$$

uniformly for  $x \in [0, o(n^{\rho/(4+2\rho)}))$  as  $n \to \infty$ . Moreover, the same equalities remain valid when  $T_n$  is replaced by  $-T_n$ .

Inequality (2.11) implies that the relative error of normal approximation for  $T_n$  tends to zero uniformly for  $x \in [0, o(n^{\rho/(4+2\rho)}))$ . Clearly, the range of validity for (2.11) coincides with the self-normalized Cramér moderate deviation result of Shao [17] for iid random variables.

By an argument similar to the proof of Corollary 2.1, Theorem 2.3 also implies the following self-normalized MDP result.

**Corollary 2.3.** Assume the conditions of Theorem 2.3. Let  $(a_n)_{n\geq 1}$  be any sequence of real numbers satisfying  $a_n \to \infty$  and  $a_n/\sqrt{n} \to 0$  as  $n \to \infty$ . Then for each Borel set B,

$$-\inf_{x\in B^o}\frac{x^2}{2} \le \liminf_{n\to\infty}\frac{1}{a_n^2}\ln\mathbb{P}\left(\frac{T_n}{a_n}\in B\right) \le \limsup_{n\to\infty}\frac{1}{a_n^2}\ln\mathbb{P}\left(\frac{T_n}{a_n}\in B\right) \le -\inf_{x\in\overline{B}}\frac{x^2}{2},\tag{2.12}$$

where  $B^{o}$  and  $\overline{B}$  denote the interior and the closure of B, respectively.

From Theorem 2.3, we get the following self-normalized Berry-Esseen bound for  $T_n$ .

Corollary 2.4. Assume the conditions of Theorem 2.3. Then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T_n \le x) - \Phi(x) \right| \le \frac{C_{\rho}}{n^{\rho/2}},\tag{2.13}$$

where  $C_{\rho}$  does not depend on n.

Clearly, the convergence rate for the Berry-Esseen bound of Corollary 2.4 is consistent with the classical case of iid random variables (cf. Bentkus and Götze [3]), and therefore it is optimal under the stated conditions.

**Remark 2.2.** Following the proof of Theorem 2.3, the results (2.9)-(2.13) remain true when  $T_n$  is replaced by

$$\widetilde{T}_n = \frac{Z_n \left( Z_{n+1} / Z_n - m \right)}{\sqrt{\sum_{i=1}^{Z_n} (X_{n,i} - m)^2}},$$

## 3. Applications

Cramér moderate deviations certainly have a lot of applications in statistics.

## 3.1. p-value for hypothesis testing

Self-normalized Cramér moderate deviations can be applied to hypothesis testing of m for the Galton-Watson processes. When  $(Z_k)_{k=n_0,\ldots,n_0+n}$  can be observed, we can make use of Theorem 2.1 to estimate p-value. Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $0 < \rho \leq 1$ , and that m > 1. Let  $(z_k)_{k=n_0,\ldots,n_0+n}$  be an observation of  $(Z_k)_{k=n_0,\ldots,n_0+n}$ . In order to estimate the offspring mean m, we can make use of the Harris estimator [2] given by

$$\widehat{m}_n = \frac{\sum_{k=n_0}^{n_0+n-1} Z_{k+1}}{\sum_{k=n_0}^{n_0+n-1} Z_k}.$$

Then observation for the Harris estimator is

$$\widehat{m}_n = \frac{\sum_{k=n_0}^{n_0+n-1} z_{k+1}}{\sum_{k=n_0}^{n_0+n-1} z_k}.$$

By Theorem 2.1, it is easy to see that

$$\frac{\mathbb{P}(M_{n_0,n} \ge x)}{1 - \Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}(M_{n_0,n} \le -x)}{1 - \Phi(x)} = 1 + o(1) \tag{3.1}$$

uniformly for  $x \in [0, o(n^{\rho/(4+2\rho)}))$ . Notice that  $1 - \Phi(x) = \Phi(-x)$ . Thus, by (3.1), the probability  $\mathbb{P}(M_{n_0,n} > |\widetilde{m}_n|)$  is almost equal to  $2\Phi(-|\widetilde{m}_n|)$ , where

$$\widetilde{m}_n = \frac{\sum_{k=n_0}^{n_0+n-1} \sqrt{z_k} (z_{k+1}/z_k - \widehat{m}_n)}{\sqrt{\sum_{k=n_0}^{n_0+n-1} z_k (z_{k+1}/z_k - \widehat{m}_n)^2}}$$

## 3.2. Construction of confidence intervals

## 3.2.1. The data $(Z_k)_{k>0}$ can be observed

Cramér moderate deviations can be also applied to construction of confidence intervals of m. We make use of Theorem 2.1 to construct confidence intervals.

**Proposition 3.1.** Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Let  $\kappa_n \in (0,1)$ . Assume that

$$\left|\ln\kappa_n\right| = o\left(n^{\rho/(2+\rho)}\right). \tag{3.2}$$

Let

$$a_{n_0,n} = \left(\sum_{k=n_0}^{n_0+n-1} \sqrt{Z_k}\right)^2 - \left(\Phi^{-1}(1-\kappa_n/2)\right)^2 \sum_{k=n_0}^{n_0+n-1} Z_k,$$
  

$$b_{n_0,n} = 2\left(\Phi^{-1}(1-\kappa_n/2)\right)^2 \sum_{k=n_0}^{n_0+n-1} Z_{k+1} - 2\left(\sum_{k=n_0}^{n_0+n-1} \frac{Z_{k+1}}{\sqrt{Z_k}}\right) \left(\sum_{k=n_0}^{n_0+n-1} \sqrt{Z_k}\right),$$
  

$$c_{n_0,n} = \left(\sum_{k=n_0}^{n_0+n-1} \frac{Z_{k+1}}{\sqrt{Z_k}}\right)^2 - \left(\Phi^{-1}(1-\kappa_n/2)\right)^2 \sum_{k=n_0}^{n_0+n-1} \frac{Z_{k+1}^2}{Z_k}.$$

Then  $[A_{n_0,n}, B_{n_0,n}]$ , with

$$A_{n_0,n} = \frac{-b_{n_0,n} - \sqrt{b_{n_0,n}^2 - 4a_{n_0,n}c_{n_0,n}}}{2a_{n_0,n}}$$

and

$$B_{n_0,n} = \frac{-b_{n_0,n} + \sqrt{b_{n_0,n}^2 - 4a_{n_0,n}c_{n_0,n}}}{2a_{n_0,n}},$$

is a  $1 - \kappa_n$  confidence interval for m, for n large enough.

*Proof.* Notice that  $1 - \Phi(x) = \Phi(-x)$ . Theorem 2.1 implies that

$$\frac{\mathbb{P}(M_{n_0,n} \ge x)}{1 - \Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}(M_{n_0,n} \le -x)}{1 - \Phi(x)} = 1 + o(1)$$
(3.3)

uniformly for  $0 \le x = o(n^{\rho/(4+2\rho)})$ , see (2.4). When  $\kappa_n$  satisfies the condition (3.2), the upper  $(\kappa_n/2)$ th quantile of a standard normal distribution satisfies

$$\Phi^{-1}(1 - \kappa_n/2) = O(\sqrt{|\ln \kappa_n|}),$$

which is of order  $o(n^{\rho/(4+2\rho)})$ . Then applying (3.3) to the last equality, we complete the proof of Proposition 3.1. Notice that  $A_{n_0,n}$  and  $B_{n_0,n}$  are solutions of the following equation

$$\frac{\sum_{k=n_0}^{n_0+n-1}\sqrt{Z_k}(Z_{k+1}/Z_k-x)}{\sqrt{\sum_{k=n_0}^{n_0+n-1}Z_k(Z_{k+1}/Z_k-x)^2}} = \Phi^{-1}(1-\kappa_n/2).$$

This completes the proof of Proposition 3.1.

3.2.2. The data  $(X_{n,i})_{1 \leq i \leq Z_n}$  can be observed

When  $(X_{n,i})_{1 \le i \le Z_n}$  can be observed, we can make use of Corollary 2.2 to construct confidence intervals.

**Proposition 3.2.** Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Let  $\kappa_n \in (0,1)$ . Assume that

$$\left|\ln\kappa_n\right| = o\left(n^{\rho/(2+\rho)}\right). \tag{3.4}$$

Let

$$\Delta_n = \frac{\Phi^{-1}(1 - \kappa_n/2)}{Z_n} \sqrt{\sum_{i=1}^{Z_n} (X_{n,i} - \frac{Z_{n+1}}{Z_n})^2}.$$

Then  $[A_n, B_n]$ , with

$$A_n = \frac{Z_{n+1}}{Z_n} - \Delta_n$$
 and  $B_n = \frac{Z_{n+1}}{Z_n} + \Delta_n$ ,

is a  $1 - \kappa_n$  confidence interval for m, for n large enough.

*Proof.* Corollary 2.2 implies that

$$\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}(T_n \le -x)}{1 - \Phi(x)} = 1 + o(1) \tag{3.5}$$

uniformly for  $0 \le x = o(n^{\rho/(4+2\rho)})$ . When  $\kappa_n$  satisfies the condition (3.2), the upper  $(\kappa_n/2)$ th quantile of a standard normal distribution satisfies  $\Phi^{-1}(1-\kappa_n/2) = O(\sqrt{|\ln \kappa_n|})$ , which is of order  $o(n^{\rho/(4+2\rho)})$ . Then applying (3.5) to the last equality, we complete the proof of Proposition 3.2.

When the risk probability  $\kappa_n$  goes to 0, we have the following more general result.

**Proposition 3.3.** Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Let  $\kappa_n \in (0,1)$  such that  $k_n \to 0$ . Assume that

$$\left|\ln\kappa_n\right| = o\left(\sqrt{n}\right).\tag{3.6}$$

Let

$$\Delta_n = \frac{\sqrt{2|\ln(\kappa_n/2)|}}{Z_n} \sqrt{\sum_{i=1}^{Z_n} (X_{n,i} - \frac{Z_{n+1}}{Z_n})^2}.$$

Then  $[A_n, B_n]$ , with

$$A_n = \frac{Z_{n+1}}{Z_n} - \Delta_n$$
 and  $B_n = \frac{Z_{n+1}}{Z_n} + \Delta_n$ ,

is a  $1 - \kappa_n$  confidence interval for m, for n large enough.

*Proof.* By Theorem 2.3, we have

$$\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)} = \exp\left\{\theta C \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} \quad \text{and} \quad \frac{\mathbb{P}(T_n \le -x)}{1 - \Phi(x)} = \exp\left\{\theta C \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} \tag{3.7}$$

uniformly for  $0 \le x = o(\sqrt{n})$ , where  $\theta \in [-1, 1]$ . Notice that

$$1 - \Phi(x_n) \sim \frac{1}{x_n \sqrt{2\pi}} e^{-x_n^2/2} = \exp\left\{-\frac{x_n^2}{2} \left(1 + \frac{2}{x_n^2} \ln(x_n \sqrt{2\pi})\right)\right\}, \ x_n \to \infty.$$

Since  $k_n \to 0$ , the last line implies that the upper  $(\kappa_n/2)$ th quantile of the distribution

$$1 - \left(1 - \Phi\left(x\right)\right) \exp\left\{\theta C \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\}$$

converges to  $\sqrt{2|\ln(\kappa_n/2)|}$ , which is of order  $o(\sqrt{n})$  as  $n \to \infty$ . Then applying (3.7) to  $T_n$ , we complete the proof of Proposition 3.3.

3.2.3. The parameter  $v^2$  is known

When  $v^2$  is known, we can apply normalized Berry-Esseen bounds (cf. Theorem 2.2) to construct confidence intervals.

**Proposition 3.4.** Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Let  $\kappa_n \in (0,1)$ . Assume that  $|\ln \kappa_n| = o(\log n)$ . (3.8)

Then  $[A_n, B_n]$ , with

$$A_n = \frac{\sum_{k=n_0}^{n_0+n} Z_{k+1} / \sqrt{Z_k} - \sqrt{n} v \Phi^{-1} (1 - \kappa_n/2)}{\sum_{k=n_0}^{n_0+n} \sqrt{Z_k}}$$

and

$$B_n = \frac{\sum_{k=n_0}^{n_0+n} Z_{k+1} / \sqrt{Z_k} + \sqrt{n} v \Phi^{-1} (1 - \kappa_n/2)}{\sum_{k=n_0}^{n_0+n} \sqrt{Z_k}},$$

is a  $1 - \kappa_n$  confidence interval for m, for n large enough.

*Proof.* Theorem 2.2 implies that

$$\frac{\mathbb{P}(H_{n_0,n} \ge x)}{1 - \Phi(x)} = 1 + o(1) \quad \text{and} \quad \frac{\mathbb{P}(H_{n_0,n} \le -x)}{1 - \Phi(x)} = 1 + o(1) \tag{3.9}$$

uniformly for  $0 \leq x = o(\sqrt{\log n})$ . The upper  $(\kappa_n/2)$ th quantile of a standard normal distribution satisfies

$$\Phi^{-1}(1-\kappa_n/2) = O(\sqrt{|\ln \kappa_n|}),$$

which, by (3.8), is of order  $o(\sqrt{\log n})$ . Proposition 3.4 follows from applying (3.9) to  $H_{n_0,n}$ .

## 3.3. An infectious disease model

An infectious disease model  $(Z_n)_{n\geq 0}$  may be described as follows:

$$Z_0 = 1, \quad Z_{n+1} = Z_n + \sum_{i=1}^{Z_n} Y_{n,i}, \quad \text{for } n \ge 0,$$
 (3.10)

where  $Z_n$  stands for the total population of patients with infectious disease at time n, and  $Y_{n,i}$  is the number of patients infected by the *i*-th individual of  $Z_n$  in a unit time (for instance, one day). Moreover, we assume that the random variables  $(Y_{n,i})_{i\geq 1}$  are iid random variables with common distribution law

$$\mathbb{P}(Y_{n,i}=k) = p_k, \quad k \in \mathbb{N}, \tag{3.11}$$

and are also independent to  $Z_n$ . Denote by r the average number of patients infected by an individual patient in a unite time, that is

$$r = \mathbb{E}Y_{n,i} = \sum_{k=0}^{\infty} k \, p_k.$$

Denote by v the standard variance of  $Y_{n,i}, n, i \ge 1$ , then v is also the standard variance of  $Z_1$ , that is

$$v^2 = \mathbb{E}(Z_1 - m)^2.$$

To avid triviality, assume that v > 0. We are interested in the estimation of r.

**Proposition 3.5.** Assume that  $\mathbb{E}Z_1^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Let  $\kappa_n \in (0,1)$ . Assume that

$$\left|\ln\kappa_{n}\right| = o\left(n^{\rho/(2+\rho)}\right). \tag{3.12}$$

Let  $A_{n_0,n}$  and  $B_{n_0,n}$  be defined in Proposition 3.1. Then  $[A_{n_0,n}-1, B_{n_0,n}-1]$  is a  $1-\kappa_n$  confidence interval for r, for n large enough.

*Proof.* It is easy to see that (3.10) can be rewritten in the form of (1.1), with  $X_{n,i} = 1 + Y_{n,i}$ . Thus, we have m = 1 + r. Then Proposition 3.5 follows by Proposition 3.1.

### 4. Proof of Theorem 2.1

In the proof of Theorem 2.1, we will make use of the following lemma (cf. Corollary 2.3 of Fan et al. [9]), which gives self-normalized Cramér moderate deviations for martingales.

**Lemma 4.1.** Let  $(\eta_k, \mathcal{F}_k)_{k=1,...,n}$  be a finite sequence of martingale differences. Assume that there exist a constant  $\rho \in (0,1]$  and numbers  $\gamma_n > 0$  and  $\delta_n \ge 0$  satisfying  $\gamma_n, \delta_n \to 0$  such that for all  $1 \le i \le n$ ,

$$\mathbb{E}[|\eta_k|^{2+\rho}|\mathcal{F}_{k-1}] \le \gamma_n^{\rho} \mathbb{E}[\eta_k^2|\mathcal{F}_{k-1}]$$
(4.1)

and

$$\left\|\sum_{k=1}^{n} \mathbb{E}[\eta_k^2 | \mathcal{F}_{k-1}] - 1\right\|_{\infty} \le \delta_n^2 \quad a.s.$$

$$(4.2)$$

Denote

$$V_n = \frac{\sum_{k=1}^n \eta_k}{\sqrt{\sum_{k=1}^n \eta_k^2}}$$

and

$$\widehat{\gamma}_n(x,\rho) = \frac{\gamma_n^{\rho(2-\rho)/4}}{1+x^{\rho(2+\rho)/4}}.$$

[i] If  $\rho \in (0,1)$ , then for all  $0 \le x = o(\gamma_n^{-1})$ ,

$$\left|\ln\frac{\mathbb{P}(V_n \ge x)}{1 - \Phi(x)}\right| \le C_\rho \left(x^{2+\rho}\gamma_n^\rho + x^2\delta_n^2 + (1+x)\left(\delta_n + \widehat{\gamma}_n(x,\rho)\right)\right).$$
(4.3)

**[ii]** If  $\rho = 1$ , then for all  $0 \le x = o(\gamma_n^{-1})$ ,

$$\left|\ln\frac{\mathbb{P}(V_n \ge x)}{1 - \Phi(x)}\right| \le C\left(x^3\gamma_n + x^2\delta_n^2 + (1 + x)\left(\delta_n + \gamma_n|\ln\gamma_n| + \widehat{\gamma}_n(x, 1)\right)\right).$$
(4.4)

Now, we are in position to prove Theorem 2.1. Denote

$$\hat{\xi}_{k+1} = \sqrt{Z_k} (Z_{k+1}/Z_k - m)$$

 $\mathfrak{F}_{n_0} = \{\emptyset, \Omega\}$  and  $\mathfrak{F}_{k+1} = \sigma\{Z_i : n_0 \leq i \leq k+1\}$  for all  $k \geq n_0$ . Notice that  $X_{k,i}$  is independent of  $Z_k$ . Then it is easy to verify that

$$\mathbb{E}[\hat{\xi}_{k+1}|\mathfrak{F}_{k}] = Z_{k}^{-1/2}\mathbb{E}[Z_{k+1} - mZ_{k}|\mathfrak{F}_{k}] = Z_{k}^{-1/2}\sum_{i=1}^{Z_{k}}\mathbb{E}[X_{k,i} - m|\mathfrak{F}_{k}]$$

$$= Z_{k}^{-1/2}\sum_{i=1}^{Z_{k}}\mathbb{E}[X_{k,i} - m]$$

$$= 0.$$
(4.5)

Thus  $(\hat{\xi}_k, \mathfrak{F}_k)_{k=n_0+1,\dots,n_0+n}$  is a finite sequence of martingale differences. Notice that  $X_{k,i} - m, i \ge 1$ , are centered and independent random variables. Thus, the following equalities hold

$$\sum_{k=n_0}^{n_0+n-1} \mathbb{E}[\hat{\xi}_{k+1}^2 | \mathfrak{F}_k] = \sum_{k=n_0}^{n_0+n-1} Z_k^{-1} \mathbb{E}[(Z_{k+1} - mZ_k)^2 | \mathfrak{F}_k] = \sum_{k=n_0}^{n_0+n-1} Z_k^{-1} \mathbb{E}[(\sum_{i=1}^{Z_k} (X_{k,i} - m))^2 | \mathfrak{F}_k]$$

$$= \sum_{k=n_0}^{n_0+n-1} Z_k^{-1} Z_k \mathbb{E}[(X_{k,i} - m)^2]$$

$$= nv^2. \tag{4.6}$$

Moreover, it is easy to see that

$$\mathbb{E}[|\hat{\xi}_{k+1}|^{2+\rho}|\mathfrak{F}_{k}] = Z_{k}^{-1-\rho/2}\mathbb{E}[|Z_{k+1} - mZ_{k}|^{2+\rho}|\mathfrak{F}_{k}] \\ = Z_{k}^{-1-\rho/2}\mathbb{E}[|\sum_{i=1}^{Z_{k}}(X_{k,i} - m)|^{2+\rho}|\mathfrak{F}_{k}].$$
(4.7)

By Rosenthal's inequality, we have

$$\mathbb{E}[|\sum_{i=1}^{Z_k} (X_{k,i} - m)|^{2+\rho} |\mathfrak{F}_k] \leq C'_{\rho} \left( \left( \sum_{i=1}^{Z_k} \mathbb{E} (X_{k,i} - m)^2 \right)^{1+\rho/2} + \sum_{i=1}^{Z_k} \mathbb{E} |X_{k,i} - m|^{2+\rho} \right) \\ \leq C'_{\rho} \left( Z_k^{1+\rho/2} v^{2+\rho} + Z_k \mathbb{E} |Z_1 - m|^{2+\rho} \right).$$

Since the set of extinction of the process  $(Z_k)_{k\geq 0}$  is negligible with respect to the annealed law  $\mathbb{P}$ , we have  $Z_k \geq 1$  for any k. From (4.7), by the last inequality and the fact  $Z_k \geq 1$ , we deduce that

$$\mathbb{E}[|\hat{\xi}_{k+1}|^{2+\rho}|\mathfrak{F}_{k}] \leq C_{\rho}'(v^{\rho} + \mathbb{E}|Z_{1} - m|^{2+\rho}/v^{2})v^{2} \\
= C_{\rho}'(v^{\rho} + \mathbb{E}|Z_{1} - m|^{2+\rho}/v^{2})\mathbb{E}[\hat{\xi}_{k+1}^{2}|\mathfrak{F}_{k}] \\
= C_{\rho}(v^{\rho} + \mathbb{E}Z_{1}^{2+\rho}/v^{2})\mathbb{E}[\hat{\xi}_{k+1}^{2}|\mathfrak{F}_{k}].$$
(4.8)

Let  $\eta_k = \hat{\xi}_{n_0+k}/\sqrt{n}v$  and  $\mathcal{F}_k = \mathfrak{F}_{n_0+k}$ . Then  $(\eta_k, \mathcal{F}_k)_{k=1,\dots,n}$  is a martingale difference sequences and satisfies the conditions (4.1) and (4.2) with  $\delta_n = 0$  and  $\gamma_n = (C_\rho (v^\rho + \mathbb{E}Z_1^{2+\rho}/v^2))^{1/\rho}/\sqrt{n}v$ . Clearly, it holds

$$M_{n_0,n} = \frac{\sum_{k=1}^n \eta_k}{\sqrt{\sum_{k=1}^n \eta_k^2}}.$$

Applying Lemma 4.1 to  $(\eta_k, \mathcal{F}_k)_{k=1,\dots,n}$ , we obtain the desired inequalities.

#### 5. Proof of Corollary 2.1

We first show that for any Borel set  $B \subset \mathbb{R}$ ,

$$\limsup_{n \to \infty} \frac{1}{a_n^2} \ln \mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \le -\inf_{x \in \overline{B}} \frac{x^2}{2}.$$
(5.1)

When  $B = \emptyset$ , the last inequality is obvious, with  $-\inf_{x \in \emptyset} \frac{x^2}{2} = -\infty$ . Thus, we may assume that  $B \neq \emptyset$ . Let  $x_0 = \inf_{x \in B} |x|$ . Clearly, we have  $x_0 \ge \inf_{x \in \overline{B}} |x|$ . Then, by Theorem 2.1, it follows that for  $a_n = o(\sqrt{n})$ ,

$$\mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \leq \mathbb{P}\left(|M_{n_0,n}| \ge a_n x_0\right) \\
\leq 2\left(1 - \Phi\left(a_n x_0\right)\right) \exp\left\{C_{\rho}\left(\frac{(a_n x_0)^{2+\rho}}{n^{\rho/2}} + \frac{\ln n}{\sqrt{n}} + \frac{(1 + a_n x_0)^{1-\rho(2+\rho)/4}}{n^{\rho(2-\rho)/8}}\right)\right\}.$$

Using the following inequalities

$$\frac{1}{\sqrt{2\pi}(1+x)}e^{-x^2/2} \le 1 - \Phi(x) \le \frac{1}{\sqrt{\pi}(1+x)}e^{-x^2/2}, \quad x \ge 0,$$
(5.2)

and the fact that  $a_n \to \infty$  and  $a_n/\sqrt{n} \to 0$ , we obtain

$$\limsup_{n \to \infty} \frac{1}{a_n^2} \ln \mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \leq -\frac{x_0^2}{2} \leq -\inf_{x \in \overline{B}} \frac{x^2}{2},$$

which gives (5.1).

Next, we prove that

$$\liminf_{n \to \infty} \frac{1}{a_n^2} \ln \mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \ge -\inf_{x \in B^o} \frac{x^2}{2}.$$
(5.3)

When  $B^o = \emptyset$ , the last inequality is obvious, with  $-\inf_{x \in \emptyset} \frac{x^2}{2} = -\infty$ . Thus, we may assume that  $B^o \neq \emptyset$ . Since  $B^o$  is an open set, for any given small  $\varepsilon_1 > 0$ , there exists an  $x_0 \in B^o$ , such that

$$0 < \frac{x_0^2}{2} \le \inf_{x \in B^o} \frac{x^2}{2} + \varepsilon_1.$$

Again by the fact that  $B^o$  is an open set, for  $x_0 \in B^o$  and all small enough  $\varepsilon_2 \in (0, |x_0|]$ , it holds  $(x_0 - \varepsilon_2, x_0 + \varepsilon_2] \subset B^o$ . Without loss of generality, we may assume that  $x_0 > 0$ . Clearly, we have

$$\mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \geq \mathbb{P}\left(M_{n_0,n} \in (a_n(x_0 - \varepsilon_2), a_n(x_0 + \varepsilon_2)]\right) \\
= \mathbb{P}\left(M_{n_0,n} \geq a_n(x_0 - \varepsilon_2)\right) - \mathbb{P}\left(M_{n_0,n} \geq a_n(x_0 + \varepsilon_2)\right).$$
(5.4)

Again by Theorem 2.1, it is easy to see that for  $a_n \to \infty$  and  $a_n = o(\sqrt{n})$ ,

$$\lim_{n \to \infty} \frac{\mathbb{P}(M_{n_0,n} \ge a_n(x_0 + \varepsilon_2))}{\mathbb{P}(M_{n_0,n} \ge a_n(x_0 - \varepsilon_2))} = 0$$

From (5.4), by the last line and Theorem 2.1, it holds for all n large enough and  $a_n = o(\sqrt{n})$ ,

$$\mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \geq \frac{1}{2} \mathbb{P}\left(M_{n_0,n} \geq a_n(x_0 - \varepsilon_2)\right) \\
\geq \frac{1}{2} \left(1 - \Phi\left(a_n(x_0 - \varepsilon_2)\right)\right) \exp\left\{-C_{\rho}\left(\frac{(a_n x_0)^{2+\rho}}{n^{\rho/2}} + \frac{\ln n}{\sqrt{n}} + \frac{(1 + a_n x_0)^{1-\rho(2+\rho)/4}}{n^{\rho(2-\rho)/8}}\right)\right\}.$$

Using (5.2) and the fact that  $a_n \to \infty$  and  $a_n/\sqrt{n} \to 0$ , after some calculations, we get

$$\liminf_{n \to \infty} \frac{1}{a_n^2} \ln \mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \ge -\frac{1}{2}(x_0 - \varepsilon_2)^2.$$

Letting  $\varepsilon_2 \to 0$ , we deduce that

$$\liminf_{n \to \infty} \frac{1}{a_n^2} \ln \mathbb{P}\left(\frac{M_{n_0,n}}{a_n} \in B\right) \geq -\frac{x_0^2}{2} \geq -\inf_{x \in B^o} \frac{x^2}{2} - \varepsilon_1.$$

Since that  $\varepsilon_1$  can be arbitrarily small, we get (5.3). Combining (5.1) and (5.3) together, we complete the proof of Corollary 2.1.

## 6. Proof of Theorem 2.2

In the proof of Theorem 2.2, we will make use of the following lemma (cf. Theorem 2.1 of Fan [8]), which gives exact Berry-Esseen's bounds for martingales.

**Lemma 6.1.** Assume the conditions of Lemma 4.1.

[i] If  $\rho \in (0, 1)$ , then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\sum_{k=1}^{n} \eta_k \le x) - \Phi(x) \right| \le C_\rho \Big(\gamma_n^\rho + \delta_n\Big).$$
(6.1)

**[ii]** If  $\rho = 1$ , then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(\sum_{k=1}^{n} \eta_k \le x) - \Phi(x) \right| \le C\left(\gamma_n |\log \gamma_n| + \delta_n\right).$$
(6.2)

Recall the martingale differences  $(\eta_k, \mathcal{F}_k)_{k=1,...,n}$  defined in the proof of Theorem 2.1. Then  $\eta_k$  satisfies the conditions (4.1) and (4.2) with  $\delta_n = 0$  and  $\gamma_n = (C_\rho (v^\rho + \mathbb{E}Z_1^{2+\rho}/v^2))^{1/\rho}/\sqrt{n}v$ . Clearly, it holds  $H_{n_0,n} = \sum_{k=1}^n \eta_k$ . Applying Lemma 6.1 to  $(\eta_k, \mathcal{F}_k)_{k=1,...,n}$ , we obtain the desired inequalities.  $\Box$ 

#### 7. Proof of Theorem 2.3

Define the generating function of  $Z_n$  as  $f_n(s) = \mathbb{E}s^{Z_n}$ ,  $|s| \leq 1$ . We have the following lemma, see Athreya [1].

**Lemma 7.1.** If  $p_1 > 0$  then

$$\lim_{n \to \infty} \frac{f_n(s)}{p_1^n} = \sum_{k=1}^{\infty} q_k s^k,\tag{7.1}$$

where  $(q_k, k \ge 1)$  is defined via the generating function  $Q(s) = \sum_{k=1}^{\infty} q_k s^k, 0 \le s < 1$ , the unique solution of the functional equation

$$Q(f(s)) = p_1 Q(s),$$
 where  $f(s) = \sum_{j=1}^{\infty} p_j s^j, \ 0 \le s < 1,$ 

subject to

$$Q(0) = 0,$$
  $Q(1) = \infty,$   $Q(s) < \infty \text{ for } 0 \le s < 1.$ 

Lemma 7.2. It holds

$$\mathbb{P}(Z_n \le n) \le C_1 \exp\{-nc_0\}.$$
(7.2)

*Proof.* When  $p_1 > 0$ , using Markov's inequality and Lemma 7.1, we have for  $s_0 = \frac{1+p_1}{2} \in (0,1)$ ,

$$\sum_{k=1}^{n-1} \mathbb{P}(Z_n = k) I_k(x) \leq \mathbb{P}(Z_n \leq n) = \mathbb{P}(s_0^{Z_n} \geq s_0^n) \leq s_0^{-n} f_n(s_0)$$
$$\leq C(\frac{p_1}{s_0})^n Q(s_0)$$
$$= C_1 \exp\{-nc_0\},$$
(7.3)

where  $C_1 = CQ(s_0)$  and  $c_0 = \ln(s_0/p_1)$ . Notice that  $s_0 \in (p_1, 1)$ , thus  $c_0 > 0$ . Recall that  $p_0 = 0$ . When  $p_1 = 0$ , we have  $Z_n \ge 2^n$ , and (7.2) holds obviously for all n large enought.

In the proof of Theorem 2.3, we need the following technical lemma of Jing, Shao and Wang [10], which gives a self-normalized Cramér moderate deviation result for iid random variables.

**Lemma 7.3.** Let  $(Y_i)_{i\geq 1}$  be a sequence of iid and centered random variables. Assume that  $\mathbb{E}|Y_1|^{2+\rho} < \infty$  for some  $\rho \in (0,1]$ . Let  $S_n = \sum_{i=1}^n Y_i$  and  $V_n^2 = \sum_{i=1}^n Y_i^2$ . Then

$$\left| \ln \frac{\mathbb{P}(S_n / V_n \ge x)}{1 - \Phi(x)} \right| \le C_{\rho} \frac{1 + x^{2+\rho}}{n^{\rho/2}}$$
(7.4)

uniformly for  $0 \le x = o(\sqrt{n})$  as  $n \to \infty$ .

## 7.1. Proof of the theorem

Now, we are in a position to prove Theorem 2.3. Recalling that  $Z_n$  is the number of individuals of the BPRE in generation n, and  $X_{n,i}$ ,  $1 \le i \le Z_n$ , is the number of the offspring of the *i*th individual in generation n. Denote

$$V(n)^{2} = \sum_{i=1}^{Z_{n}} (X_{n,i} - m)^{2}, \qquad \bar{X}(n) = \frac{Z_{n+1}}{Z_{n}}, \qquad \bar{Y}_{n} = \frac{Z_{n+1}}{n}.$$
(7.5)

Then we have

$$\sum_{i=1}^{Z_n} (X_{n,i} - \bar{X}(n))^2 = \sum_{i=1}^{Z_n} \left( (X_{n,i} - m) + (m - \bar{X}(n))^2 - V(n)^2 - Z_n(m - \bar{X}(n))^2 \right).$$
(7.6)

By (7.6), it is easy to see that  $T_n$  can be rewritten as follows:

$$T_n = \frac{\sum_{i=1}^{Z_n} (X_{n,i} - m)}{\sqrt{V(n)^2 - Z_n (m - \bar{X}(n))^2}}.$$

Notice that  $X_{n,i}$ ,  $1 \le i \le Z_n$ , have the same distribution as  $Z_1$ , and that  $Z_n$  is independent of  $\xi_n$ . By the total probability formula and the independence of  $Z_n$  and  $(X_{n,i})_{i\ge 1}$ , we obtain, for all  $x \ge 0$ ,

$$\mathbb{P}(T_n \ge x) = \mathbb{P}\left(\sum_{i=1}^{Z_n} (X_{n,i} - m) \ge x\sqrt{V(n)^2 - Z_n(m - \bar{X}(n))^2}\right) \\
= \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k) \mathbb{P}\left(\sum_{i=1}^k (X_{n,i} - m) \ge x\sqrt{V_k^2 - k(m - \bar{Y}_k)^2}\right) \\
= \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k) \mathbb{P}\left(\sum_{i=1}^k (X_{n,i} - m) \ge x\sqrt{V_k^2 - k(m - \bar{Y}_k)^2}\right) \\
=: \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k) I_k(x).$$
(7.7)

By Lemma 7.1, we have

$$\sum_{k=1}^{n-1} \mathbb{P}(Z_n = k) I_k(x) \le \mathbb{P}(Z_n \le n) \le C_1 \exp\{-nc_0\},$$
(7.8)

For  $k \ge n$ , the tail probability  $I_k(x)$  can be divided into two parts: for all  $x \ge 0$ ,

$$I_{k}(x) = \mathbb{P}\left(\sum_{i=1}^{k} (X_{n,i} - m) \ge x\sqrt{V_{k}^{2} - k(m - \bar{Y}_{k})^{2}}, \ k(m - \bar{Y}_{k})^{2} < V_{k}^{2}(1 + x^{\rho})/k^{\rho/2}\right) \\ + \mathbb{P}\left(\sum_{i=1}^{k} (X_{n,i} - m) \ge x\sqrt{V_{k}^{2} - k(m - \bar{Y}_{k})^{2}}, \ k(m - \bar{Y}_{k})^{2} \ge V_{k}^{2}(1 + x^{\rho})/k^{\rho/2}\right) \\ \le \mathbb{P}\left(\sum_{i=1}^{k} (X_{n,i} - m) \ge xV_{k}\sqrt{1 - (1 + x^{\rho})/k^{\rho/2}}\right) + \mathbb{P}\left(k(m - \bar{Y}_{k})^{2} \ge V_{k}^{2}(1 + x^{\rho})/k^{\rho/2}\right) \\ =: I_{k,1}(x) + I_{k,2}(x).$$
(7.9)

We first give an estimation for  $I_{k,1}(x)$ . Notice that  $(X_{n,i}-m)_{i\geq 1}$  are conditional independent with respect to  $\xi_n$ . When  $k \geq n$ , by self-normalized moderate deviations for centered random variables  $(X_{n,i}-m)_{i\geq 1}$  (cf. Lemma 7.3), we have, for all  $0 \leq x = o(\sqrt{n})$ ,

$$\left| \ln \frac{I_{k,1}(x)}{1 - \Phi\left(x\sqrt{1 - (1 + x^{\rho})/k^{\rho/2}}\right)} \right| \le C_2 \frac{1 + x^{2+\rho}}{k^{\rho/2}} \le C_2 \frac{1 + x^{2+\rho}}{n^{\rho/2}}.$$

Using (5.2), we deduce that, for all  $x \ge 0$  and  $0 \le \varepsilon \le 1$ ,

$$\frac{1 - \Phi\left(x\sqrt{1 - \varepsilon}\right)}{1 - \Phi\left(x\right)} = 1 + \frac{\int_{x\sqrt{1 - \varepsilon}}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^{2}/2} dt}{1 - \Phi\left(x\right)} \le 1 + \frac{\frac{1}{\sqrt{2\pi}} e^{-x^{2}(1 - \varepsilon)/2} x\varepsilon}{\frac{1}{\sqrt{2\pi}(1 + x)} e^{-x^{2}/2}} \le 1 + C(1 + x^{2})\varepsilon e^{x^{2}\varepsilon/2} \le \exp\left\{C(1 + x^{2})\varepsilon\right\}.$$
(7.10)

Using the last inequality, we get, for all  $k \ge n$  and all  $0 \le x = o(\sqrt{n})$ ,

$$I_{k,1}(x) \leq \left(1 - \Phi(x\sqrt{1 - (1 + x^{\rho})/k^{\rho/2}})\right) \exp\left\{C_2 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\}$$
  
$$\leq \left(1 - \Phi(x)\right) \exp\left\{C_2 \frac{1 + x^{2+\rho}}{n^{\rho/2}} + C(1 + x^2) \frac{1 + x^{\rho}}{k^{\rho/2}}\right\}$$
  
$$\leq \left(1 - \Phi(x)\right) \exp\left\{C_3 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\},$$
(7.11)

which gives an estimation for  $I_{k,1}(x)$ .

Next we give an estimation for  $I_{k,2}(x)$ . Notice that

$$k(m - \bar{Y}_k)^2 = \frac{1}{k} \left( \sum_{i=1}^k (X_{n,i} - m) \right)^2.$$

Thus, we have

$$I_{k,2}(x) = \mathbb{P}\left(\left(\sum_{i=1}^{k} (X_{n,i} - m)\right)^2 \ge k^{1 - \rho/2} V_k^2 (1 + x^{\rho})\right)$$
$$= \mathbb{P}\left(\left|\sum_{i=1}^{k} (X_{n,i} - m)\right| \ge V_k \sqrt{k^{1 - \rho/2} (1 + x^{\rho})}\right).$$

Applying (7.4) to the centered random variables  $(\pm (X_{n,i} - m))_{i \ge 1}$ , we obtain, for all  $k \ge n$  and all  $0 \le x = o(\sqrt{n})$ ,

$$\begin{split} I_{k,2}(x) &\leq 2\Big(1 - \Phi(\sqrt{k^{1-\rho/2}(1+x^{\rho})})\Big) \exp\left\{C\frac{1 + (\sqrt{k^{1-\rho/2}(1+x^{\rho})})^{2+\rho}}{\sqrt{k}}\right\} \\ &\leq 2\exp\left\{-\frac{1}{4}k^{1-\rho/2}(1+x^{\rho})\right\}, \end{split}$$

where the last line follows by (5.2). Again by (5.2), we have, for all  $k \ge n$  and all  $0 \le x = o(\sqrt{n})$ ,

$$I_{k,2}(x) \leq 2 \exp\left\{-\frac{1}{4}n^{1-\rho/2}(1+x^{\rho})\right\} \\ \leq C\frac{1+x}{n}\left(1-\Phi(x)\right),$$
(7.12)

which gives an estimation for  $I_{k,2}(x)$ . Combining (7.9), (7.11) and (7.12) together, we get, for all  $k \ge n$  and all  $0 \le x = o(\sqrt{n})$ ,

$$I_k(x) \leq \left(1 - \Phi(x)\right) \exp\left\{C_4 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\}.$$
 (7.13)

Returning to (7.7), using the last inequality and (7.8), we deduce that, for all  $0 \le x = o(\sqrt{n})$ ,

$$\mathbb{P}(T_n \ge x) \le \sum_{k=1}^{n-1} \mathbb{P}(Z_n = k) I_k(x) + \sum_{k=n}^{\infty} \mathbb{P}(Z_n = k) I_k(x) \\
\le C_1 \exp\{-C_0 n\} + \sum_{k=n}^{\infty} \mathbb{P}(Z_n = k) \left(1 - \Phi(x)\right) \exp\left\{C_4 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} \\
\le C_1 \exp\{-C_0 n\} + \sum_{k=1}^{\infty} \mathbb{P}(Z_n = k) \left(1 - \Phi(x)\right) \exp\left\{C_4 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} \\
= C_1 \exp\{-C_0 n\} + \left(1 - \Phi(x)\right) \exp\left\{C_4 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} \\
\le \left(1 - \Phi(x)\right) \exp\left\{C_5 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\},$$
(7.14)

where the last line follows by (5.2).

Next, we consider the lower bound of  $\mathbb{P}(T_n \ge x)$ . For  $I_k(x)$ , we have the following estimation: for all  $k \ge n$  and all  $0 \le x = o(\sqrt{n})$ ,

$$I_{k}(x) = \mathbb{P}\left(\sum_{i=1}^{k} (X_{n,i} - m) \ge x \sqrt{V_{k}^{2} - k(m - \bar{Y}_{k})^{2}}\right)$$
  
$$\ge \mathbb{P}\left(\sum_{i=1}^{k} (X_{n,i} - m) \ge x V_{k}\right).$$
(7.15)

When  $k \ge n$ , by self-normalized moderate deviations for iid random variables (cf. Lemma 7.3), we have, for all  $0 \le x = o(\sqrt{n})$ ,

$$I_k(x) \ge \left(1 - \Phi(x)\right) \exp\left\{-C_6 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\}.$$

Returning to (7.7), we deduce that, for all  $0 \le x = o(\sqrt{n})$ ,

$$\mathbb{P}(T_n \ge x) \ge \sum_{k=n}^{\infty} \mathbb{P}(Z_n = k) I_k(x)$$
  
$$\ge (1 - \Phi(x)) \exp\left\{-C_6 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} \sum_{k=n}^{\infty} \mathbb{P}(Z_n = k)$$
  
$$\ge (1 - \Phi(x)) \exp\left\{-C_6 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} (1 - \mathbb{P}(Z_n \le n))$$

Using Lemma 7.2, we get, for all  $0 \le x = o(\sqrt{n})$ ,

$$\mathbb{P}(T_n \ge x) \ge (1 - \Phi(x)) \exp\left\{-C_6 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\} (1 - C_1 e^{-C_0 n}) \\
\ge (1 - \Phi(x)) \exp\left\{-C_9 \frac{1 + x^{2+\rho}}{n^{\rho/2}}\right\}.$$
(7.16)

Combining (7.14) and (7.16) together, we obtain the desired inequality.

Applying (2.9) to  $(m - X_{n,k})_{k \ge 1}$ , we find that (2.9) remains valid when  $\frac{\mathbb{P}(T_n \ge x)}{1 - \Phi(x)}$  is replaced by  $\frac{\mathbb{P}(T_n \le -x)}{\Phi(-x)}$ . This completes the proof of Theorem 2.3.

# 8. Proof of Corollary 2.4

Clearly, it holds

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(T_n \le x) - \Phi(x) \right| \\ \le \sup_{x > n^{\rho/(8+4\rho)}} \left| \mathbb{P}(T_n \le x) - \Phi(x) \right| + \sup_{0 \le x \le n^{\rho/(8+4\rho)}} \left| \mathbb{P}(T_n \le x) - \Phi(x) \right| \\ + \sup_{-n^{\rho/(8+4\rho)} \le x \le 0} \left| \mathbb{P}(T_n \le x) - \Phi(x) \right| + \sup_{x < -n^{\rho/(8+4\rho)}} \left| \mathbb{P}(T_n \le x) - \Phi(x) \right| \\ =: TH_1 + TH_2 + TH_3 + TH_4.$$
(8.1)

By Theorem 2.3 and (5.2), it is easy to see that

$$TH_{1} = \sup_{x > n^{\rho/(8+4\rho)}} \left| \mathbb{P}(T_{n} > x) - (1 - \Phi(x)) \right|$$

$$\leq \sup_{x > n^{\rho/(8+4\rho)}} \mathbb{P}(T_{n} > x) + \sup_{x > n^{\rho/(8+4\rho)}} (1 - \Phi(x))$$

$$\leq \mathbb{P}(T_{n} > n^{\rho/(8+4\rho)}) + (1 - \Phi(n^{\rho/(8+4\rho)}))$$

$$\leq (1 - \Phi(n^{\rho/(8+4\rho)}))e^{C} + \exp\left\{-\frac{1}{2}(n^{\rho/(8+4\rho)})^{2}\right\}$$

$$\leq \frac{C_{1}}{n^{\rho/2}}$$

and

$$TH_{4} \leq \sup_{x < -n^{\rho/(8+4\rho)}} \mathbb{P}(T_{n} \leq x) + \sup_{x < -n^{\rho/(8+4\rho)}} \Phi(x)$$
  
$$\leq \mathbb{P}(T_{n} \leq -n^{\rho/(8+4\rho)}) + \Phi(-n^{\rho/(8+4\rho)})$$
  
$$\leq \Phi(-n^{\rho/(8+4\rho)})e^{C} + \exp\left\{-\frac{1}{2}(n^{\rho/(8+4\rho)})^{2}\right\}$$
  
$$\leq \frac{C_{2}}{n^{\rho/2}}.$$

By Theorem 2.3 and the inequality  $|e^x - 1| \le |x|e^{|x|}$ , we have

$$TH_{2} = \sup_{0 \le x \le n^{\rho/(8+4\rho)}} \left| \mathbb{P}(T_{n} > x) - (1 - \Phi(x)) \right|$$
  
$$\leq \sup_{0 \le x \le n^{\rho/(8+4\rho)}} (1 - \Phi(x)) \left| e^{C(1 + x^{2+\rho})/n^{\rho/2}} - 1 \right|$$
  
$$\leq \frac{C}{n^{\rho/2}} \sup_{0 \le x \le n^{\rho/(8+4\rho)}} (1 - \Phi(x)) (1 + x^{2+\rho}) e^{C(1 + x^{2+\rho})/n^{\rho/2}}$$
  
$$\leq \frac{C_{3}}{n^{\rho/2}}$$

and, similarly,

$$TH_{3} = \sup_{-n^{1/8} \le x \le 0} \left| \mathbb{P}(T_{n} \le x) - \Phi(x) \right|$$
  
$$\leq \sup_{-n^{1/8} \le x \le 0} \Phi(x) \left| e^{C(1+|x|^{3})(\ln n)/\sqrt{n}} - 1 \right|$$
  
$$\leq \frac{C_{4}}{n^{\rho/2}}.$$

Applying the bounds of  $TH_1, TH_2, TH_3$  and  $TH_4$  to (8.1), we obtain the desired inequality. This completes the proof of Corollary 2.4.

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