# Self-normalized Cramér moderate deviations for a supercritical Galton-Watson process 

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#### Abstract

Let $\left(Z_{n}\right)_{n \geq 0}$ be a supercritical Galton-Watson process. Consider the Lotka-Nagaev estimator for the offspring mean. In this paper, we establish self-normalized Cramér type moderate deviations and Berry-Esseen's bounds for the Lotka-Nagaev estimator. The results are believed to be optimal or near optimal.


Keywords: Lotka-Nagaev estimator; offspring mean; Self-normalized processes; Cramér moderate deviations; Berry-Esseen's bounds

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## 1. Introduction

A Galton-Watson process can be described as follows

$$
\begin{equation*}
Z_{0}=1, \quad Z_{n+1}=\sum_{i=1}^{Z_{n}} X_{n, i}, \quad \text { for } n \geq 0 \tag{1.1}
\end{equation*}
$$

where $X_{n, i}$ is the offspring number of the $i$-th individual of the generation $n$. Moreover, the random variables $\left(X_{n, i}\right)_{i \geq 1}$ are independent of each other with common distribution law

$$
\begin{equation*}
\mathbb{P}\left(X_{n, i}=k\right)=p_{k}, \quad k \in \mathbb{N} \tag{1.2}
\end{equation*}
$$

and are also independent of $Z_{n}$.
An important task in statistical inference of Galton-Watson processes is to estimate the average offspring number of an individual $m$, usually termed the offspring mean. Clearly, it holds

$$
m=\mathbb{E} Z_{1}=\mathbb{E} X_{n, i}=\sum_{k=0}^{\infty} k p_{k}
$$

Denote $v$ the standard variance of $Z_{1}$, that is

$$
\begin{equation*}
v^{2}=\mathbb{E}\left(Z_{1}-m\right)^{2} \tag{1.3}
\end{equation*}
$$

To avoid triviality, assume that $v>0$. For estimation of the offspring mean $m$, the Lotka-Nagaev [11, 12] estimator $Z_{n+1} / Z_{n}$ plays an important role. For the Galton-Watson processes, Athreya [1]
has established large deviations for the normalized Lotka-Nagaev estimator (see also Chu [4] for selfnormalized large deviations); Ney and Vidyashankar [14, 15] obtained sharp rate estimates for the large deviation behavior of the Lotka-Nagaev estimator; Bercu and Touati [2] proved an exponential inequalities for the Lotka-Nagaev estimator via self-normalized martingale method. The main purpose of this paper is to establish self-normalized Cramér moderate deviations for the Lotka-Nagaev estimator $Z_{n+1} / Z_{n}$ for the Galton-Watson processes.

The paper is organized as follows. In Section 2, we present Cramér moderate deviations for the self-normalized Lotka-Nagaev estimator, provided that $\left(Z_{n}\right)_{n \geq 0}$ or $\left(X_{n, i}\right)_{1 \leq i \leq Z_{n}}$ can be observed. In Section 3, we present some applications of our results in statistics. The rest sections devote to the proofs of theorems.

## 2. Main results

## 2.1. $\left(Z_{k}\right)_{k \geq 0}$ can be observed

Assume that the total populations $\left(Z_{k}\right)_{k \geq 0}$ of all generations can be observed. For any $n_{0} \geq 0$, we define

$$
\begin{equation*}
M_{n_{0}, n}=\frac{\sum_{k=n_{0}}^{n_{0}+n-1} \sqrt{Z_{k}}\left(\frac{Z_{k+1}}{Z_{k}}-m\right)}{\sqrt{\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}\left(\frac{Z_{k+1}}{Z_{k}}-m\right)^{2}}} . \tag{2.1}
\end{equation*}
$$

We assume that the set of extinction of the process $\left(Z_{k}\right)_{k \geq 0}$ is negligible with respect to the annealed law $\mathbb{P}$. Then $M_{n_{0}, n}$ is well defined $\mathbb{P}$-a.s. As $\left(Z_{k}\right)_{k=n_{0}, \ldots, n_{0}+n}$ can be observed, $M_{n_{0}, n}$ can be regarded as a time type self-normalized process for the Lotka-Nagaev estimator $Z_{k+1} / Z_{k}$. The following theorem gives a self-normalized Cramér moderate deviation result for the Galton-Watson processes.
Theorem 2.1. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$.
[i] If $\rho \in(0,1)$, then for all $x \in[0, o(\sqrt{n}))$,

$$
\begin{equation*}
\left|\ln \frac{\mathbb{P}\left(M_{n_{0}, n} \geq x\right)}{1-\Phi(x)}\right| \leq C_{\rho}\left(\frac{x^{2+\rho}}{n^{\rho / 2}}+\frac{(1+x)^{1-\rho(2+\rho) / 4}}{n^{\rho(2-\rho) / 8}}\right), \tag{2.2}
\end{equation*}
$$

where $C_{\rho}$ depends only on the constants $\rho, v$ and $\mathbb{E} Z_{1}^{2+\rho}$.
[ii] If $\rho=1$, then for all $x \in[0, o(\sqrt{n}))$,

$$
\begin{equation*}
\left|\ln \frac{\mathbb{P}\left(M_{n_{0}, n} \geq x\right)}{1-\Phi(x)}\right| \leq C\left(\frac{x^{3}}{\sqrt{n}}+\frac{\ln n}{\sqrt{n}}+\frac{(1+x)^{1 / 4}}{n^{1 / 8}}\right), \tag{2.3}
\end{equation*}
$$

where $C$ depends only on the constants $v$ and $\mathbb{E} Z_{1}^{3}$.
In particular, the inequalities (2.2) and (2.3) together implies that

$$
\begin{equation*}
\frac{\mathbb{P}\left(M_{n_{0}, n} \geq x\right)}{1-\Phi(x)}=1+o(1) \tag{2.4}
\end{equation*}
$$

uniformly for $n_{0} \in \mathbb{N}$ and for $x \in\left[0, o\left(n^{\rho /(4+2 \rho)}\right)\right)$ as $n \rightarrow \infty$. Moreover, the same inequalities remain valid when $\frac{\mathbb{P}\left(M_{n_{0}, n} \geq x\right)}{1-\Phi(x)}$ is replaced by $\frac{\mathbb{P}\left(M_{n_{0}, n} \leq-x\right)}{\Phi(-x)}$.

Notice that $C_{\rho}$ and $C$ do not depend on $n_{0}$. Thus (2.4) holds uniformity in $n_{0}$, which is of particular interesting in applications. For instance, due to the uniformity, in (2.4) we can take $n_{0}$ as a function of $n$.

Equality (2.4) implies that $\mathbb{P}\left(M_{n_{0}, n} \leq x\right) \rightarrow \Phi(x)$ as $n$ tends to $\infty$. Thus Theorem 2.1 implies the central limit theory for $M_{n_{0}, n}$. Moreover, equality (2.4) states that the relative error of normal approximation for $M_{n_{0}, n}$ tends to zero uniformly for $x \in\left[0, o\left(n^{\rho /(4+2 \rho)}\right)\right)$ as $n \rightarrow \infty$.

Theorem 2.1 implies the following moderate deviation principle (MDP) result for the time type self-normalized Lotka-Nagaev estimator.

Corollary 2.1. Assume the conditions of Theorem 2.1. Let $\left(a_{n}\right)_{n \geq 1}$ be any sequence of real numbers satisfying $a_{n} \rightarrow \infty$ and $a_{n} / \sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$. Then for each Borel set $B$,

$$
\begin{equation*}
-\inf _{x \in B^{\circ}} \frac{x^{2}}{2} \leq \liminf _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \leq \limsup _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \leq-\inf _{x \in \bar{B}} \frac{x^{2}}{2}, \tag{2.5}
\end{equation*}
$$

where $B^{o}$ and $\bar{B}$ denote the interior and the closure of $B$, respectively.
Remark 2.1. From (2.2) and (2.3), it is easy to derive the following Berry-Esseen bound for the self-normalized Lotka-Nagaev estimator:

$$
\begin{equation*}
\left|\mathbb{P}\left(M_{n_{0}, n} \leq x\right)-\Phi(x)\right| \leq \frac{C_{\rho}}{n^{\rho(2-\rho) / 8}}, \tag{2.6}
\end{equation*}
$$

where $C_{\rho}$ depends only on the constants $\rho, v$ and $\mathbb{E} Z_{1}^{2+\rho}$. When $\rho>1$, by the self-normalized BerryEsseen bound for martingales in Fan and Shao [7], we can get a Berry-Esseen bound of order $n^{-\frac{\rho}{6+2 \rho}}$.

The last remark gives a self-normalized Berry-Esseen bound for the Lotka-Nagaev estimator, while the next theorem presents a normalized Berry-Esseen bound for the Lotka-Nagaev estimator. Denote

$$
H_{n_{0}, n}=\frac{1}{\sqrt{n} v} \sum_{k=n_{0}}^{n_{0}+n-1} \sqrt{Z_{k}}\left(\frac{Z_{k+1}}{Z_{k}}-m\right)
$$

Notice that the random variables $\left(X_{k, i}\right)_{1 \leq i \leq Z_{k}}$ have the same distribution as $Z_{1}$, and that $\left(X_{k, i}\right)_{1 \leq i \leq Z_{k}}$ are independent of $Z_{k}$. Then for the Galton-Watson processes, it holds

$$
\mathbb{E}\left[\left(Z_{k+1}-m Z_{k}\right)^{2} \mid Z_{k}\right]=\mathbb{E}\left[\left(\sum_{i=1}^{Z_{k}}\left(X_{k, i}-m\right)\right)^{2} \mid Z_{k}\right]=Z_{k} v^{2}
$$

It is easy to see that $H_{n_{0}, n}=\sum_{k=n_{0}}^{n_{0}+n-1} \frac{1}{\sqrt{n v^{2} / Z_{k}}}\left(\frac{Z_{k+1}}{Z_{k}}-m\right)$. Thus $H_{n_{0}, n}$ can be regarded as a normalized process for the Lotka-Nagaev estimator $Z_{k+1} / Z_{k}$. We have the following normalized Berry-Esseen bounds for the Galton-Watson processes.

Theorem 2.2. Assume the conditions of Theorem 2.1 are satisfied.
[i] If $\rho \in(0,1)$, then

$$
\begin{equation*}
\sup _{x \in \mathbb{R}}\left|\mathbb{P}\left(H_{n_{0}, n} \leq x\right)-\Phi(x)\right| \leq \frac{C_{\rho}}{n^{\rho / 2}}, \tag{2.7}
\end{equation*}
$$

where $C_{\rho}$ depends only on $\rho, v$ and $\mathbb{E} Z_{1}^{2+\rho}$.
[ii] If $\rho=1$, then

$$
\begin{equation*}
\sup _{x \in \mathbb{R}}\left|\mathbb{P}\left(H_{n_{0}, n} \leq x\right)-\Phi(x)\right| \leq C \frac{\ln n}{\sqrt{n}}, \tag{2.8}
\end{equation*}
$$

where $C$ depends only on $v$ and $\mathbb{E} Z_{1}^{3}$.
Moreover, the same inequalities remain valid when $H_{n_{0}, n}$ is replaced by $-H_{n_{0}, n}$.
The convergence rates of (2.7) and (2.8) are same to the best possible convergence rates of the Berry-Esseen bounds for martingales, see Theorem 2.1 of Fan [8] and its comment. Notice that $H_{n_{0}, n}$ is a martingale with respect to the natural filtration.

## 2.2. $\left(X_{n, i}\right)_{1 \leq i \leq Z_{n}}$ can be observed for some $n$

Assume that the offspring numbers $\left(X_{n, i}\right)_{1 \leq i \leq Z_{n}}$ of each individual in some generation $n$ can be observed. Denote

$$
T_{n}=\frac{Z_{n}\left(\frac{Z_{n+1}}{Z_{n}}-m\right)}{\sqrt{\sum_{i=1}^{Z_{n}}\left(X_{n, i}-\frac{Z_{n+1}}{Z_{n}}\right)^{2}}}
$$

the space type self-normalized process for the Lotka-Nagaev estimator $Z_{n+1} / Z_{n}$. The following theorem gives a Cramér moderate deviation result for the space type self-normalized Lotka-Nagaev estimator $T_{n}$.

Theorem 2.3. Assume that $p_{0}=0$ and $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$. Then

$$
\begin{equation*}
\left|\ln \frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}\right|=O\left(\frac{1+x^{2+\rho}}{n^{\rho / 2}}\right) \tag{2.9}
\end{equation*}
$$

uniformly for $x \in[0, o(\sqrt{n}))$ as $n \rightarrow \infty$. Moreover, the same equality remains valid when $\frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}$ is replaced by $\frac{\mathbb{P}\left(T_{n} \leq-x\right)}{\Phi(-x)}$.

The condition $p_{0}=0$ means that each individual has at least one offspring. Moreover, it also implies that $Z_{n} \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Then by law of large numbers, we have $\frac{Z_{n+1}}{Z_{n}}$ tends to $m$ a.s. as $n \rightarrow \infty$.

For the Galton-Watson processes, we refer to [1] for closely related results of Theorem 2.3, where Athreya has established a precise large deviation rate for the Lotka-Nagaev estimator $Z_{n+1} / Z_{n}$.

Using the inequality $\left|e^{x}-1\right| \leq e^{C}|x|$ valid for $|x| \leq C$, from Theorem 2.3, we obtain the following estimation for the relative error of normal approximation.

Corollary 2.2. Assume the conditions of Theorem 2.3. Then

$$
\begin{equation*}
\frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}=1+O\left(\frac{1+x^{2+\rho}}{n^{\rho / 2}}\right) \tag{2.10}
\end{equation*}
$$

uniformly for $x \in\left[0, O\left(n^{\rho /(4+2 \rho)}\right)\right)$ as $n \rightarrow \infty$. In particular, it implies that

$$
\begin{equation*}
\frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}=1+o(1) \tag{2.11}
\end{equation*}
$$

uniformly for $x \in\left[0, o\left(n^{\rho /(4+2 \rho)}\right)\right)$ as $n \rightarrow \infty$. Moreover, the same equalities remain valid when $T_{n}$ is replaced by $-T_{n}$.

Inequality (2.11) implies that the relative error of normal approximation for $T_{n}$ tends to zero uniformly for $x \in\left[0, o\left(n^{\rho /(4+2 \rho)}\right)\right)$. Clearly, the range of validity for (2.11) coincides with the selfnormalized Cramér moderate deviation result of Shao [17] for iid random variables.

By an argument similar to the proof of Corollary 2.1, Theorem 2.3 also implies the following self-normalized MDP result.

Corollary 2.3. Assume the conditions of Theorem 2.3. Let $\left(a_{n}\right)_{n \geq 1}$ be any sequence of real numbers satisfying $a_{n} \rightarrow \infty$ and $a_{n} / \sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$. Then for each Borel set $B$,

$$
\begin{equation*}
-\inf _{x \in B^{\circ}} \frac{x^{2}}{2} \leq \liminf _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{T_{n}}{a_{n}} \in B\right) \leq \limsup _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{T_{n}}{a_{n}} \in B\right) \leq-\inf _{x \in \bar{B}} \frac{x^{2}}{2} \tag{2.12}
\end{equation*}
$$

where $B^{o}$ and $\bar{B}$ denote the interior and the closure of $B$, respectively.
From Theorem 2.3, we get the following self-normalized Berry-Esseen bound for $T_{n}$.
Corollary 2.4. Assume the conditions of Theorem 2.3. Then

$$
\begin{equation*}
\sup _{x \in \mathbb{R}}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right| \leq \frac{C_{\rho}}{n^{\rho / 2}}, \tag{2.13}
\end{equation*}
$$

where $C_{\rho}$ does not depend on $n$.
Clearly, the convergence rate for the Berry-Esseen bound of Corollary 2.4 is consistent with the classical case of iid random variables (cf. Bentkus and Götze [3]), and therefore it is optimal under the stated conditions.

Remark 2.2. Following the proof of Theorem 2.3, the results (2.9)-(2.13) remain true when $T_{n}$ is replaced by

$$
\widetilde{T}_{n}=\frac{Z_{n}\left(Z_{n+1} / Z_{n}-m\right)}{\sqrt{\sum_{i=1}^{Z_{n}}\left(X_{n, i}-m\right)^{2}}} .
$$

## 3. Applications

Cramér moderate deviations certainly have a lot of applications in statistics.

## 3.1. p-value for hypothesis testing

Self-normalized Cramér moderate deviations can be applied to hypothesis testing of $m$ for the Galton-Watson processes. When $\left(Z_{k}\right)_{k=n_{0}, \ldots, n_{0}+n}$ can be observed, we can make use of Theorem 2.1 to estimate $p$-value. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $0<\rho \leq 1$, and that $m>1$. Let $\left(z_{k}\right)_{k=n_{0}, \ldots, n_{0}+n}$ be an observation of $\left(Z_{k}\right)_{k=n_{0}, \ldots, n_{0}+n}$. In order to estimate the offspring mean $m$, we can make use of the Harris estimator [2] given by

$$
\widehat{m}_{n}=\frac{\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k+1}}{\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}} .
$$

Then observation for the Harris estimator is

$$
\widehat{m}_{n}=\frac{\sum_{k=n_{0}}^{n_{0}+n-1} z_{k+1}}{\sum_{k=n_{0}}^{n_{0}+n-1} z_{k}} .
$$

By Theorem 2.1, it is easy to see that

$$
\begin{equation*}
\frac{\mathbb{P}\left(M_{n_{0}, n} \geq x\right)}{1-\Phi(x)}=1+o(1) \quad \text { and } \quad \frac{\mathbb{P}\left(M_{n_{0}, n} \leq-x\right)}{1-\Phi(x)}=1+o(1) \tag{3.1}
\end{equation*}
$$

uniformly for $x \in\left[0, o\left(n^{\rho /(4+2 \rho)}\right)\right)$. Notice that $1-\Phi(x)=\Phi(-x)$. Thus, by (3.1), the probability $\mathbb{P}\left(M_{n_{0}, n}>\left|\widetilde{m}_{n}\right|\right)$ is almost equal to $2 \Phi\left(-\left|\widetilde{m}_{n}\right|\right)$, where

$$
\widetilde{m}_{n}=\frac{\sum_{k=n_{0}}^{n_{0}+n-1} \sqrt{z_{k}}\left(z_{k+1} / z_{k}-\widehat{m}_{n}\right)}{\sqrt{\sum_{k=n_{0}}^{n_{0}+n-1} z_{k}\left(z_{k+1} / z_{k}-\widehat{m}_{n}\right)^{2}}} .
$$

3.2. Construction of confidence intervals
3.2.1. The data $\left(Z_{k}\right)_{k \geq 0}$ can be observed

Cramér moderate deviations can be also applied to construction of confidence intervals of $m$. We make use of Theorem 2.1 to construct confidence intervals.

Proposition 3.1. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$. Let $\kappa_{n} \in(0,1)$. Assume that

$$
\begin{equation*}
\left|\ln \kappa_{n}\right|=o\left(n^{\rho /(2+\rho)}\right) . \tag{3.2}
\end{equation*}
$$

Let

$$
\begin{aligned}
& a_{n_{0}, n}=\left(\sum_{k=n_{0}}^{n_{0}+n-1} \sqrt{Z_{k}}\right)^{2}-\left(\Phi^{-1}\left(1-\kappa_{n} / 2\right)\right)^{2} \sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}, \\
& b_{n_{0}, n}=2\left(\Phi^{-1}\left(1-\kappa_{n} / 2\right)\right)^{2} \sum_{k=n_{0}}^{n_{0}+n-1} Z_{k+1}-2\left(\sum_{k=n_{0}}^{n_{0}+n-1} \frac{Z_{k+1}}{\sqrt{Z_{k}}}\right)\left(\sum_{k=n_{0}}^{n_{0}+n-1} \sqrt{Z_{k}}\right), \\
& c_{n_{0}, n}=\left(\sum_{k=n_{0}}^{n_{0}+n-1} \frac{Z_{k+1}}{\sqrt{Z_{k}}}\right)^{2}-\left(\Phi^{-1}\left(1-\kappa_{n} / 2\right)\right)^{2} \sum_{k=n_{0}}^{n_{0}+n-1} \frac{Z_{k+1}^{2}}{Z_{k}} .
\end{aligned}
$$

Then $\left[A_{n_{0}, n}, B_{n_{0}, n}\right]$, with

$$
A_{n_{0}, n}=\frac{-b_{n_{0}, n}-\sqrt{b_{n_{0}, n}^{2}-4 a_{n_{0}, n} c_{n_{0}, n}}}{2 a_{n_{0}, n}}
$$

and

$$
B_{n_{0}, n}=\frac{-b_{n_{0}, n}+\sqrt{b_{n_{0}, n}^{2}-4 a_{n_{0}, n} c_{n_{0}, n}}}{2 a_{n_{0}, n}}
$$

is a $1-\kappa_{n}$ confidence interval for $m$, for $n$ large enough.
Proof. Notice that $1-\Phi(x)=\Phi(-x)$. Theorem 2.1 implies that

$$
\begin{equation*}
\frac{\mathbb{P}\left(M_{n_{0}, n} \geq x\right)}{1-\Phi(x)}=1+o(1) \quad \text { and } \quad \frac{\mathbb{P}\left(M_{n_{0}, n} \leq-x\right)}{1-\Phi(x)}=1+o(1) \tag{3.3}
\end{equation*}
$$

uniformly for $0 \leq x=o\left(n^{\rho /(4+2 \rho)}\right)$, see (2.4). When $\kappa_{n}$ satisfies the condition (3.2), the upper ( $\kappa_{n} / 2$ )th quantile of a standard normal distribution satisfies

$$
\Phi^{-1}\left(1-\kappa_{n} / 2\right)=O\left(\sqrt{\left|\ln \kappa_{n}\right|}\right)
$$

which is of order $o\left(n^{\rho /(4+2 \rho)}\right)$. Then applying (3.3) to the last equality, we complete the proof of Proposition 3.1. Notice that $A_{n_{0}, n}$ and $B_{n_{0}, n}$ are solutions of the following equation

$$
\frac{\sum_{k=n_{0}}^{n_{0}+n-1} \sqrt{Z_{k}}\left(Z_{k+1} / Z_{k}-x\right)}{\sqrt{\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}\left(Z_{k+1} / Z_{k}-x\right)^{2}}}=\Phi^{-1}\left(1-\kappa_{n} / 2\right) .
$$

This completes the proof of Proposition 3.1.
3.2.2. The data $\left(X_{n, i}\right)_{1 \leq i \leq Z_{n}}$ can be observed

When $\left(X_{n, i}\right)_{1 \leq i \leq Z_{n}}$ can be observed, we can make use of Corollary 2.2 to construct confidence intervals.

Proposition 3.2. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$. Let $\kappa_{n} \in(0,1)$. Assume that

$$
\begin{equation*}
\left|\ln \kappa_{n}\right|=o\left(n^{\rho /(2+\rho)}\right) . \tag{3.4}
\end{equation*}
$$

Let

$$
\Delta_{n}=\frac{\Phi^{-1}\left(1-\kappa_{n} / 2\right)}{Z_{n}} \sqrt{\sum_{i=1}^{Z_{n}}\left(X_{n, i}-\frac{Z_{n+1}}{Z_{n}}\right)^{2}}
$$

Then $\left[A_{n}, B_{n}\right]$, with

$$
A_{n}=\frac{Z_{n+1}}{Z_{n}}-\Delta_{n} \quad \text { and } \quad B_{n}=\frac{Z_{n+1}}{Z_{n}}+\Delta_{n}
$$

is a $1-\kappa_{n}$ confidence interval for $m$, for $n$ large enough.

Proof. Corollary 2.2 implies that

$$
\begin{equation*}
\frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}=1+o(1) \quad \text { and } \quad \frac{\mathbb{P}\left(T_{n} \leq-x\right)}{1-\Phi(x)}=1+o(1) \tag{3.5}
\end{equation*}
$$

uniformly for $0 \leq x=o\left(n^{\rho /(4+2 \rho)}\right)$. When $\kappa_{n}$ satisfies the condition (3.2), the upper ( $\left.\kappa_{n} / 2\right)$ th quantile of a standard normal distribution satisfies $\Phi^{-1}\left(1-\kappa_{n} / 2\right)=O\left(\sqrt{\left|\ln \kappa_{n}\right|}\right)$, which is of order $o\left(n^{\rho /(4+2 \rho)}\right)$. Then applying (3.5) to the last equality, we complete the proof of Proposition 3.2.

When the risk probability $\kappa_{n}$ goes to 0 , we have the following more general result.
Proposition 3.3. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$. Let $\kappa_{n} \in(0,1)$ such that $k_{n} \rightarrow 0$. Assume that

$$
\begin{equation*}
\left|\ln \kappa_{n}\right|=o(\sqrt{n}) . \tag{3.6}
\end{equation*}
$$

Let

$$
\Delta_{n}=\frac{\sqrt{2\left|\ln \left(\kappa_{n} / 2\right)\right|}}{Z_{n}} \sqrt{\sum_{i=1}^{Z_{n}}\left(X_{n, i}-\frac{Z_{n+1}}{Z_{n}}\right)^{2}}
$$

Then $\left[A_{n}, B_{n}\right]$, with

$$
A_{n}=\frac{Z_{n+1}}{Z_{n}}-\Delta_{n} \quad \text { and } \quad B_{n}=\frac{Z_{n+1}}{Z_{n}}+\Delta_{n}
$$

is a $1-\kappa_{n}$ confidence interval for $m$, for $n$ large enough.
Proof. By Theorem 2.3, we have

$$
\begin{equation*}
\frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}=\exp \left\{\theta C \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \quad \text { and } \quad \frac{\mathbb{P}\left(T_{n} \leq-x\right)}{1-\Phi(x)}=\exp \left\{\theta C \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \tag{3.7}
\end{equation*}
$$

uniformly for $0 \leq x=o(\sqrt{n})$, where $\theta \in[-1,1]$. Notice that

$$
1-\Phi\left(x_{n}\right) \sim \frac{1}{x_{n} \sqrt{2 \pi}} e^{-x_{n}^{2} / 2}=\exp \left\{-\frac{x_{n}^{2}}{2}\left(1+\frac{2}{x_{n}^{2}} \ln \left(x_{n} \sqrt{2 \pi}\right)\right)\right\}, x_{n} \rightarrow \infty
$$

Since $k_{n} \rightarrow 0$, the last line implies that the upper $\left(\kappa_{n} / 2\right)$ th quantile of the distribution

$$
1-(1-\Phi(x)) \exp \left\{\theta C \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\}
$$

converges to $\sqrt{2\left|\ln \left(\kappa_{n} / 2\right)\right|}$, which is of order $o(\sqrt{n})$ as $n \rightarrow \infty$. Then applying (3.7) to $T_{n}$, we complete the proof of Proposition 3.3.

### 3.2.3. The parameter $v^{2}$ is known

When $v^{2}$ is known, we can apply normalized Berry-Esseen bounds (cf. Theorem 2.2) to construct confidence intervals.
Proposition 3.4. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$. Let $\kappa_{n} \in(0,1)$. Assume that

$$
\begin{equation*}
\left|\ln \kappa_{n}\right|=o(\log n) . \tag{3.8}
\end{equation*}
$$

Then $\left[A_{n}, B_{n}\right]$, with

$$
A_{n}=\frac{\sum_{k=n_{0}}^{n_{0}+n} Z_{k+1} / \sqrt{Z_{k}}-\sqrt{n} v \Phi^{-1}\left(1-\kappa_{n} / 2\right)}{\sum_{k=n_{0}}^{n_{0}+n} \sqrt{Z_{k}}}
$$

and

$$
B_{n}=\frac{\sum_{k=n_{0}}^{n_{0}+n} Z_{k+1} / \sqrt{Z_{k}}+\sqrt{n} v \Phi^{-1}\left(1-\kappa_{n} / 2\right)}{\sum_{k=n_{0}}^{n_{0}+n} \sqrt{Z_{k}}}
$$

is a $1-\kappa_{n}$ confidence interval for $m$, for $n$ large enough.
Proof. Theorem 2.2 implies that

$$
\begin{equation*}
\frac{\mathbb{P}\left(H_{n_{0}, n} \geq x\right)}{1-\Phi(x)}=1+o(1) \quad \text { and } \quad \frac{\mathbb{P}\left(H_{n_{0}, n} \leq-x\right)}{1-\Phi(x)}=1+o(1) \tag{3.9}
\end{equation*}
$$

uniformly for $0 \leq x=o(\sqrt{\log n})$. The upper $\left(\kappa_{n} / 2\right)$ th quantile of a standard normal distribution satisfies

$$
\Phi^{-1}\left(1-\kappa_{n} / 2\right)=O\left(\sqrt{\left|\ln \kappa_{n}\right|}\right)
$$

which, by (3.8), is of order $o(\sqrt{\log n})$. Proposition 3.4 follows from applying (3.9) to $H_{n_{0}, n}$.

### 3.3. An infectious disease model

An infectious disease model $\left(Z_{n}\right)_{n \geq 0}$ may be described as follows:

$$
\begin{equation*}
Z_{0}=1, \quad Z_{n+1}=Z_{n}+\sum_{i=1}^{Z_{n}} Y_{n, i}, \quad \text { for } n \geq 0 \tag{3.10}
\end{equation*}
$$

where $Z_{n}$ stands for the total population of patients with infectious disease at time $n$, and $Y_{n, i}$ is the number of patients infected by the $i$-th individual of $Z_{n}$ in a unit time (for instance, one day). Moreover, we assume that the random variables $\left(Y_{n, i}\right)_{i \geq 1}$ are iid random variables with common distribution law

$$
\begin{equation*}
\mathbb{P}\left(Y_{n, i}=k\right)=p_{k}, \quad k \in \mathbb{N}, \tag{3.11}
\end{equation*}
$$

and are also independent to $Z_{n}$. Denote by $r$ the average number of patients infected by an individual patient in a unite time, that is

$$
r=\mathbb{E} Y_{n, i}=\sum_{k=0}^{\infty} k p_{k} .
$$

Denote by $v$ the standard variance of $Y_{n, i}, n, i \geq 1$, then $v$ is also the standard variance of $Z_{1}$, that is

$$
v^{2}=\mathbb{E}\left(Z_{1}-m\right)^{2}
$$

To avid triviality, assume that $v>0$. We are interested in the estimation of $r$.

Proposition 3.5. Assume that $\mathbb{E} Z_{1}^{2+\rho}<\infty$ for some $\rho \in(0,1]$. Let $\kappa_{n} \in(0,1)$. Assume that

$$
\begin{equation*}
\left|\ln \kappa_{n}\right|=o\left(n^{\rho /(2+\rho)}\right) \tag{3.12}
\end{equation*}
$$

Let $A_{n_{0}, n}$ and $B_{n_{0}, n}$ be defined in Proposition 3.1. Then $\left[A_{n_{0}, n}-1, B_{n_{0}, n}-1\right]$ is a $1-\kappa_{n}$ confidence interval for $r$, for $n$ large enough.

Proof. It is easy to see that (3.10) can be rewritten in the form of (1.1), with $X_{n, i}=1+Y_{n, i}$. Thus, we have $m=1+r$. Then Proposition 3.5 follows by Proposition 3.1.

## 4. Proof of Theorem 2.1

In the proof of Theorem 2.1, we will make use of the following lemma (cf. Corollary 2.3 of Fan et al. [9]), which gives self-normalized Cramér moderate deviations for martingales.

Lemma 4.1. Let $\left(\eta_{k}, \mathcal{F}_{k}\right)_{k=1, \ldots, n}$ be a finite sequence of martingale differences. Assume that there exist a constant $\rho \in(0,1]$ and numbers $\gamma_{n}>0$ and $\delta_{n} \geq 0$ satisfying $\gamma_{n}, \delta_{n} \rightarrow 0$ such that for all $1 \leq i \leq n$,

$$
\begin{equation*}
\mathbb{E}\left[\left|\eta_{k}\right|^{2+\rho} \mid \mathcal{F}_{k-1}\right] \leq \gamma_{n}^{\rho} \mathbb{E}\left[\eta_{k}^{2} \mid \mathcal{F}_{k-1}\right] \tag{4.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\sum_{k=1}^{n} \mathbb{E}\left[\eta_{k}^{2} \mid \mathcal{F}_{k-1}\right]-1\right\|_{\infty} \leq \delta_{n}^{2} \quad \text { a.s. } \tag{4.2}
\end{equation*}
$$

Denote

$$
V_{n}=\frac{\sum_{k=1}^{n} \eta_{k}}{\sqrt{\sum_{k=1}^{n} \eta_{k}^{2}}}
$$

and

$$
\widehat{\gamma}_{n}(x, \rho)=\frac{\gamma_{n}^{\rho(2-\rho) / 4}}{1+x^{\rho(2+\rho) / 4}}
$$

[i] If $\rho \in(0,1)$, then for all $0 \leq x=o\left(\gamma_{n}^{-1}\right)$,

$$
\begin{equation*}
\left|\ln \frac{\mathbb{P}\left(V_{n} \geq x\right)}{1-\Phi(x)}\right| \leq C_{\rho}\left(x^{2+\rho} \gamma_{n}^{\rho}+x^{2} \delta_{n}^{2}+(1+x)\left(\delta_{n}+\widehat{\gamma}_{n}(x, \rho)\right)\right) . \tag{4.3}
\end{equation*}
$$

[ii] If $\rho=1$, then for all $0 \leq x=o\left(\gamma_{n}^{-1}\right)$,

$$
\begin{equation*}
\left|\ln \frac{\mathbb{P}\left(V_{n} \geq x\right)}{1-\Phi(x)}\right| \leq C\left(x^{3} \gamma_{n}+x^{2} \delta_{n}^{2}+(1+x)\left(\delta_{n}+\gamma_{n}\left|\ln \gamma_{n}\right|+\widehat{\gamma}_{n}(x, 1)\right)\right) . \tag{4.4}
\end{equation*}
$$

Now, we are in position to prove Theorem 2.1. Denote

$$
\hat{\xi}_{k+1}=\sqrt{Z_{k}}\left(Z_{k+1} / Z_{k}-m\right)
$$

$\mathfrak{F}_{n_{0}}=\{\emptyset, \Omega\}$ and $\mathfrak{F}_{k+1}=\sigma\left\{Z_{i}: n_{0} \leq i \leq k+1\right\}$ for all $k \geq n_{0}$. Notice that $X_{k, i}$ is independent of $Z_{k}$. Then it is easy to verify that

$$
\begin{align*}
\mathbb{E}\left[\hat{\xi}_{k+1} \mid \mathfrak{F}_{k}\right] & =Z_{k}^{-1 / 2} \mathbb{E}\left[Z_{k+1}-m Z_{k} \mid \mathfrak{F}_{k}\right]=Z_{k}^{-1 / 2} \sum_{i=1}^{Z_{k}} \mathbb{E}\left[X_{k, i}-m \mid \mathfrak{F}_{k}\right] \\
& =Z_{k}^{-1 / 2} \sum_{i=1}^{Z_{k}} \mathbb{E}\left[X_{k, i}-m\right] \\
& =0 . \tag{4.5}
\end{align*}
$$

Thus $\left(\hat{\xi}_{k}, \mathfrak{F}_{k}\right)_{k=n_{0}+1, \ldots, n_{0}+n}$ is a finite sequence of martingale differences. Notice that $X_{k, i}-m, i \geq 1$, are centered and independent random variables. Thus, the following equalities hold

$$
\begin{align*}
\sum_{k=n_{0}}^{n_{0}+n-1} \mathbb{E}\left[\hat{\xi}_{k+1}^{2} \mid \mathfrak{F}_{k}\right] & =\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}^{-1} \mathbb{E}\left[\left(Z_{k+1}-m Z_{k}\right)^{2} \mid \widetilde{\mathfrak{F}}_{k}\right]=\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}^{-1} \mathbb{E}\left[\left(\sum_{i=1}^{Z_{k}}\left(X_{k, i}-m\right)\right)^{2} \mid \mathfrak{F}_{k}\right] \\
& =\sum_{k=n_{0}}^{n_{0}+n-1} Z_{k}^{-1} Z_{k} \mathbb{E}\left[\left(X_{k, i}-m\right)^{2}\right] \\
& =n v^{2} \tag{4.6}
\end{align*}
$$

Moreover, it is easy to see that

$$
\begin{align*}
\mathbb{E}\left[\left|\hat{\xi}_{k+1}\right|^{2+\rho} \mid \widetilde{\mathfrak{F}}_{k}\right] & =Z_{k}^{-1-\rho / 2} \mathbb{E}\left[\left|Z_{k+1}-m Z_{k}\right|^{2+\rho} \mid \mathfrak{F}_{k}\right] \\
& =Z_{k}^{-1-\rho / 2} \mathbb{E}\left[\left|\sum_{i=1}^{Z_{k}}\left(X_{k, i}-m\right)\right|^{2+\rho} \mid \widetilde{\mathfrak{F}}_{k}\right] . \tag{4.7}
\end{align*}
$$

By Rosenthal's inequality, we have

$$
\begin{aligned}
\mathbb{E}\left[\left|\sum_{i=1}^{Z_{k}}\left(X_{k, i}-m\right)\right|^{2+\rho} \mid \mathfrak{F}_{k}\right] & \leq C_{\rho}^{\prime}\left(\left(\sum_{i=1}^{Z_{k}} \mathbb{E}\left(X_{k, i}-m\right)^{2}\right)^{1+\rho / 2}+\sum_{i=1}^{Z_{k}} \mathbb{E}\left|X_{k, i}-m\right|^{2+\rho}\right) \\
& \leq C_{\rho}^{\prime}\left(Z_{k}^{1+\rho / 2} v^{2+\rho}+Z_{k} \mathbb{E}\left|Z_{1}-m\right|^{2+\rho}\right) .
\end{aligned}
$$

Since the set of extinction of the process $\left(Z_{k}\right)_{k \geq 0}$ is negligible with respect to the annealed law $\mathbb{P}$, we have $Z_{k} \geq 1$ for any $k$. From (4.7), by the last inequality and the fact $Z_{k} \geq 1$, we deduce that

$$
\begin{align*}
\mathbb{E}\left[\left|\hat{\xi}_{k+1}\right|^{2+\rho} \mid \mathfrak{F}_{k}\right] & \leq C_{\rho}^{\prime}\left(v^{\rho}+\mathbb{E}\left|Z_{1}-m\right|^{2+\rho} / v^{2}\right) v^{2} \\
& =C_{\rho}^{\prime}\left(v^{\rho}+\mathbb{E}\left|Z_{1}-m\right|^{2+\rho} / v^{2}\right) \mathbb{E}\left[\hat{\xi}_{k+1}^{2} \mid \mathfrak{F}_{k}\right] \\
& =C_{\rho}\left(v^{\rho}+\mathbb{E} Z_{1}^{2+\rho} / v^{2}\right) \mathbb{E}\left[\hat{\xi}_{k+1}^{2} \mid \mathfrak{F}_{k}\right] . \tag{4.8}
\end{align*}
$$

Let $\eta_{k}=\hat{\xi}_{n_{0}+k} / \sqrt{n} v$ and $\mathcal{F}_{k}=\mathfrak{F}_{n_{0}+k}$. Then $\left(\eta_{k}, \mathcal{F}_{k}\right)_{k=1, \ldots, n}$ is a martingale difference sequences and satisfies the conditions (4.1) and (4.2) with $\delta_{n}=0$ and $\gamma_{n}=\left(C_{\rho}\left(v^{\rho}+\mathbb{E} Z_{1}^{2+\rho} / v^{2}\right)\right)^{1 / \rho} / \sqrt{n} v$. Clearly, it holds

$$
M_{n_{0}, n}=\frac{\sum_{k=1}^{n} \eta_{k}}{\sqrt{\sum_{k=1}^{n} \eta_{k}^{2}}} .
$$

Applying Lemma 4.1 to $\left(\eta_{k}, \mathcal{F}_{k}\right)_{k=1, \ldots, n}$, we obtain the desired inequalities.

## 5. Proof of Corollary 2.1

We first show that for any Borel set $B \subset \mathbb{R}$,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \leq-\inf _{x \in \bar{B}} \frac{x^{2}}{2} . \tag{5.1}
\end{equation*}
$$

When $B=\emptyset$, the last inequality is obvious, with $-\inf _{x \in \emptyset} \frac{x^{2}}{2}=-\infty$. Thus, we may assume that $B \neq \emptyset$. Let $x_{0}=\inf _{x \in B}|x|$. Clearly, we have $x_{0} \geq \inf _{x \in \bar{B}}|x|$. Then, by Theorem 2.1, it follows that for $a_{n}=o(\sqrt{n})$,

$$
\begin{aligned}
\mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) & \leq \mathbb{P}\left(\left|M_{n_{0}, n}\right| \geq a_{n} x_{0}\right) \\
& \leq 2\left(1-\Phi\left(a_{n} x_{0}\right)\right) \exp \left\{C_{\rho}\left(\frac{\left(a_{n} x_{0}\right)^{2+\rho}}{n^{\rho / 2}}+\frac{\ln n}{\sqrt{n}}+\frac{\left(1+a_{n} x_{0}\right)^{1-\rho(2+\rho) / 4}}{n^{\rho(2-\rho) / 8}}\right)\right\}
\end{aligned}
$$

Using the following inequalities

$$
\begin{equation*}
\frac{1}{\sqrt{2 \pi}(1+x)} e^{-x^{2} / 2} \leq 1-\Phi(x) \leq \frac{1}{\sqrt{\pi}(1+x)} e^{-x^{2} / 2}, \quad x \geq 0 \tag{5.2}
\end{equation*}
$$

and the fact that $a_{n} \rightarrow \infty$ and $a_{n} / \sqrt{n} \rightarrow 0$, we obtain

$$
\limsup _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \leq-\frac{x_{0}^{2}}{2} \leq-\inf _{x \in \bar{B}} \frac{x^{2}}{2}
$$

which gives (5.1).
Next, we prove that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \geq-\inf _{x \in B^{o}} \frac{x^{2}}{2} . \tag{5.3}
\end{equation*}
$$

When $B^{o}=\emptyset$, the last inequality is obvious, with $-\inf _{x \in \emptyset} \frac{x^{2}}{2}=-\infty$. Thus, we may assume that $B^{o} \neq \emptyset$. Since $B^{o}$ is an open set, for any given small $\varepsilon_{1}>0$, there exists an $x_{0} \in B^{o}$, such that

$$
0<\frac{x_{0}^{2}}{2} \leq \inf _{x \in B^{\circ}} \frac{x^{2}}{2}+\varepsilon_{1}
$$

Again by the fact that $B^{o}$ is an open set, for $x_{0} \in B^{o}$ and all small enough $\varepsilon_{2} \in\left(0,\left|x_{0}\right|\right]$, it holds $\left(x_{0}-\varepsilon_{2}, x_{0}+\varepsilon_{2}\right] \subset B^{o}$. Without loss of generality, we may assume that $x_{0}>0$. Clearly, we have

$$
\begin{align*}
\mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) & \geq \mathbb{P}\left(M_{n_{0}, n} \in\left(a_{n}\left(x_{0}-\varepsilon_{2}\right), a_{n}\left(x_{0}+\varepsilon_{2}\right)\right]\right) \\
& =\mathbb{P}\left(M_{n_{0}, n} \geq a_{n}\left(x_{0}-\varepsilon_{2}\right)\right)-\mathbb{P}\left(M_{n_{0}, n} \geq a_{n}\left(x_{0}+\varepsilon_{2}\right)\right) \tag{5.4}
\end{align*}
$$

Again by Theorem 2.1, it is easy to see that for $a_{n} \rightarrow \infty$ and $a_{n}=o(\sqrt{n})$,

$$
\lim _{n \rightarrow \infty} \frac{\mathbb{P}\left(M_{n_{0}, n} \geq a_{n}\left(x_{0}+\varepsilon_{2}\right)\right)}{\mathbb{P}\left(M_{n_{0}, n} \geq a_{n}\left(x_{0}-\varepsilon_{2}\right)\right)}=0
$$

From (5.4), by the last line and Theorem 2.1, it holds for all $n$ large enough and $a_{n}=o(\sqrt{n})$,

$$
\begin{aligned}
& \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \geq \frac{1}{2} \mathbb{P}\left(M_{n_{0}, n} \geq a_{n}\left(x_{0}-\varepsilon_{2}\right)\right) \\
& \quad \geq \frac{1}{2}\left(1-\Phi\left(a_{n}\left(x_{0}-\varepsilon_{2}\right)\right)\right) \exp \left\{-C_{\rho}\left(\frac{\left(a_{n} x_{0}\right)^{2+\rho}}{n^{\rho / 2}}+\frac{\ln n}{\sqrt{n}}+\frac{\left(1+a_{n} x_{0}\right)^{1-\rho(2+\rho) / 4}}{n^{\rho(2-\rho) / 8}}\right)\right\} .
\end{aligned}
$$

Using (5.2) and the fact that $a_{n} \rightarrow \infty$ and $a_{n} / \sqrt{n} \rightarrow 0$, after some calculations, we get

$$
\liminf _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \geq-\frac{1}{2}\left(x_{0}-\varepsilon_{2}\right)^{2} .
$$

Letting $\varepsilon_{2} \rightarrow 0$, we deduce that

$$
\liminf _{n \rightarrow \infty} \frac{1}{a_{n}^{2}} \ln \mathbb{P}\left(\frac{M_{n_{0}, n}}{a_{n}} \in B\right) \geq-\frac{x_{0}^{2}}{2} \geq-\inf _{x \in B^{o}} \frac{x^{2}}{2}-\varepsilon_{1}
$$

Since that $\varepsilon_{1}$ can be arbitrarily small, we get (5.3). Combining (5.1) and (5.3) together, we complete the proof of Corollary 2.1.

## 6. Proof of Theorem 2.2

In the proof of Theorem 2.2, we will make use of the following lemma (cf. Theorem 2.1 of Fan [8]), which gives exact Berry-Esseen's bounds for martingales.

Lemma 6.1. Assume the conditions of Lemma 4.1.
[i] If $\rho \in(0,1)$, then

$$
\begin{equation*}
\sup _{x \in \mathbb{R}}\left|\mathbb{P}\left(\sum_{k=1}^{n} \eta_{k} \leq x\right)-\Phi(x)\right| \leq C_{\rho}\left(\gamma_{n}^{\rho}+\delta_{n}\right) . \tag{6.1}
\end{equation*}
$$

[ii] If $\rho=1$, then

$$
\begin{equation*}
\sup _{x \in \mathbb{R}}\left|\mathbb{P}\left(\sum_{k=1}^{n} \eta_{k} \leq x\right)-\Phi(x)\right| \leq C\left(\gamma_{n}\left|\log \gamma_{n}\right|+\delta_{n}\right) \tag{6.2}
\end{equation*}
$$

Recall the martingale differences $\left(\eta_{k}, \mathcal{F}_{k}\right)_{k=1, \ldots, n}$ defined in the proof of Theorem 2.1. Then $\eta_{k}$ satisfies the conditions (4.1) and (4.2) with $\delta_{n}=0$ and $\gamma_{n}=\left(C_{\rho}\left(v^{\rho}+\mathbb{E} Z_{1}^{2+\rho} / v^{2}\right)\right)^{1 / \rho} / \sqrt{n} v$. Clearly, it holds $H_{n_{0}, n}=\sum_{k=1}^{n} \eta_{k}$. Applying Lemma 6.1 to $\left(\eta_{k}, \mathcal{F}_{k}\right)_{k=1, \ldots, n}$, we obtain the desired inequalities.

## 7. Proof of Theorem 2.3

Define the generating function of $Z_{n}$ as $f_{n}(s)=\mathbb{E} s^{Z_{n}},|s| \leq 1$. We have the following lemma, see Athreya [1].

Lemma 7.1. If $p_{1}>0$ then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{f_{n}(s)}{p_{1}^{n}}=\sum_{k=1}^{\infty} q_{k} s^{k} \tag{7.1}
\end{equation*}
$$

where ( $q_{k}, k \geq 1$ ) is defined via the generating function $Q(s)=\sum_{k=1}^{\infty} q_{k} s^{k}, 0 \leq s<1$, the unique solution of the functional equation

$$
Q(f(s))=p_{1} Q(s), \quad \text { where } f(s)=\sum_{j=1}^{\infty} p_{j} s^{j}, 0 \leq s<1,
$$

subject to

$$
Q(0)=0, \quad Q(1)=\infty, \quad Q(s)<\infty \text { for } 0 \leq s<1 .
$$

Lemma 7.2. It holds

$$
\begin{equation*}
\mathbb{P}\left(Z_{n} \leq n\right) \leq C_{1} \exp \left\{-n c_{0}\right\} \tag{7.2}
\end{equation*}
$$

Proof. When $p_{1}>0$, using Markov's inequality and Lemma 7.1, we have for $s_{0}=\frac{1+p_{1}}{2} \in(0,1)$,

$$
\begin{align*}
\sum_{k=1}^{n-1} \mathbb{P}\left(Z_{n}=k\right) I_{k}(x) & \leq \mathbb{P}\left(Z_{n} \leq n\right)=\mathbb{P}\left(s_{0}^{Z_{n}} \geq s_{0}^{n}\right) \leq s_{0}^{-n} f_{n}\left(s_{0}\right) \\
& \leq C\left(\frac{p_{1}}{s_{0}}\right)^{n} Q\left(s_{0}\right) \\
& =C_{1} \exp \left\{-n c_{0}\right\} \tag{7.3}
\end{align*}
$$

where $C_{1}=C Q\left(s_{0}\right)$ and $c_{0}=\ln \left(s_{0} / p_{1}\right)$. Notice that $s_{0} \in\left(p_{1}, 1\right)$, thus $c_{0}>0$. Recall that $p_{0}=0$. When $p_{1}=0$, we have $Z_{n} \geq 2^{n}$, and (7.2) holds obviously for all $n$ large enought.

In the proof of Theorem 2.3, we need the following technical lemma of Jing, Shao and Wang [10], which gives a self-normalized Cramér moderate deviation result for iid random variables.

Lemma 7.3. Let $\left(Y_{i}\right)_{i \geq 1}$ be a sequence of iid and centered random variables. Assume that $\mathbb{E}\left|Y_{1}\right|^{2+\rho}<$ $\infty$ for some $\rho \in(0,1]$. Let $S_{n}=\sum_{i=1}^{n} Y_{i}$ and $V_{n}^{2}=\sum_{i=1}^{n} Y_{i}^{2}$. Then

$$
\begin{equation*}
\left|\ln \frac{\mathbb{P}\left(S_{n} / V_{n} \geq x\right)}{1-\Phi(x)}\right| \leq C_{\rho} \frac{1+x^{2+\rho}}{n^{\rho / 2}} \tag{7.4}
\end{equation*}
$$

uniformly for $0 \leq x=o(\sqrt{n})$ as $n \rightarrow \infty$.

### 7.1. Proof of the theorem

Now, we are in a position to prove Theorem 2.3. Recalling that $Z_{n}$ is the number of individuals of the BPRE in generation $n$, and $X_{n, i}, 1 \leq i \leq Z_{n}$, is the number of the offspring of the $i$ th individual in generation $n$. Denote

$$
\begin{equation*}
V(n)^{2}=\sum_{i=1}^{Z_{n}}\left(X_{n, i}-m\right)^{2}, \quad \bar{X}(n)=\frac{Z_{n+1}}{Z_{n}}, \quad \bar{Y}_{n}=\frac{Z_{n+1}}{n} . \tag{7.5}
\end{equation*}
$$

Then we have

$$
\begin{align*}
\sum_{i=1}^{Z_{n}}\left(X_{n, i}-\bar{X}(n)\right)^{2} & =\sum_{i=1}^{Z_{n}}\left(\left(X_{n, i}-m\right)+(m-\bar{X}(n))^{2}\right. \\
& =V(n)^{2}-Z_{n}(m-\bar{X}(n))^{2} \tag{7.6}
\end{align*}
$$

By (7.6), it is easy to see that $T_{n}$ can be rewritten as follows:

$$
T_{n}=\frac{\sum_{i=1}^{Z_{n}}\left(X_{n, i}-m\right)}{\sqrt{V(n)^{2}-Z_{n}(m-\bar{X}(n))^{2}}} .
$$

Notice that $X_{n, i}, 1 \leq i \leq Z_{n}$, have the same distribution as $Z_{1}$, and that $Z_{n}$ is independent of $\xi_{n}$. By the total probability formula and the independence of $Z_{n}$ and $\left(X_{n, i}\right)_{i \geq 1}$, we obtain, for all $x \geq 0$,

$$
\begin{align*}
\mathbb{P}\left(T_{n} \geq x\right) & =\mathbb{P}\left(\sum_{i=1}^{Z_{n}}\left(X_{n, i}-m\right) \geq x \sqrt{V(n)^{2}-Z_{n}(m-\bar{X}(n))^{2}}\right) \\
& =\sum_{k=1}^{\infty} \mathbb{P}\left(Z_{n}=k\right) \mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x \sqrt{V_{k}^{2}-k\left(m-\bar{Y}_{k}\right)^{2}}\right) \\
& =\sum_{k=1}^{\infty} \mathbb{P}\left(Z_{n}=k\right) \mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x \sqrt{V_{k}^{2}-k\left(m-\bar{Y}_{k}\right)^{2}}\right) \\
& =: \sum_{k=1}^{\infty} \mathbb{P}\left(Z_{n}=k\right) I_{k}(x) . \tag{7.7}
\end{align*}
$$

By Lemma 7.1, we have

$$
\begin{equation*}
\sum_{k=1}^{n-1} \mathbb{P}\left(Z_{n}=k\right) I_{k}(x) \leq \mathbb{P}\left(Z_{n} \leq n\right) \leq C_{1} \exp \left\{-n c_{0}\right\} \tag{7.8}
\end{equation*}
$$

For $k \geq n$, the tail probability $I_{k}(x)$ can be divided into two parts: for all $x \geq 0$,

$$
\begin{align*}
I_{k}(x)= & \mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x \sqrt{V_{k}^{2}-k\left(m-\bar{Y}_{k}\right)^{2}}, k\left(m-\bar{Y}_{k}\right)^{2}<V_{k}^{2}\left(1+x^{\rho}\right) / k^{\rho / 2}\right) \\
& +\mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x \sqrt{V_{k}^{2}-k\left(m-\bar{Y}_{k}\right)^{2}}, k\left(m-\bar{Y}_{k}\right)^{2} \geq V_{k}^{2}\left(1+x^{\rho}\right) / k^{\rho / 2}\right) \\
\leq & \mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x V_{k} \sqrt{1-\left(1+x^{\rho}\right) / k^{\rho / 2}}\right)+\mathbb{P}\left(k\left(m-\bar{Y}_{k}\right)^{2} \geq V_{k}^{2}\left(1+x^{\rho}\right) / k^{\rho / 2}\right) \\
= & I_{k, 1}(x)+I_{k, 2}(x) . \tag{7.9}
\end{align*}
$$

We first give an estimation for $I_{k, 1}(x)$. Notice that $\left(X_{n, i}-m\right)_{i \geq 1}$ are conditional independent with respect to $\xi_{n}$. When $k \geq n$, by self-normalized moderate deviations for centered random variables $\left(X_{n, i}-m\right)_{i \geq 1}(\mathrm{cf}$. Lemma 7.3), we have, for all $0 \leq x=o(\sqrt{n})$,

$$
\left|\ln \frac{I_{k, 1}(x)}{1-\Phi\left(x \sqrt{1-\left(1+x^{\rho}\right) / k^{\rho / 2}}\right)}\right| \leq C_{2} \frac{1+x^{2+\rho}}{k^{\rho / 2}} \leq C_{2} \frac{1+x^{2+\rho}}{n^{\rho / 2}} .
$$

Using (5.2), we deduce that, for all $x \geq 0$ and $0 \leq \varepsilon \leq 1$,

$$
\begin{align*}
\frac{1-\Phi(x \sqrt{1-\varepsilon})}{1-\Phi(x)} & =1+\frac{\int_{x \sqrt{1-\varepsilon}}^{x} \frac{1}{\sqrt{2 \pi}} e^{-t^{2} / 2} d t}{1-\Phi(x)} \leq 1+\frac{\frac{1}{\sqrt{2 \pi}} e^{-x^{2}(1-\varepsilon) / 2} x \varepsilon}{\frac{1}{\sqrt{2 \pi}(1+x)} e^{-x^{2} / 2}} \\
& \leq 1+C\left(1+x^{2}\right) \varepsilon e^{x^{2} \varepsilon / 2} \\
& \leq \exp \left\{C\left(1+x^{2}\right) \varepsilon\right\} . \tag{7.10}
\end{align*}
$$

Using the last inequality, we get, for all $k \geq n$ and all $0 \leq x=o(\sqrt{n})$,

$$
\begin{align*}
I_{k, 1}(x) & \leq\left(1-\Phi\left(x \sqrt{1-\left(1+x^{\rho}\right) / k^{\rho / 2}}\right)\right) \exp \left\{C_{2} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \\
& \leq(1-\Phi(x)) \exp \left\{C_{2} \frac{1+x^{2+\rho}}{n^{\rho / 2}}+C\left(1+x^{2}\right) \frac{1+x^{\rho}}{k^{\rho / 2}}\right\} \\
& \leq(1-\Phi(x)) \exp \left\{C_{3} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\}, \tag{7.11}
\end{align*}
$$

which gives an estimation for $I_{k, 1}(x)$.
Next we give an estimation for $I_{k, 2}(x)$. Notice that

$$
k\left(m-\bar{Y}_{k}\right)^{2}=\frac{1}{k}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right)\right)^{2} .
$$

Thus, we have

$$
\begin{aligned}
I_{k, 2}(x) & =\mathbb{P}\left(\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right)\right)^{2} \geq k^{1-\rho / 2} V_{k}^{2}\left(1+x^{\rho}\right)\right) \\
& =\mathbb{P}\left(\left|\sum_{i=1}^{k}\left(X_{n, i}-m\right)\right| \geq V_{k} \sqrt{k^{1-\rho / 2}\left(1+x^{\rho}\right)}\right)
\end{aligned}
$$

Applying (7.4) to the centered random variables $\left( \pm\left(X_{n, i}-m\right)\right)_{i \geq 1}$, we obtain, for all $k \geq n$ and all $0 \leq x=o(\sqrt{n})$,

$$
\begin{aligned}
I_{k, 2}(x) & \leq 2\left(1-\Phi\left(\sqrt{k^{1-\rho / 2}\left(1+x^{\rho}\right)}\right)\right) \exp \left\{C \frac{1+\left(\sqrt{k^{1-\rho / 2}\left(1+x^{\rho}\right)}\right)^{2+\rho}}{\sqrt{k}}\right\} \\
& \leq 2 \exp \left\{-\frac{1}{4} k^{1-\rho / 2}\left(1+x^{\rho}\right)\right\}
\end{aligned}
$$

where the last line follows by (5.2). Again by (5.2), we have, for all $k \geq n$ and all $0 \leq x=o(\sqrt{n})$,

$$
\begin{align*}
I_{k, 2}(x) & \leq 2 \exp \left\{-\frac{1}{4} n^{1-\rho / 2}\left(1+x^{\rho}\right)\right\} \\
& \leq C \frac{1+x}{n}(1-\Phi(x)), \tag{7.12}
\end{align*}
$$

which gives an estimation for $I_{k, 2}(x)$.
Combining (7.9), (7.11) and (7.12) together, we get, for all $k \geq n$ and all $0 \leq x=o(\sqrt{n})$,

$$
\begin{equation*}
I_{k}(x) \leq(1-\Phi(x)) \exp \left\{C_{4} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \tag{7.13}
\end{equation*}
$$

Returning to (7.7), using the last inequality and (7.8), we deduce that, for all $0 \leq x=o(\sqrt{n})$,

$$
\begin{align*}
\mathbb{P}\left(T_{n} \geq x\right) & \leq \sum_{k=1}^{n-1} \mathbb{P}\left(Z_{n}=k\right) I_{k}(x)+\sum_{k=n}^{\infty} \mathbb{P}\left(Z_{n}=k\right) I_{k}(x) \\
& \leq C_{1} \exp \left\{-C_{0} n\right\}+\sum_{k=n}^{\infty} \mathbb{P}\left(Z_{n}=k\right)(1-\Phi(x)) \exp \left\{C_{4} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \\
& \leq C_{1} \exp \left\{-C_{0} n\right\}+\sum_{k=1}^{\infty} \mathbb{P}\left(Z_{n}=k\right)(1-\Phi(x)) \exp \left\{C_{4} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \\
& =C_{1} \exp \left\{-C_{0} n\right\}+(1-\Phi(x)) \exp \left\{C_{4} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \\
& \leq(1-\Phi(x)) \exp \left\{C_{5} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \tag{7.14}
\end{align*}
$$

where the last line follows by (5.2).

Next, we consider the lower bound of $\mathbb{P}\left(T_{n} \geq x\right)$. For $I_{k}(x)$, we have the following estimation: for all $k \geq n$ and all $0 \leq x=o(\sqrt{n})$,

$$
\begin{align*}
I_{k}(x) & =\mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x \sqrt{V_{k}^{2}-k\left(m-\bar{Y}_{k}\right)^{2}}\right) \\
& \geq \mathbb{P}\left(\sum_{i=1}^{k}\left(X_{n, i}-m\right) \geq x V_{k}\right) . \tag{7.15}
\end{align*}
$$

When $k \geq n$, by self-normalized moderate deviations for iid random variables (cf. Lemma 7.3), we have, for all $0 \leq x=o(\sqrt{n})$,

$$
I_{k}(x) \geq(1-\Phi(x)) \exp \left\{-C_{6} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} .
$$

Returning to (7.7), we deduce that, for all $0 \leq x=o(\sqrt{n})$,

$$
\begin{aligned}
\mathbb{P}\left(T_{n} \geq x\right) & \geq \sum_{k=n}^{\infty} \mathbb{P}\left(Z_{n}=k\right) I_{k}(x) \\
& \geq(1-\Phi(x)) \exp \left\{-C_{6} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} \sum_{k=n}^{\infty} \mathbb{P}\left(Z_{n}=k\right) \\
& \geq(1-\Phi(x)) \exp \left\{-C_{6} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\}\left(1-\mathbb{P}\left(Z_{n} \leq n\right)\right) .
\end{aligned}
$$

Using Lemma 7.2 , we get, for all $0 \leq x=o(\sqrt{n})$,

$$
\begin{align*}
\mathbb{P}\left(T_{n} \geq x\right) & \geq(1-\Phi(x)) \exp \left\{-C_{6} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\}\left(1-C_{1} e^{-C_{0} n}\right) \\
& \geq(1-\Phi(x)) \exp \left\{-C_{9} \frac{1+x^{2+\rho}}{n^{\rho / 2}}\right\} . \tag{7.16}
\end{align*}
$$

Combining (7.14) and (7.16) together, we obtain the desired inequality.
Applying (2.9) to $\left(m-X_{n, k}\right)_{k \geq 1}$, we find that (2.9) remains valid when $\frac{\mathbb{P}\left(T_{n} \geq x\right)}{1-\Phi(x)}$ is replaced by $\frac{\mathbb{P}\left(T_{n} \leq-x\right)}{\Phi(-x)}$. This completes the proof of Theorem 2.3.

## 8. Proof of Corollary 2.4

Clearly, it holds

$$
\begin{align*}
& \sup _{x \in \mathbb{R}}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right| \\
& \leq \sup _{x>n^{\rho /(8+4 \rho)}}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right|+\sup _{0 \leq x \leq n^{\rho /(8+4 \rho)}}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right| \\
& \quad+\sup _{-n^{\rho /(8+4 \rho)} \leq x \leq 0}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right|+\sup _{x<-n^{\rho /(8+4 \rho)}}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right| \\
& =: T H_{1}+T H_{2}+T H_{3}+T H_{4} . \tag{8.1}
\end{align*}
$$

By Theorem 2.3 and (5.2), it is easy to see that

$$
\begin{aligned}
T H_{1} & =\sup _{x>n^{\rho /(8+4 \rho)}}\left|\mathbb{P}\left(T_{n}>x\right)-(1-\Phi(x))\right| \\
& \leq \sup _{x>n^{\rho /(8+4 \rho)}} \mathbb{P}\left(T_{n}>x\right)+\sup _{x>n^{\rho /(8+4 \rho)}}(1-\Phi(x)) \\
& \leq \mathbb{P}\left(T_{n}>n^{\rho /(8+4 \rho)}\right)+\left(1-\Phi\left(n^{\rho /(8+4 \rho)}\right)\right) \\
& \leq\left(1-\Phi\left(n^{\rho /(8+4 \rho)}\right)\right) e^{C}+\exp \left\{-\frac{1}{2}\left(n^{\rho /(8+4 \rho)}\right)^{2}\right\} \\
& \leq \frac{C_{1}}{n^{\rho / 2}}
\end{aligned}
$$

and

$$
\begin{aligned}
T H_{4} & \leq \sup _{x<-n^{\rho /(8+4 \rho)}} \mathbb{P}\left(T_{n} \leq x\right)+\sup _{x<-n^{\rho /(8+4 \rho)}} \Phi(x) \\
& \leq \mathbb{P}\left(T_{n} \leq-n^{\rho /(8+4 \rho)}\right)+\Phi\left(-n^{\rho /(8+4 \rho)}\right) \\
& \leq \Phi\left(-n^{\rho /(8+4 \rho)}\right) e^{C}+\exp \left\{-\frac{1}{2}\left(n^{\rho /(8+4 \rho)}\right)^{2}\right\} \\
& \leq \frac{C_{2}}{n^{\rho / 2}}
\end{aligned}
$$

By Theorem 2.3 and the inequality $\left|e^{x}-1\right| \leq|x| e^{|x|}$, we have

$$
\begin{aligned}
T H_{2} & =\sup _{0 \leq x \leq n^{\rho /(8+4 \rho)}}\left|\mathbb{P}\left(T_{n}>x\right)-(1-\Phi(x))\right| \\
& \leq \sup _{0 \leq x \leq n^{\rho /(8+4 \rho)}}(1-\Phi(x))\left|e^{C\left(1+x^{2+\rho}\right) / n^{\rho / 2}}-1\right| \\
& \leq \frac{C}{n^{\rho / 2}} \sup _{0 \leq x \leq n^{\rho /(8+4 \rho)}}(1-\Phi(x))\left(1+x^{2+\rho}\right) e^{C\left(1+x^{2+\rho}\right) / n^{\rho / 2}} \\
& \leq \frac{C_{3}}{n^{\rho / 2}}
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
T H_{3} & =\sup _{-n^{1 / 8} \leq x \leq 0}\left|\mathbb{P}\left(T_{n} \leq x\right)-\Phi(x)\right| \\
& \leq \sup _{-n^{1 / 8} \leq x \leq 0} \Phi(x)\left|e^{C\left(1+|x|^{3}\right)(\ln n) / \sqrt{n}}-1\right| \\
& \leq \frac{C_{4}}{n^{\rho / 2}} .
\end{aligned}
$$

Applying the bounds of $T H_{1}, T H_{2}, T H_{3}$ and $T H_{4}$ to (8.1), we obtain the desired inequality. This completes the proof of Corollary 2.4.

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