Extrema of multinomial assignment process

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Abstract

We study the asymptotic behavior of the expectation of the maxima and minima of random assignment process generated by a large matrix with multinomial entries. A variety of results is obtained for different sparsity regimes.

Key words. Expected maxima, minima, multinomial distribution, random assignment process.

AMS subject classifications. 60C05 (Primary), 05C70, 60K30 (Secondary).

1 Introduction and main results

1.1 Random assignment problem

We consider the following random assignment problem. Let (X_{ij}) be an $n \times n$ random matrix and let [1..n] denote the set $\{1, 2, \ldots, n\}$. Let S_n denote the group of permutations $\sigma : [1..n] \mapsto [1..n]$. For every $\sigma \in S_n$, let

$$S(\sigma) = \sum_{i=1}^{n} X_{i\sigma(i)}$$

The process $\{S(\sigma), \sigma \in S_n\}$ is called a *random assignment process*. The problem consists in the study of the asymptotic behaviour of its extrema, in particular,

$$\mathbb{E} \max_{\sigma \in S_n} S(\sigma) \quad \text{and} \quad \mathbb{E} \min_{\sigma \in S_n} S(\sigma), \quad \text{as } n \to \infty.$$
(1)

We refer to [5, 11] for many applications of assignment processes and their extrema in various fields of mathematics.

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There are many remarkable results in the area, including a famous result of Aldous [1] who proved a conjecture by Mézard and Parisi claiming that

$$\lim_{n \to \infty} \mathbb{E} \min_{\sigma \in S_n} S(\sigma) = \frac{\pi^2}{6}$$

when the X_{ij} are i.i.d. standard exponential. Actually, he showed that, when the random variables considered are nonnegative, the distribution of X_{ij} affects the limit in the minimisation problem only through the value of its probability density function at 0.

In the mentioned case, the common distribution is bounded from below. The situation is very different when one deals with the variables having unbounded distributions. For obvious reasons, it is more convenient to illustrate this phenomenon for maxima instead of minima. If the common law of the entries is not bounded from above, then the expectation of maxima does not tend anymore to a finite limit but grows to infinity and the problem consists in evaluation of the corresponding growth order. In this direction, Mordant and Segers [9] showed that if X_{ij} are i.i.d. standard Gaussian, then

$$\mathbb{E}\max_{\sigma\in\mathcal{S}_n} S(\sigma) = n\sqrt{2\log n}(1+o(1)).$$

Some rather general results of this type were recently obtaind by Cheng et al. [4] and Lifshits and Tadevosian [8].

Not so much is known for the assignment problem in the discrete setting. One may mention the case of i.i.d. Poisson random variables studied in [8] and a work of Parviainen [10] who considered uniform distributions on [1..n], or on $[1..n^2]$, random permutations of [1..n] for each row, and those of $[1..n^2]$ for the whole matrix.

In this article, we study (1) for random matrices $X = (X_{ij})_{1 \le i,j \le n}$ with the joint *multinomial* distribution of entries $\mathcal{M}(m, n^2)$. Therefore, the matrix entries are integer-valued, negatively dependent random variables with common binomial distribution $\mathcal{B}(m, p)$ with success probability $p = n^{-2}$ and number of trials m. We allow the dependence m = m(n). As one will see, the presence of this extra parameter m creates a space for a variety of asymptotic behaviors for the expectation of the extrema.

1.2 A motivating example

Let us give an example showing how the studied problem emerges in information transmission. Let $\mathcal{A} = (a_1, ..., a_n)$ be an alphabet of n letters. If u and v are two independent uniformly distributed words of length m, the $n \times n$ matrix X defined by

$$X_{ij} := \sum_{k=1}^{m} \mathbf{1}_{\{u_k = a_i, v_k = a_j\}}, \qquad 1 \le i, j \le n,$$

is distributed according to the multinomial law $\mathcal{M}(m, n^2)$. Recall that Hamming distance between the words is defined by

$$d_H(u,v) := \sum_{k=1}^m \mathbf{1}_{\{u_k \neq v_k\}} = m - \sum_{k=1}^m \mathbf{1}_{\{u_k = v_k\}} = m - \sum_{i=1}^n X_{ii}.$$

Assume that we have received a word v through a noisy channel and we have to decide whether v is just a random word or a word u that passed through an unknown coding $\sigma : \mathcal{A} \mapsto \mathcal{A}$. The answer should clearly depend on the quantity

$$\min_{\sigma} d_H(\sigma(u), v) = \min_{\sigma} \left(m - \sum_{i=1}^n X_{i\sigma(i)} \right) = m - \max_{\sigma} \sum_{i=1}^n X_{i\sigma(i)}.$$

1.3 Results

Our setting is an asymptotic one, i.e., we let $n \to \infty$ and allow $m = m_n$ to be a function of n. The results depend heavily on the relation between n and m. Therefore, we consider separately several zones gradually going down from large m's to the smaller ones. Everywhere we use the notation $p = p_n := n^{-2}$ for the probability which is naturally related to our basic multinomial law $\mathcal{M}(m, n^2)$. All limits are meant for $n \to \infty$.

Quasi-Gaussian zone

This zone is defined by assumption

$$\frac{mp}{\log n} \to \infty \tag{2}$$

which essentially means that all entries X_{ij} are sufficiently large to be heuristically approximated with Gaussian variables.

Theorem 1. Under assumption (2) it is true that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = \frac{m}{n} (1+o(1)),$$
$$\mathbb{E} \min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = \frac{m}{n} (1+o(1)).$$

Critical zone

The critical zone is described by assumption

$$\frac{mp}{\log n} \to c \tag{3}$$

with some c > 0. Unlike to the quasi-Gaussian case, the expectation behavior of maxima and minima is not the same anymore.

Theorem 2. Under assumption (3) for all c > 0 it is true that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = c H_* n \log n (1 + o(1)),$$

where $H_* = H_*(c)$ is the unique solution of equation

$$\begin{cases} H \log H - (H - 1) = \frac{1}{c}, \\ H > 1, \end{cases}$$
(4)

and for all c > 1 it is true that

$$\mathbb{E} \min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = c \widetilde{H}_* n \log n \left(1 + o(1)\right),$$

where $\widetilde{H}_* = \widetilde{H}_*(c)$ is the unique solution of equation

$$\begin{cases} H \log H - (H - 1) = \frac{1}{c}, \\ 0 < H < 1. \end{cases}$$
(5)

For c < 1 equation (5) has no solution and the result for the minimum is completely different, as stated in the next theorem.

Theorem 3. Let c < 1 and

$$\limsup \frac{mp}{\log n} \le c. \tag{6}$$

Then,

$$\lim \mathbb{P}\left(\min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = 0\right) = 1.$$

Remark 4. The intermediate case c = 1 admits a similar treatment but the result is less attractive. For example, one may replace assumption (6) with

$$\frac{mp}{\log n} \le 1 - \frac{\log(b\log n)}{\log n}, \qquad b > 1.$$

Quasi-Poissonian zone

The quasi-Poissonian zone is described by the assumptions

$$\frac{mp}{\log n} \to 0 \tag{7}$$

while, for every $\delta > 0$,

$$mp \gg n^{-\delta}.$$
 (8)

In this zone all entries X_{ij} are well approximated by Poissonian variables with intensity parameter mp. This zone includes moderately growing intensities mp, the constant mp and even a narrow zone of mp slowly decreasing to zero, e.g., with logarithmic speed. **Theorem 5.** Under assumptions (7) and (8) it is true that

$$\mathbb{E} \max_{\sigma} \sum_{i=i}^{n} X_{i\sigma(i)} = \frac{n \log n}{\log\left(\frac{\log n}{mp}\right)} (1 + o(1)).$$
(9)

Remark 6. Note that if $\log(mp) \ll \log \log n$ we obtain asymptotics $\frac{n \log n}{\log \log n}$ as in the Poisson i.i.d. case with constant intensity [8].

Rather sparse matrices

In this zone, we go below (8) and assume that

$$mp = c n^{-a} (1 + o(1)), \qquad a \in (0, 1).$$
 (10)

Consider first a regular case.

Theorem 7. Assume that (10) holds and

$$a \notin \left\{\frac{1}{k}, \ k \in \mathbb{N}\right\}.$$
 (11)

Then, there exists a unique positive integer k such that

$$\frac{1}{k+1} < a < \frac{1}{k} \tag{12}$$

and

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = k n \left(1 + o(1)\right).$$
(13)

Let us now briefly discuss the irregular case $a = \frac{1}{k}$ for some integer $k \ge 2$. Since the lower bound $a > \frac{1}{k+1}$ is still true, one may obtain again

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \le k n (1 + o(1)).$$

However, the opposite bound breaks down and we are only able to prove that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \ge (k-1) n(1+o(1)).$$

To summarise, for the assignment process, we have in this case that

$$(k-1) n (1+o(1)) \le \mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \le k n (1+o(1))$$

and conjecture that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = (k - \kappa) n(1 + o(1)),$$

for some $\kappa \in [0, 1]$ depending on a and c. Proving this and finding κ is beyond the reach of current techniques.

Very sparse matrices

This zone is determined by

$$1 \ll m \ll n. \tag{14}$$

Notice that $m \approx n$ is equivalent to $mp \approx n^{-1}$, thus the current zone is just below the previous one.

Theorem 8. Under assumption (14) it is true that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = m \left(1 + o(1)\right).$$

2 Proofs

Proof of Theorem 1. Let X be a $\mathcal{B}(m,p)$ -distributed random variable. Then,

$$\mathbb{E} \exp(\gamma X) = (1 + p(e^{\gamma} - 1))^m, \qquad \gamma \in \mathbb{R}.$$
 (15)

Let now $X_j, 1 \leq j \leq n$, be $\mathcal{B}(m, p)$ -distributed random variables. We do not assume any independence. Then, for every $\gamma > 0$, we have

$$\mathbb{E} \exp(\gamma \max_{1 \le j \le n} X_j) \le \mathbb{E} \sum_{j=1}^n \exp(\gamma X_j) = n \left(1 + p(e^{\gamma} - 1)\right)^m.$$

By Jensen inequality,

$$\exp\left(\gamma \mathbb{E}\max_{1 \le j \le n} X_j\right) \le \mathbb{E} \exp(\gamma \max_{1 \le j \le n} X_j) \le n \left(1 + p(e^{\gamma} - 1)\right)^m.$$

It follows that

$$\mathbb{E} \max_{1 \le j \le n} X_j \le \gamma^{-1} \left(\log n + m \log(1 + p(e^{\gamma} - 1)) \right)$$
$$\le \gamma^{-1} \left(\log n + mp(e^{\gamma} - 1) \right).$$

We choose $\gamma := (\frac{2 \log n}{mp})^{1/2}$. By (2) we have $\gamma \to 0$. Using the expansion $e^{\gamma} - 1 = \gamma + \gamma^2 (1 + o(1))/2$, we obtain

$$\mathbb{E} \max_{1 \le j \le n} X_j \le \gamma^{-1} \left(\log n + mp[\gamma + \gamma^2(1 + o(1))/2] \right)$$

= $mp + \gamma^{-1} \log n + mp \gamma(1 + o(1))/2$
= $mp + (2mp \log n)^{1/2} (1 + o(1)).$

Furthermore, by (2) the second term is negligible and we obtain

$$\mathbb{E} \max_{1 \le j \le n} X_j \le m p (1 + o(1)).$$

The same approach applies to the minima. With the same notation we have for every $\gamma>0$

$$\mathbb{E} \exp(-\gamma \min_{1 \le j \le n} X_j) \le \mathbb{E} \sum_{j=1}^n \exp(-\gamma X_j) = n \left(1 + p(e^{-\gamma} - 1)\right)^m.$$

By Jensen inequality,

$$\exp\left(-\gamma \mathbb{E} \min_{1 \le j \le n} X_j\right) \le \mathbb{E} \exp(-\gamma \min_{1 \le j \le n} X_j) \le n \left(1 + p(e^{-\gamma} - 1)\right)^m.$$

It follows that

$$\mathbb{E} \min_{1 \le j \le n} X_j \ge -\gamma^{-1} \left(\log n + m \log(1 + p(e^{-\gamma} - 1)) \right).$$

We still use $\gamma := (\frac{2\log n}{mp})^{1/2} \to 0$. The expansion $e^{-\gamma} - 1 = -\gamma + \gamma^2 (1 + o(1))/2$ yields

$$\log(1 + p(e^{-\gamma} - 1)) = p(e^{-\gamma} - 1)(1 + o(1)) = -p\gamma(1 + o(1)) + p\gamma^2(1 + o(1))/2$$

From this we get

$$\mathbb{E} \min_{1 \le j \le n} X_j \ge -\gamma^{-1} \left(\log n + mp[-\gamma(1+o(1)) + \gamma^2(1+o(1))/2] \right)$$

= $mp(1+o(1)) - \gamma^{-1} \log n - mp\gamma(1+o(1))/2$
= $mp(1+o(1)) - (2mp\log n)^{1/2}(1+o(1)).$

By (2) the second term is negligible and we obtain

$$\mathbb{E} \min_{1 \le j \le n} X_j \ge mp (1 + o(1)).$$

Let us now apply these results to the multinomial assignment process. Here the joint law of the entries X_{ij} is $\mathcal{M}(m, n^2)$ and every X_{ij} follows Binomial law $\mathcal{B}(m, p)$ with $p = n^{-2}$. Our bound for the maxima yields

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \leq \sum_{i=1}^{n} \mathbb{E} \max_{1 \leq j \leq n} X_{ij} = n \cdot \mathbb{E} \max_{1 \leq j \leq n} X_{1j} \leq \frac{m}{n} (1 + o(1)),$$

while the bound for the minima yields

$$\mathbb{E} \min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \ge \sum_{i=1}^{n} \mathbb{E} \min_{1 \le j \le n} X_{ij} = n \cdot \mathbb{E} \min_{1 \le j \le n} X_{1j} \ge \frac{m}{n} (1 + o(1)).$$

It follows that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = \frac{m}{n} (1 + o(1)),$$
$$\mathbb{E} \min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = \frac{m}{n} (1 + o(1)),$$

as required.

Proof of Theorem 2. Let (X_j) be negatively associated random variables following the Bernoulli law $\mathcal{B}(m, p)$. We claim that for every c > 0 under (3) and under the additional assumption

$$p \log n \to 0,$$
 (16)

it is true that

$$\mathbb{E} \max_{1 \le j \le n} X_j = cH_* \log n \, (1 + o(1)), \qquad \text{as } n \to \infty.$$
(17)

Further, for every c > 1,

$$\mathbb{E} \min_{1 \le j \le n} X_j = c \widetilde{H}_* \log n \, (1 + o(1)), \qquad \text{as } n \to \infty.$$
(18)

The upper bound in (17) and the lower bound in (18). Let $H > H_*$. Then

$$H\log H - (H-1) > \frac{1}{c}.$$
 (19)

Let r := Hp. Then, by (3), $mr = Hmp = c H \log n (1 + o(1))$.

Applying the exponential Chebyshev inequality for every j and every v > 0, we obtain

$$\mathbb{P}(X_j \ge cH \log n + v) = \mathbb{P}(X_j \ge mr + v)$$

$$\le \frac{\mathbb{E} e^{\gamma X_j}}{e^{\gamma(mr+v)}} = \left[\frac{1 + p(e^{\gamma} - 1)}{e^{\gamma r}}\right]^m e^{-\gamma v}.$$
(20)

By choosing the optimal $\gamma := \log \left(\frac{(1-p)r}{p(1-r)}\right)$, we have

$$\frac{1+p(e^{\gamma}-1)}{e^{\gamma r}} = \left(\frac{p}{r}\right)^r \exp\left((1-r)\log(1-p) - (1-r)\log(1-r)\right)$$
$$= H^{-Hp}\exp\left(-p + r + O(p^2)\right)$$
$$= \exp\left(-(H\log H - (H-1))p + O(p^2)\right).$$

Hence,

$$\left[\frac{1 + p(e^{\gamma} - 1)}{e^{\gamma r}} \right]^m = \exp\left(-(H\log H - (H - 1) + o(1)) mp \right)$$

= $\exp\left(-(H\log H - (H - 1)) c \log n(1 + o(1)) \right)$
:= $n^{-\beta + o(1)}$,

where by (19) it is true that

 $\beta = (H \log H - (H - 1))c > 1.$

Substituting the above results in (20) we obtain

$$\mathbb{P}(X_j \ge cH\log n + v) \le n^{-\beta + o(1)} e^{-\gamma v}.$$

It is now trivial that

$$\mathbb{P}(\max_{1 \le j \le n} X_j \ge cH \log n + v) \le n^{-(\beta-1)+o(1)} e^{-\gamma v}.$$

It follows that

$$\mathbb{E} \max_{1 \le j \le n} X_j - cH \log n = \mathbb{E} \left(\max_{1 \le j \le n} X_j - cH \log n \right)$$

$$\leq \mathbb{E} \left(\max_{1 \le j \le n} X_j - cH \log n \right)_+$$

$$= \int_0^\infty \mathbb{P} (\max_{1 \le j \le n} X_j \ge cH \log n + v) \, dv$$

$$\leq n^{-(\beta - 1) + o(1)} \int_0^\infty e^{-\gamma v} \, dv = n^{-(\beta - 1) + o(1)} \frac{1}{\gamma}$$

$$= n^{-(\beta - 1) + o(1)} \frac{1}{\log H} (1 + o(1)) \to 0.$$

Therefore,

$$\mathbb{E} \max_{1 \le j \le n} X_j \le cH \log n + o(1).$$

By letting $H \searrow H_*$ we obtain the upper bound in (17).

The lower bound in (18) is obtained in exactly the same way through the Chebyshev inequality for the lower tails.

Converse bounds. The lower bound in (17) is reached in a few steps. We give a Poissonian approximation of Binomial laws, then provide a lower bound for this Poissonian approximation. This bound provides a lower bound for the maximum's expectation of *independent* Binomial i.i.d. random variables. Finally, using negative association argument, we reduce the claim to the independence case.

Step 1. Let X be a Binomial $\mathcal{B}(m, p)$ -distributed random variable. Elementary calculations show that Poissonian approximation

$$\mathbb{P}(X = k) = e^{-mp} \frac{(mp)^k}{k!} \ (1 + o(1))$$

is valid if $p^2m \to 0$, $pk \to 0$, and $\frac{k^2}{m} \to 0$.

Step 2. Let c > 0 and H > 1. Let $k = [cH \log n + 1]$ and $\lambda = c \log n(1 + o(1))$. Then an elementary evaluation of Poissonian probabilities yields

$$e^{-\lambda} \frac{\lambda^k}{k!} = n^{-\beta + o(1)}$$

where

$$\beta := c(H \log H - (H - 1)).$$
(21)

Now we combine the results of the two steps. Note that with (3), (16) and for $k = c H \log n (1 + o(1))$, all three assumptions of Step 1 are verified and, with $\lambda = mp$, we obtain

$$\mathbb{P}(X \ge cH \log n) \ge \mathbb{P}(X = k) = n^{-\beta + o(1)}.$$

If $1 < H < H_*$, then $\beta < 1$.

Step 3. Let $(\widetilde{X}_j)_{1 \leq j \leq n}$ be independent copies of X. Then

$$\mathbb{P}(\max_{1 \le j \le n} \widetilde{X}_j \le c H \log n) = \mathbb{P}(X \le c H \log n)^n \le (1 - n^{-\beta + o(1)})^n \le \exp(-n^{1 - \beta + o(1)}) \to 0.$$
(22)

It follows that

$$\mathbb{E} \max_{1 \le j \le n} \widetilde{X}_j \ge cH \log n(1 + o(1)).$$
(23)

Step 4. From the desintegration theorem for negatively associated variables, due to Christofides and Vaggelatou [3], see also Bulinski and Shashkin [2, Chapter 2, Theorem 2.6 and Lemma 2.2], one has

$$\mathbb{E} \max_{1 \le j \le n} X_j \ge \mathbb{E} \max_{1 \le j \le n} \widetilde{X}_j.$$
(24)

Combining this estimate with the result of Step 3, for every $H < H_*$ we obtain

$$\mathbb{E} \max_{1 \le j \le n} X_j \ge cH \log n(1 + o(1)).$$

Letting $H \nearrow H_*$, we obtain the lower bound in (17).

The upper bound in (18) follows in a similar way. Let now $k := [cH \log n]$. By using Poissonian approximation and Poissonian asymptotics we obtain

$$\mathbb{P}(X \le cH \log n) \ge \mathbb{P}(X = k) = n^{-\beta + o(1)}$$

with the same β from (21). If $\widetilde{H}_* < H < 1$, then $\beta < 1$.

As before, for independent variables we obtain

$$\mathbb{P}\left(\min_{1\leq j\leq n} \widetilde{X}_j \geq cH\log n\right) \leq \exp\left(-n^{1-\beta+o(1)}\right)$$

It follows that

 $\mathbb E$

$$\begin{split} \min_{1 \le j \le n} \widetilde{X}_j &= \mathbb{E} \left[\min_{1 \le j \le n} \widetilde{X}_j \mathbf{1}_{\{\min_{1 \le j \le n} \widetilde{X}_j \le cH \log n\}} \right] \\ &+ \mathbb{E} \left[\min_{1 \le j \le n} \widetilde{X}_j \mathbf{1}_{\{\min_{1 \le j \le n} \widetilde{X}_j > cH \log n\}} \right] \\ &\leq cH \log n + \sum_{j=1}^n \mathbb{E} \left[X_j \mathbf{1}_{\{\min_{1 \le i \le n, i \ne j} \widetilde{X}_i > cH \log n\}} \right] \\ &= cH \log n + n \mathbb{E} \widetilde{X}_1 \mathbb{P} \left(\min_{2 \le i \le n} \widetilde{X}_i > cH \log n \right) \\ &\leq cH \log n + n \cdot c \log n \left(1 + o(1)\right) \exp(-n^{1 - \beta + o(1)}) \\ &= cH \log n + o(1). \end{split}$$

The final negative association argument reads as follows. Since (X_j) are negatively associated, so are $(-X_j)$, too. From the desintegration theorem cited above it follows that

$$\mathbb{E} \max_{1 \le j \le n} (-X_j) \ge \mathbb{E} \max_{1 \le j \le n} (-\widetilde{X}_j)$$

which is equivalent to

$$\mathbb{E} \min_{1 \le j \le n} X_j \le \mathbb{E} \min_{1 \le j \le n} \widetilde{X}_j.$$

By combining the obtained results, we have

$$\mathbb{E} \min_{1 \le j \le n} X_j \le c H \log n(1 + o(1)).$$

Finally, letting $H \searrow \widetilde{H}_*$ we obtain the upper bound in (18).

The estimates for assignment process. Recall that a multinomial distribution is *negatively associated*, see Joag-Dev and Proschan [7] and Bulinski and Shashkin [2, Chapter 1, Theorem 1.27]. Furthermore, with $p = n^{-2}$, the assumption (16) is also valid.

Therefore, the bounds (17) and (18) apply to the sums of the entries X_{ij} . They yield, respectively,

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \leq \sum_{i=1}^{n} \mathbb{E} \max_{1 \leq j \leq n} X_{ij} \leq c H_* n \log n (1+o(1)),$$
$$\mathbb{E} \min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \geq \sum_{i=1}^{n} \mathbb{E} \min_{1 \leq j \leq n} X_{ij} \geq c \widetilde{H}_* n \log n (1+o(1)).$$

The opposite bounds follow by the "greedy method" introduced in [9] (and used in [8]) that we recall now. This method allows to construct a quasi-optimal permutation σ^* that provides sufficiently large value or sufficiently small value of the assignment process. Recall that $[1..i] := \{1, 2, ..., i\}$. Define

$$\sigma^*(1) := \arg \max_{j \in [1..n]} X_{1j},$$

and let for all $i = 2, \ldots, n$

$$\sigma^*(i) := \arg \max_{j \notin \sigma^*([1..i-1])} X_{ij}.$$

It is natural to call this strategy greedy, because at every step we consider the row i, take the maximum of its available elements (without considering the influence of this choice on subsequent steps) and then forget the row i and the corresponding column $\sigma^*(i)$. The number of variables used at consequent steps is decreasing from n to 1.

By using the greedy method, we have

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \geq \mathbb{E} \sum_{i=1}^{n} X_{i\sigma^{*}(i)} = \sum_{i=1}^{n} \mathbb{E} \max_{j \notin \sigma^{*}([1..i-1])} X_{ij}$$
$$= \sum_{i=1}^{n} \mathbb{E} \max_{1 \le j \le n-i+1} X_{ij}.$$
(25)

The latter equality may seem surprising because the index sets $[n] \setminus \sigma^*([1..i-1])$ are random and depend on the matrix X. However, it is justified by the following lemma.

Lemma 9. Let $N_1, N_2 > 0$ be positive integers and let a random vector $X := (X_j)_{1 \leq j \leq N_1+N_2}$ be distributed according to a multinomial law \mathcal{M}_{m,N_1+N_2} . Let $X^{(1)} := (X_j)_{1 \leq j \leq N_1}$ and $X^{(2)} := (X_j)_{N_1 < j \leq N_2}$. Let $1 \leq q \leq N_2$ and let $\mathcal{J} \subset (N_1, N_1 + N_2]$ be a random set of size q determined by $X^{(1)}$. Then the variables $\max_{j \in \mathcal{J}} X_j$ and $\max_{N_1 < j \leq N_1+q} X_j$ are equidistributed.

By applying the asymptotic expression (17) to each term of the sum (25) and using that the function $n \mapsto \log n$ is slowly varying we obtain the desired lower bound

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \ge c H_* n \log n (1+o(1)).$$

Replacing maxima by minima in the greedy method and using (18) yields the remaining upper bound

$$\mathbb{E} \min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \le c \widetilde{H}_* n \log n (1+o(1)).$$

This completes the proof of Theorem 2 except for the postponed proof of Lemma 9. $\hfill \Box$

Proof of Lemma 9. Let

$$S = S(X^{(1)}) := \sum_{j=1}^{N_1} X_j.$$

Recall that the conditional distribution of $X^{(2)}$ w.r.t. $X^{(1)}$ is \mathcal{M}_{m-S,N_2} . This means that for all $x_1 \in \mathbb{N}^{N_1}, x_2 \in \mathbb{N}^{N_2}$ it is true that

$$\mathbb{P}(X^{(2)} = x_2, X^{(1)} = x_1) = \mathbb{P}(X^{(1)} = x_1) \mathcal{M}_{m-S(x_1),N_2}(x_2).$$

For every fixed set $J \subset (N_1, N_1 + N_2]$ of size q, it holds that

$$\mathbb{P}(X^{(2)} = x_2, \mathcal{J} = J) = \sum_{s=0}^{m} \mathbb{P}(\mathcal{J} = J, S = s) \mathcal{M}_{m-s, N_2}(x_2),$$

by summing up over $x_1 \in \mathcal{J}^{-1}(J)$. Now, for every non-negative integer μ , by summing up over x_2 such that $\max_{j \in J} x_{2j} = \mu$, we obtain

$$\mathbb{P}(\max_{j \in J} X_j = \mu, \mathcal{J} = J) = \sum_{s=0}^{m} \mathbb{P}(\mathcal{J} = J, S = s) \ \mathcal{M}_{m-s,N_2}(x_2 : \max_{j \in J} x_{2j} = \mu).$$

The latter factor does not depend on a particular set J due to exchangeability property of the multinomial law. We thus may denote

$$\mathcal{M}_{m-s,N_2}(x_2 : \max_{j \in J} x_{2j} = \mu) =: F(m-s, N_2, q, \mu)$$

and obtain

$$\mathbb{P}(\max_{j\in J} X_j = \mu, \mathcal{J} = J) = \sum_{s=0}^m \mathbb{P}(\mathcal{J} = J, S = s) \ F(m-s, N_2, q, \mu).$$

By summing up over all sets J of size q we see that

$$\mathbb{P}(\max_{j\in\mathcal{J}}X_j=\mu) = \sum_{s=0}^m \mathbb{P}(S=s) \ F(m-s, N_2, q, \mu)$$

does not depend on the specific choice of \mathcal{J} , and the claim of lemma follows. \Box

Proof of Theorem 3. We are going to use an old result by Erdős and Rényi [6] about the existence of perfect matching in a random bipartite graph. Let G be a uniformly distributed n + n bipartite graph with m = m(n) edges. If

$$\lim\left(\frac{m}{n} - \log n\right) = \infty,\tag{26}$$

then with probability tending to one, as $n \to \infty, \, G$ has a perfect matching.

In the matrix form, this result asserts the following. Let $Y = Y(n, m) = \{Y_{ij}\}_{1 \le i,j \le n}$ be a uniformly distributed random $n \times n$ matrix with entries taking values in $\{0,1\}$ and satisfying $\sum_{i,j=1}^{n} Y_{ij} = m$. If (26) holds, then

$$\lim \mathbb{P}\left(\max_{\sigma} \sum_{i=1}^{n} Y_{i\sigma(i)} = n\right) = 1.$$
(27)

Let now $X = (X_{ij})$ be our matrix following the multinomial law $\mathcal{M}(m, n^2)$. Introduce the matrix \widetilde{Y} by

$$\widetilde{Y}_{ij} := \begin{cases} 0, & X_{ij} > 0, \\ 1, & X_{ij} = 0. \end{cases}$$

Note that

$$\mathbb{P}(\widetilde{Y}_{ij} = 1) = \mathbb{P}(X_{ij} = 0) = (1 - p)^m = \exp(-m p (1 + o(1))).$$

Let $S := \sum_{i,j=1}^{n} \widetilde{Y}_{ij}$ be the number of empty cells in our matrix X. Observe that, conditioned on S, the matrix \widetilde{Y} has the same distribution as Y(n, S). Taking into account that the probability in (27) is non-decreasing as a function of m, we have for every positive integer M

$$\mathbb{P}\left(\min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = 0\right) = \mathbb{P}\left(\max_{\sigma} \sum_{i=1}^{n} \widetilde{Y}_{i\sigma(i)} = n\right)$$

$$\geq \mathbb{P}(S \ge M) \mathbb{P}\left(\max_{\sigma} \sum_{i=1}^{n} Y(n, M)_{i\sigma(i)} = n\right).$$
(28)

We choose $M = n^{\beta}$ with $\beta \in (1, 2 - c)$ and show that both probabilities in the latter product tend to one as $n \to \infty$.

For the first one, using (6), we have

$$\mathbb{E} S = n^2 \mathbb{E} \widetilde{Y}_{11} = n^2 \exp\left(-m \, p \left(1 + o(1)\right)\right) \ge n^{2 - c(1 + o(1))}.$$

Furthermore, since the variables \widetilde{Y}_{ij} are negatively correlated, we have

$$\operatorname{Var} S \le n^2 \operatorname{Var} \widetilde{Y}_{11} \le n^2 \operatorname{\mathbb{E}} \widetilde{Y}_{11} = \operatorname{\mathbb{E}} S.$$

Finally, using $\beta < 2 - c$, by Chebyshev inequality,

$$\begin{split} \mathbb{P}(S \leq n^{\beta}) &\leq \mathbb{P}(|S - \mathbb{E}S| \geq \mathbb{E}S - n^{\beta}) = \mathbb{P}(|S - \mathbb{E}S| \geq \mathbb{E}S(1 + o(1))) \\ &\leq \frac{\operatorname{Var}S}{(\mathbb{E}S)^2(1 + o(1))} \leq \frac{\mathbb{E}S}{(\mathbb{E}S)^2(1 + o(1))} \to 0. \end{split}$$

On the other hand, since $\beta > 1$, the assumption (26) with $m := M = n^{\beta}$ is true. Therefore, the second probability in the product (28) tends to one by Erdős–Rényi result. We obtain from (28) that

$$\lim \mathbb{P}\left(\min_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = 0\right) = 1,$$

which is the desired claim.

Proof of Theorem 5. The proof goes along the same lines as the one of Theorem 2. Instead of the key relation (17), we prove the following claim. Let (X_j) be negatively associated random variables following Bernoulli law $\mathcal{B}(m, p)$. Then under assumptions (7) and (8) it is true that

$$\mathbb{E}\max_{1\leq j\leq n} X_j = \frac{\log n}{\log\left(\frac{\log n}{mp}\right)} (1+o(1)), \quad \text{as } n \to \infty.$$
⁽²⁹⁾

Upper bound. For the upper bound in (29) that we are going to prove now, no lower bound on mp is needed; we only use (7).

Let $\beta > 1$, $y := \frac{\beta \log n}{mp}$, $r := \frac{y}{\log y}$. Notice that under (7) we have $y, r \to \infty$. Next, for a Binomisal $\mathcal{B}(m, p)$ random variable X and for every v > 0 it is true that

$$\begin{split} \mathbb{P}\left(X \ge \frac{\beta \log n}{\log\left(\frac{\log n}{mp}\right)} + v\right) &\leq \mathbb{P}\left(X \ge \frac{\beta \log n}{\log\left(\frac{\beta \log n}{mp}\right)} + v\right) \\ &= \mathbb{P}\left(X \ge \frac{\beta \frac{\log n}{mp}}{\log\left(\frac{\beta \log n}{mp}\right)} mp + v\right) \\ &= \mathbb{P}\left(X \ge \frac{y}{\log y} mp + v\right) = \mathbb{P}\left(X \ge r mp + v\right) \end{split}$$

In the next calculation we use the Poisson version of the bound for exponential moment

$$\mathbb{E} \exp(\gamma X) \le \exp(mp(e^{\gamma} - 1))$$

that immediately follows from the exact formula (15). By applying Chebyshev inequality with Poisson-optimal parameter $\gamma = \log r$ we obtain

$$\mathbb{P}(X \ge rmp + v) \le \mathbb{E} \exp(\gamma X) \exp(-\gamma (rmp + v)) \\ \le \exp(-mp(\gamma r - e^{\gamma} + 1) - \gamma v) \\ = \exp(-mp(r\log r - r + 1) - \gamma v).$$

Since $r \to \infty$, we have

$$r\log r - r + 1 \sim r\log r \sim y = \frac{\beta\log n}{mp}$$

It follows that

$$\mathbb{P}(X \ge rmp + v) \le \exp(-\beta \log n(1 + o(1)) - \gamma v)$$

= $n^{-\beta(1+o(1))} \exp(-\gamma v).$

and

$$\mathbb{P}\left(\max_{1\leq j\leq n} X_j \geq rmp + v\right) \leq n \ \mathbb{P}\left(X \geq rmp + v\right) \leq n^{-(\beta-1)(1+o(1))} \exp(-\gamma v).$$

Hence,

$$\mathbb{E} \max_{1 \le j \le n} X_j \le rmp + n^{-(\beta - 1)(1 + o(1))} \int_0^\infty \exp(-\gamma v) dv$$

= $rmp + n^{-(\beta - 1)(1 + o(1))} \gamma^{-1}.$

Note that

$$rmp\gamma = r\log r\,mp \sim y\,mp = \beta\log n \rightarrow \infty$$

hence we conclude that $n^{-(1-\beta)(1+o(1))}\gamma^{-1}$ is negligible compared to rmp, thus

$$\mathbb{E} \max_{1 \le j \le n} X_j \le rmp(1 + o(1)) \sim \frac{\beta \log n}{\log\left(\frac{\log n}{mp}\right)}$$

and the required upper bound follows by letting $\beta \searrow 1$.

Lower bound. Let $\beta \in (0,1)$, $y := \frac{\beta \log n}{mp}$, $r := \frac{y}{\log y}$, and

$$k := rmp = \frac{y}{\log y} mp = \frac{\beta \log n}{\log y}.$$

Assumption (7) yields $y \to \infty$, $k = o(\log n)$, $e^k = n^{o(1)}$, $e^{mp} = n^{o(1)}$.

On the other hand, under assumption (8) we have $|\log(mp)| \ll \log n$, which yields $\log y \ll \log n$, hence $k \to \infty$.

Therefore, by using Poissonian approximation, we obtain

$$\mathbb{P}(X \ge k) \ge \mathbb{P}(X = k) \sim e^{-mp} \frac{(mp)^k}{k!} \sim e^{-mp} e^k (2\pi k)^{-1/2} \left(\frac{mp}{k}\right)^k$$

= $n^{o(1)} r^{-k} = n^{o(1)} r^{-rmp} = n^{o(1)} \exp(-r\log r mp)$
= $n^{o(1)} \exp(-y(1+o(1))mp) = n^{-\beta+o(1)}.$

By repeating the arguments from (22), (23), and (24) we obtain

$$\mathbb{E} \max_{1 \le j \le n} X_j \ge k(1+o(1)) = \frac{y}{\log y} mp(1+o(1)) = \frac{\beta \log n}{\log y} (1+o(1))$$

. .

and letting $\beta \nearrow 1$ provides the required lower bound in (29).

Once (29) is proved, the proof of Theorem 5 is completed by the same simple arguments (including the greedy method) as that of Theorem 2. \Box

Proof of Theorem 7. Upper bound. We have

$$\mathbb{E} \max_{1 \le j \le n} X_j$$

$$= \mathbb{E} \left[\max_{1 \le j \le n} X_j \mathbf{1}_{\{\max_{1 \le j \le n} X_j \le k\}} \right] + \mathbb{E} \left[\max_{1 \le j \le n} X_j \mathbf{1}_{\{\max_{1 \le j \le n} X_j > k\}} \right]$$

$$\leq k + \sum_{j=1}^n \mathbb{E} \left[X_j \mathbf{1}_{\{X_j > k\}} \right] = k + n \mathbb{E} \left[X_1 \mathbf{1}_{\{X_1 > k\}} \right].$$

Furthermore, since the law of X_1 is $\mathcal{B}(m, p)$, it is true that

$$\mathbb{P}(X_1 = \ell) = \frac{m!}{(m-\ell)!} \frac{p^{\ell}}{\ell!} (1-p)^{m-\ell} \le \frac{m^{\ell} p^{\ell}}{\ell!}, \qquad 0 \le \ell \le m.$$

Hence,

$$\mathbb{E}\left[X_1 \mathbf{1}_{\{X_1 > k\}}\right] \leq \sum_{\ell=k+1}^{\infty} \frac{(mp)^{\ell}}{(\ell-1)!} = \sum_{q=0}^{\infty} \frac{(mp)^{k+1+q}}{(k+q)!} \leq (mp)^{k+1} \exp(mp) = (mp)^{k+1} (1+o(1)).$$

Therefore,

$$\mathbb{E} \max_{1 \le j \le n} X_j \le k + c^{k+1} n^{1-a(k+1)} \left(1 + o(1)\right) = k + o(1), \tag{30}$$

where we used the lower bound in (12) at the last step.

Turning to the lower bound, for every positive integer v in the *independent* case, we have

$$\mathbb{P}\left(\max_{1 \le j \le v} X_{j} < k\right) = \mathbb{P}\left(X_{1} < k\right)^{v} = (1 - \mathbb{P}\left(X_{1} \ge k\right))^{v} \\
\le (1 - \mathbb{P}\left(X_{1} = k\right))^{v} \\
= \exp\{-v \ \mathbb{P}\left(X_{1} = k\right) \ (1 + o(1))\} \\
= \exp\left\{-v \ \frac{c^{k} n^{-ak}}{k!} \ (1 + o(1))\right\}.$$
(31)

Let us fix some small $\delta \in (0, 1)$. By letting $v = [\delta n]$ and using the upper bound in (12) we obtain

$$\mathbb{P}\left(\max_{1 \le j \le [\delta n]} X_j < k\right) \to 0.$$

It follows that

$$\mathbb{E} \max_{1 \le j \le [\delta n]} X_j \ge k \mathbb{P} \left(\max_{1 \le j \le [\delta n]} X_j \ge k \right) = k \left(1 + o(1) \right),$$

By using negative association argument (24), we also obtain

$$\mathbb{E} \max_{1 \le j \le [\delta n]} X_j \ge k \left(1 + o(1)\right) \tag{32}$$

in the multinomial setting.

Finally, by using (30) and the greedy method based on (32), we conclude that in the regular case (11) for the assignment process it is true that

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} = k n (1 + o(1)).$$

Proof of Theorem 14. The upper bound

$$\max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \le m$$

is trivial; it remains to prove the lower bound.

Let us denote $(u_i, v_i)_{1 \le i \le m}$ the coordinates of the particles thrown on the square table. All u_i and all v_i are i.i.d. random variables uniformly distributed on integers [1..n]. Let $U_0 = V_0 = \emptyset$,

$$U_k := \{u_i, 1 \le i \le k\}, \quad V_k := \{v_i, 1 \le i \le k\}, \qquad 1 \le k \le m,$$

and introduce the events

$$A_k := \{ u_k \notin U_{k-1}, v_k \notin V_{k-1} \}, \quad 1 \le k \le m.$$

It is obvious that for each k

$$\mathbb{P}(A_k) \ge 1 - \frac{2m}{n},$$

hence by $m \ll n$

$$\mathbb{E}\left(\sum_{k=1}^{m} \mathbf{1}_{\{A_k\}}\right) \ge m\left(1 - \frac{2m}{n}\right) = m\left(1 + o(1)\right).$$

On the other hand, we have

$$\max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \ge \sum_{k=1}^{m} \mathbf{1}_{\{A_k\}},\tag{33}$$

which entails the desired

$$\mathbb{E} \max_{\sigma} \sum_{i=1}^{n} X_{i\sigma(i)} \ge m \left(1 + o(1)\right).$$

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References

- Aldous, D. J. (2001) The ζ(2) limit in the random assignment problem. Random Structures & Algorithms 18, No.4, 381–418.
- [2] Bulinski, A. V. and Shashkin A. P. (2007) Limit theorems for associated random fields and related systems. Advanced Series on Statistical Science & Applied Probability, vol. 10, World Scientific.

- [3] Christofides, T. C. and Vaggelatou, E. (2004) A connection between supermodular ordering and positive/negative association. J. Multivar. Anal. 88, No.1, 138– 151.
- [4] Cheng, Y., Liu, Y., Tkocz, T. and Xu A. (2021) Typical values of extremal-weight combinatorial structures with independent symmetric weights. Preprint.
- [5] Coppersmith, D. and Sorkin, G. B. (1999) Constructive bounds and exact expectations for the random assignment problem. *Random Structures & Algorithms* 15, No.2, 113–144.
- [6] Erdős, P. and Rényi, A. (1964) On random matrices. Publ. Math. Inst. Hungar. Acad. Sci. 8, 455–461.
- [7] Joag-Dev, K. and Proschan, F. (1983) Negative association of random variables with applications. Ann. Statist. 11, No.1, 286–295.
- [8] Lifshits, M. and Tadevosian, A. (2022) On the maximum of random assignment process. *Statist. Probab. Letters* 187, 109530, 1–6.
- [9] Mordant, G. and Segers, J. (2021) Maxima and near-maxima of a Gaussian random assignment field. *Statistics & Probability Letters* 173 109087.
- [10] Parviainen, R. (2004) Random assignment with integer costs. Combinatorics, Probability and Computing 13, No.1, 103–113.
- [11] Steele, J. M., (1997). Probability theory and combinatorial optimization. Ser.: CBMS-NSF Regional Conference Series in Applied Mathematics, Vol.69.