

Generically stable regular types

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Abstract

We study non-orthogonality of symmetric, regular types and show that it preserves generic stability and is an equivalence relation on the set of all generically stable, regular types. We will also prove that some of the nice properties from the stable context hold in general. In the case of strongly regular types we will relate $\not\perp$ to the global Rudin-Keisler order.

The concept of (strong) regularity for global, invariant types in an arbitrary first-order theory was introduced in Section 3 of [8]. The definition there was motivated by and extends that of regular (stationary) and strongly regular types in stable theories. Intuitively, it can be described as follows: Fix a global, invariant type \mathbf{p} . Consider all the formulas in \mathbf{p} as defining "large" subsets of the monster and their negations as defining "small" ones. Then it is natural to define: $\text{cl}_{\mathbf{p}}(X)$ is the union of all small subsets definable over X . It turned out that the regularity of \mathbf{p} means precisely that $\text{cl}_{\mathbf{p}}$ is a closure operation on the locus of $\mathbf{p}|_A$ (for almost all A over which \mathbf{p} is invariant). There are two kinds of regular types:

- \mathbf{p} is symmetric. Morley sequences are totally indiscernible and the closure operation is a pregeometry operation inducing the dimension function.
- \mathbf{p} is asymmetric. The closure operation is induced by a definable partial ordering which totally orders Morley sequences.

Stable regular types are symmetric, while asymmetric regular types may exist only in theories with the strict order property. For example, the type

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of an infinite element in a theory of dense linear orders without endpoints is strongly regular. Interesting examples of both kinds of strongly regular types are heirs of "generic" types of minimal and quasi-minimal groups (and fields); they were recently studied in [6] and [2].

Asymmetric regular types are studied in detail in the forthcoming paper [7], and in this article we will concentrate on non-orthogonality of symmetric, regular types. For generically stable regular types we will prove that non-orthogonality is an equivalence relation.

Theorem 1. *Generic stability is preserved under non-orthogonality of regular types. Non-orthogonality is an equivalence relation on the set of all regular, generically stable types.*

Next we will study generically stable, strongly regular types and prove that non-orthogonality is strongly related to the global version of Lascar's Rudin-Keisler order which was originally defined in the \aleph_0 -stable context:

Theorem 2. *Suppose that \mathfrak{p} is generically stable and strongly regular. Then:*

- (1) \mathfrak{p} is RK-minimal in the global RK-order.
- (2) If \mathfrak{q} is invariant then: $\mathfrak{p} \not\leq_{RK} \mathfrak{q}$ if and only if $\mathfrak{p} \leq_{RK} \mathfrak{q}$.

The next theorem is probably more surprising than the previous, because its original proof in the \aleph_0 -stable case (see e.g [5]) relied heavily on the existence of prime models over arbitrary sets.

Theorem 3. *Suppose that $(\mathfrak{p}(x), \phi_{\mathfrak{p}}(x))$ and $(\mathfrak{q}(x), \phi_{\mathfrak{q}}(x))$ are M -invariant, strongly regular and generically stable. Then the following conditions are all equivalent:*

- (1) $\mathfrak{p} \perp \mathfrak{q}$;
- (2) $\mathfrak{p}|_M \perp^w \mathfrak{q}|_M$;
- (3) For some $C \supseteq M$: $\mathfrak{p}|_C \perp^w \mathfrak{q}|_C$.

The paper is organized as follows: Section 1 contains preliminaries. In Section 2 we will (slightly) re-define regularity for global invariant types. The re-definition is needed due to the fact that in Remark 3.1 in [8] it was noted (without proof) that the regularity condition in the definition does not depend on the particular choice of the parameter set over which the type is invariant. This is correct if the type is strongly regular but we do not know if that holds for an arbitrary regular type. Fortunately, the lapsus did not

affect proofs of main results, only minor rephrasing of some of the statements is needed: replacement of " \mathfrak{p} is regular and A -invariant" by " \mathfrak{p} is regular over A "; the regularity of \mathfrak{p} over a set is introduced in Definition 2.1 below. In Section 3 we study non-orthogonality of generically stable regular types and prove Theorem 1. The main technical fact used in the proof, stating that generically stable regular types have weight one, is proved in Proposition 3.3. Section 4 deals with strongly regular types and there we prove Theorems 2 and 3. As an application of the results from Section 3, in Section 5 we prove that one can vary dimensions of generically stable regular types in countable models as in the stable case:

Theorem 4. *Suppose that T and A are countable and $\{\mathfrak{p}_i \mid i \in I\}$ is a countable family of pairwise orthogonal, regular over A , generically stable types. Also assume that each $\mathfrak{p}_i \upharpoonright_A$ is non-isolated. Then for any function $f : I \rightarrow \omega$ there exists a countable $M_f \supseteq A$ such that $\dim_{\mathfrak{p}_i}(M_f/A) = f(i)$ for all $i \in I$.*

1 Preliminaries

The notation is mainly standard, the only exception is the convention on the product of invariant types. We fix a complete first-order theory T and operate in its monster model \bar{M} . By $a, b, c, \dots, \bar{a}, \bar{b}, \bar{c}, \dots$ we will denote elements and tuples of elements, by A, B, C, \dots small subsets of the monster, while M, M', \dots will denote small elementary submodels. Global types will be denoted by $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \dots$. A global type \mathfrak{p} is *A -invariant* if whenever $\bar{b}_1 \equiv \bar{b}_2 (A)$ then $(\phi(\bar{x}, \bar{b}_1) \leftrightarrow \phi(\bar{x}, \bar{b}_2)) \in \mathfrak{p}(\bar{x})$ for all $\phi(\bar{x}; \bar{y})$ with parameters from A . \mathfrak{p} is *invariant* if it is A -invariant over some small A . $\mathfrak{p} \upharpoonright_A$ will denote the restriction of \mathfrak{p} to A and $(a_i \mid i < \alpha)$ is a *Morley sequence in \mathfrak{p} over A* if $a_i \models \mathfrak{p} \upharpoonright_{Aa_{<i}}$ for all $i < \alpha$.

Assume for a while that \mathfrak{p} is A -invariant. Then Morley sequences in \mathfrak{p} over A are indiscernible. We will occasionally go out of \bar{M} (into a larger monster) in order to get realizations of global types; these will be also denoted by \bar{a}, \bar{b}, \dots , in which case $\mathfrak{p} \upharpoonright_{\bar{M}\bar{a}}$ will be well-defined due to the invariance of \mathfrak{p} . Thus global Morley sequences are also well-defined, as well as the powers \mathfrak{p}^α (types of Morley sequences of length α) are. Let $\bar{a}_1, \bar{a}_2 \models \mathfrak{p}^2$. If $\text{tp}(\bar{a}_1, \bar{a}_2/\bar{M}) = \text{tp}(\bar{a}_2, \bar{a}_1/\bar{M})$ then we will say that \mathfrak{p} is *symmetric*; otherwise, it is *asymmetric*. If \mathfrak{p} is symmetric, $(\bar{a}_i \mid i < \alpha) \models \mathfrak{p}^\alpha$ and π is a permutation of α then $(\bar{a}_{\pi(i)} \mid i < \alpha) \models \mathfrak{p}^\alpha$.

Products of invariant types were introduced in [3]. Here we will reverse the order in the definition: if \mathfrak{p} and \mathfrak{q} are invariant then their *product* $\mathfrak{p}(\bar{x}) \otimes \mathfrak{q}(\bar{y})$ is defined as follows: if $\bar{a} \models \mathfrak{p}$ and $\bar{b} \models \mathfrak{q}_{\upharpoonright \bar{M}\bar{a}}$ then $\mathfrak{p}(\bar{x}) \otimes \mathfrak{q}(\bar{y}) = \text{tp}_{\bar{x}, \bar{y}}(\bar{a}, \bar{b}/\bar{M})$; thus our $\mathfrak{p}(\bar{x}) \otimes \mathfrak{q}(\bar{y})$ is the original $\mathfrak{q}(\bar{y}) \otimes \mathfrak{p}(\bar{x})$. This change was suggested by Ludomir Newelski due to the fact that it is natural to have the equivalence: (\bar{a}_1, \bar{a}_2) is a Morley sequence in \mathfrak{p} over A if and only if $\bar{a}_1, \bar{a}_2 \models \mathfrak{p} \otimes \mathfrak{p}$. The product is associative, but not commutative. We say that \mathfrak{p} and \mathfrak{q} commute if $\mathfrak{p}(\bar{x}) \otimes \mathfrak{q}(\bar{y}) = \mathfrak{q}(\bar{y}) \otimes \mathfrak{p}(\bar{x})$.

Complete types p, q over the same domain are *weakly orthogonal*, or $p \perp^w q$, if $p(\bar{x}) \cup q(\bar{y})$ determines a complete type. Global types \mathfrak{p} and \mathfrak{q} are *orthogonal*, or $\mathfrak{p} \perp \mathfrak{q}$, if they are weakly orthogonal. It is possible that both $\mathfrak{p} \not\perp \mathfrak{q}$ and $\mathfrak{p}_{\upharpoonright A} \perp^w \mathfrak{q}_{\upharpoonright A}$ hold for A -invariant types (even regular in a superstable theory). The opposite situation, $\mathfrak{p} \perp \mathfrak{q}$ and $\mathfrak{p}_{\upharpoonright A} \not\perp^w \mathfrak{q}_{\upharpoonright A}$ may occur in an unstable theory, but not in a stable one. In a stable theory $\mathfrak{p}_{\upharpoonright A} \not\perp^w \mathfrak{q}_{\upharpoonright A}$ implies $\mathfrak{p}_{\upharpoonright C} \not\perp^w \mathfrak{q}_{\upharpoonright C}$ for all $C \supseteq A$ and, in particular, $\mathfrak{p} \not\perp \mathfrak{q}$. We will see in Proposition 5.2 that this holds for any regular, generically stable type. For definable types over a model we have:

Fact 1.1. *Suppose that \mathfrak{p} and \mathfrak{q} are both M -invariant and definable, and $\mathfrak{p}_{\upharpoonright M} \not\perp^w \mathfrak{q}_{\upharpoonright M}$. Then $\mathfrak{p}_{\upharpoonright C} \not\perp^w \mathfrak{q}_{\upharpoonright C}$ for all $C \supseteq M$; in particular, $\mathfrak{p} \not\perp \mathfrak{q}$.*

A non-algebraic global type $\mathfrak{p}(\bar{x})$ is *generically stable* if, for some small A , it is A -invariant and:

if α is infinite and $(\bar{a}_i : i < \alpha)$ is a Morley sequence in \mathfrak{p} over A then for any formula $\phi(\bar{x})$ (with parameters from \bar{M}) $\{i : \models \phi(\bar{a}_i)\}$ is either finite or co-finite.

Using compactness it is straightforward to check that \mathfrak{p} is generically stable if the condition holds for $\alpha = \omega + \omega$. Also, if \mathfrak{p} is generically stable then as a witness-set A in the definition we can take any small A over which \mathfrak{p} is invariant. Generically stable types are definable and symmetric. They commute with all invariant types. A power of a generically stable type may not be generically stable; an example the reader can find in [1]. However, this cannot happen if \mathfrak{p} is in addition regular.

Let \mathcal{C} be any subset of the monster. A partial type $\pi(x)$ is *finitely satisfiable in \mathcal{C}* if any finite subtype has infinitely many realization in \mathcal{C} , in which case we also say that $\pi(x)$ is a \mathcal{C} -type. By a \mathcal{C} -sequence over A we will mean a sequence $(a_i \mid i \leq n)$ such that $\text{tp}(a_i/A\bar{a}_{<i})$ is a \mathcal{C} -type for each

$i \leq n$. Elements of a \mathcal{C} -sequence can realize distinct types because there may exist many distinct \mathcal{C} -types, so a \mathcal{C} -sequence may not be indiscernible. The following well known fact guaranties existence of \mathcal{C} -extensions.

Fact 1.2. *Suppose that a partial type $\pi(x)$ is defined over A and finitely satisfiable in \mathcal{C} . Then for any $B \supseteq A$ there exists a \mathcal{C} -type in $S(B)$ extending $\pi(x)$.*

Non-isolation of $p \in S_n(A)$ can be expressed in terms of satisfiability: Fix $\phi(x) \in p$ and let $\mathcal{C} = \phi(\bar{M}) \setminus p(\bar{M})$. Then p is non-isolated if and only if it is a \mathcal{C} -type. Thus, isolation of a type is a strong negation of its finite satisfiability. The next fact follows from Fact 1.2.

Fact 1.3. *Suppose that $p \in S_1(A)$ is non-isolated, $\phi(x) \in p$, and $A \subseteq B$. Then p has an extension in $S_1(B)$ which is finitely satisfiable in $\mathcal{C} = \{c \in \bar{M} \mid \phi(x) \in \text{tp}(c/A) \neq p\}$.*

A weak negation of satisfiability is the semi-isolation: \bar{b} is *semi-isolated* by \bar{a} over A (or \bar{a} *semi-isolates* \bar{b} over A), denoted also by $\bar{b} \in \text{Sem}_A(\bar{a})$, iff there is a formula $\phi(\bar{y}, \bar{x}) \in \text{tp}(\bar{b}, \bar{a}/A)$ such that $\phi(\bar{a}, \bar{x}) \vdash \text{tp}(\bar{b}/A)$; $\phi(\bar{x}, \bar{y})$ is said to witness the semi-isolation. Semi-isolation is transitive: if $\bar{b} \in \text{Sem}_A(\bar{a})$ is witnessed by $\phi(\bar{y}, \bar{a})$ and $\bar{c} \in \text{Sem}_A(\bar{b})$ is witnessed by $\psi(\bar{z}, \bar{b})$, then $\bar{c} \in \text{Sem}_A(\bar{a})$ is witnessed by $\exists \bar{y}(\phi(\bar{y}, \bar{a}) \wedge \psi(\bar{z}, \bar{y}))$.

Fact 1.4. *Suppose that $\text{tp}(a/A) = p$ is non-algebraic and $\phi(x) \in p$. Let $\mathcal{C} = \{c \in \bar{M} \mid \phi(x) \in \text{tp}(c/A) \neq p\}$ and assume $\mathcal{C} \neq \emptyset$. Then:*

(1) $\psi(x, \bar{b}) \in \text{tp}(a/A\bar{b})$ witnesses $a \in \text{Sem}_A(\bar{b})$ if and only if it is not satisfied in \mathcal{C} .

(2) $a \notin \text{Sem}_A(\bar{b})$ if and only if $\text{tp}(a/A\bar{b})$ is a \mathcal{C} -type.

The Rudin-Keisler order on complete types was introduced by Lascar in [4]. In an \aleph_0 -stable theory it was related to strong regularity; a nice exposition of the material can be found in Lascar's book [5], or in Poizat's book [9]. For non-algebraic types $p, q \in S(M)$ define $p \leq_{RK} q$ iff every model $M' \supset M$ which realizes p also realizes q . \leq_{RK} is a quasi-order which particularly well behaves in an \aleph_0 -stable theory; there, due to the existence of prime models over arbitrary sets, omitting types is much easier than in general. In an \aleph_0 -stable theory RK-minimal elements exist and they are precisely the strongly regular types. Some of equivalent ways of defining $p \leq_{RK} q$ in the \aleph_0 -context are:

1. p is realized in $M(q)$ (the model prime over M and a realization of q);
2. There are $\bar{a} \models p$ and $\bar{b} \models q$ such that $\text{tp}(\bar{a}/M\bar{b})$ is isolated;
3. There are $\bar{a} \models p$ and $\bar{b} \models q$ such that $\bar{a} \in \text{Sem}_M(\bar{b})$.

In this article we will consider a variant of the RK-order, defined only for global types. In the unstable context only the third equivalent is adequate.

Definition 1.5. $\mathfrak{p} \leq_{RK} \mathfrak{q}$ if and only if there are $\bar{a} \models \mathfrak{p}$ and $\bar{b} \models \mathfrak{q}$ such that $\bar{a} \in \text{Sem}_{\bar{M}}(\bar{b})$.

Transitivity of semi-isolation implies that we have a quasi-order. However, the order can be quite trivial: take the theory of the random graph and notice that no two distinct global 1-types are \leq_{RK} -comparable.

If both $\mathfrak{p} \leq_{RK} \mathfrak{q}$ and $\mathfrak{q} \leq_{RK} \mathfrak{p}$ hold then we say that \mathfrak{p} and \mathfrak{q} are *RK-equivalent* and denote it by $\mathfrak{p} \sim_{RK} \mathfrak{q}$. \mathfrak{p} and \mathfrak{q} are *strongly RK-equivalent*, or $\mathfrak{p} \equiv_{RK} \mathfrak{q}$, if there are $\bar{a} \models \mathfrak{p}$ and $\bar{b} \models \mathfrak{q}$ such that both $\bar{a} \in \text{Sem}_{\bar{M}}(\bar{b})$ and $\bar{b} \in \text{Sem}_{\bar{M}}(\bar{a})$ hold. RK-equivalent types may not be strongly RK-equivalent.

2 Regularity

In this section we will re-define regularity for global invariant types. To simplify notation we define it for global 1-types only. This will not affect the generality because we can always switch to an appropriate sort in \bar{M}^{eq} . The definition given here slightly differs in that we first define when \mathfrak{p} is regular over A (here \mathfrak{p} is A -invariant), and then repeat the original one: \mathfrak{p} is regular if such a small set A exists. Concerning strong regularity, the definition remains unchanged.

Definition 2.1. Let $\mathfrak{p}(x)$ be a global non-algebraic type and let A be small.

- (i) $\mathfrak{p}(x)$ is said to be *regular over A* if it is A -invariant and for any $B \supseteq A$ and $a \models \mathfrak{p} \upharpoonright_A$: either $a \models \mathfrak{p} \upharpoonright_B$ or $\mathfrak{p} \upharpoonright_B \vdash \mathfrak{p} \upharpoonright_{Ba}$.
- (ii) \mathfrak{p} is *regular* if \mathfrak{p} is regular over some small set.

Clearly, if \mathfrak{p} is regular over A and $A \subseteq B$ then \mathfrak{p} is regular over B , too. The same observation holds for strong regularity. But, before defining strong regularity it is convenient to introduce the following notation: we will say that $(\mathfrak{p}(x), \phi(x))$ is an *A -invariant pair* if \mathfrak{p} is A -invariant and $\phi(x) \in \mathfrak{p} \upharpoonright_A$.

Definition 2.2. (i) $(\mathfrak{p}(x), \phi(x))$ is *strongly regular* if for some small A it is an A -invariant pair and:

for all $B \supseteq A$ and a satisfying $\phi(x)$: either $a \models \mathfrak{p}|_B$ or $\mathfrak{p}|_B \vdash \mathfrak{p}|_{Ba}$.

(ii) \mathfrak{p} is *strongly regular* if $(\mathfrak{p}, \phi(x))$ is strongly regular for some $\phi(x) \in \mathfrak{p}$.

We will prove in Proposition 2.6 that as a witness set A in the previous definition we can take any small A for which (\mathfrak{p}, ϕ) is A -invariant. For, we need to label a local regularity condition.

Definition 2.3. Suppose that $p \in S(A)$ and $\pi \subseteq p$. We say that (p, π) satisfies *the weak orthogonality condition*, or (WOR) for short, if:

$$p \perp^w \text{tp}(\bar{b}/A) \quad \text{for all } \bar{b} \in \pi(\bar{M}) \setminus p(\bar{M}).$$

WOR is a technical property of locally strongly regular types, see Definition 7.1 in [8]. Examples of such types are "generic" types of minimal and quasi-minimal structures. Recall that M is a minimal structure iff any definable subset with parameters is either finite or co-finite. In a minimal structure there is a unique non-algebraic type $p \in S_1(M)$. $(p(x), x = x)$ satisfies WOR, so p is locally strongly regular via $x = x$. By Corollary 7.1 from [8] the same is true if p is the "generic" type of a quasi-minimal structure (the type containing all the formulas with a co-countable solution set).

Remark 2.4. Some of equivalent ways of expressing the fact that (p, π) satisfies WOR are:

1. $p \vdash p|A\bar{b}$ for all $\bar{b} \in \pi(\bar{M}) \setminus p(\bar{M})$
2. $p \vdash p|A \cup (\pi(\bar{M}) \setminus p(\bar{M}))$

Lemma 2.5. *Suppose that $(\mathfrak{p}(x), \phi(x))$ is A -invariant, $A \subseteq B$, and $(\mathfrak{p}|_B, \phi)$ satisfies WOR. Then $(\mathfrak{p}|_A, \phi)$ satisfies WOR, too.*

Proof. Suppose, on the contrary, that $\bar{c} \in \phi(\bar{M}) \setminus \mathfrak{p}|_A(\bar{M})$ is such that $\mathfrak{p}|_A \not\vdash \mathfrak{p}|_{A\bar{c}}$. Then there are a', a'' realizing $\mathfrak{p}|_A$ and a formula $\varphi(x, \bar{y})$ (over A) such that $\models \varphi(a', \bar{c}) \wedge \neg\varphi(a'', \bar{c})$. Let B', B'' be such that $B \equiv B' \equiv B'' (A)$, $a' \models \mathfrak{p}|_{B'}$ and $a'' \models \mathfrak{p}|_{B''}$. Note that the A -invariance of \mathfrak{p} implies that both $(\mathfrak{p}|_{B'}, \phi)$ and $(\mathfrak{p}|_{B''}, \phi)$ satisfy WOR. Since $\bar{c} \in \phi(\bar{M}) \setminus \mathfrak{p}|_A(\bar{M})$ both $\mathfrak{p}|_{B'} \vdash \mathfrak{p}|_{B'\bar{c}}$ and $\mathfrak{p}|_{B''} \vdash \mathfrak{p}|_{B''\bar{c}}$ hold. In particular:

$$\mathfrak{p}|_{B'}(x) \vdash \varphi(x, \bar{c}) \quad \text{and} \quad \mathfrak{p}|_{B''}(x) \vdash \neg\varphi(x, \bar{c})$$

Let a realize $\mathfrak{p}|_{B'B''}$. Then $\models \varphi(a, \bar{c}) \wedge \neg\varphi(a, \bar{c})$. A contradiction. \square

Proposition 2.6. (i) *An A -invariant pair $(\mathfrak{p}(x), \phi(x))$ is strongly regular if and only if $(\mathfrak{p}|_B, \phi)$ satisfies WOR for all $B \supseteq A$.*

(ii) *An A -invariant pair $(\mathfrak{p}(x), \phi(x))$ is strongly regular if and only if: for all $B \supseteq A$ and a satisfying $\phi(x)$: either $a \models \mathfrak{p}|_B$ or $\mathfrak{p}|_B \vdash \mathfrak{p}|_{Ba}$. Therefore, as a witness set A in the definition of strong regularity we can take any small set A over which (\mathfrak{p}, ϕ) is invariant.*

Proof. (i) \Leftarrow) is easy, so we prove only \Rightarrow). Suppose that $(\mathfrak{p}(x), \phi(x))$ is strongly regular. Let $A_1 \supseteq A$ be such that the regularity condition holds:

$$\text{for all } B_1 \supseteq A_1 \text{ and } a \text{ satisfying } \phi(x): \text{ either } a \models \mathfrak{p}|_{B_1} \text{ or } \mathfrak{p}|_{B_1} \vdash \mathfrak{p}|_{B_1 a}.$$

Fix $B_1 \supseteq A_1$ and we will show that (p, ϕ) satisfies WOR (where $p = \mathfrak{p}|_{B_1}$). Suppose that $b_1 \dots b_n = \bar{b} \subset \phi(\bar{M}) \setminus p(\bar{M})$. Apply the regularity condition to (b_1, B_1) : $b_1 \in \phi(\bar{M}) \setminus \mathfrak{p}|_{B_1}(\bar{M})$ so $p(x) \vdash \mathfrak{p}|_{B_1 b_1}(x)$; then apply it to $(b_2, B_1 b_1)$: $b_2 \in \phi(\bar{M}) \setminus \mathfrak{p}|_{B_1}(\bar{M})$ so $\mathfrak{p}|_{B_1 b_1}(x) \vdash \mathfrak{p}|_{B_1 b_1 b_2}(x)$; continuing in this way we get:

$$p(x) \vdash \mathfrak{p}|_{B_1 b_1}(x) \vdash \mathfrak{p}|_{B_1 b_1 b_2}(x) \vdash \dots \vdash \mathfrak{p}|_{B_1 b_1 \dots b_n}(x)$$

Thus $p(x) \vdash \mathfrak{p}|_{B_1 \bar{b}}(x)$ and (p, ϕ) satisfies WOR. Now let $B \supseteq A$. Then $(\mathfrak{p}|_{BA_1}, \phi)$ satisfies WOR so, by Lemma 2.5, $(\mathfrak{p}|_B, \phi)$ satisfies WOR, too.

(ii) Follows from part (i). \square

Remark 2.7. Suppose that $\mathfrak{p}(x)$ is non-algebraic and A -invariant. As in the proof of Proposition 2.6(i) one checks that \mathfrak{p} is regular over A if and only if \mathfrak{p} is A -invariant and $(\mathfrak{p}|_B, \mathfrak{p}|_A)$ satisfies WOR for any (finite) extension $B \supseteq A$. The stronger equivalence, like the one that we have established for strongly regular types in Proposition 2.6, would be: an A -invariant type is regular iff \mathfrak{p} is regular over A . However that does not seem to hold: it is likely that there is a regular, A -invariant type which is not regular over A (but we don't know of an example).

The following fact, suggested by Anand Pillay, shows that the stronger equivalence holds for regular types under additional assumptions:

Proposition 2.8. *Suppose that $\mathfrak{p}(x)$ is definable and M -invariant. Then \mathfrak{p} is regular if and only if it is regular over M .*

Proof. Only \Rightarrow) requires a proof. Suppose that \mathfrak{p} is regular and let $A \supseteq M$ be such that \mathfrak{p} is regular over A . We claim that \mathfrak{p} is regular over M , too. Otherwise, for some \bar{c} there are a, b realizing $\mathfrak{p}_{\upharpoonright M}$ such that $a \models \mathfrak{p}_{\upharpoonright M\bar{c}}$, $b \not\models \mathfrak{p}_{\upharpoonright M\bar{c}}$ and $a \not\models \mathfrak{p}_{\upharpoonright M\bar{c}b}$. Let $A_1 \equiv A(M)$ be such that $\text{tp}(a, b, \bar{c}/A_1)$ is a heir of $\text{tp}(a, b, \bar{c}/M)$. Then both a and b realize $\mathfrak{p}_{\upharpoonright A_1}$, because \mathfrak{p} is a heir of $\mathfrak{p}_{\upharpoonright M}$.

Suppose $a \not\models \mathfrak{p}_{\upharpoonright A_1\bar{c}}$ and find $\bar{d} \in A_1$ and φ such that $\models \varphi(a, \bar{c}, \bar{d}) \wedge d_{\mathfrak{p}} \neg \varphi(t, \bar{c}, \bar{d})$. Since $\text{tp}(\bar{d}/\bar{c}aM)$ is a coheir, there exists $\bar{d}' \in M$ such that $\models \varphi(a, \bar{c}, \bar{d}') \wedge d_{\mathfrak{p}} \neg \varphi(t, \bar{c}, \bar{d}')$; hence $a \not\models \mathfrak{p}_{\upharpoonright M\bar{c}}$. A contradiction. We conclude $a \models \mathfrak{p}_{\upharpoonright A_1\bar{c}}$.

Therefore $a \models \mathfrak{p}_{\upharpoonright A_1\bar{c}}$, $b \models \mathfrak{p}_{\upharpoonright A_1}$ and $a \not\models \mathfrak{p}_{\upharpoonright A_1b\bar{c}}$, so \mathfrak{p} is not regular over A_1 . A contradiction. \square

Suppose that \mathfrak{p} is A -invariant and let $p = \mathfrak{p}_{\upharpoonright A}$. Define $\text{cl}_{\mathfrak{p},A}$ as an operation on the power set of $p(\bar{M})$:

$$\text{cl}_{\mathfrak{p},A}(X) = \{b \in p(\bar{M}) \mid b \not\models \mathfrak{p}_{\upharpoonright AX}\} \quad \text{for all } X \subset p(\bar{M}).$$

If \mathfrak{p} is regular over A then, by Lemma 3.1(iii) from [8], $\text{cl}_{\mathfrak{p},A}$ a closure operator on $p(\bar{M})$. The proof of this fact does not depend on Remark 3.1 there, neither does the proof of Theorem 3.1 there, which is a dichotomy theorem for regular types. Here we state only a restricted version and we will use only the first part:

Theorem 2.9. *Suppose that \mathfrak{p} is regular over A . Then $\text{cl}_{\mathfrak{p},B}$ is a closure operator on $\mathfrak{p}_{\upharpoonright B}(\bar{M})$ for all $B \supseteq A$. We have two kinds of regular types:*

(1) *Symmetric (\mathfrak{p} is symmetric). Then $\text{cl}_{\mathfrak{p},B}$ is a pregeometry operator on $p(\bar{M})$ for all $B \supseteq A$.*

(2) *Asymmetric. Then there exists a finite extension A_0 of A and an A_0 -definable partial order \leq such that every Morley sequence in p over A_0 is strictly increasing; $\text{cl}_{\mathfrak{p},A_0}$ is not a pregeometry.*

In this paper we will deal only with symmetric regular types. Then the pregeometry describes the independence: $(a_i \mid i \in \alpha)$ is a Morley sequence in \mathfrak{p} over A if and only if it is $\text{cl}_{\mathfrak{p},A}$ -independent. In particular, maximal Morley sequences in any $M \supseteq A$ have the same cardinality, so $\dim_{\mathfrak{p}}(M/A)$ is a well defined cardinal number.

3 Orthogonality

In this section we study orthogonality of regular symmetric types. Our goal is to prove Theorem 1. We start by mentioning a result from [7]; it will not be used further in the text:

Theorem 3.1. *A regular asymmetric type is orthogonal to any symmetric invariant type. In particular, symmetry is preserved under non-orthogonality of regular types.*

Question 1. *Is $\not\perp$ an equivalence relation on the set of all regular symmetric types?*

Below, a positive answer will be given for generically stable types.

Lemma 3.2. *Suppose that \mathfrak{p} and \mathfrak{q} are A -invariant and that \mathfrak{p} is regular over A and symmetric. Also suppose that $\bar{b} \models \mathfrak{q}_{\uparrow A} = q$ and $a \models \mathfrak{p}_{\uparrow A} = p$ are such that a does not realize $\mathfrak{p}_{\uparrow A\bar{b}}$ and let $\phi(x, \bar{b}) \in \text{tp}(a/A\bar{b}) \setminus \mathfrak{p}_{\uparrow A\bar{b}}$. Then exactly one of the following two conditions holds:*

(A) *Whenever $\{a_i \mid i \in \omega + \omega\}$ is a Morley sequence in \mathfrak{p} over A then*

$$q(\bar{y}) \cup \{\phi(a_i, \bar{y}) \mid i \in \omega\} \cup \{\neg\phi(a_{\omega+i}, \bar{y}) \mid i \in \omega\} \text{ is consistent.}$$

(B) *$p(x) \cup \mathfrak{q}_{\uparrow A}^2(\bar{y}_1, \bar{y}_2) \cup \{\phi(x, \bar{y}_1) \wedge \phi(x, \bar{y}_2)\}$ is inconsistent.*

Moreover, if \mathfrak{p} is generically stable then (B) holds.

Proof. Suppose that neither (A) nor (B) are satisfied and work for a contradiction. The failure of (A), by compactness, implies that for some n

$$q(\bar{y}) \cup \{\phi(a_i, \bar{y}) \mid 1 \leq i \leq n\} \cup \{\neg\phi(a_j, \bar{y}) \mid n < j \leq 2n\} \text{ is inconsistent; (1)}$$

We claim that $q(\bar{y}) \cup \{\phi(a_i, \bar{y}) \mid i \leq n\}$ is inconsistent. Otherwise it would be satisfied by some \bar{b}' and whenever $(a'_{n+1}, \dots, a'_{2n})$ is a Morley sequence in \mathfrak{p} over $A\bar{b}'_{a \leq n}$ we would have $\models \bigwedge_i \neg\phi(a'_i, \bar{b}')$; this is justified by $\neg\phi(x, \bar{b}') \in \mathfrak{p}$ which is implied by: $\neg\phi(x, \bar{b}) \in \mathfrak{p}$, the A -invariance of \mathfrak{p} , and $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}'/A) = q$. Therefore \bar{b}' realizes

$$q(\bar{y}) \cup \{\phi(a_i, \bar{y}) \mid i \leq n\} \cup \{\neg\phi(a'_j, \bar{y}) \mid n < j \leq 2n\}$$

which is in contradiction with (1).

Let n_ϕ be maximal such that $q(\bar{y}) \cup \{\phi(a_i, \bar{y}) \mid i < n_\phi\}$ is consistent and, without loss of generality, assume that \bar{b} realizes the type. The maximality of n_ϕ implies that no element of $\phi(\bar{M}, \bar{b}) \cap p(\bar{M})$ realizes $\mathfrak{p}_{\uparrow A\bar{c}}$, where \bar{c} denotes $a_0 \dots a_{n_\phi-1}$. Hence $\phi(\bar{M}, \bar{b}) \cap p(\bar{M}) \subseteq \text{cl}_{\mathfrak{p}, A}(\bar{c})$.

Fix $m > n_\phi$ and let $\bar{c}_0, \bar{c}_1, \dots, \bar{c}_m$ be a Morley sequence in $\mathfrak{p}_{\uparrow A}^{n_\phi}$. For each $i \leq m$ choose \bar{b}_i such that $\bar{c}_i \bar{b}_i \equiv \bar{c} \bar{b}(A)$; note that $\phi(\bar{M}, \bar{b}_i) \cap p(\bar{M}) \subseteq \text{cl}_{\mathfrak{p}, A}(\bar{c}_i)$ holds. Let \bar{b}' realize $\mathfrak{q}_{\uparrow A\bar{c} \leq m \bar{b} \leq m}$. (\bar{b}', \bar{b}_i) is a Morley sequence in \mathfrak{q} over A for each $i \leq m$, so the failure of (B) implies that $p(x) \cup \{\phi(x, \bar{b}') \wedge \phi(x, \bar{b}_i)\}$ is consistent; let d_i realize it. Then $\models \phi(d_i, \bar{b}_i)$ implies $d_i \in \text{cl}_{\mathfrak{p}, A}(\bar{c}_i)$. Since \mathfrak{p} is symmetric, $\text{cl}_{\mathfrak{p}, A}$ is a pregeometry so the $\text{cl}_{\mathfrak{p}, A}$ -independence of \bar{c}_i 's implies that d_0, d_1, \dots, d_m is a Morley sequence in $\mathfrak{p}_{\uparrow A}$. Thus \bar{b}' realizes $q(\bar{y}) \cup \{\phi(d_i, \bar{y}) \mid i \leq m\}$; this contradicts the maximality of n_ϕ . \square

In the next proposition we will prove that generically stable regular types "have weight 1 with respect to the \otimes -independence".

Proposition 3.3. *Suppose that $\mathfrak{p}, \mathfrak{q}$ and \mathfrak{r} are (not necessarily distinct) A -invariant types and that (\mathfrak{p}, A) is regular and generically stable. Let $\bar{b} \bar{c} \models \mathfrak{q} \otimes \mathfrak{r}_{\uparrow A}$ and $a \models \mathfrak{p}_{\uparrow A}$. Then at least one of $a \models \mathfrak{p}_{\uparrow A\bar{b}}$ and $a \models \mathfrak{p}_{\uparrow A\bar{c}}$ holds.*

Proof. Suppose that neither of them holds and choose $\varphi(x, \bar{b}) \in \text{tp}(a/A\bar{b}) \setminus \mathfrak{p}_{\uparrow A\bar{b}}$ and $\psi(x, \bar{c}) \in \text{tp}(a/A\bar{c}) \setminus \mathfrak{p}_{\uparrow A\bar{c}}$. Let $\phi(x, \bar{b}, \bar{c})$ be $\varphi(x, \bar{b}) \vee \psi(x, \bar{c})$. Let $\mathfrak{q}' = \mathfrak{q} \otimes \mathfrak{r}$, $\bar{b}' = \bar{b} \bar{c}$. Then we have:

- \mathfrak{p} and \mathfrak{q}' are A -invariant and \mathfrak{p} is regular and symmetric;
- $a \models \mathfrak{p}_{\uparrow A}$ and $\bar{b}' \models \mathfrak{q}'_{\uparrow A}$;
- a does not realize $\mathfrak{p}_{\uparrow A\bar{b}'}$ and $\phi(x, \bar{b}') \in \text{tp}(a/A\bar{b}') \setminus \mathfrak{p}_{\uparrow A\bar{b}'}$.

Therefore $\mathfrak{p}, \mathfrak{q}', a, \bar{b}'$ and $\phi(x, \bar{b}')$ satisfy assumptions of Lemma 3.2 in place of $\mathfrak{p}, \mathfrak{q}, a, \bar{b}$. Since \mathfrak{p} is generically stable option (B) holds:

$$\mathfrak{p}_{\uparrow A}(x) \cup (\mathfrak{q}')_{\uparrow A}^2(\bar{y}_1, \bar{y}_2) \cup \{\phi(x, \bar{y}_1) \wedge \phi(x, \bar{y}_2)\} \text{ is inconsistent.} \quad (1)$$

Let $\bar{b} \bar{c}, \bar{b}_1 \bar{c}_1$ be a Morley sequence in $\mathfrak{q} \otimes \mathfrak{r}$ over A and let a_1 be such that $a_1 \bar{c}_1 \equiv a \bar{c}(A\bar{b})$. Then $\models \varphi(a_1, \bar{b})$ implies $\models \phi(a_1, \bar{b}, \bar{c})$, and $\models \psi(a_1, \bar{c}_1)$ implies $\models \phi(a_1, \bar{b}_1, \bar{c}_1)$. Summing up, we have:

$$a_1 \models \mathfrak{p}_{\uparrow A}(x), \quad (\mathfrak{q}')_{\uparrow A}(\bar{b} \bar{c}, \bar{b}_1 \bar{c}_1) \text{ and } \models \phi(a_1, \bar{b}, \bar{c}) \wedge \phi(a_1, \bar{b}_1, \bar{c}_1)$$

which contradicts (1). This proves the proposition. \square

Corollary 3.4. *Suppose that \mathfrak{p} , \mathfrak{q} and \mathfrak{r} are invariant and that \mathfrak{p} is regular and generically stable. Then $\mathfrak{p} \perp \mathfrak{q} \otimes \mathfrak{r}$ if and only if $\mathfrak{p} \perp \mathfrak{q}$ and $\mathfrak{p} \perp \mathfrak{r}$.*

Lemma 3.5. *Suppose that \mathfrak{p} , \mathfrak{q} , \mathfrak{r} are A -invariant, \mathfrak{p} and \mathfrak{q} are regular over A , and that \mathfrak{p} is generically stable. Further, suppose that a, b, \bar{c} are realizations of p, q, r (where $x = \mathfrak{r}|_A$) respectively such that $b \not\models \mathfrak{q}|_{A\bar{c}}$ and $a \not\models \mathfrak{p}|_{Ab}$. Then:*

(i) $a \not\models \mathfrak{p}|_{A\bar{c}}$.

(ii) *For all $\phi(y, \bar{c}) \in \text{tp}(b/A\bar{c})$ witnessing $b \not\models \mathfrak{q}|_{A\bar{c}}$ and $\theta(b, x) \in \text{tp}(a/Ab)$ witnessing $a \not\models \mathfrak{p}|_{Ab}$ there exist $\varphi_q(y) \in \mathfrak{q}|_A$ and $\varphi_p(x) \in \mathfrak{p}|_A$ such that*

$$\exists y(\varphi_p(x) \wedge \varphi_q(y) \wedge \phi(y, \bar{c}) \wedge \theta(y, x)) \notin \mathfrak{p}|_{A\bar{c}}(x) .$$

Proof. (i) Suppose on the contrary that $a \models \mathfrak{p}|_{A\bar{c}}$. Let b' realize $\mathfrak{q}|_{A\bar{c}}$ and let a' be a realization of p such that $ab \equiv a'b'(A)$. We claim that $a' \models \mathfrak{p}|_{A\bar{c}}$ holds. Otherwise we have

$$a' \not\models \mathfrak{p}|_{A\bar{c}} \quad a' \not\models \mathfrak{p}|_{Ab'} \quad \text{and} \quad \bar{c}b' \models \mathfrak{r} \otimes \mathfrak{q}|_A$$

which contradicts Proposition 3.3. The claim implies $a' \equiv a(A\bar{c})$. Now choose b'' such that $a'b'' \equiv ab(A\bar{c})$. Then $b' \models \mathfrak{q}|_{A\bar{c}}$ and $b'' \not\models \mathfrak{q}|_{A\bar{c}}$, by regularity of $(\mathfrak{q}, A\bar{c})$, imply $b' \models \mathfrak{q}|_{Ab''\bar{c}}$. Hence (b'', b') is a Morley sequence in \mathfrak{q} over A . By Proposition 3.3 at least one of $a' \models \mathfrak{p}|_{Ab'}$ and $a' \models \mathfrak{p}|_{Ab''}$ holds. The first is not possible because $a'b' \equiv ab(A\bar{c})$, and the second because $a'b'' \equiv ab(A\bar{c})$. A contradiction.

(ii) By interpreting assumptions of the lemma and what we have just proved in part (i), we have:

$$\mathfrak{p}|_A(x) \cup \mathfrak{q}|_A(y) \cup \{\phi(y, \bar{c}) \wedge \theta(y, x)\} \cup \mathfrak{p}|_{A\bar{c}}(x) \text{ is inconsistent}$$

By compactness there are $\varphi_p(x) \in \mathfrak{p}|_A$, $\varphi_q(y) \in \mathfrak{q}|_A$ and $\sigma(x, \bar{c}) \in \mathfrak{p}|_{A\bar{c}}$ such that:

$$\varphi_p(x) \wedge \varphi_q(y) \wedge \phi(y, \bar{c}) \wedge \theta(y, x) \vdash \neg\sigma(x, \bar{c})$$

Therefore $\exists y(\varphi_p(x) \wedge \varphi_q(y) \wedge \phi(y, \bar{c}) \wedge \theta(y, x)) \vdash \neg\sigma(x, \bar{c})$ and

$$\exists y(\varphi_p(x) \wedge \varphi_q(y) \wedge \phi(y, \bar{c}) \wedge \theta(y, x)) \notin \mathfrak{p}|_{A\bar{c}}(x) .$$

This proves (ii). □

Proposition 3.6. *Generic stability is preserved under non-orthogonality of symmetric, regular types.*

Proof. Suppose that \mathfrak{p} and \mathfrak{q} are both regular, non-orthogonal and that \mathfrak{p} is generically stable. Choose a small model M such that both \mathfrak{p} and \mathfrak{q} are regular over M and $p \not\equiv^w q$ holds for their corresponding restrictions. Let $a \models p$, $b \models q$ and $\theta(y, x) \in \text{tp}(b, a/M)$ be such that $\theta(b, x) \notin \mathfrak{p}$.

Suppose for a contradiction that \mathfrak{q} is not generically stable. Then for a suitably chosen larger M , $\phi(y, \bar{z})$ over M , and a Morley sequence $\{b_i \mid i \in \omega + \omega\}$ in \mathfrak{q} over M there exists \bar{c} realizing

$$\{\phi(b_n, \bar{z}) \mid i \in \omega\} \cup \{\neg\phi(b_{\omega+n}, \bar{z}) \mid i \in \omega\}$$

Since \mathfrak{q} is symmetric, after possibly replacing the first and the second ω -part of the sequence, we may assume $\neg\phi(x, \bar{c}) \in \mathfrak{q}$. Also, after replacing the second ω -part by a Morley sequence in \mathfrak{q} over $M \bar{b}_{<\omega} \bar{c}$ we may assume that each $b_{\omega+n}$ realizes $\mathfrak{q}_{\uparrow M \bar{b}_{<\omega+n} \bar{c}}$.

For each $i \in \omega + \omega$ choose a_i such that $a_i b_i \equiv ab(M)$. Then $\theta(b_i, x) \notin \mathfrak{p}$ witnesses that $a_i \not\models \mathfrak{p}_{\uparrow M b_i}$. We claim that:

$$a_i \models \mathfrak{p}_{\uparrow M \bar{b}_{<i}} \text{ holds for all } i \in \omega + \omega. \quad (2)$$

To prove it note that $\bar{b}_{<i} b_i \models (\mathfrak{p}^{<i} \otimes \mathfrak{p})_{\uparrow M}$ so, by Proposition 3.3, at least one of $a_i \models \mathfrak{p}_{\uparrow M \bar{b}_{<i}}$ and $a_i \models \mathfrak{p}_{\uparrow M b_i}$ holds. Since $\models \theta(b_i, a_i)$ implies $a_i \not\models \mathfrak{p}_{\uparrow M b_i}$ we conclude that $a_i \models \mathfrak{p}_{\uparrow M \bar{b}_{<i}}$ holds, proving the claim. Combining (2) with $\bar{a}_{<i} \subseteq \text{cl}_{\mathfrak{p}, M}(\bar{b}_{<i})$ and the regularity of $(\mathfrak{p}, M \bar{b}_{<i})$ we derive:

$$a_i \models \mathfrak{p}_{\uparrow M \bar{b}_{<i} \bar{a}_{<i}} \text{ holds for all } i \in \omega + \omega.. \quad (3)$$

In particular, $(a_i \mid i \in \omega + \omega)$ is a Morley sequence in \mathfrak{p} over M .

Continuing the proof of the proposition we first note that a_0, b_0 and \bar{c} satisfy assumptions of the Lemma 3.5: let \mathfrak{r} be any global coheir of $\text{tp}(\bar{c}/M)$. Then $b_0 \not\models \mathfrak{q}_{\uparrow M \bar{c}}$ (witnessed by $\models \phi(b_0, \bar{c})$) and $a_0 \not\models \mathfrak{p}_{\uparrow M b}$ (witnessed by $\models \theta(b_0, a_0)$). So we apply Lemma 3.5(ii) and consider the formula

$$\exists y(\phi(y, \bar{c}) \wedge \varphi_{\mathfrak{q}}(y) \wedge \varphi_{\mathfrak{p}}(x) \wedge \theta(y, x)) \notin \mathfrak{p} .$$

Denote it by $\psi(x, \bar{c})$. Then $\models \psi(a_n, \bar{c})$ holds for all $n \in \omega$: the existential quantifier is witnessed by b_n . On the other hand, $a_{\omega+n} \models \mathfrak{p}_{\uparrow M \bar{c}}$ implies that $\models \neg\psi(a_{\omega+n}, \bar{c})$ holds for all $n \in \omega$. Therefore, \bar{c} realizes

$$\{\psi(a_n, \bar{z}) \mid n \in \omega\} \cup \{\neg\psi(a_{\omega+n}, \bar{z}) \mid n \in \omega\}$$

and \mathfrak{p} is not generically stable. A contradiction. \square

Proof of Theorem 1. It remains to prove that $\not\sim$ is an equivalence relation on the set of all generically stable regular types. Only transitivity needs verification, so assume that $\mathfrak{p} \not\sim \mathfrak{q}$ and $\mathfrak{q} \not\sim \mathfrak{r}$ are regular and generically stable. Let a, b, c realize $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ respectively be such that $b \not\models \mathfrak{q}_{\upharpoonright \bar{M}c}$ and $a \not\models \mathfrak{p}_{\upharpoonright \bar{M}b}$. Then, by Lemma 3.5(i), $a \not\models \mathfrak{p}_{\upharpoonright \bar{M}c}$ holds, so $\mathfrak{p} \not\sim \mathfrak{r}$ and $\not\sim$ is transitive. \square

4 Strong regularity

In this section we study non-orthogonality of an invariant type and a strongly regular type. We will show that the dependence of their realizations is witnessed by semi-isolation.

Lemma 4.1. *Suppose that $(\mathfrak{p}, \phi_{\mathfrak{p}})$ is A -invariant, definable and strongly regular and that a, b are realizations of $p = \mathfrak{p}_{\upharpoonright A}$.*

(i) *If $\text{tp}(a/Ab)$ is finitely satisfiable in $\mathcal{C}_{\mathfrak{p}} = \phi_{\mathfrak{p}}(\bar{M}) \setminus \mathfrak{p}_{\upharpoonright A}(\bar{M})$ then (a, b) is a Morley sequence in \mathfrak{p} over A .*

(ii) *If (a_0, \dots, a_n) is a $\mathcal{C}_{\mathfrak{p}}$ -sequence of realizations of p , then (a_n, \dots, a_0) is a Morley sequence in \mathfrak{p} over A .*

(iii) *If p is non-isolated then: $b \not\models \mathfrak{p}_{\upharpoonright Aa}$ iff $a \in \text{Sem}_A(b)$.*

Proof. (i) Assuming that (a, b) is not a Morley sequence we will show that $\text{tp}(a/Ab)$ is not a $\mathcal{C}_{\mathfrak{p}}$ -type. Let $\varphi(x, y) \in \text{tp}(a, b/A)$ witness $b \not\models \mathfrak{p}_{\upharpoonright Aa}$; then $\models \varphi(a, b) \wedge d_{\mathfrak{p}}t\varphi(a, t)$ holds. We claim that $\varphi(x, b) \wedge d_{\mathfrak{p}}t\varphi(x, t)$ is not satisfied in $\mathcal{C}_{\mathfrak{p}}$: otherwise, for some $c \in \mathcal{C}_{\mathfrak{p}}$ we would have $\models \varphi(c, b) \wedge d_{\mathfrak{p}}t\varphi(c, t)$ which implies $b \not\models \mathfrak{p}_{\upharpoonright Ac}$ and $p \not\sim^w \text{tp}(c/A)$. This is impossible because, by Lemma 2.5, $(p, \phi_{\mathfrak{p}})$ satisfies WOR.

(ii) Follows from part (i) by induction.

(iii) To prove the \Rightarrow part assume $b \not\models \mathfrak{p}_{\upharpoonright Aa}$. Then (a, b) is not a Morley sequence and, by part (i), $\text{tp}(a/Ab)$ is not a $\mathcal{C}_{\mathfrak{p}}$ -type. Choose $\theta(x, b) \in \text{tp}(a/Ab)$ which is not satisfied in $\mathcal{C}_{\mathfrak{p}}$. Then $\theta(x, b) \wedge \phi_{\mathfrak{p}}(x) \vdash p(x)$ witnesses $a \in \text{Sem}_A(b)$. This proves the \Rightarrow part.

For the \Leftarrow part assume $b \models \mathfrak{p}_{\upharpoonright Aa}$. Then (a, b) is a Morley sequence over A . Since p is non-isolated, it is finitely satisfiable in $\mathcal{C}_{\mathfrak{p}}$ so, by Fact 1.3, it has an extension in $S_1(Ab)$ which is finitely satisfiable in $\mathcal{C}_{\mathfrak{p}}$; let a' realize it. By part (i) (a', b) is a Morley sequence over A so $\text{tp}(a, b/A) = \text{tp}(a', b/A)$. Hence $\text{tp}(a/Ab)$ is finitely satisfiable in $\mathcal{C}_{\mathfrak{p}}$ and $a \notin \text{Sem}_A(b)$. \square

Proposition 4.2. *Suppose that $(\mathfrak{p}, \phi_{\mathfrak{p}})$ is A -invariant, strongly regular and generically stable. Further, suppose that \mathfrak{q} is A -invariant and that $a \models \mathfrak{p}_{\upharpoonright A}$ and $\bar{b} \models \mathfrak{q}_{\upharpoonright A}$ are such that $a \not\models \mathfrak{p}_{\upharpoonright A\bar{b}}$. Then \bar{b} semi-isolates a over A .*

Proof. Let p, q denote $\mathfrak{p}_{\upharpoonright A}$ and $\mathfrak{q}_{\upharpoonright A}$ respectively. Choose $\theta(x, \bar{b}) \in \text{tp}(a/A\bar{b})$ witnessing $a \not\models \mathfrak{p}_{\upharpoonright A\bar{b}}$ and, without loss of generality, assume $\models \theta(x, \bar{b}) \Rightarrow \phi_{\mathfrak{p}}(x)$. Suppose that the conclusion of the proposition fails: \bar{b} does not semi-isolate a over A . Then $\text{tp}(a/A\bar{b})$ is finitely satisfiable in $\mathcal{C} = \theta(\bar{M}, \bar{b}) \setminus p(\bar{M})$. Let $(a_i \mid i \leq n)$ be a \mathcal{C} -sequence of realizations of $\text{tp}(a/A\bar{b})$. Since $\mathcal{C} \subseteq \mathcal{C}_{\mathfrak{p}} = \phi_{\mathfrak{p}}(\bar{M}) \setminus p(\bar{M})$ $(a_i \mid i \leq n)$ is also a $\mathcal{C}_{\mathfrak{p}}$ -sequence. By Lemma 4.1 $(a_n, a_{n-1}, \dots, a_0)$ is a Morley sequence in \mathfrak{p} over A . Since $\models \theta(a_i, \bar{b})$ holds for all $i \leq n$

$$(\mathfrak{p}^n)_{\upharpoonright A}(x_1, \dots, x_n) \cup \{\theta(x_i, \bar{b}) \mid a \leq i \leq n\}$$

is consistent for all n . Let $(a'_i \mid i \in \omega)$ be an infinite Morley sequence in \mathfrak{p} over A . By compactness $q(\bar{y}) \cup \{\theta(a'_i, \bar{y}) \mid i \in \omega\}$ is consistent so, without loss of generality, assume that \bar{b} realizes that type. Let $(a'_{\omega+i} \mid i \in \omega)$ be an infinite Morley sequence in \mathfrak{p} over $A\bar{a}'_{<\omega}\bar{b}$. Then $\neg\theta(x, \bar{b}) \in \mathfrak{p}$ implies that $\models \neg\theta(a'_{\omega+i}, \bar{b})$ holds for all $i \in \omega$. Therefore \bar{b} realizes

$$q(\bar{y}) \cup \{\theta(a'_i, \bar{y}) \mid i \in \omega\} \cup \{\neg\theta(a'_{\omega+i}, \bar{y}) \mid i \in \omega\}$$

and \mathfrak{p} is not generically stable. A contradiction. \square

Lemma 4.3. $\mathfrak{p} \leq_{RK} \mathfrak{q}$ implies $\mathfrak{p} \not\perp \mathfrak{q}$.

Proof. Choose \bar{a} and \bar{b} realizing \mathfrak{p} and \mathfrak{q} respectively such that $\bar{a} \in \text{Sem}_{\bar{M}}(\bar{b})$; then $\text{tp}(\bar{a}/\bar{M}\bar{b})$ is not a coheir of \mathfrak{p} . Therefore $\text{tp}(\bar{a}/\bar{M}\bar{b})$ and a coheir are two distinct global extensions of $\mathfrak{p}(x)$, so $\mathfrak{p} \not\perp \mathfrak{q}$ holds. \square

The following is a technical version of Theorem 2:

Theorem 4.4. (1) *Strongly regular, generically stable types are minimal in the Rudin-Keisler global order. Moreover, if \mathfrak{p} is strongly regular and generically stable then for any \mathfrak{q} : $\mathfrak{p} \not\perp \mathfrak{q}$ iff $\mathfrak{p} \leq_{RK} \mathfrak{q}$.*

(2) *Non-orthogonal strongly regular, generically stable types are strongly RK-equivalent. Moreover, whenever a, b realize over \bar{M} such types \mathfrak{p} and \mathfrak{q} and $a \not\models \mathfrak{p}_{\upharpoonright \bar{M}b}$ holds, then: $b \not\models \mathfrak{q}_{\upharpoonright \bar{M}a}$, $b \in \text{Sem}_{\bar{M}}(a)$ and $a \in \text{Sem}_{\bar{M}}(b)$.*

Proof. (1) Suppose that \mathfrak{p} is strongly regular and generically stable. Then, by Lemma 4.3, $\mathfrak{p} \leq_{RK} \mathfrak{q}$ implies $\mathfrak{p} \not\perp \mathfrak{q}$. To prove the other direction assume $\mathfrak{q} \not\perp \mathfrak{p}$ and let $a \models \mathfrak{p}$ and $\bar{b} \models \mathfrak{q}$ be such that $a \not\models \mathfrak{p}_{\upharpoonright \bar{M}\bar{b}}$. Then, by Proposition 4.2, $a \in \text{Sem}_{\bar{M}}(\bar{b})$ holds, so $\mathfrak{p} \leq_{RK} \mathfrak{q}$. This proves the other implication.

(2) Suppose that both \mathfrak{p} and \mathfrak{q} are strongly regular, generically stable and that $a \models \mathfrak{p}$ and $b \models \mathfrak{q}$ are such that $a \not\models \mathfrak{p}_{\upharpoonright \bar{M}b}$. Then, by Proposition 4.2, $a \in \text{Sem}_{\bar{M}}(b)$ holds. Let $a_1 \models \mathfrak{p}$ be such that $b \not\models \mathfrak{p}_{\upharpoonright \bar{M}a_1}$. By Proposition 4.2 again we have $b \in \text{Sem}_{\bar{M}}(a_1)$. By transitivity $a \in \text{Sem}_{\bar{M}}(a_1)$ so (a_1, a) is not a Morley sequence in \mathfrak{p} . Then, by symmetry, (a, a_1) is not a Morley sequence so $a_1 \not\models \mathfrak{p}_{\upharpoonright \bar{M}a}$. By Lemma 4.1(iii) $a_1 \in \text{Sem}_{\bar{M}}(a)$ and, by transitivity, $b \in \text{Sem}_{\bar{M}}(a)$ holds. This proves the "moreover" part and strong RK-equivalence of \mathfrak{p} and \mathfrak{q} follows. \square

We have just proved that strongly regular types are RK-minimal. The converse is not true: Take the theory of the random graph. There distinct global 1-types are \leq_{RK} -incomparable, so every 1-type is RK-minimal. However, none of them is strongly regular. We now proceed towards proving Theorem 3.

Lemma 4.5. *Suppose that $(\mathfrak{p}(x), \phi_{\mathfrak{p}}(x))$ and $(\mathfrak{q}(x), \phi_{\mathfrak{q}}(x))$ are strongly regular, generically stable and non-orthogonal. Let $\mathfrak{r} \neq \mathfrak{q}$ be any global type containing $\phi_{\mathfrak{q}}(x)$. Then $\mathfrak{p} \perp \mathfrak{r}$.*

Proof. Suppose that the conclusion fails: $\mathfrak{p} \not\perp \mathfrak{r}$. Theorem 4.4(1) implies $\mathfrak{p} \leq_{RK} \mathfrak{r}$. $\mathfrak{p} \not\perp \mathfrak{q}$, by Theorem 4.4(2), implies $\mathfrak{p} \equiv_{RK} \mathfrak{q}$. Combining the two we conclude $\mathfrak{q} \leq_{RK} \mathfrak{r}$ and, by Lemma 4.3, $\mathfrak{q} \not\perp \mathfrak{r}$, contradicting the strong regularity of $(\mathfrak{q}(x), \phi_{\mathfrak{q}}(x))$. \square

Proof of Theorem 3. Suppose that $(\mathfrak{p}(x), \phi_{\mathfrak{p}}(x))$ and $(\mathfrak{q}(x), \phi_{\mathfrak{q}}(x))$ are M -invariant, strongly regular and generically stable. We will prove that the following conditions are all equivalent:

- (1) $\mathfrak{p} \not\perp \mathfrak{q}$;
- (2) $\mathfrak{p}_{\upharpoonright M} \not\perp^w \mathfrak{q}_{\upharpoonright M}$;
- (3) For all $C \supseteq M$: $\mathfrak{p}_{\upharpoonright C} \not\perp^w \mathfrak{q}_{\upharpoonright C}$.

(2) \Rightarrow (3) holds by Fact 1.1 (or by Proposition 5.2), and (3) \Rightarrow (1) is obvious. We will prove (1) \Rightarrow (2). So assume $\mathfrak{p} \not\perp \mathfrak{q}$ and let \bar{c} be such that $\mathfrak{p}_{\upharpoonright M\bar{c}} \not\perp^w \mathfrak{q}_{\upharpoonright M\bar{c}}$. Choose $a \models \mathfrak{p}_{\upharpoonright M\bar{c}}$ and $b \models \mathfrak{q}_{\upharpoonright M\bar{c}}$ such that $a \not\models \mathfrak{p}_{\upharpoonright M\bar{c}b}$. Suppose that $\varphi(x, b, \bar{c}) \notin \mathfrak{p}_{\upharpoonright M\bar{c}}$ is satisfied by a . Then:

$$\models d_{\mathfrak{p}}x \exists y (\phi_{\mathfrak{q}}(y) \wedge \varphi(x, y, \bar{c}) \wedge \neg d_{\mathfrak{p}t} \varphi(t, y, \bar{c})).$$

Let $\bar{c}' \in M$ satisfy the formula in place of \bar{c} . Then:

$$\models \exists y(\phi_{\mathfrak{q}}(y) \wedge \varphi(a, y, \bar{c}') \wedge \neg d_{\mathfrak{p}} t \varphi(t, y, \bar{c}')).$$

Let $\psi(x, y)$ be the formula $\phi_{\mathfrak{q}}(y) \wedge \varphi(x, y, \bar{c}') \wedge \neg d_{\mathfrak{p}} t \varphi(t, y, \bar{c}')$. We *claim* $\psi(a, y) \vdash \mathfrak{q}_{\uparrow M}(y)$. Suppose for a contradiction that b' satisfies $\models \psi(a, b')$ and $\text{tp}(b'/M) \neq \mathfrak{q}_{\uparrow M}$. Choose a formula $\theta(y) \in \text{tp}(b'/M)$ which is not in \mathfrak{q} and implies $\phi_{\mathfrak{q}}(y)$. Then $\models \psi(a, b') \wedge \theta(b')$ so $\exists y(\psi(x, y) \wedge \theta(y)) \in \mathfrak{p}$. Let $a_0 \models \mathfrak{p}$ and let b_0 be such that $\models \psi(a_0, b_0) \wedge \theta(b_0)$. $\models \psi(a_0, b_0)$ implies $\models \varphi(a_0, b_0, \bar{c}') \wedge \neg d_{\mathfrak{p}} t \varphi(t, b_0, \bar{c}')$ so $a_0 \not\models \mathfrak{p}_{\uparrow M b_0}$. Therefore $\mathfrak{p} \not\perp \mathfrak{r}$, where $\mathfrak{r} = \text{tp}(b_0/M)$. Now $\theta(y) \in \mathfrak{r}$ implies $\phi_{\mathfrak{q}}(y) \in \mathfrak{r} \neq \mathfrak{q}$ so, by Lemma 4.5, $\mathfrak{p} \perp \mathfrak{r}$. A contradiction. Thus $\psi(a, y) \vdash \mathfrak{q}_{\uparrow M}(y)$.

To finish the proof we note that $\mathfrak{q}_{\uparrow M}(y)$ has at least two distinct extensions in $S(Ma)$: one that contains $\psi(a, y)$ and the coheir. Therefore $\mathfrak{p}_{\uparrow M} \not\perp^w \mathfrak{q}_{\uparrow M}$. \square

5 Omitting types

Lemma 5.1. *Suppose that \mathfrak{p} is generically stable and regular over A , and that $\mathfrak{p}_{\uparrow A}$ is non-isolated. Then $\mathfrak{p}_{\uparrow B}$ is non-isolated for all $B \supseteq A$.*

Proof. Otherwise there is \bar{b} such that $\mathfrak{p}_{\uparrow A\bar{b}}$ is isolated, by $\varphi(x, \bar{b})$ say. Since $\mathfrak{p}_{\uparrow A}$ is non-isolated it has a non-isolated extension in $S_1(A\bar{b})$, so $\mathfrak{p}_{\uparrow A}(x) \cup \{\neg\varphi(x, \bar{b})\}$ is consistent. Let n be the length of the longest possible Morley sequence a_1, a_2, \dots, a_n in \mathfrak{p} over A satisfying $\models \bigwedge_{i=1}^n \neg\varphi(a_i, \bar{b})$; it exists because $\varphi(x, \bar{b}) \in \mathfrak{p}$ and \mathfrak{p} is generically stable. Since \mathfrak{p} is regular over $A\bar{b}$ and none of the a_i 's realize $\mathfrak{p}_{\uparrow A\bar{b}}$ we have $\mathfrak{p}_{\uparrow A\bar{b}}(x) \vdash \mathfrak{p}_{\uparrow A\bar{b}a_1 \dots a_n}(x)$. Let a realize $\mathfrak{p}_{\uparrow Aa_1 \dots a_n}$. The maximality of n implies $\text{tp}_{\bar{y}}(\bar{b}/A) \cup \{\bigwedge_{i=1}^n \neg\varphi(a_i, \bar{y})\} \vdash \varphi(a, \bar{y})$. Let $\phi(\bar{y}) \in \text{tp}_{\bar{y}}(\bar{b}/A)$ be such that:

$$\models \forall \bar{y}((\phi(\bar{y}) \wedge \bigwedge_{i=1}^n \neg\varphi(a_i, \bar{y})) \Rightarrow \varphi(a, \bar{y}))$$

Denote this formula by $\psi(a, a_1, \dots, a_n)$. Then $\models d_{\mathfrak{p}^n z_1 \dots z_n} \psi(a, z_1, \dots, z_n)$. We will reach the contradiction by showing that $d_{\mathfrak{p}^n z_1 \dots z_n} \psi(x, z_1, \dots, z_n)$ isolates $\mathfrak{p}_{\uparrow A}$. So let a' satisfy the formula and, without loss of generality, assume that (a_1, \dots, a_n) witnesses the $d_{\mathfrak{p}^n}$ quantifier: $a_1 \dots a_n \models \mathfrak{p}_{\uparrow Aa'}^n$. Then:

$$\models \forall \bar{y}((\phi(\bar{y}) \wedge \bigwedge_{i=1}^n \neg\varphi(a_i, \bar{y})) \Rightarrow \varphi(a', \bar{y})) \quad (1)$$

Let \bar{b}' be such that $\bar{b}' \equiv \bar{b}(Aa_1 \dots a_n)$. Then the left hand side of the implication in (1) is satisfied by \bar{b}' , so we derive $\models \varphi(a', \bar{b}')$. Since $\varphi(x, \bar{b}')$ isolates $\mathfrak{p}_{\upharpoonright A\bar{b}'}$ we conclude $a' \models \mathfrak{p}_{\upharpoonright A\bar{b}'}$ and, in particular, $a' \models \mathfrak{p}_{\upharpoonright A}$. \square

Proposition 5.2. *Suppose that \mathfrak{p} is generically stable and regular over A , and that \mathfrak{q} is A -invariant. Then $\mathfrak{p} \perp \mathfrak{q}$ implies $\mathfrak{p}_{\upharpoonright C} \perp^w \mathfrak{q}_{\upharpoonright C}$ for all $C \supseteq A$.*

Proof. Assume $\mathfrak{p}_{\upharpoonright A} \not\perp^w \mathfrak{q}_{\upharpoonright A}$ and let us prove that $\mathfrak{p}_{\upharpoonright A\bar{c}} \not\perp^w \mathfrak{q}_{\upharpoonright A\bar{c}}$ holds for all tuples \bar{c} . Witness $\mathfrak{p}_{\upharpoonright A} \not\perp^w \mathfrak{q}_{\upharpoonright A}$ by $a \models \mathfrak{p}_{\upharpoonright A}$, $\bar{b} \models \mathfrak{q}_{\upharpoonright A}$ and $\varphi(x, \bar{y}) \in \text{tp}(a, \bar{b}/A)$ such that $\varphi(x, \bar{b}) \notin \mathfrak{p}_{\upharpoonright A\bar{b}}$.

For each $\phi(x, \bar{c}) \in \mathfrak{p}_{\upharpoonright A\bar{c}}$ pick a Morley sequence of maximal possible length $(a_i^\phi \mid i \leq n_\phi)$ in \mathfrak{p} over A such that $\models \bigwedge_i \neg \phi(a_i^\phi, \bar{c})$. Since \mathfrak{p} is generically stable each of the chosen sequences is finite. Let D be the union of all them and, without loss of generality, assume $\bar{b} \models \mathfrak{q}_{\upharpoonright AD\bar{c}}$. We *claim* that a realizes $\mathfrak{p}_{\upharpoonright AD}$. Otherwise, there would be $\bar{d} \in D$ such that $a \not\models \mathfrak{p}_{\upharpoonright A\bar{d}}$. Moreover, such a \bar{d} can be chosen $\text{cl}_{\mathfrak{p}, A}$ -independent, i.e. realizing a Morley sequence in \mathfrak{p} over A . Then we would have:

$$\bar{d}, \bar{b} \models \mathfrak{p}^n \otimes \mathfrak{q}_{\upharpoonright A}, \quad a \not\models \mathfrak{p}_{\upharpoonright A\bar{d}} \quad \text{and} \quad a \not\models \mathfrak{p}_{\upharpoonright A\bar{b}}$$

which contradicts Proposition 3.3 and proves the claim: $a \models \mathfrak{p}_{\upharpoonright AD}$.

To complete the proof it remains to note that $\mathfrak{p}_{\upharpoonright AD} \vdash \mathfrak{p}_{\upharpoonright A\bar{c}}$ holds by our choice of D ; thus $a \models \mathfrak{p}_{\upharpoonright A\bar{c}}$. Summing up, we have: $\bar{b} \models \mathfrak{q}_{\upharpoonright A\bar{c}}$, $a \models \mathfrak{p}_{\upharpoonright A\bar{c}}$ and $a \not\models \mathfrak{p}_{\upharpoonright A\bar{b}}$. Thus $\mathfrak{p}_{\upharpoonright A\bar{c}} \not\perp^w \mathfrak{q}_{\upharpoonright A\bar{c}}$. \square

Corollary 5.3. *Suppose that \mathfrak{p} is generically stable and regular over A , and that \mathfrak{q} is A -invariant. Then $\mathfrak{p} \perp \mathfrak{q}$ implies $\mathfrak{p}_{\upharpoonright C}^\omega \perp^w \mathfrak{q}_{\upharpoonright C}^\omega$ for all $C \supseteq A$.*

Proof. Easy induction using Proposition 5.2. \square

Proof of Theorem 4. Suppose that A is countable and that $\{\mathfrak{p}_i \mid i \in I\}$ is a countable family of pairwise orthogonal, regular over A , generically stable types. Assume that each $\mathfrak{p}_i \upharpoonright A$ is non-isolated. Let $f : I \rightarrow \omega$. We will prove that there is a countable $M_f \supseteq A$ such that $\dim_{\mathfrak{p}_i}(M_f/A) = f(i)$ for all $i \in I$.

For each $i \in I$ for which $f(i) \neq 0$ holds choose a Morley sequence $J_i = (a_j^i \mid 1 \leq j \leq f(i))$ in \mathfrak{p}_i over A . Let J be the union of all the chosen sequences. By Lemma 5.1 each $p_i = \mathfrak{p}_i \upharpoonright AJ$ is non-isolated so, by the Omitting Types Theorem, there is a countable $M_f \supseteq AJ$ which omits all the p_i 's. We will prove that M_f is the desired model. $J_i \subseteq M_f$ implies $\dim_{\mathfrak{p}_i}(M_f/A) \geq f(i)$. To prove that the equality holds, it suffices to show that each $\mathfrak{p}_i \upharpoonright AJ_i$ is omitted

in M_f . By repeatedly applying Corollary 5.3 we get $\mathfrak{p}_{i \upharpoonright AJ_i} \vdash p_i$. Thus any realization of $\mathfrak{p}_{i \upharpoonright AJ_i}$ in M_f also realizes p_i . The latter type is omitted in M_f , so $\mathfrak{p}_{i \upharpoonright AJ_i}$ is omitted in M_f , too. Therefore $\dim_{\mathfrak{p}_i}(M_f/A) = f(i)$ completing the proof of the theorem. \square

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