A CLASSIFICATION OF 2-CHAINS HAVING 1-SHELL BOUNDARIES IN ROSY THEORIES

BYUNGHAN KIM, SUNYOUNG KIM, AND JUNGUK LEE

Abstract. We classify, in a non-trivial amenable collection of functors, all 2-chains up to the relation of having the same 1-shell boundary. In particular, we prove that in a rosy theory, every 1-shell of a Lascar strong type is the boundary of some 2-chain, hence making the 1st homology group trivial.

We also show that, unlike in simple theories, in rosy theories there is no upper bound on the minimal lengths of 2-chains whose boundary is a 1-shell.

1. Introduction

In [5],[6], J. Goodrick, A. Kolesnikov and the first author developed a homology theory for any amenable collection of functors in a very general context. But the most interesting examples appear in model theory. Namely, given any strong type $p \in S(A)$ in a rosy theory T, we may assign a non-trivial amenable collection of functors preserving thorn-independence and compute the corresponding homology groups. By the general theory, if T has n-complete amalgamation ($n \geq 2$) over $A = \operatorname{acl}(A)$ then the $(n-1)$-th homology group of $p \in S(A)$ consists of $(n-1)$-shells with the support $n+1 = \{0, \ldots, n\}$. Hence, in any simple T (where, due to 3-amalgamation, every 1-shell is the boundnary of some 2-simplex), the 1st homology group is trivial. But the question remained whether the same would hold in rosy theories. In this paper, we show that the answer is *yes* (as long as p is a Lascar type). A crucial ingredient in our proof is the fact that a and b realize the same Lascar type if and only if their Lascar distance is finite, i.e., $d_A(a, b) < \omega$. In the proof, the number of 2-simplices involved in a 2-chain having the 1-shell boundary is proportional to $d_A(a, b)$. Therefore one may guess that, there does not exist a uniform bound for the minimal lengths of 2-chains with 1-shell boundaries for various Lascar types in rosy theories,

The first author was supported by NRF of Korea grants 2011-0021916 and 2013R1A1A2073702. The second and third authors were supported by NRF of Korea grants 2012-030479 and 2012-044239. The third author was supported by European Community's Seventh Framework Programme [FP7/2007-2013] grant 238381.

contrary to the case of simple theories where the bound is 1, due to 3-amalgamation. A series of rosy examples in [2] where the Lascar distances increase are candidates. However in order to confirm that in each example that a candidate 2-chain has the expected minimal length, we need to rule out all other possibilities. For this goal we start to classify all the 2-chains having the same 1-shell boundary in a very general amenable context. The classification also has its own research interests. We obtain some interesting and important results in regard to the classification.

There are basically two operations on the class of 2-chains preserving the length and boundary of a chain. The first one is called the crossing (CR-)operation and the second one is called the renaming-of-support (RS-)operation. Two 2-chains are said to be equivalent if one is obtained from the other by finitely many applications of the two operations.

In the remainder of this section, we recall the definitions of an amenable family of functors and the corresponding homology groups introduced in [5],[6]. We thank Hyeung-Joon Kim and John Goodrick for their valuable suggestions and comments.

Notation. Throughout the paper, s denotes an arbitrary finite set of natural numbers. Given any subset $X \subseteq \mathcal{P}(s)$, we may view X as a category where for any $u, v \in X$, $\mathrm{Mor}(u, v)$ consists of a single morphism $\iota_{u,v}$ if $u \subseteq v$, and $\mathrm{Mor}(u, v) = \emptyset$ otherwise. If $f\colon X \to \mathcal{C}$ is any functor into some category $\mathcal{C}$ then for any $u, v \in X$ with $u \subseteq v$, we let f^u_v denote the morphism $f(\iota_{u,v}) \in \mathrm{Mor}_{\mathcal{C}}(f(u), f(v))$. We shall call $X \subseteq \mathcal{P}(s)$ a *primitive category* if X is non-empty and *downward closed*, i.e., for any $u, v \in \mathcal{P}(s)$, if $u \subseteq v$ and $v \in X$ then $u \in X$. (Note that all primitive categories have the empty set $\emptyset \subset \omega$ as an object.)

Remark/Definition 1.1. Given any primitive categories X and Y, define

$$X + Y := \{t \cup k \mid t \in X,\, k \in Y\}$$

which is clearly a primitive category itself containing X and Y as subcategories. And, for any $t \in X$, define

$$X_t := \{k \in X \mid t \cap k = \emptyset\} \quad \text{and} \quad X|_t := \{k \in X_t \mid t \cup k \in X\}$$

both of which are clearly primitive subcategories of X. Observe:

(1) $X|_t \subseteq X_t \subseteq X$
(2) $X \subseteq X_t + \mathcal{P}(t)$
(3) $X|_t = \bigcup\{\mathcal{P}(u \setminus t) \mid t \subseteq u \in X\}$.

Moreover, it is easy to check that the following are equivalent:
$X = X_t + \mathcal{P}(t) \iff X_t = X|_t \iff X = \bigcup\{\mathcal{P}(u) \mid t \subseteq u \in X\}$.

If one of these equivalent conditions holds, we shall say that X *splits at* t.

For any functor $f\colon X \to \mathcal{C}$ to some category $\mathcal{C}$ and for any $t \in X$, the *localization of* f *at* t is the functor $f|_t\colon X|_t \to \mathcal{C}$ defined as follows: for any $u \subseteq v \in X|_t$, $(f|_t)^u_v = f^{u\cup t}_{v\cup t}$ and $f|_t(v) = f(t \cup v)$.

Definition 1.2. Let $X \subseteq \mathcal{P}(s)$ and $Y \subseteq \mathcal{P}(t)$ be any primitive categories (where s, t are some finite sets of natural numbers). And let $f\colon X \to \mathcal{C}$ and $g\colon Y \to \mathcal{C}$ be any functors to some category $\mathcal{C}$.

(1) We say that f and g are *isomorphic* if there is an order-preserving bijection $\tau\colon s \to t$ such that $Y = \{\tau(u) \mid u \in X\}$ and there is a family of isomorphisms $\{h_u\colon f(u) \to g(\tau(u)) \mid u \in X\} \subseteq \mathrm{Mor}(\mathcal{C})$ such that, for any $u \subseteq v \in X$,
$$h_v \circ f^u_v = g^{\tau(u)}_{\tau(v)} \circ h_u.$$
(2) We say that f and g are *permutations of each other* if there is a bijection $\sigma\colon s \to t$ (not necessarily order-preserving) such that $Y = \{\sigma(u) \mid u \in X\}$ and, for any $u \subseteq v \in Y$, $g(v) = f(\sigma^{-1}(v))$ and $(g)^u_v = f^{\sigma^{-1}(u)}_{\sigma^{-1}(v)}$. In this case, we write $g = f \circ \sigma^{-1}$.

Note that, if f and g are permutations of each other via an *order-preserving* map $\sigma\colon s \to t$, then f and g are isomorphic.

Definition 1.3. Let $\mathcal{A}$ be a non-empty family of functors from various primitive categories into some fixed category $\mathcal{C}$. We say that $\mathcal{A}$ is *amenable* if it satisfies the following properties:

(1) (Closed under isomorphism and permutation) If $f \in \mathcal{A}$ then any functor g which is isomorphic to f or is a permutation of f also belongs to $\mathcal{A}$.
(2) (Closed under restriction and union) For any functor $f\colon X \to \mathcal{C}$ from some primitive category X into $\mathcal{C}$,
$$f \in \mathcal{A} \iff \text{for every } t \in X, f \restriction \mathcal{P}(t) \in \mathcal{A}.$$
(3) (Closed under localization) If $f\colon X \to \mathcal{C}$ is any functor in $\mathcal{A}$ then for every $t \in X$, $f|_t\colon X|_t \to \mathcal{C}$ is also in $\mathcal{A}$.
(4) (Extensions of localizations are localizations of extensions) Let $f\colon X \to \mathcal{C}$ be any functor in $\mathcal{A}$ which splits at some $t \in X$. Then whenever $f|_t$ can be extended to some functor $g\colon Z \to \mathcal{C}$ in $\mathcal{A}$ where $t \cap \bigcup Z = \emptyset$, f can be extended to some functor $h\colon \mathcal{P}(t) + Z \to \mathcal{C}$ in $\mathcal{A}$ such that $h|_t = g$.

Definition 1.4. By a (*regular*) *n-simplex* in a category $\mathcal{C}$, we mean a functor $f\colon \mathcal{P}(s) \to \mathcal{C}$ where $s \subseteq \omega$ has the size $n+1$. We call s the *support* of f and denote it by $\mathrm{supp}(f)$.

Definition 1.5. Let $\mathcal{A}$ be an amenable family of functors into some category $\mathcal{C}$. Let $B \in \mathrm{Ob}(\mathcal{C})$. If f is a functor in $\mathcal{A}$ such that $f(\emptyset) = B$, we shall say that f is *over* B. And we define:

$$S_n(\mathcal{A}; B) := \{\, f \in \mathcal{A} \mid f \text{ is a regular } n\text{-simplex over } B \,\}$$
$$C_n(\mathcal{A}; B) := \text{the free abelian group generated by } S_n(\mathcal{A}; B).$$

The elements of $C_n(\mathcal{A}; B)$ are called the *n-chains over B in $\mathcal{A}$*. For each $i = 0, \dots, n$, we define a group homomorphism

$$\partial_n^i\colon C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)$$

by letting, for any n-simplex $f\colon \mathcal{P}(s) \to \mathcal{C}$ in $S_n(\mathcal{A}; B)$ where $s = \{s_0 < \cdots < s_n\}$,

$$\partial_n^i(f) := f \restriction \mathcal{P}(s \setminus \{s_i\})$$

and then extending linearly to all n-chains in $C_n(\mathcal{A}; B)$. Then we define the *boundary map*

$$\partial_n\colon C_n(\mathcal{A}; B) \to C_{n-1}(\mathcal{A}; B)$$

by

$$\partial_n(c) := \sum_{0 \le i \le n} (-1)^i \partial_n^i(c).$$

We shall often refer to $\partial_n(c)$ as the *boundary of c*. Next, we define:

$$Z_n(\mathcal{A}; B) := \mathrm{Ker}\ \partial_n$$
$$B_n(\mathcal{A}; B) := \mathrm{Im}\ \partial_{n+1}.$$

The elements of $Z_n(\mathcal{A}; B)$ and $B_n(\mathcal{A}; B)$ are called *n-cycles* and *n-boundaries*, respectively. It is straightforward to check

$$\partial_{n-1} \circ \partial_n = 0.$$

Hence we may define

$$H_n(\mathcal{A}; B) := Z_n(\mathcal{A}; B)/B_n(\mathcal{A}; B)$$

called the n-th *(simplicial) homology group of $\mathcal{A}$ over B*.

Notation 1.6. (1) For $c \in Z_n(\mathcal{A}; B)$, $[c]$ denotes the coset of $B_n(\mathcal{A}; B)$ containing c.
(2) When n is clear from context, we shall often omit n from ∂_n^i and ∂_n, writing simply as ∂^i and ∂.

(3) When we write an n-chain $c \in C_n(\mathcal{A}; B)$ as

$$c = \sum_{i=1}^{k} n_i f_i$$

we shall assume, unless stated otherwise, that n_i's are nonzero integers and f_i's are distinct n-simplices. (This form is called the *standard form* of a chain.) For such an n-chain c, we define the *length* of c and the *support* of c as $|c| := \sum_{i=1}^{k} |n_i|$ and $\mathrm{supp}(c) := \bigcup_{i=1}^{k} \{\mathrm{supp}(f_i)\}$, respectively.

(4) For $c, d \in C_n(\mathcal{A}; B)$, we say that d is a *subchain (or subsummand) of* c if they are in the standard forms

$$c = \sum_{i=1}^{k} n_i f_i \qquad \text{and} \qquad d = \sum_{i \in J} m_i f_i,$$

where $J \subseteq \{1, \ldots, k\}$ and, for each $i \in J$, $n_i \cdot m_i > 0$ and $|m_i| \leq |n_i|$.

Remark/Definition 1.7. Let c be any n-chain and let d be a subsummand of c. For any n-chain d', we shall say that the n-chain

$$c' := c - d + d'$$

is obtained by *replacing the subsummand d in c by d'*. Note that, if $|d'| \leq |d|$ then $|c'| \leq |c|$.

Remark/Definition 1.8. Given any bijection $\sigma \colon \omega \to \omega$ (not necessarily order-preserving), we may induce an automorphism $\sigma_n^* \colon C_n(\mathcal{A}; B) \to C_n(\mathcal{A}; B)$ for each n as follows: for any n-chain $c = \sum_i n_i f_i \in C_n(\mathcal{A}, B)$, where each f_i is an n-simplex with $s_i := \mathrm{supp}(f_i) = \{s_{i,0} < \cdots < s_{i,n}\}$, we let $\sigma_i := \sigma \restriction s_i$ and $t_i := \sigma_i(s_i) = \{t_{i,0} < \cdots < t_{i,n}\}$. We define

$$\sigma^*(c) := \sum_i n_i |\sigma_i| f_i \circ \sigma_i^{-1}$$

(see Definition 1.2(2)) with $|\sigma_i| := \mathrm{sign}(\sigma_i')$ $(= \pm 1)$ where $\sigma_i' \in \mathrm{Sym}(n+1)$ such that for $j \leq n$, $\sigma_i(s_{i,j}) = t_{i,\sigma_i'(j)}$. For example

$$\sigma^*(f_i) = |\sigma_i| f_i \circ \sigma_i^{-1}.$$

Moreover, σ^* commutes with the boundary map, i.e., $\partial \circ \sigma^* = \sigma^* \circ \partial$. This can be verified inductively by first checking the case where σ is a transposition.

Next we define the amalgamation properties. For $n = \{0, \ldots, n-1\}$, we let $\mathcal{P}^-(n) := \mathcal{P}(n) \setminus \{n\}$. i.e., $\mathcal{P}^-(n)$ is the set of all the *proper* subsets of n.

Definition 1.9. Let $\mathcal{A}$ be an amenable family of functors into a category $\mathcal{C}$.

(1) $\mathcal{A}$ has *n-amalgamation* ($n \geq 1$) if every functor $f\colon \mathcal{P}^-(n) \to \mathcal{C}$ in $\mathcal{A}$ can be extended to some functor $g\colon \mathcal{P}(n) \to \mathcal{C}$ in $\mathcal{A}$.
(2) $\mathcal{A}$ has *n-complete amalgamation* (written n-CA) if it has k-amalgamation for every $1 \leq k \leq n$.
(3) $\mathcal{A}$ has *strong 2-amalgamation* if, whenever $f\colon \mathcal{P}(s) \to \mathcal{C}$ and $g\colon \mathcal{P}(t) \to \mathcal{C}$ are simplices in $\mathcal{A}$ which agree on $\mathcal{P}(s \cap t)$, then there exists some simplex $h\colon \mathcal{P}(s \cup t) \to \mathcal{C}$ in $\mathcal{A}$ extending both f and g.

Remark 1.10. It is easy to verify that, for any amenable family $\mathcal{A}$:

(1) strong 2-amalgamation $\Rightarrow$ 2-amalgamation.
(2) (1-amalgamation + strong 2-amalgamation) $\Rightarrow$ $\mathcal{A}$ has n-simplices for every $n \geq 0$.

Definition 1.11. An amenable family of functors is called *non-trivial* if it has 1-amalgamation and strong 2-amalgamation (in particular, it has 2-CA).

Definition 1.12. An n-chain $c \in C_n(\mathcal{A}; B)$ is called an *n-shell* if it is in the form

$$c = \pm \sum_{0 \leq i \leq n+1} (-1)^i f_i$$

where f_i's are n-simplices satisfying

$$\partial^i f_j = \partial^{j-1} f_i \quad \text{whenever } 0 \leq i < j \leq n+1.$$

We define $E_n(\mathcal{A}; B) := \{\, c \in C_n(\mathcal{A}; B) \mid c \text{ is an } n\text{-shell} \,\}$.

It is straightforward to verify the following proposition.

Proposition 1.13. *(1) $E_n(\mathcal{A}; B) \subset Z_n(\mathcal{A}; B)$.*

(2) For every $f \in S_n(\mathcal{A}; B)$, $\partial_n(f) \in E_{n-1}(\mathcal{A}; B)$.
(3) If $c = \pm \sum_{0 \leq i \leq n+1} (-1)^i f_i$ is any n-shell, then $|\operatorname{supp}(c)| = n+2$. Moreover, there exists a unique functor $g\colon \mathcal{P}^-(\operatorname{supp}(c)) \to \mathcal{C}$ in $\mathcal{A}$ extending all the f_i's. More precisely, if we let $\operatorname{supp}(c) = \{s_0 < \cdots < s_{n+1}\}$, then $g \restriction \mathcal{P}(\operatorname{supp}(c) \setminus \{s_i\}) = f_i$ for each i.
(4) $\mathcal{A}$ has $(n+2)$-amalgamation if and only if for any n-shell c, there exists some $(n+1)$-simplex d such $c = \pm\partial(d)$.

Definition 1.14. An amenable family of functors has *weak 3-amalgamation* if each 1-shell is the boundary of some 2-chain c with $|c| \leq 3$.

The following result due to [5], [6] illustrates the importance of the notion of shell.

Fact 1.15. [5][6] *Let $\mathcal{A}$ be any non-trivial amenable family of functors. If $\mathcal{A}$ has $(n+1)$-CA for some $n \geq 1$, then*

$$H_n(\mathcal{A}; B) = \{ [c] \mid c \in E_n(\mathcal{A}; B),\ \mathrm{supp}(c) = \{0, \ldots, n+1\} \}.$$

In particular,

(1) $H_1(\mathcal{A}; B) = 0 \Leftrightarrow E_1(\mathcal{A}; B) \subset B_1(\mathcal{A}; B)$

(2) If $\mathcal{A}$ has weak 3-amalgamation then $H_1(\mathcal{A}; B) = 0$.

In the remainder of the paper, $\mathcal{A}$ shall denote a non-trivial amenable family of functors into a category $\mathcal{C}$.

Now we begin to talk about the prototypical examples of an amenable family of functors : complete types in rosy theories. **In the sequel we work with a large saturated model $\mathcal{M} = \mathcal{M}^{\mathrm{eq}}$ and its theory T which is rosy.** Recall that a theory is called *rosy* if there is a ternary independence relation $\downarrow$ on the small sets of its model, satisfying the basic independence properties. (See [1], [4] for the precise definition.) We take $\downarrow$ here to be thorn-independence. Any simple or o-minimal theory is known to be rosy. Moreover, if a simple theory T has elimination of hyperimaginaries then non-forking independence is equal to thorn-independence. **So we assume that any simple T in this paper has elimination of hyperimaginaries**. (Of course this is just for convenience as we can work in $\mathcal{M}^{\mathrm{heq}}$ without the assumption.) In particular, we assume that 3-amalgamation holds over any algebraically closed set in simple T.

We fix any algebraically closed small subset $B \subseteq \mathcal{M}$ and consider the category $\mathcal{C}_B$ whose objects are all the small subsets of $\mathcal{M}$ containing B, and whose morphisms are elementary maps over B (i.e., fixing B pointwise). We also fix any $p(x) \in S(B)$ (where x could be an infinite tuple). When f is any functor from a primitive category X into $\mathcal{C}_B$ and $u \subseteq v \in X$, we shall abbreviate $f^u_v(f(u))$ as $f^u_v(u)$.

Definition 1.16. By a *closed independent functor in* $p(x)$, we mean a functor f from some primitive category X into $\mathcal{C}_B$ satisfying the following:

(1) Whenever $\{i\} \subset \omega$ is an object in X, we can choose a realization $b \models p(x)$ such that, if we let $C := f^{\emptyset}_{\{i\}}(\emptyset)$ then $f(\{i\}) = \mathrm{acl}(Cb)$ and $b \downarrow_B C$.

(2) Whenever $u(\neq \emptyset) \subset \omega$ is an object in X, we have
$$f(u) = \operatorname{acl}\left(\bigcup_{i \in u} f_u^{\{i\}}(\{i\})\right)$$
and $\{f_u^{\{i\}}(\{i\}) |\ i \in u\}$ is independent over $f_u^{\emptyset}(\emptyset)$.

We let $\mathcal{A}(p)$ be the family of all closed independent functors in p.

Fact 1.17. [6] *$\mathcal{A}(p)$ is a non-trivial amenable family of functors.*

Notation 1.18. We shall abbreviate $S_n(\mathcal{A}(p); B), C_n(\mathcal{A}(p); B), \dots$ as $S_n\mathcal{A}(p), C_n\mathcal{A}(p), \dots$. We shall also abbreviate $H_n(\mathcal{A}(p); B)$ simply as $H_n(p)$. Other than this, we use standard notation. For example $a \equiv_A b$ denotes $\operatorname{tp}(a/A) = \operatorname{tp}(b/A)$; and $a \equiv_A^L b$ denotes $\operatorname{Ltp}(a/A) = \operatorname{Ltp}(b/A)$, i.e., the Lascar (strong) types of a, b over A are the same.

2. $H_1(p)$ in rosy theories

If a theory T is simple then due to 3-amalgamation and Fact 1.15, we know $H_1(p) = 0$. In this section we show the same holds for any rosy T as long as p is a Lascar type.

Let $f\colon X \to \mathcal{C}_B$ be any functor in $\mathcal{A}(p)$ with $f(\emptyset) = B$. If $u \in X$ with $u = \{i_0 < \cdots < i_k\}$, we shall write $f(u) = [a_0, \dots, a_k]$, where $a_j \models p$, $f(u) = \operatorname{acl}(B, a_0 \cdots a_k)$, and $\operatorname{acl}(a_j B) = f_u^{\{i_j\}}(\{i_j\})$. Thus $\{a_0, \dots, a_k\}$ is independent over B.

Theorem 2.1. *If B is a model, then $\mathcal{A}(p)$ has weak 3-amalgamation over B (so $H_1(p) = 0$).*

Proof. Let $f = a_{12} - a_{02} + a_{01}$ be any 1-shell in $E_1\mathcal{A}(p)$ where each $a_{ij}\colon \mathcal{P}(\{i,j\}) \to \mathcal{C}_B$ is a 1-simplex. We want to find a 2-chain g with length 3 such that $\partial g = f$. For this goal there is no harm in assuming that $a_{01}(\{1\}) = [a] = a_{12}(\{1\})$ and $a_{12}(\{2\}) = [b] = a_{02}(\{2\})$. Let $a_{01}(\{0\}) := [c]$ and $a_{02}(\{0\}) := [c']$, and let q be a coheir of p over $Babcc'$. Choose any $c'' \models q$. Then $c'' \mathop{\smile\!\!\!\!\!\!|}_B abcc'$ (see [4]) and $cc'' \equiv_B c'c''$. Now let $g := a_{123} - a_{023} + a_{013}$ where a_{ij3} are 2-simplices having support $\{i,j,3\}$ extending a_{ij} such that $a_{123}(\{1,2,3\}) = [a,b,c'']$, $a_{023}(\{0,2,3\}) = [c',b,c'']$, $a_{013}(\{0,1,3\}) = [c,a,c'']$. Hence we may assume $\partial^0(a_{023}) = \partial^0(a_{123})$ and $\partial^0(a_{013}) = \partial^1(a_{123})$. But $cc'' \equiv_B c'c''$ implies that we may further assume $\partial^1(a_{013}) = \partial^1(a_{023})$. Therefore $\partial g = f$ as desired. $\square$

Remark 2.2. Of course the same proof shows that weak 3-amalgamation (over a model) holds not only in $\mathcal{A}(p)$ but more generally inside $\mathcal{M}$ (with arbitrary vertices).

Recall that, for any tuples a and b, we write $d_B(a,b) \leq n$ iff there is a sequence of tuples $c_0, \ldots, c_n$ with $c_0 = a$ and $c_n = b$, such that each $c_i c_{i+1}$ begins some B-indiscernible sequence. The smallest such n (if it exists) is denoted by $d_B(a,b)$ (called the *Lascar distance between a and b*). Recall the fact that $a \equiv^L_B b$ iff $d_B(a,b) < \omega$ in any rosy theory.

Lemma 2.3. *Let $I = \langle a_0, a_1, \ldots \rangle$ be any B-indiscernible sequence. Then for any c_0 there is $c \equiv_B c_0$ such that $c \mathop{\smile\!\!\!\!\!\!\;\,\,\mid}_B a_0 a_1$ and $ca_0 \equiv_B ca_1$.*

Proof. Extend I to I' indiscernible over B having a sufficiently large length. Then by the extension axiom there is $c' \equiv_B c_0$ such that $c' \mathop{\smile\!\!\!\!\!\!\;\,\,\mid}_B I'$. Moreover, by the pigeonhole principle, there are $a_i, a_j \in I'$ $(i < j)$ such that $c'a_i \equiv_B c'a_j$. Now, by B-indiscernibility, there is c such that $ca_0a_1 \equiv_B c'a_ia_j$. Then c is the desired tuple. □

Theorem 2.4. *Suppose that p is a Lascar strong type. Then $H_1(p) = 0$.*

Proof. For notational simplicity we let $B = \emptyset$. As in the proof of Theorem 2.1, given any 1-shell $f = a_{12} - a_{02} + a_{01}$ in $E_1\mathcal{A}(p)$ where each $a_{ij} \colon \mathcal{P}(\{i,j\}) \to \mathcal{C}_B$ is a 1-simplex, we want to find a 2-chain g such that $\partial g = f$. Again there is no harm in assuming that $a_{01}(\{1\}) = [a] = a_{12}(\{1\})$ and $a_{12}(\{2\}) = [b] = a_{02}(\{2\})$. Let $a_{01}(\{0\}) := [c]$ and $a_{02}(\{0\}) := [c']$. By extension we can further assume $\{a,b,c,c'\}$ is independent. Now $c, c' \models p$ and let $d(c,c') = n$. So there are $c = c_0, \ldots, c_n = c'$ such that $c_i c_{i+1}$ begins an indiscernible sequence, for $i < n$. We can further assume that $ab \mathop{\smile\!\!\!\!\!\!\;\,\,\mid}_{cc'} c_1 c_{n-1}$; so $ab \mathop{\smile\!\!\!\!\!\!\;\,\,\mid} c_0 \cdots c_n$. Then by Lemma 2.3, there are $e_i \models p$ $(i < n)$ such that $c_i c_{i+1} \mathop{\smile\!\!\!\!\!\!\;\,\,\mid} e_i$ and $e_i c_i \equiv e_i c_{i+1}$ (*). Again by extension we suppose $ab \mathop{\smile\!\!\!\!\!\!\;\,\,\mid}_{c_i c_{i+1}} e_i$, so that each of the $\{a, c_i, e_i\}, \{a, c_{i+1}, e_i\}$ is independent. Moreover each $\{a, e_{n-1}, b\}, \{e_{n-1}, c_n, b\}$ is independent as well (**).

Now there is $g_0 := g_0^+ - g_0^-$ where g_0^+, g_0^- are 2-simplices with support $\{0,1,3\}$ such that $g_0^+(\{0,1,3\}) = [c_0, a, e_0]$ and $g_0^-(\{0,1,3\}) = [c_1, a, e_0]$; $\partial^0 g_0^+ = \partial^0 g_0^-$; $\partial^1 g_0^+ = \partial^1 g_0^-$ (this is possible by (*)); and g_0^+ extends a_{01} (i.e., $\partial^2 g_0^+ = a_{01}$). Hence $\partial g_0 = a_{01} - \partial^2 g_0^-$.

By iteration we can find $g_i := g_i^+ - g_i^-$ $(0 < i < n-1)$ where g_i^+, g_i^- are 2-simplices with support $\{0,1,3\}$ such that $g_i^+(\{0,1,3\}) = [c_i, a, e_i]$ and $g_i^-(\{0,1,3\}) = [c_{i+1}, a, e_i]$; $\partial^0 g^+ = \partial^0 g^-$; $\partial^1 g^+ = \partial^1 g^-$ (this again is possible by (*)); and $\partial^2 g_i^+ = \partial^2 g_{i-1}^-$. Therefore we have

$$\partial(g_0 + \cdots + g_{n-2}) = a_{01} - \partial^2 g_{n-2}^-.$$

The rest of the proof is similar to that of Theorem 2.1. We put $g_{n-1} := g_{n-1}^+ - a_{023} + a_{123}$ where a_{j23} is a 2-simplex with support $\{j,2,3\}$ extending a_{j2} such that $a_{023}(\{0,2,3\}) = [c_n, b, e_{n-1}], a_{123}(\{1,2,3\}) = [a, b, e_{n-1}]$ (see (**)). Also g_{n-1}^+ is a 2-simplex with $g_{n-1}^+(\{0,1,3\}) =$

$[c_{n-1}, a, e_{n-1}]$ extending $\partial^2 g^-_{n-2}$. Moreover again by (*), we have $\partial^1 g^+_{n-1} = \partial^1 a_{023}$. Thus it follows

$$\partial g_{n-1} = \partial^2 g^+_{n-1} - a_{02} + a_{12} = \partial^2 g^-_{n-2} - a_{02} + a_{12}.$$

Therefore for $g := g_0 + \cdots + g_{n-1}$, we have $\partial g = f$ as desired. $\square$

3. A classification of 2-chains with a 1-shell boundary

In this section, we bring our attention back to a non-trivial amenable family of functors $\mathcal{A}$ and classify 2-chains of $\mathcal{A}$ having 1-shell boundaries. Basically we show that any 2-chain having a 1-shell boundary is equivalent to one of two types of 2-chains, called the *NR-type* and the *RN-type*.

We start by introducing two operations on 2-chains called the *crossing operation* and the *renaming-of-support operation*, respectively. For any distinct real numbers x and y, we shall abbreviate the open interval $(\min\{x,y\}, \max\{x,y\})$ as $[(x,y)] = [(y,x)]$.

Definition 3.1. Let $v \in C_2(\mathcal{A}; B)$ be a 2-chain and let $w := \epsilon_1\alpha_1 + \epsilon_2\alpha_2$ be a subsummand of v, where α_i's are 2-simplices with for $i = 1, 2$, $\epsilon_i = \pm 1$, $\mathrm{supp}(\alpha_i) = \{\ell_1, \ell_2, k_i\}$ (k_i, ℓ_i being all distinct numbers) such that α_1 and α_2 agree on the intersection of their domains, namely $\mathcal{P}(\{\ell_1, \ell_2\})$. Further assume that, if we let $\gamma := \alpha_i \restriction \mathcal{P}(\{\ell_1, \ell_2\})$, then γ does not appear in $\partial(w)$, i.e., the two γ terms in $\partial(w)$ have opposite signs and cancel each other.

Now by strong 2-amalgamation, there exists some 3-simplex μ extending both α_i. For $i = 1, 2$, let $\beta_i := \mu \restriction \mathcal{P}(\{k_1, k_2, \ell_i\})$ and

$$w' := \begin{cases} \epsilon_2\ \beta_1 + \epsilon_1\ \beta_2 & \text{if } \epsilon_1\epsilon_2 = -1, \text{and exactly one of } k_2, \ell_1 \text{ belongs to } [(k_1, \ell_2)] \\ \epsilon_1\ \beta_1 + \epsilon_2\ \beta_2 & \text{otherwise.} \end{cases}$$

Then the operation of replacing the subsummand w in v by w' is called the *crossing operation* (or simply *CR-operation*).

Example 3.2. Let f_0, f_1, f_2, f_3 be 2-simplices with $\mathrm{supp}(f_i) = \{0,1,2,3\} \setminus \{i\}$. Assume that f_i and f_j agree on their intersection, for every pair i, j. Consider the 2-chain $c = f_0 - f_1 + f_2$. Then we can apply the CR-operation to the subsummand $f_0 - f_1$ to obtain a new 2-chain

$$c' = (-f_2 + f_3) + f_2 \text{ or simply } f_3.$$

This example illustrates in particular that a CR-operation may not be reversible. i.e., once we apply a CR-operation to a 2-chain, we may not be able to recover the original 2-chain by applying more CR-operations (unless we allow 2-chains to be written redundantly as $f_3 - f_2 + f_2$).

Next, we define an operation on n-chains called the *renaming-of-support operation*.

Definition 3.3. Let c be an n-chain in $C_n(\mathcal{A}; B)$ and let d be a subsummand of c. Let $j \in \mathrm{supp}(d)$ such that $j \notin \mathrm{supp}(\partial_n(d))$. (In this situation, we say that d has a *vanishing support*, namely j, in its boundary.) Choose any $k \notin \mathrm{supp}(c)$ and any bijection $\sigma\colon \omega \to \omega$ which sends $j \mapsto k$ but which fixes the rest of the elements in $\mathrm{supp}(c)$. Then the operation of replacing the subsummand d in c by $\sigma_n^*(d)$ is called the *renaming-of-support operation* (or simply *RS-operation*). (See Remark/Definition 1.8 to recall the definition of σ_n^*.)

Remark 3.4. When we apply the CR- and RS-operation to some subsummand of an n-chain c, the resulting n-chain has the same boundary as c (guaranteed by the fact that σ_n^* commutes with the boundary map ∂) and has a shorter or equal length as c (by Remark/Definition 1.7 and the clear fact that σ_n^* preserves the lengths of n-chains).

Remark/Definition 3.5. A 2-chain c is called *proper* if its length $|c|$ does not change after any finitely many applications of CR/RS-operations to its subsummands. It is clear that any 2-chain may be reduced to a proper 2-chain after finitely many applications of the two operations. Any CR-operation (also RS-operation) applied to any *proper* 2-chain is in fact reversible. This allows us to define an equivalence relation $\sim$ among proper 2-chains by: $c \sim c' \Leftrightarrow c$ can be obtained from c' by finitely many applications of the CR/RS-operations to its subsummands. Note that $c \sim c'$ implies $\partial(c) = \partial(c')$ and $|c| = |c'|$.

We are now ready to introduce the notions of renameable type and non-renameable type for 2-chains having 1-shell boundaries.

Definition 3.6. Let α be a 2-chain having a 1-shell boundary.

(1) We say α is of *renameable type* (or simply *RN-type*) if some subsummand of α has a vanishing support. Otherwise, α is said to be of *non-renameable type* (or simply *NR-type*).
(2) α is called *minimal* if it is proper, and for any proper α' equivalent to α, there does not exist any subsummand β of α' such that $\partial(\beta) = 0$.

Remark 3.7. Suppose that α is a 2-chain having a 1-shell boundary.

(1) Note that α is of NR-type iff none of the CR or RS-operation is applicable to α, i.e. nothing else is equivalent to α except α itself. So an NR-type chain is minimal.

As was the case in Example 3.2, an RN-type α can sometimes be transformed to an NR-type by CR-operations. But if α is proper then its RN/NR-type is preserved under equivalence.

(2) We can always find some minimal 2-chain α' such that $\partial(\alpha) = \partial(\alpha')$. Such an α' can be obtained from α by finitely many applications of CR/RS-operations and deleting subsummands having trivial boundary.

There is a 2-chain β with $|\beta| = 5$ having a 1-shell boundary such that any subsummand of β does not have the trivial boundary but β' with $|\beta'| = 5$ obtained from β by the CR-operation has a subsummand with the boundary 0.

(3) If α is minimal then any α' equivalent to α is minimal as well (of course $|\alpha| = |\alpha'|$ and $\partial(\alpha) = \partial(\alpha')$ too).

Notation. Let f be any simplex. For any subset $\{j_0, \ldots, j_k\} \subseteq \mathrm{supp}(f)$, we shall abbreviate $f \restriction \mathcal{P}(\{j_0, \ldots, j_k\})$ as $f^{j_0,\cdots,j_k}$. Also, given a chain $c = \sum_{i\in I} n_i f_i$ (in its standard form), and any subset $\{j_0, \ldots, j_k\} \subseteq \mathrm{supp}(c)$, we shall write $c^{j_0,\ldots,j_k}$ to denote the subchain $\sum_{i\in J} n_i f_i$, where $J := \{i \in I \mid \mathrm{supp}(f_i) = \{j_0, \ldots, j_k\}\}$.

Example 3.8. Of course any 2-simplex is of NR-type. The following is an NR-type 2-chain with length 5: Let $\alpha = a_1 + a_2 + a_3 - a_4 - a_5$ be a 2-chain with 2-simplices a_i having $\mathrm{supp}(a_i) = \{0, 1, 2\}$ such that;

- $a_1^{1,2}, a_2^{1,2} = a_4^{1,2}, a_3^{1,2} = a_5^{1,2}$ are distinct;
- $a_2^{0,2}, a_1^{0,2} = a_5^{0,2}, a_3^{0,2} = a_4^{0,2}$ are distinct;
- and so are $a_3^{0,1}, a_1^{0,1} = a_4^{0,1}, a_2^{0,1} = a_5^{0,1}$.

Then α is of NR-type with a 1-shell boundary $a_1^{1,2} - a_2^{0,2} + a_3^{0,1}$.

Before stating our first main theorem of the classification, we introduce a notion called *chain-walk* which will be used in our proof.

Remark 3.9. Recall that if α is a 2-chain with a 1-shell boundary, then its length is always an odd positive number.

For the rest of this section, **we fix a 1-shell boundary** $f_{12}-f_{02}+f_{01}$ **with** $\mathrm{supp}(f_{jk}) = \{j < k\}$.

Definition 3.10. Let α be a 2-chain having the boundary $f_{12}-f_{02}+f_{01}$. A subchain $\beta = \sum_{i=0}^{m} \epsilon_i b_i$ of α (where $\epsilon_i = \pm 1$ and b_i is a 2-simplex, for each i) is called a *chain-walk in α from f_{01} to $-f_{02}$* if

(1) there are non-zero numbers $k_0, \ldots, k_{m+1}$ (not necessarily distinct) such that $k_0 = 1$, $k_{m+1} = 2$, and for $i \leq m$, $\mathrm{supp}(b_i) = \{k_i, k_{i+1}, 0\}$;

(2) $(\partial\epsilon_0 b_0)^{0,1} = f_{01}$, $(\partial\epsilon_m b_m)^{0,2} = -f_{02}$; and
(3) for $0 \leq i < m$,

$$(\partial\epsilon_i b_i)^{0,k_{i+1}} + (\partial\epsilon_{i+1} b_{i+1})^{0,k_{i+1}} = 0.$$

The sum $\sum_{i=0}^{m} \epsilon_i b_i$ with *its order* is called a *representation* of the chain-walk β. Unless said otherwise a chain-walk is written in the form of a representation. Notice that a chain-walk may have more than one representation. For example, a reordering of terms in β above may also satisfy conditions (1)-(3). By a *section* of the chain-walk β, we shall mean a subchain of β in the form

$$\beta' := \sum_{i=j}^{m'} \epsilon_i b_i \quad \text{for some } 0 \leq j < m' \leq m$$

and the sequence $\langle k_j, k_{j+1}, \ldots, k_{m'}, k_{m'+1}\rangle$ is called the *walk sequence* of β'. A chain-walk β in α is called *maximal* (in α) if it has the maximal possible length. We say α is *centered at* 0 if some (hence every) maximal chain-walk in α from f_{01} to $-f_{02}$ is, as a chain, equal to α.

We similarly define such notions as *a chain-walk in α from $-f_{02}$ to f_{12}, α is centered at* 2, and so on.

Remark 3.11. In the definition above, if β is a chain-walk in α from f_{01} to $-f_{02}$, then $0 \in \operatorname{supp}(b_i)$ for all i, but $0 \notin \operatorname{supp}(\partial\beta - f_{01} + f_{02})$; and the walk sequence of β is a sequential arrangement of $(\operatorname{supp}(b_i) \setminus \{0\})$'s without repetition of the overlapped support.

Note now that given any 2-chain α as in the definition above, since there are only finitely many 2-simplex terms in α, we can always find a chain-walk say, from f_{01} to $-f_{02}$: We start with any 2-simplex term b in α such that $\partial^2 b = f_{01}$ and then keep finding a term in α (with the coefficient) cancelling out adjacent 1-simplex boundaries. This process must stop with a term having $-f_{02}$ as its boundary.

Even if 0 is in the support of every simplex term of α, it need not be centered at 0: Let $\alpha = c_0 + c_1 - c_2$ such that $\partial c_0 = g_{12} - f_{02} + f_{01}$; $\partial c_1 = f_{12} - g_{02} + g_{01}$; and $\partial(-c_2) = -g_{12} + g_{02} - g_{01}$, where $f_{ij} \neq g_{ij}$. Then c_0 itself is a maximal chain-walk in α from f_{01} to $-f_{02}$. Note that $\alpha = c_0 + c_1 - c_2$ is not a chain-walk from f_{01} to $-f_{02}$, whereas it is a chain-walk from f_{12} to f_{01}, i.e, α is centered at 1.

Lemma 3.12. *Let α be a 2-chain with the 1-shell boundary $f_{12} - f_{02} + f_{01}$. Let $\beta = \sum_{i=0}^{m} \epsilon_i b_i$ be a chain-walk in α, say from $-f_{02}$ to f_{12}. Assume there is a section $\beta' = \sum_{i=j}^{m'} \epsilon_i b_i$ of β such that for* $\operatorname{supp}(b_i) = \{2, k_i, k_{i+1}\}$, *either* $k_i \neq k_{m'+1}$ *for all* $i = j, \dots, m'$*; or* $k_i \neq k_j$ *for all* $i = j + 1, \dots, m' + 1$*. Then by finitely many applications of CR-operations to* β'*, we obtain a simplex* c *with* $\operatorname{supp}(c) = \{2, k_j, k_{m'+1}\}$ *such that, for some* $\epsilon = \pm 1$*,* $\beta'' := \sum_{i=0}^{j-1} \epsilon_i b_i + \epsilon c + \sum_{i>m'}^{m} \epsilon_i b_i$ *is still a chain-walk from* $-f_{02}$ *to* f_{12}*.*

Proof. When $j = m'$, there is nothing to prove. Assume the lemma holds when $m' - j = n$. Let us show that the lemma holds when $m' - j = n + 1$. Assume $k_i \neq k_{m'+1}$ for all $i = j, \dots, m'$. Then we can apply the CR-operation to $\epsilon_{m'-1} b_{m'-1} + \epsilon_{m'} b_{m'}$, and we get $\epsilon'_{m'-1} b'_{m'-1} + \epsilon'_{m'} b'_{m'}$ with $\operatorname{supp}(b'_{m'-1}) = \{2, k_{m'-1}, k_{m'+1}\}$, having the same boundary. Due to the induction hypothesis applied to $\sum_{i=j}^{m'-2} \epsilon_i b_i + \epsilon'_{m'-1} b'_{m'-1}$, we are done. When $k_i \neq k_j$ for all $i = j + 1, \dots, m' + 1$, we apply the CR-operation to $\epsilon_j b_j + \epsilon_{j+1} b_{j+1}$, and similarly we are done. □

Remark/Definition 3.13. In Lemma 3.12, we call β'', a *reduct* of β. The walk sequence of β'' is also called a *reduct* of the walk sequence of β. So given a chain-walk its reducts are also chain-walks, which are obtained by the repeated applications of the CR-operation as described in Lemma 3.12.

Theorem 3.14. *Let* α *be a minimal* 2*-chain with the boundary* $f_{12} - f_{02} + f_{01}$*.*

(1) Assume α *is of NR-type. Then* $|\alpha| = 1$ *or* $|\alpha| \geq 5$*. If* $|\alpha| \geq 5$ *then any chain-walk in* α *from* f_{01} *to* $-f_{02}$ *is of the form* $\sum_{i=0}^{2n} (-1)^i a_i$ *which is as a chain equal to* α *such that* $f_{12} = a_{2j}^{1,2}$ *for some* $1 \leq j \leq n - 1$*.*

(2) α *is of RN-type iff* α *is equivalent to a* 2*-chain*

$$\alpha' = a_0 + \sum_{i=1}^{2n-1} \epsilon_i a_i + a_{2n}$$

($n \geq 1$) *which is a chain-walk from* f_{01} *to* $-f_{02}$ *such that* $\partial^0 a_{2n} = f_{12}$*,* $\partial^1(a_{2n}) = -f_{02}$ *and* $\operatorname{supp}(a_{2n}) = \{0, 1, 2\}$*. (The representation of* α' *is called* standard*.)*

Proof. (1) As mentioned in Remark 3.11, a chain-walk β in α from f_{01} to $-f_{02}$ exists. Now since α is of NR-type, $\operatorname{supp}(\alpha) = \{0, 1, 2\}$. If $|\beta| < |\alpha|$ then it follows $\alpha - \beta$ has a vanishing support 0, a contradiction. Hence $|\alpha| = |\beta|$ and $\alpha = \beta$. Suppose now that $|\alpha| = 3$. So the chain-walk is $a_0 - a_1 + a_2 = \alpha$ and either $\partial^0 a_0 = f_{12}$ or $\partial^0 a_2 = f_{12}$. Then either $\partial^0 a_1 = \partial^0 a_2$ or $\partial^0 a_0 = \partial^0 a_1$. In either case, the subchain of α has a vanishing support 1 or 2, a contradiction. Hence $|\alpha| = 1$ or ≥ 5. When $|\alpha| \geq 5$, all we need to show is that $f_{12} \neq \partial^0 a_0$ and $f_{12} \neq \partial^0 a_{2n}$. If $f_{12} = \partial^0 a_0$ then $\alpha - a_0$ has a vanishing support 1, a contradiction. Hence $f_{12} \neq \partial^0 a_0$. Similarly, we can show $f_{12} \neq \partial^0 a_{2n}$.

(2) ($\Leftarrow$) It follows $\mathrm{supp}(\partial(\alpha' - a_{2n})) = \{0,1\}$, i.e., $\alpha' - a_{2n}$ has a vanishing support, so α' is of RN-type. Since CR/RS-operations preserve the minimality and the chain types, α is also an RN-type.

($\Rightarrow$) We prove this in a series of claims. Note that $|\alpha| \geq 3$.

Claim 1. There is a 2-chain $\alpha_1 \sim \alpha$ which is centered at 2 such that $|\,\mathrm{supp}(\alpha_1)| > 3$.

Proof of Claim 1. Let $\alpha_2 := \alpha$ if $|\,\mathrm{supp}(\alpha)| > 3$. Otherwise since α is of RN-type, we can apply RS-operations to obtain some $\alpha_2 \sim \alpha$ with $|\,\mathrm{supp}(\alpha_2)| > 3$. Now there is $\beta := \sum_{i \in I} \epsilon_i b_i$, a maximal chain-walk in α_2 from $-f_{02}$ to f_{12}. If $\beta = \alpha_2$ we put $\alpha_1 := \alpha_2$ and we are done. Otherwise let $\gamma := \alpha_2 - \beta$, and then γ has a vanishing support 2 in its boundary. By applying the RS-operation to γ we find γ' with $2 \notin \mathrm{supp}(\gamma')$ such that $\alpha_2 \sim \alpha_2' := \beta + \gamma'$.

Assume now inductively we can find a desired $\alpha_1 \sim \alpha_2'$ when $|\gamma'| = m$. Let $|\gamma'| = m+1$. Note that $f_{12} - f_{02} + f_{01} \neq \partial(\beta)$, since otherwise $\partial(\gamma') = 0$ contradicting the minimality of α_2'. Hence there is $i_0 \in I$ with $\mathrm{supp}(b_{i_0}) = \{2, n_0, n_1\}$ such that $b_{i_0}^{n_0,n_1} (\neq f_{01})$ with a coefficient, stays in $\partial(\beta)$. Therefore there must be a term $\epsilon_{j_0} b_{j_0}$ ($\epsilon_{j_0} \in \{1,-1\}$) in γ' such that $(2 \notin)\,\mathrm{supp}(b_{j_0}) = \{n_0, n_1, n_2\}$ and $b_{i_0}^{n_0,n_1} = b_{j_0}^{n_0,n_1}$ is cancelled out in $\partial(\epsilon_{i_0} b_{i_0} + \epsilon_{j_0} b_{j_0})$. Now applying the CR-operation to $\epsilon_{i_0} b_{i_0} + \epsilon_{j_0} b_{j_0}$, we get $\epsilon'_{i_0} b'_{i_0} + \epsilon'_{j_0} b'_{j_0}$ with $\mathrm{supp}(b'_{i_0}) = \{2, n_0, n_2\}, \mathrm{supp}(b'_{j_0}) = \{2, n_1, n_2\}$, preserving the boundary. Then from β, we obtain β' by substituting $\epsilon'_{i_0} b'_{i_0} + \epsilon'_{j_0} b'_{j_0}$ for $\epsilon_{i_0} b_{i_0}$. Notice that β' is still a chain-walk from $-f_{02}$ to f_{12} while $\alpha_2' \sim \alpha_2'' := \beta' + (\gamma' - \epsilon_{j_0} b_{j_0})$. Hence by the induction hypothesis there is a desired $\alpha_1 \sim \alpha_2''$. We have proved Claim 1.

Claim 2. There is a 2-chain $\alpha_2 \sim \alpha_1$ that has a 1-simplex term c (with the coefficient 1) such that $\mathrm{supp}(c) = \{0,1,2\}$, and $f_{12} - f_{02} = \partial^0(c) - \partial^1(c)$.

Proof of Claim 2. For notational simplicity, let $\{0,1,2,3\} \subseteq \mathrm{supp}(\alpha_1)$, and write $\alpha_1 = \sum_{i=0}^{2n} \epsilon_i c_i$, a chain-walk from $-f_{02}$ to f_{12}. So for some $j_0 \leq 2n$, we have $\epsilon_{j_0} = 1$, $\mathrm{supp}(c_{j_0}) = \{0,1,2\}$, and $\partial^2(c_{j_0}) = f_{01}$. Let $\beta_0 := \sum_{i=0}^{j_0-1} \epsilon_i c_i$, and $\beta_1 := \sum_{i=j_0+1}^{2n} \epsilon_i c_i$. We shall find the desired c

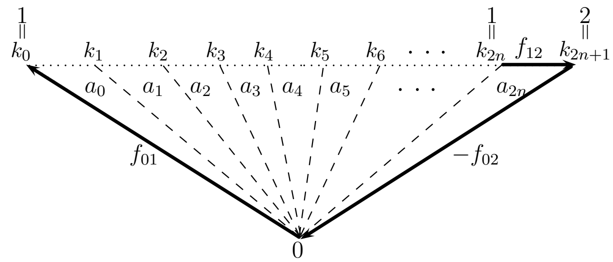

FIGURE 1. A standard RN-type 2-chain

(and α_2) by applying the process in Lemma 3.12 and finding reducts of chain-walks, starting from α_1. Each time, the reduced chain-walk together with the deleted terms is equivalent to α_1.

Case 1) $3 \notin \mathrm{supp}(\beta_1)$: So $3 \in \mathrm{supp}(\beta_0)$. Now let $I_0 := \langle 0, \dots, 3, \dots, 0\rangle$ be the walk sequence of β_0, and let $I_1 = \langle 1, \dots, 1\rangle$ be the walk sequence of β_1. So $I_0 I_1$ is the walk sequence of α_1. Now $I_0 = J_0 J_1$ such that J_1 starts with 3 but all other components $\neq 3$. Then due to Lemma 3.12 (applied to $J_1 I_1$), we can find γ_1, a reduct of α_1, whose walk sequence is $J_0\langle 3, 1\rangle$. Now $J_0 = \langle 0, \dots\rangle$.

If 3 is not in J_0 then again by Lemma 3.12, we can further find a reduct of γ_1 whose walk sequence is $\langle 0, 3, 1\rangle$, then again further reduce it with the walk sequence $\langle 0, 1\rangle$, and we are done.

If 3 is in J_0 then in general, by finding a sequence of all 3's in J_0 and applying Lemma 3.12, we can reduce J_0 to a sequence of the form $J_0' = \langle 03, k_1 3, k_2 3, \dots; k_\ell\rangle$ where each $k_i \neq 3$. If none of the k_i's is 0 then by applying Lemma 3.12 again to $J_0'\langle 3, 1\rangle$ we directly reduce it to $\langle 0, 1\rangle$ and we are done. Otherwise, one of the k_i's is 0, and we can similarly reduce J_0' to a sequence of the form $\langle 03, 03, \dots; k_\ell\rangle$. Now the reduced walk sequence is $\langle 03, 03, \dots; k_\ell; 3, 1\rangle$. If $k_\ell \neq 1$ then it can directly be reduced to $\langle 0, 1\rangle$ and we are done. If $k_\ell = 1$ then it can be reduced to $\langle 0, 1; 3, 1\rangle$ and further reduced to $\langle 0, 3, 1\rangle$ and to $\langle 0, 1\rangle$, so we are done.

Case 2) $3 \notin \mathrm{supp}(\beta_0)$: Then $3 \in \mathrm{supp}(\beta_1)$ and the proof will be similar to Case 1.

Case 3) $3 \in \mathrm{supp}(\beta_0) \cap \mathrm{supp}(\beta_1)$: By an argument similar to that in Case 1, the walk sequence of α_1 can be in general reduced to $I = \langle 03, 03, \dots; k, 3\rangle\langle 0, 1\rangle\langle 3, k'; 31, \dots, 31\rangle$. Now by the argument in the last part of the proof of Case 1, $\langle 03, 03, \dots; k, 3\rangle\langle 0, 1\rangle$ can be reduced to $\langle 0, 1\rangle$. Hence I can be reduced to $\langle 0, 1\rangle\langle 3, k'; 31, \dots, 31\rangle$. Then by the same argument it can finally be reduced to $\langle 0,1\rangle$, and we have proved Claim 2.

Now lastly we simply take a chain-walk γ from f_{01} to $-f_{02}$ in α_2 terminating with c (= the 1-simplex described in Claim 2). Then by an argument similar to that in the proof of Claim 1, we repeatedly apply the CR-operation to γ (while keeping c unchanged), and obtain a desired $\alpha' \sim \alpha_2$ centered at 0 forming a chain-walk from f_{01} to $-f_{02}$. Then we take the reverse order of the representation of the chain-walk α'. □

In an upcoming paper [7], it is shown that for any minimal 2-chain whose boundary is a 1-shell, there is an equivalent 2-chain which has the same boundary with support size three.

4. Examples

This section is devoted to exhibiting a certain family of examples of 2-chains of types in rosy theories whose boundaries are 1-shells. The existence of these examples implies that, in rosy theories, there is no uniform bound for the minimal lengths of 2-chains having 1-shell boundaries.

We recall the examples described in [2]. For a positive integer n, consider a (saturated) structure $M_n = (|M_n|; S, g_n)$, where $|M_n|$ is a circle; S is a ternary relation such that $S(a,b,c)$ holds iff a,b,c are distinct and b comes before c going around the circle clockwise starting at a; and g_n is a rotation (clockwise) by $2\pi/n$-radians. When n is obvious from context, g_n is often written as g. The following Fact 4.1, 4.2 are from [2].

Fact 4.1. *(1) $\mathrm{Th}(M_n)$ has the unique 1-complete type $p_n(x)$ over $\emptyset$, which is isolated by the formula $x = x$.*

(2) $\mathrm{Th}(M_n)$ is $\aleph_0$-categorical and has quantifier-elimination.

(3) For any subset $A \subset M_n$, $\mathrm{acl}(A) = \mathrm{dcl}(A) = \bigcup_{0\le i<n} g_n^i(A)$ (in the home-sort), where $g_n^i = \underbrace{g_n \circ \cdots \circ g_n}_{i \text{ times}}$.

(4) For each $a \in M_n$ with $n > 1$, and an integer i, $S(g^i(a), x, g^{i+1}(a))$ isolates a complete type over a.

In what follows, we assume $n > 1$.

Fact 4.2. *(1) There are $a,b \in M_n$ such that $d(a,b) > n/2$.*

(2) For any $a,b \in M_n$, the following are equivalent:

(i) a,b begin some $\emptyset$-indiscernible sequence,

(ii) *a and b have the same type over some elementary substructure of M_n,*
(iii) *$a = b \vee S(a, b, g_n(a)) \vee S(b, a, g_n(b))$ holds.*

Thus the unique 1-complete type p_n is also a Lascar type.

Theorem 4.3. *(1)* $\mathrm{Th}(M_n)$ *has weak elimination of imaginaries.*
(2) $\mathrm{Th}(M_n)$ *is rosy having thorn U-rank 1 with a trivial pregeometry.*

Proof. (1) We claim that if a set D in $(M_n)^k$ is definable over A_0 and A_1 respectively where $A_i = \mathrm{acl}(A_i) = \mathrm{dcl}(A_i)$ (in the home-sort) then it is definable over $B := A_0 \cap A_1$: We sketch the proof of the claim by freely using Fact 4.1. Let $k = 1$. Due to quantifier elimination, D is some union of finitely many arcs on M_n. Clearly each end-point of a connected component of D is in $\mathrm{dcl}(A_i)$ and so in B as well. Hence D is indeed B-definable. Now for induction, assume the claim holds for $k-1$. We want to show it holds for k. Suppose that $\varphi_i(x_1, \ldots, x_k, \bar{a}_i)$ defines D where $\bar{a}_i \in A_i$. Then, for each element b, the set D_b defined by $\varphi_i(x_1, \ldots, x_{k-1}, b, \bar{a}_i)$ is definable over Bb, by the induction hypothesis. But due to $\aleph_0$-categoricity (so there are only finitely many formulas over $\emptyset$ up to equivalence), it easily follows that for each y, $\varphi_i(x_1, \ldots, x_{k-1}, y, \bar{a}_i)$ is definable over B, i.e. D is definable over B as we wanted.

Now let $E(\bar{x}, \bar{y})$ be an $\emptyset$-definable equivalence relation on $(M_n)^k$. For $\bar{a} \in (M_n)^k$, let $\bar{a}'$ denote a finite tuple of algebraic closure of $\bar{a}$ in the home-sort. Let $\bar{b}$ be the maximal subtuple of $\bar{a}'$ which is algebraic over $\bar{a}/E$. Thus there is $\bar{a}'' \equiv_{\mathrm{acl}(\bar{a}/E)} \bar{a}'$ such that $\bar{b} = \bar{a}' \cap \bar{a}''$ as sets. Hence due to the claim, $\bar{a}/E \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$ and $\bar{b} \in \mathrm{acl}(\bar{a}/E)$. We have proved (1).

(2) Due to (1), $\mathrm{Th}(M_n)$ is rosy having thorn U-rank 1 as pointed out in [4]. Notice that M_n has the same pregeometry as the n-copies of a half-closed interval, and so M_n forms a trivial pregeometry with its algebraic closures. □

Definition 4.4. Let $a, b \in M_n$ be any elements with $\mathrm{acl}(a) \neq \mathrm{acl}(b)$.

(1) We define the *S-distance of b from a*, denoted by $\mathrm{Sd}\,(a, b)$ as follows: $\mathrm{Sd}\,(a, b) = k$ iff $M_n \models S(g^k(a), b, g^{k+1}(a))$. For integers $k < l$, we write $k \leq \mathrm{Sd}\,(a, b) \leq l$ if $M_n \models S(g^k(a), b, g^{l+1}(a))$.
(2) We define the *$\widehat{S}$-distance of b from a*, denoted by $\widehat{\mathrm{Sd}}\,(a, b)$, as similar manner as $\mathrm{Sd}\,(a, b)$, using the formula

$$\widehat{S}(x, y, z) \equiv (x \neq z \wedge S(x, y, z)) \vee (x = z \wedge x \neq y).$$

Remark 4.5. Let $x, y, z \in M_n$ have mutually disjoint algebraically closures. Then for any $k, l, m \in \mathbb{Z}$,

(1) $\operatorname{Sd}(y, x) = -\operatorname{Sd}(x, y) - 1$;
(2) (a) for $l - k \not\equiv -1, 0 \pmod n$, if $k \leq \operatorname{Sd}(x, y) \leq l - 1$, and $\operatorname{Sd}(y, z) = m$, then $m + k \leq \operatorname{Sd}(x, z) \leq m + l$;
 (b) for $l - k \equiv -1 \pmod n$, if $k \leq \operatorname{Sd}(x, y) \leq l - 1$, and $\operatorname{Sd}(y, z) = m$, then $g^{k+m}(x) \neq z$.
(1)′ $\widehat{\operatorname{Sd}}(y, x) = -\widehat{\operatorname{Sd}}(x, y) - 1$;
(2)′ for $k \not\equiv l \pmod n$, if $k \leq \widehat{\operatorname{Sd}}(x, y) \leq l - 1$, and $\widehat{\operatorname{Sd}}(y, z) = m$, then $m + k \leq \widehat{\operatorname{Sd}}(x, z) \leq m + l$.

Lemma 4.6. *Let k and $l_0, \ldots, l_m$ be fixed integers and $L_j := \sum_{i=0}^{j} l_i$. Let a and $d_0, \ldots, d_{m+1}$ $(m + 1 < n)$ be elements in M_n such that*

$$(*)_m : \widehat{\operatorname{Sd}}(a, d_0) = k,\ \widehat{\operatorname{Sd}}(d_i, d_{i+1}) = l_i,\ 0 \leq i \leq m.$$

Then

$$k + L_m \leq \widehat{\operatorname{Sd}}(a, d_{m+1}) \leq k + L_m + m + 1.$$

Moreover, by choosing appropriate elements for a and $d_0, \ldots, d_{m+1}$, the quantity $\widehat{\operatorname{Sd}}(a, d_{m+1})$ can be made to be any integer in $[\, k + L_m,\ k + L_m + m + 1\,]$ $(**)_m$.

Proof. We show this using induction on m. For $m = 0$, by Remark 4.5(2)′, it follows from $(*)_0$ that

$$k + l_0 \leq \widehat{\operatorname{Sd}}(a, d_1) \leq k + l_0 + 1.$$

Moreover it is not hard to see $(**)_0$ holds.

Now assume the lemma holds for $m-1$ with $m+1 < n$. Let us show the lemma for m. For $i \leq m + 1$, $a, d_i \in M_n$ are given which satisfy $(*)_m$. Firstly, by the induction hypothesis for $m - 1$,

$$k + L_{m-1} \leq \widehat{\operatorname{Sd}}(a, d_m) \leq k + L_{m-1} + m.$$

Since $m + 1 < n$,

$$k + L_{m-1} \leq \operatorname{Sd}(a, d_m) \leq k + L_{m-1} + m.$$

Then again by Remark 4.5(2)′,

$$k + L_m \leq \widehat{\operatorname{Sd}}(a, d_{m+1}) \leq k + L_m + m + 1.$$

Secondly, we show the moreover part. Fix $L_m \leq j \leq L_m + m + 1$ and $a' \in M_n$. If $j = L_m$, then $j - l_m = L_{m-1}$ and due to the induction hypothesis, there are $d'_0, \ldots, d'_m$ that satisfy $(*)_{m-1}$ and

$$\widehat{\operatorname{Sd}}(a', d'_m) = k + j - l_m.$$

So, $\widehat{\mathrm{Sd}}\,(a', g^{l_m}(d'_m)) = k + j$, and $M_n \models S\,(g^{l_m}(d'_m), d'_{m+1}, g^{k+j+1}(a'))$ for some $d'_{m+1} \in M_n$. Thus

$$\widehat{\mathrm{Sd}}\,(d'_m, d'_{m+1}) = l_m,\ \widehat{\mathrm{Sd}}\,(a', d'_{m+1}) = k + j.$$

So, a' and d'_i for $i \leq m+1$ satisfy the required condition. Now for $j > L_m$, the proof is similar to the case $j = L_m$ except that we replace $j - l_m$ by $j - l_m - 1$ and take d'_{m+1} in M_n such that

$$M_n \models S(g^{k+j}(a'), d'_{m+1}, g^{l_m+1}(d'_m)).$$

□

Now, let $\mathcal{A}(p_n)$ be the family of all the closed independent functors in p_n. We follow the notation given at the beginning of Section 2: given a closed independent functor f over $\emptyset$ in p_n with $u = \{i_0 < \cdots < i_k\} \in \mathrm{dom}(f)$, we write $f(u) = [a_0, \ldots, a_k]$, where $a_j \in M_n$, $f(u) = \mathrm{acl}(a_0, \ldots, a_k)$, and $\mathrm{acl}(a_j) = f_u^{\{i_j\}}(\{i_j\})$. When we write $f(u) \equiv [b_0, \ldots, b_k]$, it of course means that $[a_0, \ldots, a_k] \equiv [b_0, \ldots, b_k]$. By Theorem 4.3, it is equivalent to saying $a_0 \cdots a_k \equiv b_0 \cdots b_k$.

Remark 4.7. Let $\tau = \sum_{i=0}^{m} \epsilon_i t_i$ (t_i 2-simplex) be a chain-walk (in p_n) from f_{01} to $-f_{02}$ such that $D_i = \mathrm{supp}(t_i) = \{0, k_i, k_{i+1}\}$ with $k_0 = 1$, $k_{m+1} = 2$. Then putting together the *triangles* $t_0(D_0), \ldots, t_m(D_m)$ side by side *centered at* 0, we can find elements a and $d_0, \ldots, d_{m+1}$ in M_n such that for $0 \leq i \leq m$,

$$t_i(D_i) \equiv \begin{cases} [a, d_i, d_{i+1}] & \text{if } k_i < k_{i+1} \\ [a, d_{i+1}, d_i] & \text{if } k_i > k_{i+1}. \end{cases}$$

Combining the classification results in Section 3 and Lemma 4.6, we will show that there does not exist any finite upper bound for the minimal lengths of 2-chains with 1-shell boundaries in the types p_n.

Theorem 4.8. *Let $\mathcal{A}$ be a non-trivial amenable collection and let s be a 1-shell. Define $B(s)$, and $B(A)$ as follows:*

(1) $B(s) := \min\{\ |\tau| : \tau$ is a (minimal) 2-chain and $\partial(\tau) = s\ \}$.
(If s is not the boundary of any 2-chain, define $B(s) := -\infty$.)

(2) $B(\mathcal{A}) := \max\{B(s)\colon s$ is a 1-shell of $\mathcal{A}\ \}$.

Let $n > 1$ and let $s = s_{12} - s_{02} + s_{01}$ be a 1-shell from $\mathcal{A}(p_n)$ with $\mathrm{supp}(s_{ij}) = \{i, j\}$. Then there are a, b, c, c' in M_n and some integers k_1, k_2, k_3 with $0 \leq k_i < n$ such that,

- $\widehat{\mathrm{Sd}}\,(a, c) = k_1$, $\widehat{\mathrm{Sd}}\,(a, b) = k_2$, *and* $\widehat{\mathrm{Sd}}\,(b, c') = k_3$;
- $s_{01}(\{0, 1\}) \equiv [a, c], s_{02}(\{0, 2\}) \equiv [a, b]$, *and* $s_{12}(\{1, 2\}) \equiv [c', b]$.

Let $0 \leq k_4 (< n) \equiv k_2 - (k_1 - k_3)$ *(mod n) and let*

$$n_s := \min\{2(n - k_4) - 1,\ 2k_4 + 1\}.$$

Then

$$B(s) = n_s.$$

Moreover, taking $k_1 = 0$, $k_2 = 0$, *and* $k_3 = [\frac{n}{2}]$, *we get* $n_s \geq n - 1$ *and* $B(\mathcal{A}(p_n)) \geq n - 1$. *Therefore* $\lim_{n\to\infty} B(\mathcal{A}(p_n)) = \infty$.

Proof. (1) $B(s) \geq n_s$: By Theorem 2.4 and Corollary 3.14, there is a chain-walk $\tau = \sum_{i=0}^{2m} (-1)^i t_i$ from s_{01} to $-s_{02}$ and $\partial(\tau) = s$. We want to show $|\tau| \geq n_s$. Suppose not, i.e., $|\tau| = 2m+1 < n-1$. By Remark 4.7, there are d_i's $(0 \leq i \leq 2m+1)$ in M_n such that $ac \equiv ad_0,\ d_{2m+1} = b$; and

- $\widehat{\mathrm{S}}\mathrm{d}\,(d_0, d_1) = l_0,\ \widehat{\mathrm{S}}\mathrm{d}\,(d_{2m-1}, d_{2m}) = l_{2m}$ for some integers l_i;
- $t_0(\{0, k_0, k_1\}) \equiv [a, d_0, d_1]$, $t_{2j-1}(\{0, k_{2j-1}, k_{2j}\}) \equiv [a, d_{2j}, d_{2j-1}]$, and $t_{2j}(\{0, k_{2j}, k_{2j+1}\}) \equiv [a, d_{2j}, d_{2j+1}]$ for $1 \leq j \leq m$.

Now $\partial\tau = s$ implies $\partial^0 t_{2j_0} = s_{12}$ for some $0 \leq 2j_0 \leq 2m$; and for any $0 \leq j_1 \neq j_0 \leq m$ there is $0 \leq j_2 \neq j_0 \leq m$ (indeed a bijection) such that $\partial^0 t_{2j_1} = \partial^0 t_{2j_2+1}$. So

- $\widehat{\mathrm{S}}\mathrm{d}\,(d_{2j_0}, d_{2j_0+1}) = -k_3 - 1$; and
- $[d_{2j_1}, d_{2j_1+1}] \equiv [d_{2j_2+2}, d_{2j_2+1}]$.

By Remark 4.5(1)$'$, $\widehat{\mathrm{S}}\mathrm{d}\,(d_{2j_1}, d_{2j_1+1}) = -\widehat{\mathrm{S}}\mathrm{d}\,(d_{2j_2}, d_{2j_2+1}) - 1$. Therefore $l_{2j_0} = -k_3 - 1$ and $l_{2j_2+1} = -l_{2j_1} - 1$, so $\sum_{j=0}^{2m} l_j = -k_3 - m - 1$. Hence due to Lemma 4.6 and $2m+1 < n-1$, we have $k_1 - k_3 - m - 1 \leq \widehat{\mathrm{S}}\mathrm{d}\,(a, b) \leq k_1 - k_3 + m$. Thus

$$\widehat{\mathrm{S}}\mathrm{d}\,(a, b) = k_2;\ \text{and } k_1 - k_3 - m - 1 \leq \widehat{\mathrm{S}}\mathrm{d}\,(a, b) \leq k_1 - k_3 + m.$$

We rewrite it as

$$\widehat{\mathrm{S}}\mathrm{d}\,(g^{k_1-k_3}(a), b) = k_2 - (k_1 - k_3);\ \text{and } -m-1 \leq \widehat{\mathrm{S}}\mathrm{d}\,(g^{k_1-k_3}(a), b) \leq m.$$

We can replace $k_2 - (k_1 - k_3)$ by k_4 and we have $n - (m+1) < k_4 + 1$ or $m+1 > k_4$. In either case, we have $m \geq \min\{n - k_4 - 1,\ k_4\}$. Therefore $2m+1 \geq 2\min\{n - k_4 - 1,\ k_4\} + 1 = \min\{2(n - k_4) - 1,\ 2k_4 + 1\} = n_s$, a contradiction. We have proved $B(s) \geq n_s$.

(2) $B(s) \leq n_s$: We construct a chain-walk $\gamma = \sum_{i=0}^{n_s - 1} r_i$ with $\mathrm{supp}(\gamma) = \{0, 1, 2\}$ and $\partial\gamma = s$ as follows: Note that since n_s is odd, $m_s := (n_s - 1)/2$ is an integer. Also note that, if we let $N_1 := k_1 - k_3 - m_s - 1$ and $N_2 := k_1 - k_3 + m_s$, then $k_2 \equiv N_i \pmod n$ $(i = 1$ or $2)$. Hence we

have $\widehat{\mathrm{Sd}}\,(a,b) = N_1$ or $\widehat{\mathrm{Sd}}\,(a,b) = N_2$. Applying Lemma 4.6 with k_1 and $l_0, \ldots, l_{2m_s}$ such that $l_{2i+1} = -l_{2i} - 1$ for $0 \le i < m_s$ and $l_{2m_s} = -k_3 - 1$, we obtain $\sum_{i=0}^{2m_s} l_i = L_{2m_s} = -m_s - k_3 - 1$, and $L_{2m_s} + 2m_s + 1 = m_s - k_3$. Therefore if j is chosen to be such that $j = N_1 - k_1$ or $= N_2 - k_1$, and by applying $(**)_{2m_s}$ in Lemma 4.6, we can find $d'_0, \ldots, d'_{2m_s+1} (= d'_{n_s})$ such that

$$\widehat{\mathrm{Sd}}\,(a, d'_0) = k_1,\ \widehat{\mathrm{Sd}}\,(d'_i, d'_{i+1}) = l_i \text{ for } 0 \le i \le 2m_s, \text{ and } \widehat{\mathrm{Sd}}\,(a, d'_{n_s}) = k_2.$$

Then due to Fact 4.1(4) and Remark 4.5 $(1)'$, it follows that $ad'_0 \equiv ac$, $ad'_{n_s} \equiv ab$, $d'_{n_s-1}d'_{n_s} \equiv c'b$ and $d'_{2i}d'_{2i+1} \equiv d'_{2i+1}d'_{2i+2}$ for $0 \le i < m_s$. Hence clearly we have a desired 2-chain $\gamma = \sum_{i=0}^{n_s-1} r_i$ such that

$$r_i(\{0,1,2\}) \equiv \begin{cases} [a, d'_i, d'_{i+1}] & \text{if } i \equiv 0 \pmod 2 \\ [a, d'_{i+1}, d'_i] & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

□

Corollary 4.9. *For each $n \ge 5$, $\mathcal{A}(p_n)$ does not have weak 3-amalgamation.*

References

[1] Hans Adler. Explanations of Independence. Ph. D. Thesis, Univ. of Freiburg (2005).

[2] Enrique Casanovas, Daniel Lascar, Anand Pillay, and Martin Ziegler. Galois groups of first order theories. *Journal of Math. Logic*, 1 (2001) 305-319.

[3] Reinhard Diestel. *Graph Theory*, 2nd edition. Springer, New York (2000).

[4] Clifton Ealy and Alf Onshuus. Characterizing rosy theories. *Journal of Symbolic Logic*, 72 (2007) 919-940.

[5] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Homology groups of types in model theory and the computation of $H_2(p)$. *Journal of Symbolic Logic*, 78 (2013) 1086-1114.

[6] John Goodrick, Byunghan Kim, and Alexei Kolesnikov. Amalgamation functors and homology groups in model theory. To appear in *Proceedings of ICM 2014*.

[7] SunYoung Kim and Junguk Lee. More on classification of 2-chains having 1-shell boundaries in rosy theories. Preprint.

Department of Mathematics, Yonsei University, 50 Yonsei-Ro, Seodaemun-Gu, Seoul 120-749, Korea

E-mail address: bkim@yonsei.ac.kr

E-mail address: sy831@yonsei.ac.kr

E-mail address: ljw@yonsei.ac.kr