# Conservativity for theories of compositional truth via cut elimination

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We present a cut elimination argument that witnesses the conservativity of the compositional axioms for truth (without the extended induction axiom) over any theory interpreting a weak subsystem of arithmetic. In doing so we also fix a critical error in Halbach's original presentation. Our methods show that the admission of these axioms determines a hyper-exponential reduction in the size of derivations of truth-free statements.

## 1 OVERVIEW

We denote by  $I\Delta_0 + exp$  and  $I\Delta_0 + exp_1$  the first-order theories extending Robinson's arithmetic by  $\Delta_0$ -induction and, respectively, axioms expressing the totality of the exponentiation and hyper-exponentiation function. If S is a recursively axiomatised first-order theory interpreting  $I\Delta_0 + exp$  then by CT[S] we denote the extension of S by a fresh unary predicate T and the *compositional axioms of truth* for T.<sup>1</sup>

In this paper we provide syntactic proofs for the following theorems.

**Theorem 1.** Let S be an elementary axiomatised theory interpreting  $|\Delta_0 + \exp$ . Every theorem of CT[S] that does not contain the predicate T is a theorem of S. Moreover, this fact is verifiable in  $|\Delta_0 + \exp_1$ .

Let *p* be a fresh unary predicate symbol not present in the language  $\mathcal{L}$  of S. An  $\mathcal{L}$ -formula D is an S-*schema* if  $S \vdash D^{\neg} \sigma^{\neg} \rightarrow \sigma$  for every  $\mathcal{L}$ -formula  $\sigma$  and there exists a finite set *U* of  $\mathcal{L} \cup \{p\}$ -formulæ with at most *x* free such that  $S \vdash Dx \rightarrow \exists \psi \bigvee_{\varphi \in U} (x = \lceil \varphi[\psi/p] \rceil)$ .

**Theorem 2.** Let S be an elementary  $\mathcal{L}$ -theory interpreting  $|\Delta_0 + \exp$ . For any S-schema D, the theory  $CT[S] + \forall x(Dx \rightarrow Tx)$  is a conservative extension of S.

In the case that S is Peano arithmetic, the first part of both theorems is a consequence of the main theorems of [8, 9]. The proof is model-theoretic, however, establishing that

<sup>&</sup>lt;sup>1</sup>See definition 1 for a formal definition of CT[S].

a countable non-standard model of Peano arithmetic contains a full satisfaction class if and only if it is recursively saturated. Since every model of the Peano axioms is elementarily extended by a recursively saturated model, proof-theoretic conservativity is obtained. Halbach [7] offers a proof-theoretic approach to the first part of theorem 1. The strategy proceeds as follows. First the theory CT[S] is reformulated as a finitary sequent calculus with a cut rule and rules of inference corresponding to each of the compositional axioms for truth. A typical derivation in this calculus will involve cuts on formulæ involving the truth predicate. The elimination of all cuts is not possible as S is assumed to interpret a modicum of arithmetic. Instead, Halbach outlines a method of partial cut elimination whereby every cut on a formula involving the truth predicate is systematically replaced by a derivation without cuts on formulæ containing T. As noted in [2] and [5] however, the proof contains a critical error. An inspection of the cut elimination argument demonstrates that it does provide a method to eliminate cuts on formulæ of the form Ts provided there is a separate derivation, within say S, establishing that the logical complexity of the formula coded by *s* is bounded by some numeral.

The present paper provides the necessary link between the CT[S] and its fragment with bounded cuts. This takes the form of the following lemma.

**Lemma 1** (Bounding lemma). If  $\Gamma$  and  $\Delta$  are finite sets consisting of only truth-free and atomic formulæ, and the sequent  $\Gamma \Rightarrow \Delta$  is derivable in CT[S], then there exists a derivation of this sequent in which all cuts are either on  $\mathcal{L}$ -formulæ or bounded.

Let  $CT^*[S]$  denote the subsystem of CT[S] featuring only bounded cuts. Since this calculus permits the elimination of all cuts containing the truth predicate, the first part of theorem 1 is a consequence of the above lemma. Moreover, the proof (see §5) yields bounds on the size of the resulting derivation, from which the second part of theorem 1 can be deduced.

A particular instance of theorem 2 of interest is if D is the predicate  $Ax_S$  formalising the property of encoding an axiom of S. In this case we notice that the reduction of CT[S] to  $CT^*[S]$  also yields a reduction of  $CT[S] + \forall x (Ax_Sx \to Tx)$  to a corresponding extension of  $CT^*[S]$ . Unlike before, the latter theory does not admit cut elimination. Instead we show that the extension of  $CT^*[S]$  is relatively interpretable in CT[S], whence theorem 1 provides the desired result.

Theorem 1 has been independently proved by Enayat and Visser in [2] (the special case in which S is Peano arithmetic is also outlined in [1]). Their proof involves a refinement and extension of the original model-theoretic proof appearing in [8] that permits the argument to be formalised within a weak fragment of arithmetic. The author understands that Enayat and Visser also have a proof of theorem 2, again model-theoretic, though at the time of writing this is not in circulation.

## 1.1 OUTLINE

In the following two sections we formally define the theory CT[S] for a theory S interpreting  $I\Delta_0 + exp$  and its presentation as a sequent calculus, as well as the sub-theory

with bounded cuts,  $CT^*[S]$ . Section 4 contains the technical lemmata necessary to prove the core theorems, that every theorem of CT[S] not involving the predicate T is derivable in  $CT^*[S]$ ; the proofs of which form the content of section 5. In the final section we present applications of our analysis to questions relating to interpretability and speedup.

## 2 Preliminaries

We are interested in first-order theories that possess the mathematical resources to develop their own meta-theory. It is well-known that only a weak fragment of arithmetic is required for this task, namely  $|\Delta_0 + \exp|$ . For our purposes we therefore take the interpretability of  $|\Delta_0 + \exp|$  as representing that a theory possess the resources to express basic properties about its own syntax. For notational convenience we shall restrict ourselves exclusively to theories that extend this base theory. Our results, however, apply just as well to the general case.

Let  $\mathcal{L}$  be a recursive, first-order language containing the language of arithmetic. It will be useful to work with an extension of  $\mathcal{L}$  that includes a countable list of fresh predicate symbols  $\{p_j^i \mid i, j < \omega\}$  where  $p_j^i$  has arity *i*, plus a fresh propositional constant  $\epsilon$ . We denote this extended language by  $\mathcal{L}^+$ . We fix some standard representation of  $\mathcal{L}^+$  in  $|\Delta_0 + \exp$ , which takes the form of a fixed simple Gödel coding of  $\mathcal{L}^+$  into  $\mathcal{L}$  with:

- 1. Predicates  $\text{Term}_{\mathcal{L}}x$ ,  $\mathcal{L}x$ ,  $\text{Sent}_{\mathcal{L}}x$ , and Varx of  $\mathcal{L}$  expressing respectively the relations that x is the code of a closed term, a formula, a sentence and a variable symbol of  $\mathcal{L}^+$ .
- A Σ<sub>1</sub>-predicate val(x, y) such that val(<sup>¬</sup>t<sup>¬</sup>, t) is provable in the base theory for every term t. We view val as defining a function and write eq(r, s) in place of ∀x∀y(val(r, x) ∧ val(s, y) → x = y).
- 3. Predicates defining operations on codes; namely the binary terms  $=, \land, \lor, \neg, \lor, \dashv$ ,  $\exists, p$ , unary terms Q for each relation Q in  $\mathcal{L}$  and d, and a ternary term *sub* with:
  - $Q(\langle \lceil t_1 \rceil, \dots, \lceil t_n \rceil \rangle) = \lceil Q(t_1, \dots, t_n) \rceil$  for each  $Q \in \mathcal{L}$ ,
  - $p(\bar{\jmath}, \langle \ulcorner t_1 \urcorner, \dots, \ulcorner t_i \urcorner)) = \ulcorner p_j^i(t_1, \dots, t_i) \urcorner$ ,
  - $d(\ulcorner \alpha \urcorner) = x$  if the logical complexity of the  $\mathcal{L}^+$  formula  $\alpha$  is x, and
  - sub(x, y, z) denoting the usual substitution function that replaces in the term or formula (encoded by) x each occurrence of the variable with code y by the term with code z. We abbreviate uses of this function by writing x[z/y] in place of sub(x, y, z).

**Definition 1.** Let S be some fixed theory in a recursive language  $\mathcal{L}$  which interprets  $|\Delta_0 + \exp$ . The theory CT[S] is formulated in the language  $\mathcal{L}_T = \mathcal{L} \cup \{T\}$  and consists of the axioms of S together with

$$\begin{array}{l} \operatorname{\mathsf{Term}}_{\mathcal{L}} x \wedge \operatorname{\mathsf{Term}}_{\mathcal{L}} y \to (\operatorname{T} x = y \leftrightarrow x^{\circ} = y^{\circ})), \\ \operatorname{\mathsf{Sent}}_{\mathcal{L}} x \wedge \operatorname{\mathsf{Sent}}_{\mathcal{L}} y \to (\operatorname{T}(x \land y) \leftrightarrow \operatorname{T} x \wedge \operatorname{T} y), \\ \operatorname{\mathsf{Sent}}_{\mathcal{L}} x \wedge \operatorname{\mathsf{Sent}}_{\mathcal{L}} y \to (\operatorname{T}(x \lor y) \leftrightarrow \operatorname{T} x \lor \operatorname{T} y), \\ \operatorname{\mathsf{Sent}}_{\mathcal{L}} x \wedge \operatorname{\mathsf{Sent}}_{\mathcal{L}} y \to (\operatorname{T}(x \to y) \leftrightarrow (\operatorname{T} x \to \operatorname{T} y)), \\ \operatorname{\mathsf{Sent}}_{\mathcal{L}} x \to \operatorname{\mathsf{Sent}}_{\mathcal{L}} y \to (\operatorname{T} (x \to y) \leftrightarrow (\operatorname{T} x \to \operatorname{T} y)), \\ \operatorname{\mathsf{Sent}}_{\mathcal{L}} x \to (\operatorname{T} \neg x \leftrightarrow \neg \operatorname{T} x), \\ \operatorname{\mathsf{Var}} y \wedge \operatorname{\mathsf{Sent}}_{\mathcal{L}} x(\dot{\bar{0}}/y) \to (\operatorname{T} \not y x \leftrightarrow \exists z (\operatorname{\mathsf{Term}}_{\mathcal{L}} z \to \operatorname{T} x(z/y))), \\ \operatorname{\mathsf{Var}} y \wedge \operatorname{\mathsf{Sent}}_{\mathcal{L}} x(\dot{\bar{0}}/y) \to (\operatorname{T} \not y x \leftrightarrow \exists z (\operatorname{\mathsf{Term}}_{\mathcal{L}} z \wedge \operatorname{T} x(z/y))), \\ \operatorname{\mathsf{Term}}_{\mathcal{L}} x_1 \wedge \dots \wedge \operatorname{\mathsf{Term}}_{\mathcal{L}} x_n \to (\operatorname{T}(\operatorname{\mathsf{Q}} \langle x_1, \dots, x_n \rangle) \leftrightarrow \operatorname{\mathsf{Q}}(\operatorname{val} x_1, \dots, \operatorname{val} x_n)). \end{array}$$

for each relation Q of  $\mathcal{L}$  (with arity n). We call the formulæ above the compositional axioms for  $\mathcal{L}$  and any formula in the language of  $\mathcal{L}$  arithmetical. Moreover explicit mention of the base theory S is often omitted and we write CT and CT<sup>\*</sup> in place of CT[S] and CT<sup>\*</sup>[S] respectively.

Finally, we fix a few notational conventions for the remainder of the paper. The start of the Greek lower-case alphabet,  $\alpha$ ,  $\beta$ ,  $\gamma$ , etc., will be used to represent formulæ of  $\mathcal{L}_{T} = \mathcal{L} \cup \{T\}$ , while the end,  $\varphi$ ,  $\chi$ ,  $\psi$ ,  $\omega$ , as well as Roman lower-case symbols r, s, etc. denote terms in  $\mathcal{L}^2$  Upper-case Greek letters  $\Gamma$ ,  $\Delta$ ,  $\Sigma$  etc., are for finite sets of  $\mathcal{L}_{T}$ formulæ and boldface lower-case Greek symbols  $\varphi$ ,  $\psi$ , etc. represent finite sequences of  $\mathcal{L}$  terms. For a sequence  $\varphi = (\varphi_0, \ldots, \varphi_k)$ ,  $T\varphi$  denotes the set  $\{T\varphi_i \mid i \leq k\}$ . As usual,  $\Gamma$ ,  $\alpha$  is shorthand for  $\Gamma \cup \{\alpha\}$  and  $\Gamma$ ,  $\Delta$  for  $\Gamma \cup \Delta$ .

# 3 Two sequent calculi for compositional truth

Let S be a fixed theory extending  $|\Delta_0 + \exp$  formulated in the language  $\mathcal{L}$ . We present sequent calculi for CT[S] and CT\*[S]. In the former calculus, derivations are finite and the calculus supports the elimination of all cuts on non-atomic formulæ containing the truth predicate. The latter system replaces the cut rule of CT[S] by two restricted variants: one of these is the ordinary cut rule applicable to only formulæ not containing T; the other is a cut rule for the atomic truth predicate which is only applicable if the formula under the truth predicate subject to the cut has, provably, a fixed finite logical complexity. This second variant turns out to be admissible, so any sequent derivable in CT\*[S] has a derivation containing only arithmetical cuts. It follows therefore, that CT\*[S] is a conservative extension of S. We show that any CT[S] derivation can be transformed into a derivation in CT\*[S] and hence obtain the conservativity of CT[S] over S.

We now list the axioms and rules of CT[S] and CT<sup>\*</sup>[S].

## 3.1 Axioms

1.  $\Gamma \Rightarrow \Delta, \varphi$  if  $\varphi$  is an axiom of S,

 $<sup>^{2}</sup>$ The former list will be used exclusively as meta-variables ranging over terms encoding formulæ of  $\mathcal{L}^{+}$ .

- 2.  $\Gamma, r = s, \mathrm{T}r \Rightarrow \Delta, \mathrm{T}s$  for all terms r and s,
- 3.  $\Gamma, \mathrm{T}r \Rightarrow \mathsf{Sent}(r), \Delta$  for every r.
- 3.2 Arithmetical rules

$$\begin{array}{ccc} \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \forall v_i \alpha} \ (\forall \mathbb{R}) & \frac{\Gamma, \alpha(s/v_i) \Rightarrow \Delta}{\Gamma, \forall v_i \alpha \Rightarrow \Delta} \ (\forall \mathbb{L}) \\ \hline \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \lor \beta} \ (\lor \mathbb{R}) & \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma, \alpha \lor \beta \Rightarrow \Delta} \ (\lor \mathbb{L}) \\ \hline \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg \alpha} \ (\neg \mathbb{R}) & \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma, \neg \alpha \Rightarrow \Delta} \ (\neg \mathbb{L}) \\ \hline (\operatorname{Cut}_{\mathcal{L}}) & \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} & \Gamma \Rightarrow \Delta, \alpha \\ \hline \end{array}$$

We write  $\Gamma \Rightarrow^* \Delta$  to express that the derivation of  $\Gamma \Rightarrow \Delta$  involves only the axioms and arithmetical rules.

3.3 TRUTH RULES

$$\begin{array}{c} \frac{\Gamma \Rightarrow \Delta, \mathrm{T}\psi_{0}, \mathrm{T}\psi_{1}}{\Gamma, \psi = \psi_{0} \lor \psi_{1} \Rightarrow \Delta, \mathrm{T}\psi} (\vee_{\mathrm{T}} \mathrm{R}) & \frac{\Gamma, \mathrm{T}\psi_{0} \Rightarrow \Delta}{\Gamma, \psi = \psi_{0} \lor \psi_{1}, \mathrm{T}\psi \Rightarrow \Delta} (\vee_{\mathrm{T}} \mathrm{L}) \\ \\ \frac{\Gamma \Rightarrow \Delta, \mathrm{T}(\psi_{0}[v_{i}/s])}{\Gamma, \psi = \because s\psi_{0} \Rightarrow \Delta, \mathrm{T}\psi} (\forall_{\mathrm{T}} \mathrm{R}) & \frac{\Gamma, \mathrm{T}(\psi_{0}[t/s]) \Rightarrow \Delta}{\Gamma, \psi = \because s\psi_{0}, \mathrm{T}\psi \Rightarrow \Delta} (\forall_{\mathrm{T}} \mathrm{L}) \\ \\ \frac{\Gamma, \mathrm{T}\psi_{0} \Rightarrow \Delta}{\Gamma, \mathrm{Sent}\psi, \psi = \neg \psi_{0} \Rightarrow \Delta, \mathrm{T}\psi} (\neg_{\mathrm{T}} \mathrm{R}) & \frac{\Gamma \Rightarrow \Delta, \mathrm{T}\psi_{0}}{\Gamma, \mathrm{Sent}\psi, \psi = \neg \psi_{0}, \mathrm{T}\psi \Rightarrow \Delta} (\forall_{\mathrm{T}} \mathrm{L}) \\ \\ \frac{\Gamma \Rightarrow \Delta, \mathrm{eq}(r, s)}{\Gamma, \varphi = (r = s) \Rightarrow \Delta, \mathrm{T}\varphi} (=_{\mathrm{T}} \mathrm{R}) & \frac{\Gamma, \mathrm{eq}(r, s) \Rightarrow \Delta}{\Gamma, \varphi = (r = s), \mathrm{T}\varphi \Rightarrow \Delta} (=_{\mathrm{T}} \mathrm{L}) \end{array}$$

3.4 Additional cut rules

In CT[S]:

$$(\operatorname{Cut}_{\mathrm{T}}) \quad \frac{\Gamma, \mathrm{T}\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}, \mathrm{T}\varphi}{\Gamma \Rightarrow \Delta}$$

In CT\*[S]:

$$(\operatorname{Cut}_{\operatorname{T}}^k) \xrightarrow{\Gamma, \operatorname{T}\varphi \Rightarrow \Delta} \qquad \begin{array}{c} \Gamma \Rightarrow \Delta, \operatorname{T}\varphi \qquad \Gamma, \operatorname{Sent}\varphi \Rightarrow^* \underline{d}(\varphi) \leq \overline{k} \\ \Gamma \Rightarrow \Delta \end{array}$$

Normal eigenvariable conditions apply to four quantifier rules. We refer to the two rules  $(Cut_T)$  and  $(Cut_T^m)$  collectively as *T*-cuts.

## 3.5 DERIVATIONS

Derivations in either CT[S] or CT<sup>\*</sup>[S] are defined in the ordinary manner; the *truth depth* of a derivation is the maximum number of truth rules occurring in a path through the derivation. The *truth rank* is the least r such that for any rule (Cut<sup>m</sup><sub>T</sub>) occurring in the derivation, m < r. The *rank* of a derivation is any pair of numbers (a, r) such that a bounds the truth depth and r the truth rank of the derivation.

## 3.6 Meta-theorems for CT[S]

The key fact we require from  $I\Delta_0 + \exp$  is that the theory suffices to show that codes for  $\mathcal{L}^+$  formulæ are uniquely decomposable.

**Lemma 2** (Unique readability lemma). The sequent  $\Gamma \Rightarrow \Delta$  is derivable in  $|\Delta_0 + \exp w$  henever one of the following conditions hold.

- 1.  $\Gamma$  is a doubleton subset of  $\{x = y_0 \lor z_0, x = \forall y_1 z_1, x = (y_2 = z_2), x = \neg y_3\}$ .
- 2.  $\{\operatorname{Sent}_{\mathcal{L}}(y), \operatorname{Sent}_{\mathcal{L}}(z)\} \subset \Gamma, \Gamma \cap \{x = y \lor z, x = z \lor y, x = \exists zy, x = \neg y\} \neq \emptyset$  and  $\{\operatorname{Sent}_{\mathcal{L}}(x)\} \subseteq \Delta$ .
- *3.*  $\{y_0 = y_1 \land z_0 = z_1\} \subseteq \Delta$  and  $\Gamma$  extends:
  - a)  $\{x = y_0 \lor z_0, x = y_1 \lor z_1\};$
  - b)  $\{x = \forall y_0 z_0, x = \forall y_1 z_1\}; or$
  - c)  $\{x = (y_0 = z_0), x = (y_1 = z_1)\}.$
- 4.  $\{y_0 = y_1\} \subseteq \Delta$  and  $\{x = \neg y_0, x = \neg y_1\} \subseteq \Gamma$ .
- 5.  $\emptyset \neq \Gamma \subseteq \{x = y \lor z, x = z \lor y, x = \forall zy, x = \neg y\}$  and  $\{d(y) < d(x)\} \subseteq \Delta$ .
- 6.  $\emptyset \neq \Gamma \subseteq \{x = y \lor z, x = z \lor y, x = \forall zy, x = \neg y, x = (y = z), x = (z = y)\}$  and  $\{y < x\} \subseteq \Delta$ .
- 7.  $\{d(x) \le x\} \subseteq \Delta$ .

If S does not contain axioms containing the truth predicate then partial cut elimination is at least available in CT[S].

**Lemma 3** (Embedding lemma for CT). Suppose T does not occur in  $\mathcal{L}$  and  $CT[S] \vdash \alpha$ . Then the sequent  $\emptyset \Rightarrow \alpha$  has a derivation according to the rules of CT[S].

The next lemma demonstrates the key difference between CT and CT\*.

**Lemma 4** (Cut elimination theorem). Suppose  $\Gamma \Rightarrow \Delta$  is derivable in  $CT^*$  with cut rank (a, r + 1). Then the same sequent is derivable with rank  $(3^a, r)$ .

*Proof.* The argument follows the standard cut elimination procedure that is available for the formulation of CT in  $\omega$ -logic where the standard measure of complexity for terms encoding  $\mathcal{L}$ -sentences is available. The simplest approach to achieving cut elimination in that setting is through the use of a "reduction lemma" formalisation. In the finitary scenario, this corresponds to proving that from derivations of the sequents  $\Gamma \Rightarrow \Delta$ ,  $T\chi$  and  $\Gamma$ ,  $T\chi \Rightarrow \Delta$ , with ranks (a, r) and (b, r) respectively, and a truth-free derivation of the sequent  $\Gamma \Rightarrow \dot{q}(\chi) \leq \bar{r}$ , a derivation of the sequent  $\Gamma \Rightarrow \Delta$  can be obtained with rank  $((a + b) \cdot 2, r)$ .

As usual the proof proceeds via induction on the sum of the heights of the two derivations and we can assume that  $T\chi$  is principal in both derivations. If either sequent is an axiom, it takes the form  $\Gamma', \chi' = \chi, T\chi' \Rightarrow \Delta, T\chi$ , whence substituting  $\chi$  for  $\chi'$  in the other sequent we obtain  $\Gamma \Rightarrow \Delta$ . That leaves only the truth rules to consider. We will provide only one of the relevant cases of the proof and leave the remainder as an exercise for the reader.

Suppose the first derivation ends with an application of  $(\forall_T R)$ . Then a = a' + 1 and there are terms  $s_0$  and  $\chi_0$  such that the formula  $\chi = \frac{1}{2} s_0 \chi_0$  is a member of  $\Gamma$  and the sequent

$$\Gamma \Rightarrow \Delta, T\chi, T(\chi_0[v_i/s_0])$$

is derivable with rank (a', r). Now if any rule other than  $(\forall_T L)$  occurs as the last rule in the derivation of  $\Gamma, T\chi \Rightarrow \Delta$ , there are terms  $\chi'_0$  and  $\chi'_1$  such that either  $\{\chi = lash s_0\chi_0, \chi = \chi'_0 \lor \chi'_1\} \subseteq \Gamma$  or  $\{\chi = lash s_0\chi_0, \chi = \neg \chi'_0\} \subseteq \Gamma$ , whence  $\Gamma \Rightarrow \Delta$  follows by the unique readability lemma. Thus we may assume  $(\forall_T L)$  was applied to obtain  $\Gamma, T\chi \Rightarrow \Delta$  and so there are terms  $s_1, \chi_1$  and t such that  $\{\chi = lash s_0\chi_0, \chi = lash s_0\chi_0, \chi = lash s_0\chi_0, \chi = lash s_0\chi_0$ 

$$\Gamma, \mathrm{T}\chi, \mathrm{T}\chi_1[t/s_1] \Rightarrow \Delta$$

has a derivation with rank (b', r) for some b' < b. Then there is some r' < r for which the sequents

$$\Gamma \Rightarrow s_0 = s_1 \land \chi_0 = \chi_1 \qquad \qquad \Gamma \Rightarrow d(\chi_0[v_i/s_0]) \le \bar{r}'$$

are truth-free derivable and so by term substitution we obtain a derivation of

$$\Gamma, T\chi, T\chi_0[t/s_0] \Rightarrow \Delta,$$

with rank (b', r). Applying the induction hypothesis yields derivations of

$$\Gamma, \mathrm{T}\chi_0[t/s_0] \Rightarrow \Delta$$
  $\Gamma \Rightarrow \Delta, \mathrm{T}\chi_0[v_i/s_0]$ 

with ranks  $((a+b')\cdot 2, r)$  and  $((a'+b)\cdot 2, r)$  respectively. Substituting *t* for  $v_i$  in the second derivation and applying  $(\operatorname{Cut}_T')$  yields a derivation of  $\Gamma \Rightarrow \Delta$  with rank  $((a+b)\cdot 2, r)$ .  $\Box$ 

**Corollary 1.** *If the language of* S *does not contain* T *then*  $CT^*[S]$  *is a conservative extension of* S*.* 

## 3.7 Obstacles

It remains to embed CT into CT<sup>\*</sup>. Consider, for example, a derivation of the form

$$\frac{\vdots}{\Rightarrow T\varphi} \frac{T\varphi \Rightarrow T\varphi}{T\varphi \Rightarrow T(\varphi \lor \varphi)} (\lor_{T}R)$$
$$\xrightarrow{\Rightarrow T(\varphi \lor \varphi)} (Cut_{T})$$

If the left-most sub-derivation is cut-free then the conclusion is also trivially derivable without cuts (simply apply the rule ( $\forall_T R$ ) to the conclusion of the left sub-derivation). Thus the cut in the above derivation could be assigned a rank of 1 regardless of the logical complexity of  $\varphi$ . This can be explained by the fact that the complexity of any formula appearing under the truth predicate in the conclusion of the above cut (namely an instantiation of the term  $\varphi \lor \varphi$  by closed terms) has complexity no greater than one plus the complexity of the cut formula (that is  $\varphi$ ). It is also easy to see that this phenomenon holds for many deeper derivations. However, this manner of assigning cut rank is not sufficiently robust when it comes to derivations containing multiple cuts. We take the next derivation (the presentation of which has been intentionally simplified) as an example of the problem.

$$\begin{array}{c} \vdots \\ (\forall_{\mathrm{T}} \mathrm{L}) & \frac{\mathrm{T}\varphi(\bar{a}), \mathrm{T}\varphi(\bar{b}) \Rightarrow \Gamma}{\mathrm{T}\forall x\varphi, \mathrm{T}\varphi(\bar{b}) \Rightarrow \Gamma} & \vdots \\ (\forall_{\mathrm{T}} \mathrm{L}) & \frac{\mathrm{T}\forall x\varphi, \mathrm{T}\varphi(\bar{b}) \Rightarrow \Gamma}{\mathrm{T}\forall x\varphi \Rightarrow \Gamma} & \xrightarrow{\Rightarrow \Gamma, \mathrm{T}\varphi(\dot{x})} (\forall_{\mathrm{T}} \mathrm{R}) \\ & \Rightarrow \Gamma, \mathrm{T}\forall x\varphi & (\mathrm{Cut}_{\mathrm{T}}) \end{array}$$

The standard reduction lemma technique transforms the above derivation into the following in which cuts are on formulæ with intuitively lower complexity.

$$(\operatorname{Cut}_{\mathrm{T}}) \underbrace{\frac{\operatorname{T}\varphi(\bar{a}), \operatorname{T}\varphi(\bar{b}) \Rightarrow \Gamma \qquad \Rightarrow \Gamma, \operatorname{T}\varphi(\bar{a})}{(\operatorname{Cut}_{\mathrm{T}}) \underbrace{\frac{\operatorname{T}\varphi(\bar{b}) \Rightarrow \Gamma}{\Rightarrow \Gamma}} \qquad \stackrel{:}{\Rightarrow \Gamma, \operatorname{T}\varphi(\bar{b})}$$

The critical question is how to assign a rank to each of the two cuts in the second derivation that is strictly smaller than the rank given to the cut in the first derivation. Assuming a is different from b, the rank associated to the bottom cut must take into account the rank that is assigned to  $T\varphi(\bar{b})$  in the left sub-derivation as after an application of the cut reduction procedure to the top-most cut the intuitive complexity of the formula represented by  $\varphi(\bar{b})$  may have increased. This is especially relevant if the sub-derivation contains other applications of the cut rule to "sub-formulæ" of  $\varphi(\bar{b})$ ,  $\varphi(\bar{a})$  or  $\varphi(\dot{x})$ . Thus, if there is an appropriate way to assign ranks to occurrences of the truth predicate so the natural reduction procedure can be proven to succeed, it will require a deep analysis of the derivation as a whole.

The core idea is to provide a method to replace the term  $\varphi$  by a new term  $\lceil B_{\varphi} \rceil$  that encodes a formula of  $\mathcal{L}^+$  with bounded logical complexity. This formula will be chosen so that  $\varphi$  provably encodes a substitution instance of  $B_{\varphi}$ . In the case of the previous example, if the left-most sub-derivation is actually cut-free with height *n* then  $B_{\forall x\varphi}$  can be chosen with complexity bounded by  $|\Gamma| \cdot 2^n$ , this being the longest possible chain of terms following the sub-formula relation induced by the derivation. The complexity of  $B_{\forall x\varphi}$  will, in general, also be at least *n* so that each relevant occurrence of a sub-formula of  $\varphi$  in the derivation can be replaced by the corresponding sub-formula of  $B_{\forall x\varphi}$ . If the same choice suffices for the occurrence of  $\forall x\varphi$  in the right sub-derivation then this single occurrence of cut has been collapsed into a form available in CT<sup>\*</sup>.

## **4** Approximations

Recall the language  $\mathcal{L}^+$  which extends  $\mathcal{L}$  by countably many fresh predicate symbols

$$\mathcal{P} = \{p_i^i \mid i, j < \omega \text{ and } p_i^i \text{ is a predicate symbol of arity } i\}$$

and a new propositional constant  $\epsilon$ . The additional predicate symbols enable us to explicitly reduce the complexity of formulæ that occur under the truth predicate in CT-derivations. This is achieved by the use of *approximations*, an idea that was utilised by Kotlarski et al in [8].

An *assignment* is any function  $g: X \to \mathcal{L}^+$  such that  $X \subseteq \mathcal{P}$  is a finite set and for every  $i, j, \text{ if } p_j^i \in X$  then  $g(p_j^i)$  is a formula with arity i. Given an assignment g and an  $\mathcal{L}^+$  formula  $\varphi$ , we write  $\varphi[g]$  for the result of replacing each predicate  $p_j^i(s_1, \ldots, s_i)$  occurring in  $\varphi$  by  $g(p_j^i)(s_1, \ldots, s_i)$ , if  $g(p_j^i)$  is defined, and  $\epsilon$  otherwise. If  $\varphi = (\varphi_0, \ldots, \varphi_m)$  and  $\psi = (\psi_0, \ldots, \psi_m)$  are two sequences of closed  $\mathcal{L}^+$  formulæ we say  $\varphi$  approximates  $\psi$  if there exists an assignment g such that  $\psi_i = \varphi_i[g]$  for each  $i \leq m$ .

For a given sequence  $\varphi$  of  $\mathcal{L}^+$ , a collection of approximations of  $\varphi$  are distinguished. The *n*-th approximation of  $\varphi$ , defined below, is a particular approximation to  $\varphi$  that has logical complexity no more than  $lh(\varphi) \cdot 2^n$ , where  $lh(\varphi)$  denotes the number of elements in  $\varphi$ .

#### 4.1 Occurrences and parts

Let  $w, z, z_1, z_2, ...$  be fresh variable symbols. Given a formula  $\varphi$  of  $\mathcal{L}$  we first define a formula  $\overline{\varphi}$  of  $\mathcal{L} \cup \{w\}$  in two steps:  $\varphi^*$  is the result of replacing in  $\varphi$  every free variable by w, and  $\overline{\varphi}$  is obtained from  $\varphi^*$  by replacing each term in which the only variable that occurs is w, by w. Thus any term occurring in  $\overline{\varphi}$  is either simply the variable w or contains a bound occurrence of a variable different from w.

For each formula  $\varphi$ , we let  $O(\varphi)$  denote the set of *occurrences* of  $\varphi$ , pairs  $(\psi, s)$  such that  $\psi$  is a formula of  $\mathcal{L} \cup \{w, z\}$  in which the variable *z* occurs exactly once, *s* is a term of  $\mathcal{L} \cup \{w\}$  which is free for *z* in  $\psi$  and  $\varphi = \psi[s/z]$ . Notice that if  $(\psi, s) \in O(\bar{\varphi})$  then s = w.

The construction of  $\bar{\varphi}$  and  $O(\varphi)$  are such that for each formula  $\varphi$  of  $\mathcal{L}$  there is a uniquely determined function  $t_{\varphi} \colon O(\bar{\varphi}) \to \operatorname{Term}_{\mathcal{L}}$  for which  $\varphi$  is the result of replacing within  $\bar{\varphi}$  each occurrence of the variable w by the appropriate value of  $t_{\varphi}$ . We call two formulæ  $\varphi$ ,  $\psi$  equivalent, written  $\varphi \sim \psi$ , if  $\bar{\varphi} = \bar{\psi}$ .

**Lemma 5.** Let  $\Phi$  be a set of  $\mathcal{L}$  formulæ such that for every  $\varphi, \psi \in \Phi, \varphi \sim \psi$ . Then there is some number l and formula  $\vartheta_{\Phi}(z_1, \ldots, z_l)$ , called the template of  $\Phi$ , such that for every  $\varphi \in \Phi$  there are terms  $s_1, \ldots, s_l$  so that  $\varphi = \vartheta_{\Phi}(s_1, \ldots, s_l)$ .

*Proof.* Suppose  $\Phi$  is a set of formulæ satisfying the hypotheses of the lemma. Notice that  $O(\bar{\varphi}) = O(\bar{\psi})$  for every  $\varphi, \psi \in \Phi$ , so  $O(\Phi)$  has a natural definition as  $O(\bar{\varphi})$  for some  $\varphi \in \Phi$ . The functions  $\{t_{\varphi} \mid \varphi \in \Phi\}$  induce an equivalence relation  $E_{\Phi}$  on  $O(\Phi)$  by setting

$$(\chi, s) E_{\Phi}(\psi, t) \iff$$
 for every  $\varphi \in \Phi$ ,  $t_{\varphi}(\chi, s) = t_{\varphi}(\psi, t)$ .

Let *l* be the number of  $E_{\Phi}$ -equivalence classes in  $\Phi$ . For each  $\varphi \in \Phi$ , the function  $t_{\varphi}$  is constant on  $O(\Phi)/E_{\Phi}$ , whence  $\vartheta_{\Phi}(z_1, \ldots, z_l)$  is easily defined.

If  $\varphi = (\varphi_0, \varphi_1, \dots, \varphi_k)$  is a non-empty sequence of  $\mathcal{L}$  formulæ, then the *set of parts of*  $\varphi$ ,  $\Pi(\varphi)$ , is the collection of pairs  $(\psi, \chi)$  such that  $\psi$  is a formula of  $\mathcal{L} \cup \{\epsilon\}$  in which  $\epsilon$  occurs exactly once,  $\chi$  is a formula of  $\mathcal{L}$  and for some  $i \leq n$ ,  $\varphi_i$  is the result of replacing  $\epsilon$  by  $\chi$  in  $\psi$ . Notice that  $|\Pi(\varphi)| < k \cdot 2^{d(\varphi)}$  where  $d(\varphi)$  denotes maximal logical complexity of formulæ occurring in  $\varphi$  with atomic formulæ having depth 0.

We now define an ordering  $\prec$  on  $\Pi(\varphi)$  as follows.  $(\varphi, \chi) \prec (\varphi', \chi')$  just in case there exists  $\psi \in \mathcal{L} \cup \{\epsilon\}$  such that  $\varphi'[\psi/\epsilon] = \varphi$  and  $\psi[\chi/\epsilon] = \chi'$ . Informally, this means that  $\varphi[\chi/\epsilon] = \varphi'[\chi'/\epsilon]$  and the occurrence of  $\epsilon$  in  $\varphi$  corresponds to some sub-formula of  $\chi'$ . Note that this definition of  $\prec$  is more refined than the ordering also denoted  $\prec$  employed in [8]. The reasons for this will be highlighted later. The *depth* of a pair  $(\varphi, \chi) \in \Pi(\varphi)$ , denoted  $d(\varphi, \chi)$ , is its (reverse) order-type in  $\prec$ , that is the number of logical connectives and quantifiers between  $\varphi$  and the occurrence of  $\epsilon$  in  $\varphi$ . Making use of  $\prec$  and  $\sim$  the following sets can be defined.

$$\Pi^{0}(\boldsymbol{\varphi}, n) = \{(\boldsymbol{\varphi}, \chi) \in \Pi(\boldsymbol{\varphi}) \mid d(\boldsymbol{\varphi}, \chi) \leq n\}$$
$$\Pi^{m+1}(\boldsymbol{\varphi}, n) = \{(\boldsymbol{\varphi}, \chi) \in \Pi(\boldsymbol{\varphi}) \mid \exists (\varphi_{1}, \chi_{1}) \in \Pi^{m}(\boldsymbol{\varphi}, n) \exists (\varphi_{0}, \chi_{0}) \in \Pi^{0}(\boldsymbol{\varphi}, n)$$
$$\land \chi_{0} \sim \chi_{1} \land (\boldsymbol{\varphi}, \chi) \prec (\varphi_{1}, \chi_{1})$$
$$\land d(\boldsymbol{\varphi}, \chi) - d(\varphi_{1}, \chi_{1}) \leq n - d(\varphi_{0}, \chi_{0})\}$$

The requirement " $\exists (\varphi_0, \chi_0) \in \Pi^0(\varphi, n)$ " serves only to ensure the set  $\Pi^{m+1}(\varphi, n)$  does not grow too large. Thus  $\Pi^{m+1}(\varphi, n)$  consists of those parts of  $\varphi$  that are approximated by some  $(\varphi_1, \chi_1)$  in  $\Pi^m(\varphi, n)$  such that

- i) the template of  $\chi_1$  occurs somewhere in  $\varphi$  with depth at most n, and
- ii) the depth of  $(\varphi, \chi)$  is regulated by the depth of  $(\varphi_1, \chi_1)$ .

#### 4.2 Approximating formulæ

The first crucial observation is that if  $(\varphi, \chi) \in \Pi^m(\varphi, n)$  then there exists  $(\varphi', \chi') \in \Pi^0(\varphi, n)$  with  $\chi \sim \chi'$ . As a result, if

$$(\varphi_0, \chi_0) \prec (\varphi_1, \chi_1) \prec \cdots \prec (\varphi_k, \chi_k)$$

and  $(\varphi_i, \chi_i) \in \Pi^m(\varphi, n)$  for every  $i \leq k$  then  $k < lh(\varphi) \cdot 2^n$ , whence

$$(\varphi, \chi) \in \Pi^m(\varphi, n) \text{ implies } d(\varphi, \chi) \le lh(\varphi) \cdot 2^n$$
(1)

and so  $|\Pi^m(\varphi, n)| \leq 2^{lh(\varphi) \cdot 2^n}$  for every m. Since these bounds are independent of m, it follows there exists k such that  $\Pi^k(\varphi, n) = \Pi^{k+1}(\varphi, n)$ .

Based on the choice of k two further sets are defined:

$$\Gamma(\boldsymbol{\varphi}, n) = \{ \psi \in \mathcal{L} \mid \exists \varphi(\varphi, \psi) \in \Pi^k(\boldsymbol{\varphi}, n) \}, \\ \Gamma_I(\boldsymbol{\varphi}, n) = \{ \psi \in \mathcal{L} \mid \exists \varphi \ (\varphi, \psi) \text{ is } \prec \text{-minimal in } \Pi^k(\boldsymbol{\varphi}, n) \}.$$

Let  $\Gamma_I^{\sim}(\varphi, n)$  be the set of ~-equivalence classes of  $\Gamma_I(\varphi, n)$  and suppose  $\Phi \in \Gamma_I^{\sim}(\varphi, n)$ . We denote by  $\vartheta_{\Phi}(z_1, \ldots, z_{l_{\Phi}})$  the *template* of  $\Phi$  as determined in lemma 5, and for each  $\varphi \in \Phi$  let  $s_1^{\varphi}, \ldots, s_{l_{\Phi}}^{\varphi}$  denote the terms for which  $\varphi = \vartheta_{\Phi}(s_1^{\varphi}, \ldots, s_{l_{\Phi}}^{\varphi})$ .

Utilising this notation a function  $F_{\varphi,n} \colon \Gamma(\varphi,n) \to \mathcal{L}^+$  can be defined by recursion through  $\prec$ . Fix some enumeration  $\Phi_0, \ldots, \Phi_n$  of the elements of  $\Gamma_I^{\sim}(\varphi,n)$ , and let  $a_j$ denote the number of arguments of the template  $\vartheta_{\Phi_j}$ . If  $\psi \in \Gamma_I(\varphi,n)$  then either  $\psi$  is atomic, whence we define  $F_{\varphi,n}(\psi) = \psi$ , or  $\psi \in \Phi_j \in \Gamma_I^{\sim}(\varphi,n)$ , whence  $F_{\varphi,n}(\psi)$  is chosen to be the formula  $p_j^{a_j}(s_1^{\psi}, \ldots, s_{a_j}^{\psi})$ . In the case  $\psi \in \Gamma(\varphi, n) \setminus \Gamma_I(\varphi, n), F_{\varphi,n}(\psi)$  is defined to commute with the external connective or quantifier in  $\psi$ .

Now the *n*-th approximation of  $\varphi = (\varphi_0, \dots, \varphi_k)$  is defined to be the sequence

$$F_{\boldsymbol{\varphi},n}(\boldsymbol{\varphi}) = (F_{\boldsymbol{\varphi},n}(\varphi_0), \dots, F_{\boldsymbol{\varphi},n}(\varphi_k)).$$

These approximations have some nice features. For instance

**Lemma 6.** Let  $\varphi$  be a sequence of  $\mathcal{L}^+$ -formulæ. Then the *i*-th approximation to  $\varphi$  is an approximation of  $\varphi$  and an approximation of the *j*-th approximation whenever  $i \leq j$ .

**Lemma 7.** Every occurrence of a predicate symbol from  $\mathcal{P}$  in the *n*-th approximation of  $\varphi$  has depth at least *n* in  $\varphi$ . Moreover, every member of the *n*-th approximation of  $\varphi$  has logical depth no greater than  $lh(\varphi) \cdot 2^n$ .

**Lemma 8.** Suppose  $(\varphi', \psi')$  is an approximation of  $(\varphi, \psi)$  such every element  $\chi \in \varphi' \cup \psi'$  has logical complexity at most n. Then  $(\varphi', \psi')$  is an approximation of the *n*-th approximation of  $(\varphi, \psi)$ .

The upper bound of lemma 7 holds on account of (1). A consequence of the previous lemmas is the following.

**Lemma 9.** If  $(\varphi', \psi'_0 \lor \psi'_1)$  is the *n*-th approximation of  $(\varphi, \psi_0 \lor \psi_1)$  and m < n then the *m*-th approximation of  $(\varphi, \psi_i)$  is an approximation of  $(\varphi', \psi'_i)$ .

Similarly we obtain:

**Lemma 10.** If  $(\varphi', \neg \psi')$  is the *n*-th approximation of  $(\varphi, \neg \psi)$  and m < n then the *m*-th approximation of  $(\varphi, \psi)$  is an approximation of  $(\varphi', \psi')$ .

**Lemma 11.** If  $(\varphi', \forall x \varphi')$  is the *n*-th approximation of  $(\varphi, \forall x \varphi)$  and m < n then for every  $a < \omega$  the *m*-th approximation of  $(\varphi, \varphi[\bar{a}/x])$  is an approximation of  $(\varphi', \varphi'[\bar{a}/x])$ .

#### 4.3 Approximating sequents

We begin by noting that all the definitions and results of the previous section can be formalised and proved within  $|\Delta_0 + \exp$ . Thus we fix the following formal notation.

- 1. Gödel coding is expanded to sequences by letting  $\lceil \varphi \rceil$  denote the term  $(\lceil \varphi_0 \rceil, \dots, \lceil \varphi_m \rceil)$  if  $\varphi = (\varphi_0, \dots, \varphi_m)$ .
- 2.  $(r)_i = s$  if r encodes a sequence of length  $k \ge i$  and s is the *i*-th element of the sequence.
- 3. If  $s = (s_0, \ldots, s_m)$  and  $t = (t_0, \ldots, t_n)$  are two sequences s f expresses the sequence  $(s_0, \ldots, s_m, t_0, \ldots, t_n)$ . In the case m = n we introduce the following further abbreviations.
  - a) s = t abbreviates  $\bigwedge_{i < m} (s_i = t_i)$ ;
  - b) s[g] abbreviates the sequence of terms  $(s_0[g], \ldots, s_m[g])$ ;
  - c)  $F_{r,u}(s)$  abbreviates the sequence of terms  $(F_{r,u}(s_0), \ldots, F_{r,u}(s_m))$ ;
  - d)  $d(s) \leq u$  abbreviates the formula  $\bigwedge_{i \leq m} d(s_i) \leq u$ .
- 4. s[g] = t expresses that either g is not an assignment and s = t or g is an assignment and t is the result of replacing within the  $\mathcal{L}^+$  formula s, each occurrence of the predicate symbol  $p_i^i$  by  $g(p_i^i)$  if defined, otherwise by  $\epsilon$ .
- 5.  $F_{r,k}(s) = t$  expresses that there exists a sequence  $\varphi$  and  $\psi \in \Gamma(\varphi, k)$  such that  $r = \lceil \varphi \rceil$ ,  $s = \lceil \psi \rceil$  and  $t = \lceil F_{\varphi,k}(\psi) \rceil$ ; if there is no sequence of  $\mathcal{L}_{\mathrm{T}}$ -formulæ  $\varphi$  such that  $r = \lceil \varphi \rceil$  then s = t.

Note that the last point above expands to apply to complex equations involving multiple occurrences of sequences. So, for instance,  $F_{r,u}(s)[g] = F_{r',u'}(t)$  is shorthand for the formula  $\bigwedge_{i < m} F_{r,u}(s_i)[g] = F_{r',u'}(t_i)$ .

Collecting together the results of the previous section we have:

**Lemma 12.** *The following sequents are derivable in*  $|\Delta_0 + \exp|$ 

$$\begin{split} \emptyset &\Rightarrow (x \lor y)[z] = (x[z] \lor y[z]), \\ \emptyset &\Rightarrow (\neg x)[z] = \neg (x[z]), \\ \emptyset &\Rightarrow (\forall xy)[z] = \forall x(y[z]), \\ \emptyset &\Rightarrow (y(x/w))[z] = (y[z])(x/w), \\ (x)_i &= y \lor z \Rightarrow \Bar{F}_{x,w+1}(y \lor z) = \Bar{F}_{x,w+1}(y) \lor \Bar{F}_{x,w+1}(z), \\ (x)_i &= \neg y \Rightarrow \Bar{F}_{x,w+1}(\neg y) = \neg \Bar{F}_{x,w+1}(y), \\ (x)_i &= \forall yz \Rightarrow \Bar{F}_{x,w+1}(\forall yz) = \forall y(\Bar{F}_{x,w+1}(z)), \\ \emptyset &\Rightarrow \Bar{F}_{x,w}(y_0 \frown y_1 \frown y_2) = \Bar{F}_{x,w}(y_0 \frown y_2 \frown y_1), \\ \emptyset &\Rightarrow \Bar{H}_{x,z}(s)) \leq lh(x) \cdot 2^z. \end{split}$$

**Lemma 13.** There is a term g with variables w, x, y and z such that the following sequents are truth-free derivable in  $|\Delta_0 + \exp|$ .

The first sequent of lemma 13 formalises lemma 7, the second lemma 6, the third lemma 9, the penultimate line formalises lemma 11, expressing that the *y*-th approximation to  $(\varphi, \varphi[a/x_2])$  can be viewed as an approximation of the *z*-th approximation to  $(\varphi, \forall x \varphi)$  whenever y < z, and the final line combines lemmata 8 and 7.

Thus tying in approximations with derivations we have:

**Lemma 14.** Let  $\Gamma$ ,  $\Delta$  be sets consisting of arithmetical formulæ, and  $\varphi$ ,  $\psi$  be sequences of terms. If  $\Gamma$ ,  $T\varphi \Rightarrow \Delta$ ,  $T\psi$  is derivable then for every term g,

$$\Gamma, \mathrm{T}\varphi[g] \Rightarrow \Delta, \mathrm{T}\psi[g]$$

*is derivable with the same truth bound. Moreover, if the first derivation contains no* T*-cuts, neither does the second.* 

The lemma is not difficult to prove. However, we require a more general version that applies also to derivations featuring T-cuts. The next lemma achieves this.

**Lemma 15.** Let  $\Gamma$ ,  $\Delta$ ,  $\varphi$  and  $\psi$  be as in the statement of the previous lemma. If the sequents  $\Gamma$ ,  $T\varphi \Rightarrow \Delta$ ,  $T\psi$  and  $\Gamma \Rightarrow \dot{q}(g) < \bar{k}$  are derivable with truth ranks (a, r) and (0, 0) respectively, the sequent

$$\Gamma, \mathrm{T}\boldsymbol{\varphi}[g] \Rightarrow \Delta, \mathrm{T}\boldsymbol{\psi}[g]$$

is derivable with truth rank (a, r + k).

*Proof.* The only non-trivial case is if the last rule is  $(Cut_T^l)$  for some l < r. So suppose a = a' + 1 and we have the following derivation

$$\frac{\Gamma, T\varphi, T\chi \Rightarrow \Delta, T\psi \qquad \Gamma, T\varphi \Rightarrow \Delta, T\chi, T\psi \qquad \Gamma \Rightarrow \underline{d}(\chi) \le \overline{l}}{\Gamma, T\varphi \Rightarrow \Delta, T\psi} (Cut_{T}^{l})$$

with the two left-most premises derivable with truth rank (a', r) and the right-most with rank (0, 0). By the induction hypothesis, the sequents  $\Gamma, T\varphi[g], T\chi[g] \Rightarrow \Delta, T\psi[g]$  and  $\Gamma, T\varphi[g] \Rightarrow \Delta, T\chi[g], T\psi[g]$  are both derivable with rank (a', r + k). Since the sequent  $\Gamma \Rightarrow \dot{q}(g) \leq \bar{k}$  is derivable with rank (0, 0), so is

$$\Gamma \Rightarrow \dot{q}(\chi[g]) \le \bar{l} + \bar{k},$$

whence the rule  $(Cut_T^{l+k})$  yields the desired sequent.

#### 4.4 Approximating derivations

All that remains is to replace derivations in CT by approximations with bounded depth. Given a sequent  $\Gamma$ ,  $Ts \Rightarrow \Delta$ , Tt, its *u*-th approximation is the sequent  $\Gamma$ ,  $T(F_{s \frown t, \bar{u}}s) \Rightarrow \Delta$ ,  $T(F_{s \frown t, \bar{u}}t)$ . Let H be the function

$$H(k,n) = n \cdot 2^k.$$

By lemma 7 the *k*-th approximation of  $\varphi$  has depth at most  $H(k, lh(\varphi))$ .

The following lemmas hold for arbitrary derivations in CT<sup>\*</sup>[S].

**Lemma 16.** Suppose  $a, r, m, n, k < \omega$ ,  $\Gamma$  and  $\Delta$  are finite sets of  $\mathcal{L}$ -formulæ,  $\varphi$  and  $\psi$  are sequences of terms and  $\psi$  is a term, none of which contain x free and such that  $lh(\varphi)+lh(\psi) = n$ . If the k-th approximation to  $\Gamma, T\varphi \Rightarrow \Delta, T\psi, T(\psi(\dot{x}))$  is derivable with rank (a, r) then there is a derivation with rank (a + 1, r + H(k + 1, n + 1)) of the (k + 1)-th approximation to  $\Gamma, T\varphi \Rightarrow \Delta, T\psi, T(\forall r x^{\neg}\psi)$ .

*Proof.* Let  $\chi = \varphi^{\frown} \psi^{\frown}(\psi(\dot{x}))$ . Then assumption of the lemma is that the sequent

$$\Gamma, \mathrm{T}(F_{\boldsymbol{\chi},\bar{k}}\boldsymbol{\varphi}) \Rightarrow \Delta, \mathrm{T}(F_{\boldsymbol{\chi},\bar{k}}\boldsymbol{\psi}), \mathrm{T}(F_{\boldsymbol{\chi},\bar{k}}(\boldsymbol{\psi}(\dot{x})))$$

is derivable with rank (a, r). Let g(x, y, z) be the term given by lemma 13 and  $g' = g(\boldsymbol{\chi}, \bar{k}, \bar{k} + 1)$ . Lemma 15 implies there is a derivation with rank (a, r + H(k + 1, n + 1)) of the sequent

$$\Gamma, \mathrm{T}(\dot{F}_{\boldsymbol{\chi}',\bar{k}+1}\boldsymbol{\varphi}) \Rightarrow \Delta, \mathrm{T}(\dot{F}_{\boldsymbol{\chi}',\bar{k}+1}\boldsymbol{\psi}), \mathrm{T}(\dot{F}_{\boldsymbol{\chi},\bar{k}}(\boldsymbol{\psi}(\dot{x}))[g'])$$

where  $\chi' = \varphi \widehat{\psi} \forall x \psi$ . Combining this derivation with those of lemmata 12 and 13 and using only arithmetical cuts, yields a derivation of the sequent

$$\Gamma, \mathrm{T}(F_{\boldsymbol{\chi}',\bar{k}+1}\boldsymbol{\varphi}) \Rightarrow \Delta, \mathrm{T}(F_{\boldsymbol{\chi}',\bar{k}+1}\boldsymbol{\psi}), \mathrm{T}(F_{\boldsymbol{\chi}',\bar{k}+1}(\boldsymbol{\psi})(\dot{x}))$$

with rank (a, r + H(k + 1, n + 1)), whence  $(\forall_T R)$  and lemma 12 yield that

$$\Rightarrow \Delta, \mathrm{T}(F_{\boldsymbol{\chi}',\bar{k}+1}\boldsymbol{\psi}), \mathrm{T}(F_{\boldsymbol{\chi}',\bar{k}+1}(\forall^{\ulcorner}x^{\urcorner}\psi))$$

is derivable with rank (a + 1, r + H(k + 1, n + 1)).

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The same holds for the other scenarios:

**Lemma 17.** If the k-th approximation to  $\Gamma, T\varphi \Rightarrow \Delta, T\psi, T\psi_i$  is derivable with rank (a, r) then the (k + 1)-th approximation of  $\Gamma, T\varphi \Rightarrow \Delta, T\psi, T(\psi_0 \lor \psi_1)$  is derivable with rank (a + 1, r + H(k + 1, n)), where  $n = lh(\varphi) + lh(\psi) + 1$ .

**Lemma 18.** Let  $n = lh(\varphi) + lh(\psi)$  and suppose  $r \leq H(k, n + 1)$ . If the k-th approximation to the sequents  $\Gamma, T\varphi \Rightarrow \Delta, T\psi, T\chi$  and  $\Gamma, T\varphi, T\chi \Rightarrow \Delta, T\psi$  are derivable with rank (a, r) then the H(k, n + 1)-th approximation of  $\Gamma, T\varphi \Rightarrow \Delta, T\psi$  is derivable with rank

$$(a+1, H(k, n+1) + H(H(k, n+1), n))).$$

*Proof.* Let N = H(k, n + 1),  $\boldsymbol{\omega} = \boldsymbol{\varphi}^{\frown} \boldsymbol{\psi}$ ,  $\boldsymbol{\omega}' = \boldsymbol{\omega}^{\frown} \chi$ . By lemma 12 there is a truth-free derivation of  $\emptyset \Rightarrow d(F_{\boldsymbol{\omega}', \bar{k}}(x)) \leq \bar{N}$ , so the sequent

$$\Gamma, \mathrm{T}(F_{\omega',\bar{k}}\varphi) \Rightarrow \Delta, \mathrm{T}(F_{\omega',\bar{k}}\psi)$$

has a derivation with rank  $(a + 1, \max\{r, N\})$ . Let g be given by lemma 13 and set  $g' = g(\omega', \omega, \bar{k}, \bar{N})$ . Thus, lemma 13 entails

$$\emptyset \Rightarrow (F_{\omega',\bar{k}}\omega)[g'] = F_{\omega,\bar{N}}\omega$$

is truth-free derivable, whence we apply lemma 15 to obtain a derivation with rank  $(a + 1, \max\{r, N\} + H(N, n))$  of the sequent

$$\Gamma, \mathrm{T}(F_{\boldsymbol{\omega},\bar{N}}\boldsymbol{\varphi}) \Rightarrow \Delta, \mathrm{T}(F_{\boldsymbol{\omega},\bar{N}}\boldsymbol{\psi}).$$

# 5 Proofs of the main theorems

We now have all the ingredients for the bounding lemma, that permits the interpretation of derivations in CT[S] as derivations in  $CT^*[S]$ . The next lemma generalises the statement of lemma 1 by incorporating the relevant bounds.

**Lemma 19** (Bounding lemma). There are recursive functions  $G_1$  and  $G_2$  such that for every  $a, n < \omega$ , if  $lh(\varphi) + lh(\psi) \le n$  and the sequent  $\Gamma, T\varphi \Rightarrow \Delta, T\psi$  is derivable in CT[S] with truth depth a, then its  $G_1(a, n)$ -th approximation is derivable in CT<sup>\*</sup>[S] with rank  $(a, G_2(a, n))$ .

*Proof.* The idea is to copy the CT[S] derivation into CT<sup>\*</sup>[S] replacing the rule (Cut<sub>T</sub>) by (Cut<sub>T</sub><sup>k</sup>) for k determined inductively. The functions  $G_1$  and  $G_2$  are defined according to the bounds obtained in the previous section:

$$G_1(0,n) = 0,$$
  

$$G_1(m+1,n) = H(G_1(m,n+1), n+1),$$
  

$$G_2(m,n) = G_1(m+1, m+n).$$

We argue by induction on *a*. Suppose the last rule applied to obtain  $\Gamma$ ,  $T\varphi \Rightarrow \Delta$ ,  $T\psi$  is a non-arithmetical cut on  $T\chi$  and that this derivation has height a + 1. Let  $\omega = \varphi \frown \psi$  and  $\omega' = \omega \frown \chi$ . The induction hypothesis implies that the  $G_1(a, n + 1)$ -th approximations to

$$\Gamma, T\varphi, T\chi \Rightarrow \Delta, T\psi$$
  $\Gamma, T\varphi \Rightarrow \Delta, T\chi, T\psi$ 

are each derivable in  $CT^*[S]$  with rank  $(a, G_2(a, n+1))$ . By lemma 18 there is a derivation with height a+1 of the  $G_1(a+1, n)$ -th approximation to  $\Gamma$ ,  $T\varphi \Rightarrow \Delta$ ,  $T\psi$ . This derivation has cut rank bounded by  $G_2(a+1, n)$  so we are done. The other cases are similar and follow from applications of lemmas 16 and 17.

A combination of lemmas 14 and 19 implies that CT[S] permits the elimination of all T-cuts.

**Corollary 2.** If  $\Gamma \Rightarrow \Delta$  is derivable in CT[S] then it is derivable without T-cuts.

#### 5.1 Proof of theorem 1

Let  $\varphi$  be an arithmetical theorem of CT[S]. By the Embedding Lemma, the sequent  $\emptyset \Rightarrow \varphi$  has a derivation within CT[S]. Lemma 19 implies that the same sequent is derivable in CT\*[S] and the cut elimination theorem for CT\*[S] shows  $\emptyset \Rightarrow \varphi$  is derivable without truth cuts. But this derivation is also a derivation within S. Notice that this final derivation has height bounded by  $2^a_{2\cdot G_1(a+1,a+1)}$ , where *a* bounds the height of the original derivation of  $\emptyset \Rightarrow \varphi$  in CT[S],  $G_1$  is as defined in the proof of the Bounding Lemma, and  $2^n_m$  represents the function of hyper-exponentiation:  $2^n_0 = 2^n$  and  $2^n_{m+1} = 2^{2^n}_m$ . Thus this reduction can be formalised within  $|\Delta_0 + \exp_1$ .

## 5.2 Proof of theorem 2

Let S and D be as given in the statement of the theorem and let U be the finite set of  $\mathcal{L} \cup \{p\}$  formulæ associated with the S-schema D. We will show that the Bounding lemma naturally extends to provide a reduction of the theory  $CT[S] + \forall x(Dx \rightarrow Tx)$  into the extension of  $CT^*[S]$  by the rule

$$\frac{\Gamma \Rightarrow \Delta, \mathsf{D}s}{\Gamma \Rightarrow \Delta, \mathsf{T}s} (\mathsf{D})$$

Despite the fact that all cuts in this latter theory remain bounded, unlike CT\*[S] the theory will not in general support the cut elimination procedure. Nevertheless, conservativity over S can be achieved by considering the additional assumptions.

Suppose *d* is a derivation with truth depth *a* of the truth-free sequent  $\Gamma \Rightarrow \Delta$  in the expansion of CT[S] by the rule (D). By redefining the functions  $G_1$  and  $G_2$  so that  $G_1(0, n)$  bounds the logical depth of the (finitely many) formulæ in *U* for each *n*, the proof of the Bounding Lemma can be carried through to obtain a derivation with rank  $(a, G_2(a, 0))$  of the same sequent in the system expanding CT<sup>\*</sup>[S] by a variant of (D):

$$(\mathsf{D}_{\boldsymbol{\omega}}) \frac{\Pi, \mathrm{T}\boldsymbol{\varphi} \Rightarrow \Sigma, \mathrm{T}\boldsymbol{\psi}, \mathsf{D}\boldsymbol{\sigma}}{\Pi, \mathrm{T}\boldsymbol{\varphi} \Rightarrow \Sigma, \mathrm{T}\boldsymbol{\psi}, \mathrm{T}(\underline{F}_{\boldsymbol{\omega}, \overline{k}}\boldsymbol{\sigma})}$$

where  $\Pi$  and  $\Sigma$  are truth-free,  $k = G_1(a, 0)$  and  $\boldsymbol{\omega} = \boldsymbol{\varphi}^{\frown} \boldsymbol{\psi}^{\frown} \sigma$ .

Let  $d^*$  denote this derivation. Fix n such that for each instance of  $(\mathsf{D}_{\omega})$  occurring in  $d^*$ ,  $lh(\omega) < n$ , and set  $U^+$  to be the finite set of instantiations of formulæ from U by  $\mathcal{L}$ -formulæ that have logical depth at most  $G_2(a, n)$ . It follows that the sequent  $\mathsf{D}x, \dot{q}(x) < \overline{G_2(a, n)} \Rightarrow \{x = \lceil \varphi \rceil \mid \varphi \in U^+\}$  is derivable in S. Because the sequent  $\sigma \Rightarrow \mathrm{T}^{\ulcorner} \sigma^{\urcorner}$  is derivable in  $\mathsf{CT}[\mathsf{S}]$  for each  $\mathcal{L}$ -sentence  $\sigma$  we may deduce

$$\mathsf{D}x \Rightarrow \mathrm{T}(F_{\neg \omega},\bar{k}x)$$

is derivable in CT[S] whenever  $lh(\omega) < n$ . Thus  $d^*$  can be interpreted in CT[S] and an application of theorem 1 completes the proof.

## 6 Conservativity, interpretability and speed-up

The following instance of theorem 2 is particularly revealing:

**Corollary 3.** Let  $Ind_{\mathcal{L}}$  be the formula expressing that x is the code of the universal closure of an instance of  $\mathcal{L}$ -induction. Then  $CT[PA] + \forall x (Ind_{\mathcal{L}}x \to Tx)$  conservatively extends PA.

Corollary 3 effectively shows the limit of what principles can be conservatively added to CT[PA]. It is well known that extending CT[PA] by induction for formulæ involving the truth predicate (even only for bounded formulæ) allows the deduction of the global reflection principle  $\forall x (\text{Bew}_{\text{PA}}x \to \text{T}x)$ , and hence the schema of reflection  $\text{Bew}_{\text{PA}}^{\Gamma}\varphi^{\neg} \to \varphi$ , a statement not provable in PA.

An analogous result holds also for other first-order systems such as set theories. For example, the above corollary still holds if PA is replaced by Zermelo-Fraenkel set theory and Ind is replaced by a formula recognising instances of the separation and replacement axioms. Expanding the axiom schemata of CT[ZF] to apply also to formulæ involving the truth predicate, however, yields a non-conservative extension.<sup>3</sup>

We conclude the paper with two corollaries that are specific to a proof-theoretic treatment of CT[S].

## **Corollary 4.** CT[S] *attains at best hyper-exponential speed-up over* S.

To restate Corollary 4, every  $\mathcal{L}$ -theorem of CT[S] is derivable in S with at most hyperexponential increase in the length of the derivation. The upper-bound results from the fact the conservativeness of CT[S] over S can be established within  $|\Delta_0 + \exp_1$ .

Fischer, in [3], discusses a further consequence of a formalised conservativeness proof for CT.

**Lemma 20** (Fischer [3]). *If*  $\mathsf{PA} \vdash \forall x (\mathsf{Sent}_{\mathcal{L}} x \land \mathsf{Bew}_{\mathsf{CT}[\mathsf{S}_0]} x \to \mathsf{Bew}_{\mathsf{S}_0} x)$  for every  $\mathsf{I}\Sigma_1 \subseteq \mathsf{S}_0 \subseteq \mathsf{PA}$  then  $\mathsf{CT}[\mathsf{PA}]$  is relatively interpretable in  $\mathsf{PA}$ .<sup>4</sup>

<sup>&</sup>lt;sup>3</sup>Assuming ZF is consistent.

<sup>&</sup>lt;sup>4</sup>We refer the reader to, e.g., [3] for a definition of *relatively interpretable*.

Combining this with theorem 1 therefore yields

**Corollary 5.** *If*  $S \subseteq PA$  *then* CT[S] *is relatively interpretable in* PA*.* 

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