

SMALL UNIVERSAL FAMILIES OF GRAPHS ON $\aleph_{\omega+1}$

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ABSTRACT. We prove that it is consistent that \aleph_ω is strong limit, 2^{\aleph_ω} is large and the universality number for graphs on $\aleph_{\omega+1}$ is small. The proof uses Prikry forcing with interleaved collapsing.

1. INTRODUCTION

If μ is an infinite cardinal, a *universal graph on μ* is a graph with vertex set μ which contains an isomorphic induced copy of every such graph. More generally, a family F of graphs on μ is *jointly universal* if every graph on μ is isomorphic to an induced subgraph of some graph in F . We denote by u_μ the least size of a jointly universal family of graphs on μ , and record the easy remarks that $u_\mu \leq 2^\mu$ and that if $u_\mu \leq \mu$ then $u_\mu = 1$. If $\mu = \mu^{<\mu}$, then by standard results in model theory there exists a saturated (and hence universal) graph on μ . It follows that under GCH and the hypothesis that μ is regular, $u_\mu = 1$. A standard idea in model theory (the construction of *special models*) shows that under GCH we have $u_\mu = 1$ for singular μ as well: we fix $\langle \mu_i : i < \text{cf}(\mu) \rangle$ a sequence of regular cardinals which is cofinal in μ , build a graph \mathcal{G} which is the union of an increasing sequence of induced subgraphs \mathcal{G}_i where \mathcal{G}_i is a saturated graph on μ_i , and argue by repeated applications of saturation that \mathcal{G} is universal.

Questions about the value of u_μ when $\mu < \mu^{<\mu}$ have been investigated by several authors. We refer the reader to papers by Džamonja and Shelah [4, 3], Kojman and Shelah [6], Mekler [10] and Shelah [13].

We will consider the case when μ is a successor cardinal κ^+ and $2^\kappa > \kappa^+$. When κ is regular it is known that:

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- (1) It is possible to produce models where u_{κ^+} is arbitrarily large [6], for example by adding many Cohen subsets of κ over a model of GCH.
- (2) It is possible to produce models where $\kappa^{<\kappa} = \kappa$, 2^κ is arbitrarily large and $u_{\kappa^+} = \kappa^{++}$ [4] by iterated forcing over a model of GCH.

The question whether we can have $u_{\kappa^+} = 1$ when $2^\kappa > \kappa^+$ remains mysterious for general values of κ , though it is known [10, 13] to have a positive solution for $\kappa = \omega$.

When κ is singular then questions about u_{κ^+} become harder, since we have fewer forcing constructions available. Džamonja and Shelah [4] found a line of attack on this kind of question, where the key idea is that we will prepare a large cardinal κ by means of iterated forcing which preserves its large cardinal character, and only at the end of the construction will we force to make κ become a singular cardinal. By this method Džamonja and Shelah produced models where κ is singular strong limit of cofinality ω , 2^κ is arbitrarily large and $u_{\kappa^+} \leq \kappa^{++}$.

In [4] the final step in the construction is Prikry forcing, so that in the final model κ is still rather large by some measures, for example it is still a cardinal fixed point. In this paper we will use a forcing poset defined by Foreman and Woodin [5] which will make κ become \aleph_ω . In some joint work with Magidor and Shelah [2], we obtain similar results where the final step is a form of Radin forcing which changes the cofinality of κ to uncountable values such as ω_1 .

Our main result is this: it is consistent relative to a supercompact cardinal that \aleph_ω is strong limit, $2^{\aleph_\omega} = \aleph_{\omega+3}$, and $u_{\aleph_{\omega+1}} \leq \aleph_{\omega+2}$. In the rest of this Introduction we give an overview of the proof, and conclude with a guide to the structure of the paper.

The Foreman-Woodin poset is a variation of Prikry forcing, which adds a Prikry sequence κ_i of inaccessible cardinals cofinal in κ , and in addition collapses all but finitely many cardinals between successive points on the Prikry sequence so that κ becomes \aleph_ω . The only parameter needed to define Prikry forcing is a normal measure U_0 , but the Foreman-Woodin forcing has an additional parameter \mathcal{F} which is a filter on the set of functions representing elements of a certain complete Boolean algebra in $\text{Ult}(V, U_0)$.

We will start with a ground model V in which κ is a supercompact cardinal, which has been prepared so as to be indestructible under κ -directed closed forcing, and $2^\kappa = \kappa^{+3}$. We will define an iterated forcing poset \mathbb{Q}^* by iterating for κ^{+4} many steps with supports of size less than κ , forcing at each stage i with a poset \mathbb{Q}_i which is κ -directed closed and

has a strong form of κ^+ -cc. The cardinal κ will still be supercompact in $V^{\mathbb{Q}^*}$, and this will enable us to choose a normal measure U_0 and filter \mathcal{F} , which can be used as parameters to define a Foreman-Woodin forcing \mathbb{P} .

The key idea is that the poset \mathbb{Q}_i will anticipate the results of forcing over $V^{\mathbb{Q}^*}$ with \mathbb{P} . To be more specific, at each stage i of the construction a suitable form of diamond sequence will be used to produce “guesses” W_i and \mathcal{F}_i at the final values of U_0 and \mathcal{F} , and there will be many stages i at which these guesses are correct (in the sense that W_i and \mathcal{F}_i are the restrictions to $V^{\mathbb{P}_i}$ of U_0 and \mathcal{F}).

At stage i there is a poset \mathbb{P}_i which is computed from W_i and \mathcal{F}_i in the same way that \mathbb{P} is computed from U_0 and \mathcal{F} . If the guesses made at stage i are correct then the final \mathbb{P} -generic object will induce a \mathbb{P}_i -generic object. The poset \mathbb{Q}_i aims to add a \mathbb{P} -name for a graph on κ^+ , whose interpretation absorbs all graphs in the extension of stage i by the induced \mathbb{P}_i -generic object.

Our final model will be obtained by halting the construction at a suitable stage i^* of cofinality κ^{++} , and forcing with \mathbb{P}_{i^*} . The point here (an idea which comes from [4]) is that we can read off a universal family of size κ^{++} from a cofinal set of stages below i^* , and we are in a situation where $2^\kappa = \kappa^{+3}$.

We conclude this section with an overview of the paper and a couple of remarks:

- In Section 2 we discuss the filter \mathcal{F} which is used in defining \mathbb{P} and give an account of its main properties.
- In Section 3 we construct the forcing \mathbb{P} and prove various key facts about it using the properties of \mathcal{F} .
- In Section 4 we construct the “anticipation forcing” \mathbb{Q} and prove that it has certain properties. Most notably \mathbb{Q} is κ -compact and has a strong form of the κ^+ -chain condition.
- In Section 5 we describe the main iteration \mathbb{Q}^* and prove a key technical fact by a master condition argument.
- In Section 6 we prove the main theorem.
- In Section 7 we discuss generalisations, related work and some natural open problems.

Remark. Foreman and Woodin’s paper [5] actually defines a supercompact Radin forcing with interleaved Cohen forcing, and its projection to a Radin forcing with interleaved Cohen forcing controlled by certain filters. Our forcing \mathbb{P} here is a version of the projected forcing, with the Cohen forcing replaced by collapsing forcing and the Radin forcing simplified to the special case of Prikry forcing. \mathbb{P} is also a close relative

of the forcing poset used by Woodin to obtain the failure of SCH at \aleph_ω from optimal hypotheses, the difference being that in Woodin’s forcing poset the constraining filters are generic over the relevant ultrapowers. Our approach was dictated by the necessity to have the “approximations” \mathbb{P}_i be well-behaved forcing posets, in a context where they can neither be obtained as projections of supercompact Prikry forcing with interleaved collapsing nor constructed from filters which are generic over ultrapowers. Of course, all this work traces back ultimately to Magidor’s original model for the failure of SCH at \aleph_ω [8].

2. CONSTRAINTS AND FILTERS

We start by assuming that $2^\kappa = \kappa^{+\omega}$ for some $n < \omega$ and that κ is 2^κ -supercompact. We will fix U an ultrafilter on $P_\kappa \kappa^{+\omega}$ witnessing the 2^κ -supercompactness of κ , and let $j : V \rightarrow M = \text{Ult}(V, U)$ be the associated ultrapower map. We let U_0 be the projection of U to an ultrafilter on κ via the map $x \mapsto x \cap \kappa$. We remind the reader of some standard facts.

- (1) $U = \{A \subseteq P_\kappa \kappa^{+\omega} : j^{\kappa^{+\omega}} \in j(A)\}$, and $[F]_U = j(F)(j^{\kappa^{+\omega}})$ for every function F with $\text{dom}(F) \in U$.
- (2) U concentrates on the set of $x \in P_\kappa \kappa^{+\omega}$ such that $x \cap \kappa$ is an inaccessible cardinal less than κ and $\text{ot}(x) = (x \cap \kappa)^{+\omega}$. We will denote this set by A_{good} , and for $x \in A_{\text{good}}$ we let $\kappa_x = x \cap \kappa$ and $\lambda_x = \text{ot}(x)$.
- (3) U_0 is a normal measure on κ , and $U_0 = \{B \subseteq \kappa : \kappa \in j(B)\}$. We let $j_0 : V \rightarrow M_0 = \text{Ult}(V, U_0)$ be the associated ultrapower map, and note that $[f]_{U_0} = j_0(f)(\kappa)$ for every function f with $\text{dom}(f) \in U_0$.
- (4) There is an elementary embedding $k : M_0 \rightarrow M$ such that $k \circ j_0 = j$, which is given by the formula $k : [f]_{U_0} \mapsto j(f)(\kappa)$.

We now fix an integer m with $n < m < \omega$, and define a family of forcing posets: for α and β inaccessible with $\alpha < \beta$ we let $\mathbb{C}(\alpha, \beta) = \text{Coll}(\alpha^{+m}, < \beta)$. We note that when $\alpha < \beta < \gamma$ we have that $\mathbb{C}(\alpha, \beta) \subseteq \mathbb{C}(\alpha, \gamma)$ and the inclusion map is a complete embedding: in particular, if G is $\mathbb{C}(\alpha, \gamma)$ -generic over V then $G \cap \mathbb{C}(\alpha, \beta)$ is $\mathbb{C}(\alpha, \beta)$ -generic over V .

Definition 2.1. A U -constraint is a function H such that $\text{dom}(H) \in U$, $\text{dom}(H) \subseteq A_{\text{good}}$ and $H(x) \in \mathbb{C}(\kappa_x, \kappa)$ for all $x \in \text{dom}(H)$.

It is easy to see that $\mathbb{C}^M(\kappa, j(\kappa))$ is the set of objects of the form $[H]_U$ for some U -constraint H .

Definition 2.2. Let H and H' be U -constraints.

- (1) $H \leq H'$ if and only if $\text{dom}(H) \subseteq \text{dom}(H')$ and $H(x) \leq H'(x)$ for all $x \in \text{dom}(H)$.
- (2) $H \leq_U H'$ if and only if $\{x : H(x) \leq H'(x)\} \in U$, or equivalently $[H]_U \leq [H']_U$.

Remark. Since $m > n$, and $\kappa^{+n} M \subseteq M$ by the hypothesis that U witnesses the κ^{+n} -supercompactness of κ , it is easy to see that $\mathbb{C}^M(\kappa, j(\kappa))$ is κ^{+n+1} -closed in V . It follows that any \leq_U -decreasing sequence of U -constraints of length less than κ^{+n+1} has a \leq_U -lower bound.

We define the complete Boolean algebra $\mathbb{B}(\alpha, \beta)$ to be the regular open algebra of the forcing poset $\mathbb{C}(\alpha, \beta)$, and then let $\mathbb{B} = \mathbb{B}^M(\kappa, j(\kappa))$ and $\mathbb{B}_0 = \mathbb{B}^{M_0}(\kappa, j_0(\kappa))$. We note that for every $\alpha < \kappa$ the poset $\mathbb{C}(\alpha, \kappa)$ is κ -cc and has cardinality κ , so that $\mathbb{B}(\alpha, \kappa)$ has cardinality κ : by elementarity we see that \mathbb{B}_0 has cardinality $j_0(\kappa)$ in M_0 , so that in V we have $|\mathbb{B}_0| = 2^\kappa$.

Remark. Officially elements of $\mathbb{B}(\alpha, \kappa)$ are regular open subsets of the poset $\mathbb{C}(\alpha, \kappa)$, so that $\mathbb{B}(\alpha, \kappa)$ is not literally a subset of V_κ . However, since $\mathbb{C}(\alpha, \kappa)$ has the κ -chain condition, $\mathbb{B}(\alpha, \kappa)$ is the direct limit of the sequence of algebras $\langle \mathbb{B}(\alpha, \gamma) : \gamma < \kappa \rangle$, so that we may identify $\mathbb{B}(\alpha, \kappa)$ with a subset of V_κ . With this identification we may represent elements of \mathbb{B}_0 in the form $[h]_{U_0}$, where h is a function from κ to V_κ .

This becomes important later, when we use such functions h as components of forcing conditions in the poset \mathbb{P} . When we move to a generic extension W with the same V_κ but new subsets of κ , we will need to know that h can still be interpreted as a function which returns an element of $\mathbb{B}(\alpha, \kappa)$ on argument α .

Following Foreman and Woodin, we define a filter $\text{Fil}(H)$ on \mathbb{B}_0 from each U -constraint H .

Definition 2.3. Let H be a U -constraint and let $A \in U$. We define a function $b(H, A)$ as follows:

$$\text{dom}(b(H, A)) = \{\kappa_x : x \in \text{dom}(H) \cap A\},$$

and

$$b(H, A)(\alpha) = \bigvee \{H(x) : x \in \text{dom}(H) \cap A \text{ and } \kappa_x = \alpha\}.$$

In the definition of $b(H, A)(\alpha)$ we are forming the Boolean supremum of a nonempty subset of $\mathbb{C}(\alpha, \kappa)$, thereby defining a nonzero element of $\mathbb{B}(\alpha, \kappa)$. Since $\{\kappa_x : x \in \text{dom}(H) \cap A\} \in U_0$, the function $b(H, A)$ is defined on a U_0 -large set and so represents a nonzero element of the Boolean algebra \mathbb{B}_0 in the ultrapower M_0 .

Lemma 2.4. *Let H be a U -constraint and let $A_1, A_2 \in U$ be such that $A_2 \subseteq A_1$. Then $\text{dom}(b(H, A_2)) \subseteq \text{dom}(b(H, A_1))$ and $b(H, A_2)(\alpha) \leq b(H, A_1)(\alpha)$ for all $\alpha \in \text{dom}(b(H, A_2))$.*

Proof. Straightforward. □

It follows immediately that the set $\{[b(H, A)]_{U_0} : A \in U\}$ forms a filter base on \mathbb{B}_0 .

Definition 2.5. Let H be a U -constraint. Then $\text{Fil}(H)$ is the filter generated by $\{[b(H, A)]_{U_0} : A \in U\}$.

Lemma 2.6. *If $H_2 \leq_U H_1$ then $\text{Fil}(H_1) \subseteq \text{Fil}(H_2)$.*

Proof. Straightforward. □

Lemma 2.7. *For every U -constraint H and every Boolean value \mathfrak{b} in \mathbb{B}_0 , there is $H' \leq_U H$ such that either $\mathfrak{b} \in \text{Fil}(H')$ or $\neg\mathfrak{b} \in \text{Fil}(H')$.*

Proof. We may assume that \mathfrak{b} is non-zero. Let $\mathfrak{b} = [f]_{U_0}$, where $f(\alpha) \in \mathbb{B}(\alpha, \kappa)$ and $f(\alpha)$ is non-zero for all $\alpha \in \text{dom}(f)$. Let $A_0 = \{x \in \text{dom}(H) : \kappa_x \in \text{dom}(f)\}$ and observe that $A_0 \in U$.

For each x in A_0 , we may choose $H^*(x) \leq H(x)$ such that either $H^*(x) \leq f(\kappa_x)$ or $H^*(x) \leq \neg f(\kappa_x)$. Let $A_1 = \{x \in A_0 : H^*(x) \leq f(\kappa_x)\}$. If $A_1 \in U$ then define $H' = H^* \upharpoonright A_1$, otherwise define $H' = H^* \upharpoonright (A_0 - A_1)$.

If $A_1 \in U$ then consider the function $b(H', A_1)$. For every relevant α we see that $b(H', A_1)(\alpha)$ is computed as a Boolean supremum of values which are bounded by $f(\alpha)$, so that $b(H', A_1)(\alpha) \leq f(\alpha)$. Hence $[b(H', A_1)]_{U_0} \leq [f]_{U_0}$, and accordingly $\mathfrak{b} \in \text{Fil}(H')$. Similarly if $A_1 \notin U$ then $\neg\mathfrak{b} \in \text{Fil}(H')$. □

Lemma 2.8. *For every U -constraint H there is $H' \leq_U H$ such that $\text{Fil}(H')$ is an ultrafilter on \mathbb{B}_0 .*

Proof. This follows immediately from the preceding lemmas, the observation that $|\mathbb{B}_0| = 2^\kappa$, and the fact that any \leq_U -decreasing 2^κ -sequence of U -constraints has a lower bound, □

Lemma 2.9. *Let H' and H'' be U -constraints such that $\text{Fil}(H')$ is an ultrafilter on \mathbb{B}_0 and $H'' \leq_U H'$. Then $\text{Fil}(H') = \text{Fil}(H'')$.*

Proof. Straightforward. □

It will be convenient for the arguments of Section 5 to formulate these ideas in a slightly different language. Recall that there is an elementary embedding $k : M_0 \rightarrow M$ such that $k \circ j_0 = j$, given by the formula $k : [f]_{U_0} \mapsto j(f)(\kappa)$.

Lemma 2.10. *For any U -constraint H ,*

$$\text{Fil}(H) = \{\mathbf{b} \in \mathbb{B}_0 : [H]_U \leq_{\mathbb{B}} k(\mathbf{b})\}.$$

Proof. Let f be a typical function representing an element \mathbf{b} of \mathbb{B}_0 , that is to say $\text{dom}(f) \in U_0$ and $f(\alpha) \in \mathbb{B}(\alpha, \kappa)$ for all α . Now $k(\mathbf{b}) = j(f)(\kappa)$, and $[H]_U = j(H)(j^{\ast}\kappa^{+n})$, so that easily $[H]_U \leq j(f)(\kappa)$ if and only if $\{x \in \text{dom}(H) : H(x) \leq f(\kappa_x)\} \in U$.

If $\mathbf{b} \in \text{Fil}(H)$ then by definition there is a set $A \in U$ such that $[b(H, A)]_{U_0} \leq [f]_{U_0}$, that is to say $B =_{\text{def}} \{\alpha : b(H, A)(\alpha) \leq f(\alpha)\} \in U_0$. Now let $A' = A \cap \text{dom}(H) \cap \{x : \kappa_x \in B\}$. Clearly $A' \in U$; fix $x \in A'$ and observe that $H(x) \leq b(H, A)(\kappa_x) \leq f(\kappa_x)$, where the first inequality holds because $x \in \text{dom}(H) \cap A$ and the second one holds because $\kappa_x \in B$. We have shown that $\{x \in \text{dom}(H) : H(x) \leq f(\kappa_x)\} \in U$, so that $[H]_U \leq_{\mathbb{B}} k(\mathbf{b})$.

Conversely, if $[H]_U \leq_{\mathbb{B}} k(\mathbf{b})$ we let $A = \{x \in \text{dom}(H) : H(x) \leq f(\kappa_x)\}$. Then $\text{dom}(b(H, A)) = \{\kappa_x : x \in A\}$. For every α in this set we have that

$$b(H, A)(\alpha) = \bigvee \{H(x) : x \in A \text{ and } \kappa_x = \alpha\} \leq f(\alpha),$$

where the second claim follows since (by the definition of A) we are forming the Boolean supremum of a set of values which is bounded by $f(\alpha)$. \square

We conclude this discussion of constraints and filters by collecting some technical facts about filters of the form $\text{Fil}(H)$ which will be useful when we define the forcing poset \mathbb{P} .

Definition 2.11. A U_0 -constraint is a partial function h from κ to V_κ such that $\text{dom}(h) \in U_0$, $\text{dom}(h)$ is a set of inaccessible cardinals, and $h(\alpha) \in \mathbb{B}(\alpha, \kappa)$ for all $\alpha \in \text{dom}(h)$.

Clearly \mathbb{B}_0 is the set of objects of the form $[h]_{U_0}$ where h is a U_0 -constraint.

Definition 2.12. Let h and h' be U_0 -constraints.

- (1) $h \leq h'$ if and only if $\text{dom}(h) \subseteq \text{dom}(h')$ and $h(\alpha) \leq h'(\alpha)$ for all $\alpha \in \text{dom}(h)$.
- (2) $h \leq_{U_0} h'$ if and only if $\{\alpha : h(\alpha) \leq h'(\alpha)\} \in U_0$ or equivalently $[h]_{U_0} \leq [h']_{U_0}$.

Lemma 2.13. *Let h be a U_0 -constraint and let H be a U -constraint. If $[h]_{U_0} \in \text{Fil}(H)$, then there is $B \in U$ such that $b(H, B) \leq h$.*

Proof. Observe that by definition there is $A \in U$ such that $b(H, A) \leq_{U_0} h$, and define

$$B = \{x \in A : b(H, A)(\kappa_x) \leq h(\kappa_x)\}.$$

It is routine to check that this B works. □

We now record some crucial properties of filters of the form $\text{Fil}(H)$. In the sequel we will limit attention to the special case in which $\text{Fil}(H)$ is an ultrafilter, but only Lemma 2.20 actually requires this assumption.

Lemma 2.14 (κ -completeness Lemma). *Let H be a U -constraint, let $\eta < \kappa$ and let $\langle h_i : i < \eta \rangle$ be a sequence of U_0 -constraints such that $[h_i] \in \text{Fil}(H)$ for all i . Then there exists a U_0 -constraint h such that $[h] \in \text{Fil}(H)$ and $h \leq h_i$ for all i .*

Proof. Appealing to Lemma 2.13 we choose for each $i < \eta$ a set $B_i \in U$ such that $b(H, B_i) \leq h_i$. Let $B = \bigcap_i B_i$, then $B \in U$ and it follows from Lemma 2.4 that $b(H, B) \leq b(H, B_i) \leq h_i$ for all $i < \eta$. □

Definition 2.15. Given a set $s \in V_\kappa$ and a U_0 -constraint h , we define

$$h \downarrow s = h \upharpoonright \{\alpha : s \in V_\alpha\}.$$

Lemma 2.16 (Normality lemma). *Let H be a U -constraint, let $I \subseteq V_\kappa$ and let $\langle h_s : s \in I \rangle$ be an I -indexed family of U_0 -constraints such that $[h_s]_{U_0} \in \text{Fil}(H)$ for all s . Then there exists a U_0 -constraint h such that $[h]_{U_0} \in \text{Fil}(H)$ and $h \downarrow s \leq h_s$ for all s .*

Proof. Choose for each $s \in I$ a set $A_s \in U$ such that $b(H, A_s) \leq h_s$. By the normality of U it follows that if we set $A = \{x \in \text{dom}(H) : \forall s \in I \cap V_{\kappa_x} x \in A_s\}$ then $A \in U$. Let $h = b(H, A)$.

To show this works we fix $\alpha \in \text{dom}(h)$ and $s \in I \cap V_\alpha$. By definition

$$h(\alpha) = \bigvee_{x \in A, \kappa_x = \alpha} H(x).$$

For every x involved in this supremum we have $s \in V_{\kappa_x}$, so that $x \in A_s$. Hence easily

$$h(\alpha) = b(H, A)(\alpha) \leq b(H, A_s)(\alpha) \leq h_s(\alpha).$$

□

With a view towards the forcing construction of Section 3 we define the notion of *lower part*.

Definition 2.17. A *lower part* is a finite sequence

$$(p_0, \kappa_1, p_1, \dots, \kappa_k, p_k)$$

such that:

- (1) $k \geq 0$.
- (2) κ_i is an inaccessible cardinal less than κ for $1 \leq i \leq k$, and $\kappa_1 < \kappa_2 < \dots < \kappa_k$.
- (3) With the convention that $\kappa_0 = 0$ and $\kappa_{k+1} = \kappa$, $p_i \in \mathbb{C}(\kappa_i, \kappa_{i+1})$ for $0 \leq i \leq k$.

Given lower parts s and t with

$$s = (p_0, \kappa_1, \dots, \kappa_{i-1}, p_{i-1}),$$

we say that $t \leq s$ if and only if

$$t = (q_0, \kappa_1, \dots, \kappa_{i-1}, q_{i-1})$$

and $q_i \leq p_i$ for all i .

Definition 2.18. A set X of lower parts is *downwards closed* if and only if for all $s \in X$ and all $t \leq s$ we have $t \in X$.

Now let us fix H a U -constraint such that $\text{Fil}(H)$ is an ultrafilter.

Definition 2.19. h is an *upper part* if and only if h is a U_0 -constraint such that $[h] \in \text{Fil}(H)$.

The fact that $\text{Fil}(H)$ is maximal is at the heart of the following crucial lemma.

Lemma 2.20 (Capturing Lemma). *Let X be a downwards closed set of lower parts and let h be an upper part. Then there exists an upper part $h^+ \leq h$ such that*

- (1) For all $\alpha, \beta \in \text{dom}(h^+)$ with $\alpha < \beta$, $h^+(\alpha) \in \mathbb{C}(\alpha, \beta)$.
- (2) For all lower parts s , exactly one of the two following statements holds:
 - (a) For all $\alpha \in \text{dom}(h^+)$ such that $s \in V_\alpha$, there exists $q \leq h^+(\alpha)$ such that $s^\frown(\alpha, q) \in X$.
 - (b) For all $\alpha \in \text{dom}(h^+)$ such that $s \in V_\alpha$, there does not exist $q \leq h^+(\alpha)$ such that $s^\frown(\alpha, q) \in X$.
- (3) For all lower parts s , and all $\alpha, \beta \in \text{dom}(h^+)$ such that $s \in V_\alpha$ with $\alpha < \beta$, IF there is $q \leq h^+(\alpha)$ such that $s^\frown(\alpha, q) \in X$ THEN

$$\{q \in \mathbb{C}(\alpha, \beta) : s^\frown(\alpha, q) \in X\}$$

is dense below $h^+(\alpha)$ in $\mathbb{C}(\alpha, \beta)$.

Proof. Strengthening h if necessary, we may assume that $h = b(H, A)$ for some $A \in U$. Fix for the moment a lower part s , and let

$$A^s \subseteq \{x \in \text{dom}(H) \cap A : s \in V_{\kappa_x}\}$$

be such that $A^s \in U$ and one of the following statements holds:

(Case One) For all $x \in A_s$ there is $q \leq H(x)$ such that $s^\frown(\kappa_x, q) \in X$.

(Case Two) For no $x \in A_s$ is there $q \leq H(x)$ such that $s^\frown(\kappa_x, q) \in X$.

We now choose $H^s \leq H$ such that $\text{dom}(H^s) = A^s$, and if s falls in Case One then $s^\frown(\kappa_x, H^s(x)) \in X$ for all $x \in A^s$, and then let $h^s = b(H^s, A^s)$. By Lemma 2.9, $\text{Fil}(H^s) = \text{Fil}(H)$ and so h^s is a legitimate upper part.

Claim One: If there exist $\alpha \in \text{dom}(h^s)$ and $p \leq h^s(\alpha)$ such that $s^\frown(\alpha, p) \in X$, then

$$\{r \in \mathbb{C}(\alpha', \kappa) : s^\frown(\alpha', r) \in X\}$$

is dense below $h^s(\alpha')$ in $\mathbb{C}(\alpha', \kappa)$ for all $\alpha' \in \text{dom}(h^s)$.

Proof of Claim One: Fix some α and $p \leq h^s(\alpha)$ with $s^\frown(\alpha, p) \in X$, and recall that $h^s(\alpha) = \bigvee_{x \in A^s, \kappa_x = \alpha} H^s(x)$. It follows that there is $x \in A^s$ such that $\kappa_x = \alpha$ and p is comparable with $H^s(x)$, and we may fix $p' \leq p, H^s(x)$. Since X is downwards closed, $s^\frown(\alpha, p') \in X$. Since $x \in A^s$ and $p' \leq H^s(x) \leq H(x)$, s falls in Case One above and so $s^\frown(\kappa_x, H^s(x)) \in X$ for all $x \in A_s$.

Let $\alpha' \in \text{dom}(h^s)$, let $q \in \mathbb{C}(\alpha, \kappa)$ be arbitrary with $q \leq h^s(\alpha')$, and observe that arguing as above there is $x \in A^s$ such that $\kappa_x = \alpha'$ and q is comparable with $H^s(x)$; if we now choose $r \leq q, H^s(x)$ then it follows from the downwards closedness of X and the definition of H^s in Case One that $s^\frown(\kappa_x, r) \in X$.

Claim Two: For all $\alpha \in \text{dom}(h^s)$, if there is $p \leq h^s(\alpha)$ with $s^\frown(\alpha, p) \in X$ then there is an inaccessible cardinal $\beta^s(\alpha) < \kappa$ such that $h^s(\alpha) \in \mathbb{C}(\alpha, \beta^s(\alpha))$ and

$$\{r \in \mathbb{C}(\alpha, \beta) : s^\frown(\alpha, r) \in X\}$$

is dense below $h^s(\alpha)$ in $\mathbb{C}(\alpha, \beta)$ for all $\beta \geq \beta^s(\alpha)$.

Proof of Claim Two: By the preceding claim,

$$\{r \in \mathbb{C}(\alpha, \kappa) : s^\frown(\alpha, r) \in X\}$$

is dense below $h^s(\alpha)$ in $\mathbb{C}(\alpha, \kappa)$, and since X is downwards closed this set is open. Choose a maximal antichain A below $h^s(\alpha)$ consisting of points in this set, and then appeal to the κ -chain condition to find $\beta^s(\alpha) < \kappa$ such that $A \subseteq \mathbb{C}(\alpha, \beta^s(\alpha))$.

Let

$$B^s = \{\beta : \forall \alpha < \beta \beta^s(\alpha) < \beta\}.$$

Since U_0 is normal, $B^s \in U_0$.

To finish the proof, use Lemma 2.16 to find an upper part h^- such that $h^- \upharpoonright s \leq h^s$ for all s , and let

$$B = \{\beta : \forall s \in V_\beta \beta \in B^s \text{ and } \forall \alpha < \beta h^-(\alpha) \in V_\beta\}.$$

By normality $B \in U_0$, and so we may define an upper part $h^+ = h^- \upharpoonright B$.

Claim Three: h^+ is as required.

Proof of Claim Three: It is immediate from the definitions that $h^+ \leq h$, and Clause 1) from the conclusion is satisfied.

Towards showing Clauses 2) and 3), suppose that $\alpha \in \text{dom}(h^+)$, $s \in V_\alpha$, $q \leq h^+(\alpha)$ and $s^\frown(\alpha, q) \in X$. By construction $h^+(\alpha) \leq h^s(\alpha)$.

By Claim One above,

$$\{r \in \mathbb{C}(\alpha', \kappa) : s^\frown(\alpha', r) \in X\}$$

is dense below $h^s(\alpha')$ in $\mathbb{C}(\alpha', \kappa)$ for all $\alpha' \in \text{dom}(h^s)$. So if $s \in V_{\alpha'}$ and $\alpha' \in \text{dom}(h^+)$, then since $h^+(\alpha') \leq h^s(\alpha')$ this same set is dense below $h^+(\alpha')$ and so Clause 2) is satisfied.

By Claim Two above,

$$\{r \in \mathbb{C}(\alpha, \beta) : s^\frown(\alpha, r) \in X\}$$

is dense below $h^s(\alpha)$ for all $\beta \geq \beta^s(\alpha)$. If $\beta \in \text{dom}(h^+)$ with $\alpha < \beta$ then (since $\beta \in B$) we have that $h^+(\alpha) \in V_\beta$ and also $\beta \in B^s$, so that $\beta^s(\alpha) < \beta$ and hence

$$\{r \in \mathbb{C}(\alpha, \beta) : s^\frown(\alpha, r) \in X\}$$

is dense below $h^+(\alpha)$. This shows that Clause 3) is satisfied. \square

Definition 2.21. If X is a downwards closed set of lower parts and h^+ is an upper part satisfying the conclusion of the Capturing Lemma then we say that h^+ captures X .

3. THE FORCING \mathbb{P} AND ITS PROPERTIES

3.1. **The filter.** In the last section we used the 2^κ supercompactness of κ to show that there exists a U -constraint H such that $\text{Fil}(H)$ is an ultrafilter. We then established that if \mathcal{F} is an ultrafilter of the form $\text{Fil}(H)$ then \mathcal{F} has three properties:

- I. (κ -completeness) Let $\eta < \kappa$ and let $\langle h_i : i < \eta \rangle$ be a sequence of upper parts. Then there exists an upper part h such that $h \leq h_i$ for all i .
- II. (normality) Let I be a set of lower parts and let $\langle h_s : s \in I \rangle$ be an I -indexed family of upper parts. Then there exists an upper part h such that $h \upharpoonright s \leq h_s$ for all s .
- III. (capturing) Let X be a downwards closed set of lower parts and let h be an upper part. Then there exists an upper part $h^+ \leq h$ such that h^+ captures X .

Remark. In III above, the last part implies immediately that if there is $q \leq h^+(\alpha)$ such that $s^\frown(\alpha, q) \in X$ then

$$\{q \in \mathbb{C}(\alpha, \kappa) : s^\frown(\alpha, q) \in X\}$$

is dense below $h^+(\alpha)$ in $\mathbb{C}(\alpha, \kappa)$.

For the rest of this section we will weaken our assumptions on κ , to be precise we will assume only that:

- (1) κ is measurable, and U_0 is a normal measure on κ , with associated ultrapower map $j_0 : V \rightarrow M_0 = \text{Ult}(V, U_0)$.
- (2) $2^\kappa = \kappa^{+n}$ and $n < m < \omega$.
- (3) \mathcal{F} is an ultrafilter on $\mathbb{B}_0 = \mathbb{C}(\kappa, j_0(\kappa))^{M_0}$ with properties I-III.

3.2. The forcing. We now fix a filter \mathcal{F} satisfying properties I-III above, and use \mathcal{F} to define a forcing poset \mathbb{P} . Conditions in \mathbb{P} are pairs (s, h) such that:

- (1) s is a lower part.
- (2) h is an upper part.

When $p = (s, h)$ we will refer to s as the *stem* or *lower part* of p , and to h as the *upper part* of p .

Suppose that $p = (s, h)$ and $q = (s', h')$ are conditions where $s = (p_0, \alpha_1, p_1, \dots, \alpha_k, p_k)$ and $s' = (q_0, \beta_1, q_1, \dots, \beta_l, p_l)$. Then $q \leq p$ if and only if

- (1) $\alpha_i = \beta_i$ and $q_i \leq p_i$ for $1 \leq i \leq k$.
- (2) $\beta_i \in \text{dom}(h)$ and $q_i \leq h(\beta_i)$ for $k < i \leq l$.
- (3) $h' \leq h$.

q is a *direct extension* of p if $q \leq p$ and in addition $k = l$. We write $q \leq^* p$ in this case.

The generic object for \mathbb{P} is a sequence

$$f_0, \kappa_1, f_1, \kappa_2, f_2 \dots$$

where the κ_i form an increasing and cofinal ω -sequence of inaccessible cardinals less than κ (which will be generic for the Prikry forcing defined from U_0), f_i is $\mathbb{C}(\omega, \kappa_1)$ -generic and f_i is $\mathbb{C}(\kappa_i, \kappa_{i+1})$ -generic for $i > 0$. The condition (s, h) where $s = (p_0, \alpha_1, p_1, \dots, \alpha_k, p_k)$ carries the information that

- (1) $\kappa_i = \alpha_i$ and $p_i \in f_i$ for $1 \leq i \leq k$.
- (2) $\kappa_i \in \text{dom}(h)$ and $h(\kappa_i) \in f_i$ for $i > k$.

Lemma 3.1. *The forcing poset \mathbb{P} has the κ^+ -cc.*

Proof. Let (s, h) and (s, h') be two conditions with the same stem s . Since $[h]_{U_0}, [h']_{U_0} \in \mathcal{F}$ and \mathcal{F} is a filter, it is easy to find h'' such that $[h'']_{U_0} \leq [h]_{U_0}, [h']_{U_0}$. $h'' \leq_{U_0} h, h'$, and if we let $B = \{\alpha : h''(\alpha) \leq h(\alpha) \text{ and } h''(\alpha) \leq h'(\alpha)\}$ then $h'' \upharpoonright B \leq h, h'$. The condition $(s, h'' \upharpoonright B)$ is clearly a lower bound for (s, h) and (s, h') . \square

The following Lemma is straightforward.

Lemma 3.2. *Let $p = (s, h)$ where $s = (p_0, \alpha_1, p_1, \dots, \alpha_k, p_k)$, and let $1 \leq i \leq k$. Then the forcing poset $\mathbb{P} \downarrow p$ is isomorphic to*

$$\mathbb{D} \times (\mathbb{P}' \downarrow (t, h)),$$

where

$$\mathbb{D} = \mathbb{C}(\omega, \alpha_1) \downarrow p_0 \times \dots \times \mathbb{C}(\alpha_{i-1}, \alpha_i) \downarrow p_{i-1},$$

\mathbb{P}' is defined just like \mathbb{P} except that α_i plays the role of ω , and

$$t = (p_i, \alpha_{i+1}, \dots, \alpha_k, p_k).$$

3.3. The Prikry Lemma.

Lemma 3.3 (Prikry Lemma for \mathbb{P}). *Let Φ be a sentence in the forcing language and let $p \in \mathbb{P}$, then there is a direct extension $q \leq p$ which decides Φ .*

Proof. We begin the proof with a construction that is done uniformly for all conditions p .

For each lower part t , if there is an upper part h such that (t, h) decides Φ then we fix such an upper part h_t . Appealing to Property II for \mathcal{F} , we find $h^0 \leq h$ such that $h^0 \upharpoonright t \leq h_t$ for all relevant t . So for every t , if there exists any h such that (t, h) decides Φ then (t, h^0) decides Φ .

We now define two sets of lower parts:

$$X^+ = \{t : (t, h^0) \Vdash \Phi\},$$

and

$$X^- = \{t : (t, h^0) \Vdash \neg\Phi\}.$$

It is clear that both X^+ and X^- are downwards closed. By two appeals to Property III we obtain $h^1 \leq h^0$ such that h^1 captures both X^+ and X^- .

Now let $p = (s, h)$. As in the proof of Lemma 3.1, we may find an upper part h^* such that $h^* \leq h, h^1$. Let $(t, h^{**}) \leq (s, h^*)$ be a condition deciding Φ , with $lh(t)$ chosen minimal among all such extensions of (s, h^*) . We will show that $lh(t) = lh(s)$, establishing that (t, h^{**}) is a direct extension of (s, h) and thereby proving the Lemma.

We will assume that $(t, h^{**}) \Vdash \Phi$, the proof in the case when it forces $\neg\Phi$ is the same. Suppose for a contradiction that $lh(t) > lh(s)$, and let t be the concatenation of a shorter lower part t^- and a pair (α, q) . Since t is longer than s , we have that $\alpha \in \text{dom}(h^*)$ and $q \leq h^*(\alpha) \leq h^1(\alpha)$. By the construction of h^0 we have also that $(t, h^0) \Vdash \Phi$, so that $t \in X^+$.

We claim that $(t^-, h^{**}) \Vdash \Phi$, which will contradict the hypothesis that $lh(t)$ was chosen minimal and establish the Lemma.

Towards the claim we observe that, since h^1 captures X^+ and $q \leq h^1(\alpha)$, for every $\beta, \gamma \in \text{dom}(h^1 \upharpoonright t^-)$ with $\beta < \gamma$ the set $\{r : t^- \frown (\beta, r) \in X^+\}$ is dense below $h^1(\beta)$ in $\mathbb{C}(\beta, \gamma)$. We will use this to show that the set of conditions which force Φ is dense below (t^-, h^{**}) in \mathbb{P} , establishing the claim that $(t^-, h^{**}) \Vdash \Phi$.

It will suffice to show that any extension of (t^-, h^{**}) with a properly longer lower part can be extended to force Φ . Consider such an extension of the form $(t' \frown (\gamma_0, q_0) \frown \dots \frown (\gamma_i, q_i), h^{***})$, where $t' \leq t^-$ and without loss of generality $i > 0$. Since $q_0 \leq h^1(\gamma_0)$ and $q_0 \in \mathcal{C}(\gamma_0, \gamma_1)$, by the remarks in the preceding paragraph there is $r \leq q_0$ with $r \in \mathcal{C}(\gamma_0, \gamma_1)$ such that $(t^- \frown (\gamma_0, r), h^0) \Vdash \Phi$.

It is now easy to verify that by strengthening q_0 to r we obtain a condition $(t' \frown (\gamma_0, r) \frown \dots \frown (\gamma_i, q_i), h^{***})$ which extends $(t^- \frown (\gamma_0, q), h^0)$, and so forces Φ . This concludes the proof. \square

Remark. The proof of the Prikry Lemma extends without any change to the forcing poset \mathbb{P}' defined in Lemma 3.2.

3.4. Analysing names for bounded subsets of κ . It is clear that the forcing poset \mathbb{P} collapses all cardinals in the open intervals (ω^{+m}, κ_1) and $(\kappa_i^m, \kappa_{i+1})$ for $i > 0$. One of the main applications of the Prikry Lemma is to show that no other cardinals are collapsed, so that κ becomes \aleph_ω in the generic extension.

Lemma 3.4. *Let G be \mathbb{P} -generic and let*

$$f_0, \kappa_1, f_1, \kappa_2, f_2 \dots$$

be the generic sequence added by G . Let $x \in V[G]$ be a bounded subset of $(\kappa_i^{+m})^V$ for some $i > 0$. Then $x \in V[f_0 \times \dots \times f_{i-1}]$.

Proof. Working below a suitable condition, we may use Lemma 3.2 to view $V[G]$ as a two-step extension $V[G'][g]$ where $g = f_0 \times \dots \times f_{i-1}$ and G' is generic for \mathbb{P}' , a version of \mathbb{P} in which κ_i plays the role of ω .

Let $x = i_G(\dot{x})$, where \dot{x} is a \mathbb{P} -name for a subset of γ for some $\gamma < \kappa_i^{+m}$. We may view \dot{x} as a \mathbb{P}' -name for a \mathbb{D} -name for a subset of γ , where $\mathbb{D} = \mathbb{C}(\omega, \kappa_1) \times \dots \times \mathbb{C}(\kappa_{i-1}, \kappa_i)$.

Since \mathbb{P}' satisfies the Prikry Lemma, it is easy to see that the \mathbb{D} -name denoted by \dot{x} lies in V , so that $x \in V[f_0 \times \dots \times f_{i-1}]$ as required. \square

Using Lemma 3.4, standard chain condition and closure arguments imply that only the cardinals in the intervals (ω^{+m}, κ_1) and $(\kappa_i^m, \kappa_{i+1})$ are collapsed by \mathbb{P} . For the purposes of some later arguments, we will prove a more refined version of Lemma 3.4. The point at stake here is that *a priori* it seems that a name for a bounded subset of κ may depend on an arbitrarily large initial segment of the generic object, and this would cause major difficulties in the chain condition arguments of Section 4.

Given an increasing sequence $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_k \rangle$ of inaccessible cardinals less than κ , we define

$$\mathbb{D}(\vec{\alpha}) = \mathbb{C}(\omega, \alpha_1) \times \dots \times \mathbb{C}(\alpha_{k-1}, \alpha_k).$$

Lemma 3.5. *Let $\mu, \eta < \kappa$ and let \dot{x} be a \mathbb{P} -name for a subset of μ . Let h be an upper part and let S be the set of increasing sequences $\langle \alpha_1, \dots, \alpha_k \rangle$ where $\alpha_i < \eta$ and α_i is inaccessible.*

Then there exist an ordinal β with $\mu, \eta < \beta < \kappa$, names $\langle \dot{y}_{\vec{\alpha}} : \vec{\alpha} \in S \rangle$ and an upper part $h' \leq h$ with $\min(\text{dom}(h')) > \beta$ such that for every $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_k \rangle \in S$:

- (1) $\dot{y}_{\vec{\alpha}}$ is a $\mathbb{D}(\vec{\alpha} \smallfrown \beta)$ name for a subset of μ .
- (2) If $t = (\emptyset, \alpha_1, \emptyset, \dots, \alpha_k, \emptyset)$ then $(t, h') \Vdash \dot{x} = \dot{y}_{\vec{\alpha}}$. That is to say that if G is \mathbb{P} -generic with $(t, h') \in G$, and

$$f_0, \alpha_1, f_1, \alpha_2, f_2 \dots$$

is the corresponding generic sequence, then $i_G(\dot{x}) = i_f(\dot{y}_{\vec{\alpha}})$, where $f = f_0 \times \dots \times f_{k-1} \times (f_k \upharpoonright \beta)$.

Proof. As in the first step of the proof of the Prikry Lemma, we find $h^0 \leq h$ such that for every lower part $t = (p_0, \beta_1, \dots, \beta_k, p_k)$, if there are an upper part h' and a $\mathbb{D}(\langle \beta_1, \dots, \beta_k \rangle)$ -name \dot{y} such that $(t, h') \Vdash \dot{x} = \dot{y}$ then $(t, h^0) \Vdash \dot{x} = \dot{y}$.

For each $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_k \rangle \in S$, each inaccessible δ with $\mu, \eta < \delta < \kappa$ and each canonical $\mathbb{D}(\vec{\alpha} \smallfrown \delta)$ -name \dot{y} for a subset of μ , let $X(\vec{\alpha}, \delta, \dot{y})$ be the set of lower parts s such that

$$s = (q_0, \alpha_1, q_1, \dots, \alpha_k, q_k, \gamma, r)$$

for some $\gamma > \delta$, and $(s, h_0) \Vdash \dot{x} = \dot{y}$. Since this is a downwards closed set of lower parts, we may find $h^{\vec{\alpha}, \delta, \dot{y}} \leq h^0$ which captures it. Using Lemmas 2.14 and 2.16 we may then find an upper part h^1 such that $h^1 \upharpoonright \delta \leq h^{\vec{\alpha}, \delta, \dot{y}}$ for all $\vec{\alpha}, \delta, \dot{y}$, and also $\min(\text{dom}(h^1)) > \mu, \eta$. By shrinking $\text{dom}(h^1)$ if necessary, we will also arrange that $\text{dom}(h^1)$ consists of Mahlo cardinals.

Fix for the moment a sequence $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_k \rangle \in S$. Fix some $\gamma \in \text{dom}(h^1)$ and consider the condition

$$((\emptyset, \alpha_1, \emptyset, \dots, \emptyset, \alpha_k, \emptyset, \gamma, h^1(\gamma)), h^1).$$

Working as in the proof of Lemma 3.4, we may find $r \leq h^1(\gamma)$ and $h^* \leq h^1$ such that

$$((\emptyset, \alpha_1, \emptyset, \dots, \emptyset, \alpha_k, \emptyset, \gamma, r), h^*) \Vdash \dot{x} = \dot{y}$$

where \dot{y} is a canonical $\mathbb{D}(\vec{\alpha} \frown \gamma)$ -name for a subset of μ . Since $\mathbb{D}(\vec{\alpha} \frown \gamma)$ has the γ -cc and γ is Mahlo, \dot{y} is a canonical $\mathbb{D}(\vec{\alpha} \frown \delta)$ -name for some inaccessible δ with $\mu, \eta < \delta < \gamma$.

By construction $r \leq h^1(\gamma) \leq h^{\vec{\alpha}, \delta, \dot{y}}(\gamma)$. By the choice of h^0 , we see that

$$((\emptyset, \alpha_1, \emptyset, \dots, \emptyset, \alpha_k, \emptyset, \gamma, r), h^0) \Vdash \dot{x} = \dot{y}.$$

By the choice of h^1 , for every $\gamma_1, \gamma_2 \in \text{dom}(h^1)$ with $\delta < \gamma_1 < \gamma_2$ the set of $r^* \in \mathbb{C}(\gamma_1, \gamma_2)$ such that

$$((\emptyset, \alpha_1, \emptyset, \dots, \emptyset, \alpha_k, \emptyset, \gamma_1, r^*), h^0) \Vdash \dot{x} = \dot{y}$$

is dense below $h^1(\gamma_1)$. So for every $\gamma_1 \in \text{dom}(h_1)$ with $\delta < \gamma_1$

$$((\emptyset, \alpha_1, \emptyset, \dots, \emptyset, \alpha_k, \emptyset, \gamma_1, h^1(\gamma_1)), h^1) \Vdash \dot{x} = \dot{y},$$

which implies that

$$((\emptyset, \alpha_1, \emptyset, \dots, \emptyset, \alpha_k, \emptyset), h^1 \upharpoonright \delta) \Vdash \dot{x} = \dot{y}.$$

To record their dependence on $\vec{\alpha}$, we write $\delta_{\vec{\alpha}}$ for δ and $\dot{y}_{\vec{\alpha}}$ for \dot{y} .

Let β be the supremum of the $\delta_{\vec{\alpha}}$ for $\vec{\alpha} \in S$, and let $h' = h^1 \upharpoonright \beta$. It is now easy to see that the ordinal β , upper part h' and family of names $\langle \dot{y}_{\vec{\alpha}} : \vec{\alpha} \in S \rangle$ are as required. \square

3.5. Characterisation of genericity. We will need one more technical fact about \mathbb{P} , namely a characterisation of the generic object. Similar “geometric” characterisations for other Prikry-type forcing posets appear at many places [9, 11, 1] in the literature.

Lemma 3.6 (Genericity Lemma). *Let*

$$f_0, \kappa_1, f_1, \dots$$

be such that

- (1) f_i is $\mathbb{C}(\omega, \kappa_1)$ -generic for $i = 0$ and $\mathbb{C}(\kappa_i, \kappa_{i+1})$ -generic for $i > 0$.
- (2) For all upper parts h there is an integer s such that $\kappa_t \in \text{dom}(h)$ and $h(\kappa_t) \in f_t$ for all $t \geq s$.

Then this is a generic sequence for \mathbb{P} .

Proof. For our later convenience we define $\mathbb{C}_0 = \mathbb{C}(\omega, \kappa_1)$ and $\mathbb{C}_i = \mathbb{C}(\kappa_i, \kappa_{i+1})$ for $i > 0$. We make the remark that by an easy application of Easton's Lemma the filters f_0, \dots, f_n are mutually generic, that is $f_0 \times \dots \times f_n$ is generic over V for $\mathbb{C}_0 \times \dots \times \mathbb{C}_n$.

We now fix E a dense open set in \mathbb{P} , with the ultimate goal of showing that E meets the filter on \mathbb{P} generated by

$$f_0, \kappa_1, f_1, \dots$$

To achieve this goal we need to “canonise” E in a sense to be made precise later.

By a familiar diagonal intersection argument, there is an upper part h_0 such that for every lower part s ,

$$\exists h (s, h) \in E \iff (s, h_0 \downarrow s) \in E.$$

Since the set E is open, it is easy to see that if we let

$$X_0 = \{s : (s, h_0 \downarrow s) \in E\}$$

then X_0 is a downwards closed set of lower parts.

Applying Property III repeatedly we construct downwards closed sets X_n and upper parts h_n such that:

- (1) $h_{n+1} \leq h_n$.
- (2) h_{n+1} captures X_n .
- (3) X_{n+1} is the set of lower parts s such that for some (equivalently, for every) $\alpha \in \text{dom}(h_{n+1})$ such that $s \in V_\alpha$ there is $q \leq h_{n+1}(\alpha)$ with $s^\frown(q, \alpha) \in X_n$.

We appeal to Property I to find an upper part h_∞ such that $h_\infty \leq h_n$ for all n . By the hypotheses, we find an integer k such that $\kappa_l \text{ dom}(h_\infty)$ and $h_\infty(\kappa_l) \in f_l$ for all $l \geq k$.

Claim.

$$\{(q_0, \dots, q_{k-1}) : \exists j (q_0, \kappa_1, \dots, \kappa_{k-1}, q_{k-1}) \in X_j\}$$

is dense in $\mathbb{C}_0 \times \dots \times \mathbb{C}_{k-1}$.

Proof. Let $(p_0, \dots, p_{k-1}) \in \mathbb{C}_0 \times \dots \times \mathbb{C}_{k-1}$, and consider the condition

$$((p_0, \kappa_1, \dots, \kappa_{k-1}, p_{k-1}, \kappa_k, h_\infty(\kappa_k)), h_\infty).$$

Since E is dense there is an extension

$$((q_0, \kappa_1, \dots, \kappa_{k-1}, q_{k-1}, \bar{\kappa}_k, q_k, \dots, \bar{\kappa}_{k+j-1}, q_{k+j-1}), h) \in E$$

for some $j > 0$. Call the lower part of this extension s , and observe that by construction of h_0 we have $(s, h_0 \downarrow s) \in E$ so that $s \in X_0$.

Now observe that $\bar{\kappa}_{k+j-1} \in \text{dom}(h_\infty)$ and $q_{k+j-1} \leq h_\infty(\bar{\kappa}_{k+j-1}) \leq h_1(\bar{\kappa}_{k+j-1})$, so that

$$(q_0, \kappa_1, \dots, \kappa_{k-1}, q_{k-1}, \bar{\kappa}_k, q_k, \dots, \bar{\kappa}_{k+j-2}, q_{k+j-2}) \in X_1.$$

Stepping backwards in the obvious way we eventually obtain that

$$(q_0, \kappa_1, \dots, \kappa_{k-1}, q_{k-1}) \in X_j.$$

□

Since $f_0 \times \dots \times f_{k-1}$ is generic, we obtain conditions $q_i \in f_i$ for $i < k$ such that $t \in X_j$ where $t = (q_0, \kappa_1, \dots, \kappa_{k-1}, q_{k-1})$. Since $t \in X_j$, $\kappa_k, \kappa_{k+1} \in \text{dom}(h_j)$ and $\kappa_k < \kappa_{k+1}$,

$$\{p \in \mathbb{C}_k : t^\frown(\kappa_k, p) \in X_{j-1}\}$$

is dense below $h_j(\kappa_k)$. Also $h_j(\kappa_k) \in f_k$ because $h_\infty(\kappa_k) \in f_k$ and $h_\infty \leq h_j$. So we may find $q_j^* \in f_j$ such that

$$t^\frown(\kappa_j, q_j^*) \in X_{j-1}.$$

Repeating this argument j times we construct $q_i^* \in f_i$ for $k \leq i < k+j$ such that

$$u = t^\frown(\kappa_j, q_j^*, \dots, \kappa_{k+j-1}, q_{k+j-1}^*) \in X_0,$$

that is to say that $(u, h_0 \upharpoonright u) \in E$.

But it is now easy to verify that $(u, h_0 \upharpoonright u)$ is in the filter generated by the sequence of (f_i) : simply observe that

- (1) $q_i \in f_i$ for $i < k$.
- (2) $q_i^* \in f_i$ for $k \leq i < k+j$.
- (3) $h_0(\kappa_i) \in f_i$ for $i \geq k+j$.

This concludes the proof of the Genericity Lemma. □

4. THE FORCING \mathbb{Q} AND ITS PROPERTIES

We work throughout with the same hypotheses as in Section 3. In particular \mathcal{F} has properties I, II and III and \mathbb{P} is the Prikry-type forcing defined from \mathcal{F} . Let $2^{\kappa^+} = \lambda$, and let T be a tree of height κ^+ such that T has at least λ branches and each level of T has size at most κ^+ . Let $\langle x_\beta : \beta < \lambda \rangle$ enumerate a sequence of distinct branches, and enumerate $\text{Lev}_\alpha(T)$ as $\langle t(\alpha, i) : i < |\text{Lev}_\alpha(T)| \rangle$ for each $\alpha < \kappa^+$.

Definition 4.1. Let A be a function such that $\text{dom}(A)$ is a bounded set of inaccessible cardinals less than κ , and $A(\alpha) \in \mathbb{B}(\alpha, \kappa)$ with $A(\alpha) \neq 0$ for all $\alpha \in \text{dom}(A)$. Let $s = (q_0, \alpha_0, q_1, \dots, \alpha_k, q_k)$ be a lower part, and let $\eta < \kappa$. Then s is *harmonious with A past η* if and only if for all j such that $\alpha_j \geq \eta$ we have $\alpha_j \in \text{dom}(A)$ and $q_j \leq A(\alpha_j)$.

Let $\langle \dot{G}_\alpha : \alpha < \lambda \rangle$ enumerate all the canonical \mathbb{P} -names for graphs on κ^+ . We define a forcing poset \mathbb{Q} .

Conditions in \mathbb{Q} are quadruples (A, B, t, f) such that:

- (1) A is a function such that $\text{dom}(A)$ is a bounded set of inaccessible cardinals less than κ , and $A(\alpha) \in \mathbb{B}(\alpha, \kappa)$ with $A(\alpha) \neq 0$ for all $\alpha \in \text{dom}(A)$.
- (2) B is an upper part.
- (3) t is a triple (ρ, a, b) where $\rho < \kappa$, $a \in [\kappa^+]^{<\kappa}$ and $b \in [\lambda]^{<\kappa}$.
- (4) f is a sequence $\langle f_{\eta, \beta} : \eta < \rho, \beta \in b \rangle$ such that each function $f_{\eta, \beta}$ has domain a .
- (5) $f_{\eta, \beta}(\zeta) \in \{x_\beta \upharpoonright \zeta\} \times \kappa$ for all $\eta < \rho$, $\beta \in b$ and $\zeta \in a$.
- (6) For every $\eta \in \text{dom}(A) \cap \rho$, every lower part s harmonious with A past η , and every $\beta, \gamma \in b$ and $\zeta', \zeta \in a$ such that $f_{\eta, \beta}(\zeta') = f_{\eta, \gamma}(\zeta') \neq f_{\eta, \beta}(\zeta) = f_{\eta, \gamma}(\zeta)$,

$$(s, B) \Vdash \zeta' \dot{G}_\beta \zeta \iff \zeta' \dot{G}_\gamma \zeta.$$

Remark. In the last clause, if s is one of the relevant stems then all ordinals appearing in s are less than $\text{ssup}(\text{dom}(A))$.

Let $q = (A, B, t, f)$ and $q' = (A', B', t', f')$ be two conditions in \mathbb{Q} . Then $q' \leq q$ if and only if:

- (1) $\text{dom}(A)$ is an initial segment of $\text{dom}(A')$, and $A' \upharpoonright \text{dom}(A) = A$.
- (2) $B' \leq B$, that is $\text{dom}(B') \subseteq \text{dom}(B)$ and $B'(\alpha) \leq B(\alpha)$ for all $\alpha \in \text{dom}(B')$.
- (3) For all $\alpha \in \text{dom}(A') \setminus \text{dom}(A)$, $\alpha \in \text{dom}(B)$ and $A'(\alpha) \leq B(\alpha)$.
- (4) If we let $t = (\rho, a, b)$ and $t' = (\rho', a', b')$ then $\rho \leq \rho'$, $a \subseteq a'$ and $b \subseteq b'$.
- (5) $f'_{\eta, \beta}(\zeta) = f_{\eta, \beta}(\zeta)$ for all $\eta < \rho$, $\beta \in b$ and $\zeta \in a$.

Remark. The forcing poset \mathbb{Q} is intended to add (among other things) a generic function h from κ to V_κ of the right general form to be an upper part. If we ultimately force with some version of \mathbb{P} for which the generic function h is a legitimate upper part, then we will add a generic sequence x which eventually obeys h but we do not know past which point on x this will begin to happen. This motivates the notion of “harmonious past η ”, and also explains why each η gets its own set of functions $f_{\eta, \beta}$.

Lemma 4.2. *If $G_{\mathbb{Q}}$ is \mathbb{Q} -generic then:*

- (1) *If we let $h^{G_{\mathbb{Q}}} = \bigcup \{A^p : p \in G_{\mathbb{Q}}\}$ then $h^{G_{\mathbb{Q}}}$ is a function, $\text{dom}(h^{G_{\mathbb{Q}}})$ is unbounded in κ , and for every upper part h we have that $\alpha \in \text{dom}(h^{G_{\mathbb{Q}}})$ and $h^{G_{\mathbb{Q}}}(\alpha) \leq h(\alpha)$ for all large enough $\alpha \in \text{dom}(h)$.*

- (2) For all $\eta < \kappa$ and $\beta < \lambda$, if we let $F_{\eta,\beta}^{G_{\mathbb{Q}}} = \bigcup \{f_{\eta,\beta}^p : p \in G_{\mathbb{Q}}\}$ then $F_{\eta,\beta}^{G_{\mathbb{Q}}}$ is a function with domain κ^+ .

Proof. For the first claim, we suppose that $\nu < \kappa$, h is an upper part, and q is an arbitrary condition. Let $\mu \in \text{dom}(B^q)$ with $\mu > \nu$, and define $r = (A^r, B^r, t^r, f^r)$ as follows: $A^r = A^q \cup \{(\mu, B^q(\mu))\}$, B^r is some upper part such that $B^r \leq B^q, h$ and $\mu < \min(\text{dom}(B^r))$, $t^r = t^q$ and $f^r = f^q$.

We must verify that r is a condition and $r \leq q$. The only non-trivial point is to see that r satisfies Clause 6) in the definition of conditionhood in \mathbb{Q} . Let t be a lower part harmonious with A^r past η . There are now two cases. If t is harmonious with A^q past η then $(t, B^r) \leq (t, B^q)$, and we are done by Clause 6) for q . Otherwise $t = s \frown \langle \mu, p \rangle$ for some $p \leq B^q(\mu)$ and s harmonious with A^q past η , $(t, B^r) \leq (s, B^q)$, and again we are done by Clause 6) for q .

For the second claim, we fix ζ, η, β and then find $a \supseteq a^q$, $b \supseteq b^q$ and $\rho \geq \rho^q$ such that $\eta < \rho$, $\zeta \in a$, $\beta \in b$. We then define $r = (A^r, B^r, t^r, f^r)$ as follows: $A^r = A^q$, $B^r = B^q$, $t^r = (\rho, a, b)$ and f^r is chosen to extend f^q and to be such that the values $f_{\eta',\beta'}^r(\zeta')$ for $(\eta', \zeta', \beta') \in (\rho \times a \times b) \setminus (\rho^q \times a^q \times b^q)$ are all distinct from each other and from any of the values $f_{\eta',\beta'}^q(\zeta')$ for $(\eta', \zeta', \beta') \in \rho^q \times a^q \times b^q$. This choice ensures that Clause 6) in the definition of conditionhood holds, so that r is a condition with $r \leq q$. \square

We recall that for a regular uncountable cardinal ν , a poset \mathbb{R} is ν -compact if and only if the following condition holds: for every $X \subseteq \mathbb{R}$ with $|X| < \nu$, if every finite subset of X has a lower bound then X has a lower bound.

Lemma 4.3. \mathbb{Q} is κ -compact.

Proof. Let $\mu < \kappa$, and let $\{q_i : i < \mu\}$ be a set of conditions in \mathbb{Q} such that for any finite subset s of μ the set $\{q_i : i \in s\}$ has a lower bound. Let $q_i = (A^i, B^i, t^i, f^i)$, and choose for each finite $s \subseteq \mu$ a condition $r^s = (A^s, B^s, t^s, f^s)$ which is a lower bound for $\{q_i : i \in s\}$.

We will define $r = (A^r, B^r, t^r, f^r)$ as follows:

- $A^r = \bigcup_{i < \mu} A^i$.
- B^r is some upper part such that $\text{ssup}(\text{dom}(A^r)) < \text{dom}(B^r)$ and $B^r \leq B^s$ for all s .
- $t^r = (\rho^r, a^r, b^r)$ where $\rho^r = \bigcup_{i < \mu} \rho^i$, $a^r = \bigcup_{i < \mu} a^i$, $b^r = \bigcup_{i < \mu} b^i$.
- If there is some i such that $(\eta, \zeta, \beta) \in \rho^i \times a^i \times b^i$, then $f_{\eta,\beta}^r(\zeta) = f_{\eta,\beta}^i(\zeta)$. As in the proof of Lemma 4.2, we choose the values of $f_{\eta,\beta}^r(\zeta)$ for $(\eta, \zeta, \beta) \in \rho^r \times a^r \times b^r \setminus \bigcup_{i < \mu} (\rho^i \times a^i \times b^i)$ to be distinct

from each other and from all values in $\{f_{\eta,\beta}^r(\zeta) : (\eta, \zeta, \beta) \in \bigcup_{i < \mu} (\rho^i \times a^i \times b^i)\}$.

We note that by our hypotheses the definition of $f_{\eta,\beta}^r(\zeta)$ yields a unique value.

As usual, the main issue is to verify that Clause 6) holds. This is straightforward: if $f_{\eta,\beta}^r(\zeta) = f_{\eta,\beta'}^r(\zeta) \neq f_{\eta,\beta}^r(\zeta') = f_{\eta,\beta'}^r(\zeta')$, then for some finite s we have $f_{\eta,\beta}^s(\zeta) = f_{\eta,\beta'}^s(\zeta) \neq f_{\eta,\beta}^s(\zeta') = f_{\eta,\beta'}^s(\zeta')$, and so we are done because $B^r \leq B^s$. \square

Corollary 4.4. *The poset \mathbb{Q} is κ -directed closed and also has the following property, which was dubbed “parallel countable closure” in [2]: if $\langle q_i^0 : i < \omega \rangle$ and $\langle q_i^1 : i < \omega \rangle$ are decreasing sequences of conditions such that q_i^0 and q_i^1 are compatible for all i , then there is q such that $q \leq q_i^0, q_i^1$ for all i .*

We recall that for an regular cardinal ν , a poset \mathbb{R} is *strongly ν^+ -cc* if and only if the following condition holds: for every ν^+ -sequence $\langle r_i : i < \nu^+ \rangle$ of conditions in \mathbb{R} , there exist a club set $E \subseteq \nu^+$ and a regressive function f on $E \cap \text{cof}(\nu)$ such that for all i and j , if $f(i) = f(j)$ then r_i is compatible with r_j .

Lemma 4.5. *\mathbb{Q} is strongly κ^+ -cc.*

Proof. Let $q^i = (A^i, B^i, t^i, f^i) \in \mathbb{Q}$ for $i < \kappa^+$, and let $t^i = (\rho^i, a^i, b^i)$.

We recall that $\text{dom}(f_{\eta,\beta}^i) = a^i$ for all $\eta < \rho^i$ and $\beta \in b^i$. Let $\mu^i = \text{ot}(a^i)$. Let \dot{x}_β^i be a \mathbb{P} -name for the set of pairs (ν, ν') such that $\zeta \dot{G}_\beta \zeta'$, where ζ and ζ' are respectively the ν^{th} and ν'^{th} elements of a^i .

Appealing to Lemma 3.5 and Property I we may assume, shrinking B^i if necessary, that for every $\beta \in b^i$ there exist an ordinal $\gamma_\beta^i < \kappa$ and names $\dot{y}_{\beta,\vec{\alpha}}^i$ for every increasing finite sequence $\vec{\alpha}$ of ordinals from $\text{ssup}(\text{dom}(A^i))$, such that B^i “reduces” \dot{x}_β^i to $\dot{y}_{\beta,\vec{\alpha}}^i$ which is a name in the product of collapses $\mathbb{D}(\vec{\alpha} \frown \langle \gamma_\beta^i \rangle)$ for the edge set of a graph on the vertex set μ^i .

We will enumerate $\bigcup_{i < \kappa^+} b^i$ as $\langle \beta_j : j < \kappa^+ \rangle$. To make the rest of the proof more readable, we will observe the following notational conventions:

- (1) The letter i and its typographic variations will denote indices for conditions in \mathbb{Q} on the sequence $\langle q^i : i < \kappa^+ \rangle$.
- (2) The letter ζ and its variations will denote elements of $\bigcup_{i < \kappa^+} a^i$, and the letter β and its variations will denote elements of $\bigcup_{i < \kappa^+} b^i$.
- (3) The letter j and its variations will denote indices for ordinals less than λ on the sequence $\langle \beta_j : j < \kappa^+ \rangle$.

- (4) Given a set $x \subseteq \kappa^+$ with $|x| < \kappa$, the letter σ and its variations will denote indices for elements of x , enumerated in increasing order.
- (5) Given a set $y \subseteq \bigcup_{i < \kappa^+} b^i$ with $|y| < \kappa$, the letter τ and its variations will denote indices for elements of $\{j : \beta_j \in y\}$, again enumerated in increasing order. Note that variations of τ denote indices (in κ) for indices (in κ^+) for elements of λ .
- (6) The letter ϕ and its variations will denote indices for elements $t \in Lev_\zeta(T)$ on the sequence $\langle t(\zeta, \phi) : \phi < |Lev_\zeta(T)| \rangle$.
- (7) The letter ψ and its variations will denote the second entries in pairs drawn from $T \times \kappa$.

We define functions F_n with domain κ^+ for $n < 6$ as follows:

- (1) $F_0(i) = (\rho^i, ot(a^i), ot(\{j : \beta_j \in b^i\}))$.
- (2) $F_1(i) = a^i \cap i$.
- (3) $F_2(i) = \{j < i : \beta_j \in b^i\}$.
- (4) $F_3(i) = A^i$.
- (5) $F_4(i)$ is the set of 5-tuples $(\eta, \sigma, \tau, \phi, \psi)$ where $\eta < \rho^i$, $\sigma < ot(a^i)$, $\tau < ot(\{j : \beta_j \in b^i\})$, $\phi < i$, $\psi < \kappa$, and if we let ζ be the σ^{th} element of a^i and $\beta = \beta_j$ for j the τ^{th} element of $\{j : \beta_j \in b^i\}$ then $f_{\eta, \beta}^i(\zeta) = (t(\zeta, \phi), \psi)$.
- (6) $F_5(i)$ is the set of 3-tuples (τ, γ, Y) where $\tau < ot(\{j : \beta_j \in b^i\})$, $\gamma < \kappa$, $Y \in V_\kappa$, and if we let $\beta = \beta_j$ for j the τ^{th} element of $\{j : \beta_j \in b^i\}$ then $\gamma = \gamma_\beta^i$, and Y is the function specified by setting $Y(\vec{\alpha}) = \dot{y}_{\beta, \vec{\alpha}}^i$ for each increasing finite sequence $\vec{\alpha}$ from $\text{ssup}(\text{dom}(A^i))$.

Remark. $F_4(i)$ is best viewed as a partial function on triples (η, σ, τ) which records a code for the value of $f_{\eta, \beta}^i(\zeta)$ when this is “permissible”. The criterion for permissibility is that (after decoding σ and τ to obtain ζ and β) the first entry $(x_\beta \upharpoonright \zeta)$ in $f_{\eta, \beta}^i(\zeta)$ is enumerated before i in the enumeration of level ζ of the tree T . The point is that we are aiming ultimately to define a regressive function so we can only record limited information.

In a similar vein, F_5 is a total function which records values of γ_β^i and $\dot{y}_{\beta, \vec{\alpha}}^i$.

Now let $F(i) = (F_0(i), F_1(i), F_2(i), F_3(i), F_4(i), F_5(i))$, so that

$$F(i) \in \kappa^3 \times [i]^{<\kappa} \times [i]^{<\kappa} \times V_\kappa \times [\kappa^3 \times i \times \kappa]^{<\kappa} \times [\kappa^2 \times V_\kappa]^{<\kappa}.$$

We fix an injective map H from

$$\kappa^3 \times [\kappa^+]^{<\kappa} \times [\kappa^+]^{<\kappa} \times V_\kappa \times [\kappa^3 \times \kappa^+ \times \kappa]^{<\kappa} \times [\kappa^2 \times V_\kappa]^{<\kappa}$$

to κ^+ . Since $\kappa^{<\kappa} = \kappa$, we may fix a club set $E_0 \subseteq \kappa^+$ such that if $i \in E_0 \cap \text{cof}(\kappa)$ then

$$\text{rge}(H \upharpoonright \kappa^3 \times [i]^{<\kappa} \times [i]^{<\kappa} \times V_\kappa \times [\kappa^3 \times i \times \kappa]^{<\kappa} \times [\kappa^2 \times V_\kappa]^{<\kappa}) \subseteq i,$$

so that in particular $H \circ F$ is regressive on $E_0 \cap \text{cof}(\kappa)$.

Let E_1 be the club subset of κ^+ consisting of those i such that for all $i' < i$:

- (1) $a^{i'} \subseteq i$.
- (2) $\{j : \beta_j \in b^{i'}\} \subseteq i$.
- (3) For all $\zeta \in a^{i'}$ and $\beta \in b^{i'}$, $x_\beta \upharpoonright \zeta = t(\zeta, \phi)$ for some $\phi < i$.
- (4) For all $\beta, \beta^* \in b^{i'}$ with $\beta \neq \beta^*$, $x_\beta \upharpoonright i \neq x_{\beta^*} \upharpoonright i$.

We claim that the function $H \circ F$ and the club set $E_0 \cap E_1$ serve as a witness to the strong κ^+ -cc for \mathbb{Q} . To see this, let $i' < i$ be points in $E_0 \cap E_1 \cap \text{cof}(\kappa)$ such that $F(i') = F(i)$. We will show that $q^{i'}$ and q^i are compatible.

We start by decoding the assertion that $F_n(i') = F_n(i)$ for $n < 4$. Directly from the definition we see that:

- (1) $\rho^{i'} = \rho^i = \rho^*$ say.
- (2) $\text{ot}(a^{i'}) = \text{ot}(a^i) = \mu^*$ say.
- (3) $\text{ot}(\{j : \beta_j \in b^{i'}\}) = \text{ot}(\{j : \beta_j \in b^i\}) = \epsilon^*$ say.
- (4) $a^{i'} \cap i' = a^i \cap i = r_0$ say. Since $a^{i'} \subseteq i$ by Clause 1) in the definition of E_1 , $a^{i'} \setminus i'$ and $a^i \setminus i$ are disjoint and $a^{i'} \cap a^i = r_0$.
- (5) $\{j < i' : \beta_j \in b^{i'}\} = \{j < i : \beta_j \in b^i\} = r_1$ say. As in the last claim, $\{j \geq i' : \beta_j \in b^{i'}\}$ and $\{j \geq i : \beta_j \in b^i\}$ are disjoint and $\{j : \beta_j \in b^i \cap b^{i'}\} = r_1$.
- (6) $A^{i'} = A^i = A^*$ say.

Claim. When both sides are defined, $f_{\eta,\beta}^{i'}(\zeta) = f_{\eta,\beta}^i(\zeta)$.

Proof. We will use the fact that $F_4(i') = F_4(i)$. Since both sides are defined, $\eta < \rho^*$, $\zeta \in a^{i'} \cap a^i$ and $\beta \in b^{i'} \cap b^i$. By the remarks in the preceding paragraph, $\zeta \in r_0$ and $\beta = \beta_j$ for some $j \in r_1$.

Since r_0 is the common initial segment of $a^{i'}$ and a^i , we have $\text{ot}(a^{i'} \cap \zeta) = \text{ot}(a^i \cap \zeta) = \sigma$ say. Similarly j has the same index (say τ) in the increasing enumerations of $\{j : \beta_j \in b^{i'}\}$ and $\{j : \beta_j \in b^i\}$.

Now let $f_{\eta,\beta}^{i'}(\zeta) = (x_\beta \upharpoonright \zeta, \psi')$, let $f_{\eta,\beta}^i(\zeta) = (x_\beta \upharpoonright \zeta, \psi)$, and let $x_\beta \upharpoonright \zeta = t(\beta, \phi)$. Since $i \in E_1$, $i' < i$, $\beta \in b^{i'}$ and $\zeta \in a^{i'}$, it follows from Clause 3) in the definition of E_1 that $\phi < i$.

By the definition of F_4 , the set $F_4(i)$ contains the tuple $(\eta, \sigma, \tau, \phi, \psi)$. This is the unique tuple in $F_4(i)$ which begins with (η, σ, τ) , and since $F_4(i') = F_4(i)$ this tuple also appears in $F_4(i')$. It follows that $\psi = \psi'$ and so $f_{\eta,\beta}^{i'}(\zeta) = f_{\eta,\beta}^i(\zeta)$. \square

We will now define $q^* = (A^*, B^*, t^*, f^*)$, which will be a lower bound for q^i and $q^{i'}$.

- Recall that $\rho^* = \rho^{i'} = \rho^i$. We set $a^* = a^{i'} \cup a^i$, $b^* = b^{i'} \cup b^i$, $t^* = (\rho^*, a^*, b^*)$.
- Recall that $A^* = A^i = A^{i'}$.
- Let B^* be some upper part such that $B^* \leq B^{i'}, B^i$.
- We define $f_{\eta, \beta}^*(\zeta)$ for all $\eta < \rho^*$, $\zeta \in a^*$ and $\beta \in b^*$. Naturally we set $f_{\eta, \beta}^*(\zeta) = f_{\eta, \beta}^{i'}(\zeta)$ when $\zeta \in a^{i'}$ and $\eta \in b^{i'}$, and similarly we set $f_{\eta, \beta}^*(\zeta) = f_{\eta, \beta}^i(\zeta)$ when $\zeta \in a^i$ and $\eta \in b^i$. As we argued already the sequences $f^{i'}$ and f^i agree sufficiently for this to make sense.

To define $f_{\eta, \beta}^*(\zeta)$ for $\eta < \rho^*$ and $(\zeta, \beta) \in a^* \times b^* \setminus (a^i \times b^i \cup a^{i'} \times b^{i'})$, we will proceed as in the proof of Lemma 4.3. That is to say we will choose suitable values whose second coordinates are distinct from each other, and also distinct from any value appearing as a second coordinate of $f_{\eta, \beta}^*(\zeta)$ for $\eta < \rho^*$ and $(\zeta, \beta) \in a^i \times b^i \cup a^{i'} \times b^{i'}$.

It is routine to check that if q^* is a condition then it is a common refinement of $q^{i'}$ and q^i , and also that q^* satisfies all the clauses in the definition of \mathbb{Q} except possibly for the final Clause 6). With a view to verifying this clause, suppose that $\eta \in \text{dom}(A^*) \cap \rho^*$, s is a lower part harmonious with A^* past η , and

$$f_{\eta, \beta'}^*(\zeta') = f_{\eta, \beta}^*(\zeta') \neq f_{\eta, \beta'}^*(\zeta) = f_{\eta, \beta}^*(\zeta).$$

where $\zeta' < \zeta$ and $\beta' = \beta_{j'}$, $\beta = \beta_j$ for some $j' < j$.

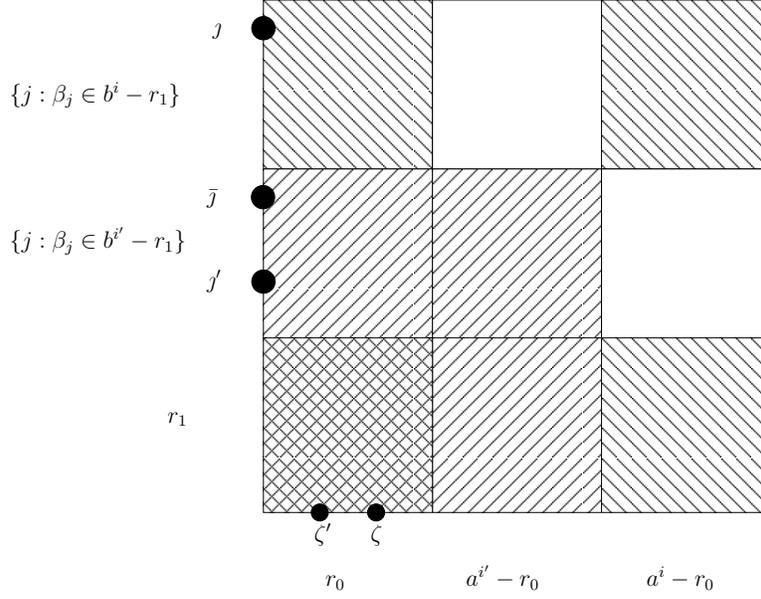
By the construction of f^* , it is immediate that all four of the pairs in $\{\zeta', \zeta\} \times \{\beta', \beta\}$ lie in the set $a^{i'} \times b^{i'} \cup a^i \times b^i$.

If all four pairs above lie in $a^{i'} \times b^{i'}$, then we are done by Clause 6) for $q^{i'}$ and the fact that $B^* \leq B^{i'}$. A similar argument works if all four pairs lie in $a^i \times b^i$. From this point we assume that we are not in either of these cases.

Now recall that $a^* = a^{i'} \cup a^i = r_0 \cup (a^{i'} \setminus r_0) \cup (a^i \setminus r_0)$, where $r_0 < a^{i'} \setminus r_0 < a^i \setminus r_0$. Similarly if we let $s = \{j : \beta_j \in b^{i'} \cup b^i\}$, then $s = r_1 \cup (\{j : \beta_j \in b^{i'}\} \setminus r_1) \cup (\{j : \beta_j \in b^i\} \setminus r_1)$, where $r_1 < \{j : \beta_j \in b^{i'}\} \setminus r_1 < \{j : \beta_j \in b^i\} \setminus r_1$.

An easy case analysis ¹ shows that there are only two possibilities:

¹In figure 1, all pairs (ζ^*, j^*) with $(\zeta^*, \beta_{j^*}) \in a^{i'} \times b^{i'}$ lie in the region shaded with forward-sloping diagonal lines, and all pairs (ζ^*, j^*) with $(\zeta^*, \beta_{j^*}) \in a^i \times b^i$ lie in the region shaded with backward-sloping diagonal lines. Points in $\{\zeta' \zeta\} \times \{j', j\}$ must all lie in the shaded region, and must not all lie in subregions shaded in a single direction.

FIGURE 1. Subcase 2b: \bar{j} is the “clone” of j .

Case 1: j' and j are both in r_1 , $\zeta' \in a^{i'} \setminus r_0$, $\zeta \in a^i \setminus r_0$.

Case 2: ζ' and ζ are both in r_0 , $j' \in b^{i'} \setminus r_1$, $j \in b^i \setminus r_1$.

We will first show that Case 1 is not possible. To dismiss Case 1, assume that we are in this case and recall that

$$f_{\eta, \beta'}^*(\zeta') = f_{\eta, \beta}^*(\zeta') \neq f_{\eta, \beta'}^*(\zeta) = f_{\eta, \beta}^*(\zeta),$$

from which it follows that $x_{\beta'} \upharpoonright \zeta = x_{\beta} \upharpoonright \zeta$. Since $i \in E_1$, $i' < i$ and $\beta', \beta \in b^{i'}$, it follows from clause 4 in the definition of E_1 that $x_{\beta'} \upharpoonright i \neq x_{\beta} \upharpoonright i$. By the case assumption we have $\zeta \in a^i \setminus r_0 = a^i \setminus (a^i \cap i)$, so that in particular $\zeta \geq i$. This is a contradiction, so Case 1 does not occur.

We now assume that we are in Case 2. In order to use the information coded in the equality of $F(i')$ and $F(i)$, we make some definitions:

- σ' is the index of ζ' in the increasing enumeration of $a^{i'} \cap a^i$.
- σ is the index of ζ in the increasing enumeration of $a^{i'} \cap a^i$.
- τ' is the index of j' in the increasing enumeration of $\{j : \beta_j \in a^{i'}\}$.
- τ is the index of j in the increasing enumeration of $\{j : \beta_j \in a^i\}$.

By definition, $F_5(i')$ (which is equal to $F_5(i)$) contains the tuples $(\tau', \gamma_{\beta'}^{i'}, Y')$ and $(\tau, \gamma_{\beta}^i, Y)$, where $Y'(\vec{\alpha}) = \dot{y}_{\beta', \vec{\alpha}}^{i'}$ and $Y(\vec{\alpha}) = \dot{y}_{\beta, \vec{\alpha}}^i$ for α any increasing finite sequence from $\text{ssup}(\text{dom}(A))$.

Recall that s is a lower part harmonious with A^* past η , and $\eta \in \text{dom}(A^*) \cap \rho^*$. Let

$$s = (p_0, \alpha_1, p_1, \dots, \alpha_k, p_k),$$

and let $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_k \rangle$. By the choice of $B^{i'}$ and the definitions of the names $\dot{x}'_{\beta'}$ and $\dot{y}'_{\beta', \vec{\alpha}}$, $(s, B^{i'})$ reduces the truth value of $\zeta' \dot{\mathcal{G}}_{\beta'} \zeta$ (a Boolean value for \mathbb{P}) to the truth value of $\sigma' \dot{y}'_{\beta', \vec{\alpha}} \sigma$ (a Boolean value for the product of collapses $\mathbb{D}(\vec{\alpha} \frown \gamma_{\beta'}^{i'})$). Similarly (s, B^i) reduces the truth value of $\zeta' \dot{\mathcal{G}}_{\beta} \zeta$ to the truth value of $\sigma' \dot{y}_{\beta, \vec{\alpha}} \sigma$.

Subcase 2a: $\tau' = \tau$.

In this subcase $\gamma_{\beta'}^{i'} = \gamma_{\beta}^i = \gamma^*$ say, and $Y' = Y$, so that in particular $\dot{y}'_{\beta', \vec{\alpha}} = \dot{y}_{\beta, \vec{\alpha}}$. It is then immediate from the preceding discussion that, since (s, B^*) is a common refinement of $(s, B^{i'})$ and (s, B^i) , $(s, B^*) \Vdash \zeta' \dot{\mathcal{G}}_{\beta'} \zeta \iff \zeta' \dot{\mathcal{G}}_{\beta} \zeta$.

Subcase 2b: $\tau' \neq \tau$.

In this subcase we will consider a “cloned” version $\bar{\beta}$ of β lying in $b^{i'}$, which we define by setting $\bar{\beta} = \beta_{\bar{j}}$ for \bar{j} the element with index τ in the increasing enumeration of $\{j : \beta_j \in b^{i'}\}$. The argument from subcase 2a shows that $(s, B^*) \Vdash \zeta' \dot{\mathcal{G}}_{\bar{\beta}} \zeta \iff \zeta' \dot{\mathcal{G}}_{\beta} \zeta$.

Since $\beta' \in b^{i'}$ and $\zeta' \in a^{i'}$, and also $i' < i$ and $i \in E_1$, it follows from Clause 3 in the definition of E_1 that $x_{\beta'} \upharpoonright \zeta' = t(\zeta', \phi)$ for some $\phi < i$. Now since $f_{\eta, \beta'}^{i'}(\zeta') = f_{\eta, \beta}^i(\zeta')$, $x_{\beta'} \upharpoonright \zeta' = x_{\beta} \upharpoonright \zeta = t(\zeta', \phi)$, so that the set $F_4(i)$ contains some tuple $(\eta, \sigma', \tau, \phi, \psi)$ coding the statement “ $f_{\eta, \beta}^i(\zeta') = (t(\zeta', \phi), \psi)$ ”. This tuple is also in $F_4(i')$, and decoding its meaning we find that $f_{\eta, \bar{\beta}}^{i'}(\zeta') = (t(\zeta', \phi), \psi) = f_{\eta, \beta}^i(\zeta')$. Since $\zeta \in a^{i'}$ also, a similar argument shows that $f_{\eta, \bar{\beta}}^{i'}(\zeta) = f_{\eta, \beta}^i(\zeta)$.

So now we have $f_{\eta, \beta'}^{i'}(\zeta') = f_{\eta, \bar{\beta}}^{i'}(\zeta')$ and $f_{\eta, \beta'}^{i'}(\zeta) = f_{\eta, \bar{\beta}}^{i'}(\zeta)$, so that (by Clause 6) for the condition $q^{i'}$ $(s, B^{i'}) \Vdash \zeta' \dot{\mathcal{G}}_{\beta'} \zeta \iff \zeta' \dot{\mathcal{G}}_{\bar{\beta}} \zeta$.

So $(s, B^*) \Vdash \zeta' \dot{\mathcal{G}}_{\beta'} \zeta \iff \zeta' \dot{\mathcal{G}}_{\beta} \zeta$ and we are done. \square

5. THE MAIN CONSTRUCTION

We will start with a model V_0 in which GCH holds and κ is supercompact. In this model we define in the standard way [7] a “Laver preparation” forcing \mathbb{L} , and let $V_1 = V_0[G_0]$ where G_0 is \mathbb{L} -generic over V_0 . Let \mathbb{A} be the poset $\text{Add}(\kappa^+, \kappa^{+3})^{V_1}$, let G_1 be \mathbb{A} -generic over V_1 , and let $V = V_1[G_1]$.

Let T be the complete binary tree of height κ^+ as defined in V . Clearly T has κ^{+3} branches and every level of T has size at most κ^+ . For use later we fix an enumeration $\langle x_\beta : \beta < \kappa^{+3} \rangle$ of a set of distinct branches, and enumerations $\langle t(\alpha, i) : i < |Lev_\alpha(T)| \rangle$ of the levels $Lev_\alpha(T)$ of T .

Since $2^{\kappa^{+3}} = \kappa^{+4}$ in V , a theorem of Shelah [14] implies that in V we have $\diamond_{\kappa^{+4}}(\text{cof}(\kappa^{++}))$.

Working in V , we will define a forcing iteration with $< \kappa$ -supports of length κ^{+4} . Each iterand \mathbb{Q}_i will either be trivial forcing or will be κ -closed, parallel countably closed (in the sense of Corollary 4.4) and strongly κ^+ -cc. By a suitable adaptation of arguments of Shelah [12], this is sufficient to show that the whole iteration will be κ -directed closed and strongly κ^+ -cc. We refer the reader to [2] for a detailed account of the chain condition proof, noting (for the experts) that the property “parallel countably closed” follows from the property “countably closed plus well-met” used in [12] and is sufficient to make the proof from that paper work.

The cardinality of the final iteration \mathbb{Q}^* will be κ^{+4} . We will have $2^\kappa = \kappa^{+4}$ in $V^{\mathbb{Q}^*}$, while $2^\kappa = \kappa^{+3}$ in the intermediate models of the iteration. We note that by the closure of \mathbb{Q}^* , the terms “ V_κ ” and “ $P_{\kappa\mu}$ ” have the same meanings in V , $V^{\mathbb{Q}^*}$, and every intermediate model.

As we build the iteration \mathbb{Q}^* , we will also (using the diamond from V) construct a sequence of names \dot{S}_i such that

- \dot{S}_i is a $\mathbb{Q}^* \restriction i$ -name for every $i < \kappa^{+4}$.
- \dot{S}_i names a pair (W_i, F_i) where $W_i \subseteq P(\kappa)$, F_i is a family of partial functions from κ to V_κ , and $\text{dom}(H) \in W_i$ for all $H \in F_i$.
- If G^* is \mathbb{Q}^* -generic, and $(W, F) \in V[G^*]$ with $W \subseteq P(\kappa)$ and F a family of functions from sets in W to V_κ , then

$$\{i \in \kappa^{+4} \cap \text{cof}(\kappa^{++}) : W \cap V[G^* \restriction i] = W_i \text{ and } F \cap V[G^* \restriction i] = F_i\}$$

is stationary in $V[G^*]$.

This is possible because:

- Pairs (W, F) as above in the extension by \mathbb{Q}^* may be coded as subsets of κ^{+4} , and names for them may be coded as subsets of $\mathbb{Q}^* \times \kappa^{+4}$.
- If we enumerate the conditions in \mathbb{Q}^* as $\langle q_j : j < \kappa^{+4} \rangle$, then $\mathbb{Q}^* \restriction i = \{q_j : j < i\}$ for almost all $i \in \kappa^{+4} \cap \text{cof}(\kappa^{++})$.
- \mathbb{Q}^* preserves stationary subsets of κ^{+4} , by virtue of being κ^+ -cc.

We refer the reader to [2] for a more detailed discussion of this kind of construction.

We observe that by κ^+ -cc, if $i < \kappa^{+4}$ with $\text{cf}(i) > \kappa$ then every subset of V_κ in the extension by $\mathbb{Q}^* \upharpoonright i$ is in the extension by $\mathbb{Q}^* \upharpoonright j$ for some $j < i$. We observe also that each of the properties I-III can be formulated as $\forall\exists$ assertions about the power set of V_κ .

The considerations in the last paragraph imply a crucial reflection statement for Properties I-III: If G^* is \mathbb{Q}^* -generic, in $V[G^*]$ we have a normal measure U_0 and filter \mathcal{F} with properties I-III, and we set $F = \{h : [h]_{U_0} \in \mathcal{F}\}$, then for almost all i with $\text{cf}(i) > \kappa$ we have that:

- $U_0 \cap V[G^* \upharpoonright i]$ and $F \cap V[G^* \upharpoonright i]$ are elements of $V[G^* \upharpoonright i]$.
- In $V[G^* \upharpoonright i]$, $U_0 \cap V[G^* \upharpoonright i]$ is a normal measure and the functions in $F \cap V[G^* \upharpoonright i]$ represent a filter with properties I-III.

After these preliminaries we can specify the iterands \mathbb{Q}_i of the iteration \mathbb{Q}^* . We assume that $G^* \upharpoonright i$ is $\mathbb{Q}^* \upharpoonright i$ -generic and that (W_i, F_i) is the realisation of \dot{S}_i , and work in $V[G^* \upharpoonright i]$. We will set \mathbb{Q}_i to be trivial forcing unless we have the conditions:

- $\text{cf}(i) = \kappa^{++}$.
- W_i is a normal measure on κ .
- $\{[h]_{W_i} : h \in F_i\}$ is an ultrafilter satisfying properties I-III.

In this case we will let \mathbb{Q}_i be the forcing \mathbb{Q} from Section 4, defined in $V[G^* \upharpoonright i]$ from the parameters $W_i, F_i, \langle x_\beta : \beta < \kappa^{+3} \rangle$, and a suitable enumeration $\langle \dot{g}_\beta^i : \beta < \kappa^{+3} \rangle$ of canonical names for graphs.

We recall from the Introduction we will ultimately force over $V[G^*]$ with a poset \mathbb{P} of the type discussed in Section 3. The forcing \mathbb{P} will be defined from some normal measure U_0 and ultrafilter \mathcal{F} , and the point of the diamond machinery in the definition of \mathbb{Q}_i is to anticipate the poset \mathbb{P} (and in particular \mathbb{P} -names for graphs on κ^+). To be a bit more precise, we will actually anticipate U_0 and F where $F = \{h : [h]_{U_0} \in \mathcal{F}\}$, or to put it another way F is the set of upper parts for the poset \mathbb{P} .

Recall further from Section 4 that in the case when \mathbb{Q}_i is not trivial forcing, part of the generic object for \mathbb{Q}_i will be a partial function h_i from κ to V_κ such that

- $\text{dom}(h_i)$ is an unbounded set of inaccessible cardinals
- $\text{dom}(h_i)$ is eventually contained in each measure one set for the measure W_i .
- For all $h \in F_i$, $h_i(\alpha) \leq h(\alpha)$ for all large enough $\alpha \in \text{dom}(h)$.

The following Lemma will be used in Section 6 to show that often enough \mathbb{Q}_i does its job, by adding a \mathbb{P} -name for a graph which will absorb all graphs whose names lie in $V[G^* \upharpoonright i]$.

Lemma 5.1. *Let G^* be \mathbb{Q}^* -generic. Then in $V[G^*]$ there is an ultrafilter U on $P_{\kappa}\kappa^{+4}$ such that if U_0 is the projection of U to κ , and we perform the construction of Section 2 to produce an ultrafilter $\mathcal{F} = \text{Fil}(H)$ for some U -constraint H , then there are stationarily many $i < \kappa^{+4}$ such that:*

- (1) $U_0 \cap V[G^* \upharpoonright i] = W_i$.
- (2) $F \cap V[G^* \upharpoonright i] = F_i$, where $F = \{h : [h]_{U_0} \in \mathcal{F}\}$.
- (3) $h_i \in F$.

Before starting the proof, we emphasise that the diamond property ensures that there many i where the first two clauses are satisfied. What takes work is arranging that the the third clause is also satisfied.

Proof. We will construct U as in the standard proof of Laver’s indestructibility result [7], with the proviso that we will be very careful about the construction of the master condition.

We begin by falling back to the initial model V_0 , where we will choose an embedding $j : V_0 \rightarrow M$ with critical point κ witnessing that κ is μ -supercompact for some very large μ , and with the additional properties that the forcing poset $\mathbb{A} * \mathbb{Q}^*$ is the iterand at stage κ in the iteration $j(\mathbb{L})$, and that the least point greater than κ in the support of the iteration $j(\mathbb{L})$ is greater than μ .

Recall that $V = V_0[G_0][G_1]$ where G_0 is \mathbb{L} -generic and G_1 is \mathbb{A} -generic. By standard arguments, for any choice of a generic object H_{tail} for $j(\mathbb{L})/G_0 * G_1 * G^*$ over the model $V[G^*]$, we have $j^{\text{``}}G_0 \subseteq G_0 * G_1 * G^* * H_{\text{tail}}$. We may therefore lift j to obtain a generic embedding $j : V_0[G_0] \rightarrow M[G_0 * G_1 * G^* * H_{\text{tail}}]$. In order to lift further, we will need to construct master conditions.

We will now work in $V[G^*]$ and perform a recursive construction of length κ^{+4} , choosing a decreasing sequence of conditions (r_i, a_i, q_i) with $(r_i, a_i, q_i) \in j(\mathbb{L})/(G_0 * G_1 * G^*) * j(\mathbb{A}) * j(\mathbb{Q}^* \upharpoonright i)$. We will arrange that

$$r_i \Vdash a_i \leq j^{\text{``}}G_1,$$

so that forcing below (r_i, a_i) we obtain $H_{\text{tail}} * H_1$ such that j can be lifted to $j : V_0[G_0][G_1] \rightarrow M[G_0 * G_1 * G^* * H_{\text{tail}} * H_1]$. Keeping this in mind, we will also arrange that

$$(r_i, a_i) \Vdash q_i \leq j^{\text{``}}(G^* \upharpoonright i).$$

Using the hypothesis that j witnesses μ -supercompactness and the remark that $G_1 \in M[j(G_0)]$, we may argue that for any choice of H_{tail} we have $j^{\text{``}}G_1 \in M[G_0 * G_1 * G^* * H_{\text{tail}}]$. Since $j(\mathbb{A})$ is $j(\kappa^+)$ -directed closed we may find a “strong master condition” $a \in j(\mathbb{A})$ with $a \leq j^{\text{``}}G_1$. We will therefore choose r_0 to be the trivial condition in

$j(\mathbb{L})/(G_0 * G_1 * G^*)$, a_0 to be (a name for) a condition $a \in j(\mathbb{A})$ with $a \leq j^{\ulcorner}G_1$, and q_0 as (a name for) the empty sequence.

The limit stages are straightforward, since the choice of μ and j gives enough closure to take lower bounds. If \mathbb{Q}_i is trivial it is easy to define suitable r_{i+1} , a_{i+1} and q_{i+1} . so we assume that \mathbb{Q}_i is non-trivial.

Forcing below (r_i, a_i, q_i) we can obtain a generic object $H_{\text{tail}} * H_1 * H_i^*$ such that there is a lifted embedding $j : V[G^* \upharpoonright i] \rightarrow M[j(G_0 * G_1) * H_i^*]$, where $j(G_0 * G_1) = G_0 * G_1 * G^* * H_{\text{tail}} * H_1$. Let g_i be the \mathbb{Q}_i -generic filter added at stage i by G^* , and let h_i be the partial function from κ to V_κ added by g_i .

To take the next step, we ask whether it is possible that the set $\{j(h)(\kappa) : h \in F_i\}$ has a non-zero lower bound: more formally, we ask whether there is a condition extending (r_i, a_i, q_i) which forces this set to have a non-zero lower bound, and define (r', a', q') to be such a condition if it exists and to be (r_i, a_i, q_i) otherwise. In the case that (r', a', q') forces that $\{j(h)(\kappa) : h \in F_i\}$ has a non-zero lower bound, we let \mathfrak{b} name the Boolean greatest lower bound for this set. In either case we force below (r', a', q') , lift j and work in $M[j(G_0 * G_1) * H_i^*]$ to define a condition in $j(\mathbb{Q}_i)$.

Let $\langle f_{\eta, \beta}^i : \eta < \kappa, \beta < \kappa^{+3} \rangle$ be the family of functions added by g_i . We define $Q = (A^Q, B^Q, t^Q, f^Q)$ as follows:

- $t^Q = \kappa \times j^{\ulcorner} \kappa^+ \times j^{\ulcorner} \kappa^{+3}$.
- For all $\eta < \kappa$, $\alpha < \kappa^+$ and $\beta < \kappa^{+3}$, $f_{\eta, j(\beta)}^Q(j(\alpha)) = j(f_{\eta, \beta}^i(\alpha))$.
- If the Boolean value \mathfrak{b} is not defined, then:
 - (1) $A^Q = h_i$.
 - (2) B^Q is some upper part such that $\kappa \cap \text{dom}(B^Q) = 0$ and $B^Q \leq j(B)$ for all upper parts $B \in F_i$.
 If \mathfrak{b} is defined, then:
 - (1) $A^Q = h_i \cup \{(\kappa, \mathfrak{b})\}$.
 - (2) B^Q is some upper part such that $(\kappa + 1) \cap \text{dom}(B^Q) = 0$ and $B^Q \leq j(B)$ for all upper parts $B \in F_i$.

In the case when the Boolean value \mathfrak{b} is not defined, it is routine to check that Q is a condition in $j(\mathbb{Q}_i)$ and $Q \leq j^{\ulcorner}g_i$. This is essentially the argument of Lemma 4.3 applied to the directed (hence linked) set $j^{\ulcorner}g_i$. In the case that \mathfrak{b} is defined, the definition of \mathfrak{b} ensures that we will still have $Q \leq j^{\ulcorner}g_i$ so long as we can verify that Q is a condition. As usual the only issue is Clause 6) in the definition of conditionhood.

So suppose that $\eta \in \text{dom}(h_i) \cap \kappa$,

$$f_{\eta, j(\beta)}^Q(j(\zeta)) = f_{\eta, j(\beta')}^Q(j(\zeta)) \neq f_{\eta, j(\beta)}^Q(j(\zeta')) = f_{\eta, j(\beta')}^Q(j(\zeta')),$$

and y is a lower part which is harmonious with A^Q past η . Let x be the largest initial segment of y which lies in V_κ , so that either $y = x$ or $y = x \frown \langle \kappa, p \rangle$ where $p \leq \mathfrak{b}$.

By elementarity and the definition of Q ,

$$f_{\eta,\beta}^i(\zeta) = f_{\eta,\beta'}^i(\zeta) \neq f_{\eta,\beta}^i(\zeta') = f_{\eta,\beta'}^i(\zeta').$$

We now choose $q \in g_i$ such that $\eta < \rho^q$, $\zeta, \zeta' \in a^q$, $\beta, \beta' \in b^q$, and $\text{dom}(A^q)$ contains every ordinal in $[\eta, \kappa)$ which is mentioned in y . It is easy to see that x is harmonious with A^q past η , and hence (as q is a condition) $(x, B^q) \Vdash \zeta' \dot{G}_\beta \zeta \iff \zeta' \dot{G}_{\beta'} \zeta$.

By elementarity $(x, j(B^q)) \Vdash j(\zeta') j(\dot{G}_\beta) j(\zeta) \iff j(\zeta') j(\dot{G}_{\beta'}) j(\zeta)$. To finish we just observe that by definition (and the choice of \mathfrak{b} in the case when it is defined, which ensures that $\mathfrak{b} \leq j(B^q)(\kappa)$) $(y, B^Q) \leq (x, j(B^q))$.

Having chosen Q as above, we let $r_{i+1} = r'$, $a_{i+1} = a'$, and q_{i+1} be the unique condition such that $q_{i+1} \upharpoonright j(i) = q'$ and $q_{i+1}(j(i)) = Q$.

At the end of the construction, we obtain $(r^*, a^*, q^*) \in j(\mathbb{L})/(G_0 * G_1 * G^*) * j(\mathbb{A}) * j(\mathbb{Q}^*)$ such that

$$r^* \Vdash a^* \leq j \text{``} G_1,$$

and

$$(r_i, a_i) \Vdash q^* \leq j \text{``} G^*.$$

Forcing below (r^*, a^*, q^*) we obtain a generic object $H_{\text{tail}} * H_1 * H^*$ and a lifted embedding $j : V[G] \rightarrow M[j(G_0 * G_1) * H^*]$. Following the idea of the Laver construction we define $U = \{A \in (P_\kappa \kappa^{+4})^{V[G]} : j \text{``} \kappa^{+4} \in j(A)\}$. Since $H_{\text{tail}} * H_1 * H^*$ is generic over $V[G]$ for highly closed forcing we have $U \in V[G]$, and so U is an ultrafilter witnessing the κ^{+4} supercompactness of κ in $V[G]$. By the results in Section 2, we may use U to define a U -constraint H such that $\text{Fil}(H)$ is an ultrafilter. It is easy to check that if U_0 is the projection of U to a normal measure on κ , and F is the set of upper parts associated with $\text{Fil}(H)$, then

$$F = \{h : h \text{ is a } U_0\text{-constraint and } j(h)(\kappa) \geq j(H)(j \text{``} \kappa^{+4})\}.$$

By the diamond property, there is a stationary set of $i \in \kappa^{+4} \cap \text{cof}(\kappa^{+2})$ such that $U_0 \cap V[G \upharpoonright i] = W_i$ and $F \cap V[G \upharpoonright i] = F_i$. For each such i , we observe that for all h

$$h \in F_i \implies h \in F \implies j(h)(\kappa) \geq j(H)(j \text{``} \kappa^{+4}).$$

So $\{j(h)(\kappa) : h \in F_i\}$ has a nonzero lower bound, and by the Truth Lemma there is a condition in $H_{\text{tail}} * H_1 * H_i^*$ which extends (r_i, a_i, q_i) and forces this. So when we chose q_{i+1} , we arranged that $j(h_i)(\kappa)$ is the Boolean infimum of $\{j(h)(\kappa) : h \in F_i\}$. Since $j(H)(j \text{``} \kappa^{+4})$ is a

lower bound for this set, $j(h_i)(\kappa) \geq j(H)(j^{\kappa+4})$, and hence $h_i \in F$ as required. \square

6. UNIVERSAL GRAPHS

We are now ready to prove the main result. By the results of Section 5, we will assume that we have in $V[G]$ a measure U_0 on κ , a filter \mathcal{F} (with associated set of upper parts F) and a stationary set S such that for every $i \in S$:

- (1) $U_0 \cap V[G \upharpoonright i] = W_i$.
- (2) $F \cap V[G \upharpoonright i] = F_i$.
- (3) $h_i \in F$.

We now let \mathbb{P} be the forcing poset defined from U_0 and \mathcal{F} as in Section 3, and force with \mathbb{P} over $V[G]$, obtaining a generic sequence

$$x = f_0, \kappa_1, f_1, \kappa_2, f_2 \dots$$

By the characterisation of genericity from Lemma 3.6, we see that for every $i \in S$ x is \mathbb{P}_i -generic over $V[G \upharpoonright i]$, where \mathbb{P}_i is the forcing defined in $V[G \upharpoonright i]$ from W_i and F_i .

We now define for each $i \in S$ a graph $\mathcal{U}_i \in V[G][x]$, which will embed every graph on κ^+ in $V[G \upharpoonright i][x]$. We begin by using the criterion for genericity to choose some j such that $\kappa_k \in \text{dom}(h_i)$ and $h_i(\kappa_k) \in f_k$ for all $k \geq j$. We set $\eta = \kappa_j$.

The underlying set of the graph \mathcal{U}_i is $T \times \kappa$, and the edges are defined as follows:

$(z, \delta)\mathcal{U}_i(z', \delta')$ if and only if there exist a lower part t and a condition $q \in \mathbb{Q}_i$ such that

- (1) $q \in g_i$.
- (2) (t, B^q) is in the generic filter on \mathbb{P}_i corresponding to the generic sequence x .
- (3) t is harmonious with A^q past η .
- (4) There exist $\beta \in b^q$ and distinct $\zeta, \zeta' \in a^q$ such that:
 - (a) $f_{\eta, \beta}^q(\zeta) = (z, \delta)$, $f_{\eta, \beta}^q(\zeta') = (z', \delta')$, and $(t, B^q) \Vdash \zeta \dot{\mathcal{G}}_\beta^i \zeta'$.

Since $\langle \dot{\mathcal{G}}_\beta^i : \beta < \kappa^{+3} \rangle$ enumerates all \mathbb{P}_i -names for graphs on κ^+ , it will suffice to verify that the generic function $f_{\eta, \beta}^i$ is an embedding of \mathcal{G}_β^i (the realisation of the name $\dot{\mathcal{G}}_\beta^i$) into the graph \mathcal{U}_i . One direction is easy: if $f_{\eta, \beta}^i(\zeta)\mathcal{U}_i f_{\eta, \beta}^i(\zeta')$ then by definition there is (t, B^q) in the generic filter on \mathbb{P}_i induced by x such that $(t, B^q) \Vdash \zeta \dot{\mathcal{G}}_\beta^i \zeta'$, and so by the Truth Lemma $\zeta \mathcal{G}_\beta^i \zeta'$.

For the converse direction, suppose that $\zeta \mathcal{G}_\beta^i \zeta'$. We may find a condition (s, B) in the generic filter induced by x on \mathbb{P}_i , such that $(s, B) \Vdash \zeta \dot{\mathcal{G}}_\beta^i \zeta'$. Let $s = q_0, \kappa_1, q_1, \dots, \kappa_n, q_n$. Extending the condition (s, B) if need be, we may assume that $n \geq j$. Since (s, B) is in the generic filter induced by x , we have that $q_m \in f_m$ for $m \leq n$, while $\kappa_m \in \text{dom}(B)$ and $B(\kappa_m) \in f_m$ for $m > n$.

By the properties of the forcing poset \mathbb{Q}_i , we may find a condition q in g_i such that $\text{ssup}(\text{dom}(A^q)) > \kappa_n$, $\text{dom}(B^q) \subseteq \text{dom}(B)$, $B^q(\alpha) \leq B(\alpha)$ for all $\alpha \in \text{dom}(B)$, $\beta \in b^q$ and $\zeta, \zeta' \in a^q$.

Recall now that $\kappa_k \in \text{dom}(h_i)$ and $h_i(\kappa_k) \in f_k$ for all $k \geq j$, and also that $\eta = \kappa_j$ and $n \geq j$. Let \bar{n} be the largest k such that $\kappa_k < \text{ssup}(\text{dom}(A^q))$.

Define a lower part t as follows:

$$t = q'_0, \kappa_1, q'_1, \dots, \kappa_{\bar{n}}, q'_{\bar{n}}$$

where:

- (1) $q'_k = q_k$ for $k < j$.
- (2) $q'_k = q_k \cup A^q(\kappa_k)$ for $j \leq k \leq n$.
- (3) $q'_k = A^q(\kappa_k) \cup B(\kappa_k)$ for $n < k \leq \bar{n}$.

We note that since $q \in g_i$ and h_i is added by g_i , A^q is an initial segment of h_i and $h_i(\alpha) \leq B^q(\alpha)$ for all $\alpha \in \text{dom}(h_i) \setminus \text{dom}(A^q)$. For $j \leq k \leq n$ we have that $q_k \in f_k$ and $A^q(\kappa_k) = h_i(\kappa_k) \in f_k$, so that $q_k \cup A^q(\kappa_k)$ is a condition and lies in f_k . For $n < k \leq \bar{n}$, again $A^q(\kappa_k) = h_i(\kappa_k) \in f_k$ and also $B(\kappa_k) \in f_k$, so that $A^q(\kappa_k) \cup B(\kappa_k)$ is a condition and lies in f_k .

We will verify that t is harmonious with A^q past η , (t, B^q) extends (s, B) , and (t, B^q) is in the filter generated by x . This will suffice, since it will then be clear that t and q will serve as witnesses that $f_{\eta, \beta}^i(\zeta) \mathcal{U}_i f_{\eta, \beta}^i(\zeta')$.

The harmoniousness is immediate from the definitions. (t, B^q) extends (s, B) because $q'_k \leq q_k$ for $k \leq n$, $q'_k \leq B(\kappa_k)$ for $n < k \leq \bar{n}$, and $B^q \leq B$. We already checked that $q'_k \in f_k$ for all $k \leq \bar{n}$, so to finish we just need to see that $B^q(\kappa_k) \in f_k$ for all $k > \bar{n}$; this is immediate because $h_i(\kappa_k) \leq B^q(\kappa_k)$ for all such k , and also $h_i(\kappa_k) \in f_k$ for all $k \geq j$.

To finish the construction of a small family of universal graphs, we will fix $i^* \in S$ which is a limit of points of S , and an increasing κ^{++} -sequence of points $i_\eta \in S$ which is cofinal in i^* . By routine chain condition arguments, every \mathbb{P}_{i^*} -name for a graph on κ^+ may be viewed as a \mathbb{P}_{i_η} -name for some $\eta < \kappa^{++}$. We now consider the model $V[G \upharpoonright$

$i^*[x]$. The family of graphs $\{\mathcal{U}_{i_\eta} : \eta < \kappa^{++}\}$ is universal in this model, where $2^\kappa = 2^{\kappa^+} = \kappa^{+3}$ and of course $\kappa = \aleph_\omega$.

We have proved:

Theorem 6.1. *It is consistent from large cardinals that \aleph_ω is strong limit, $2^{\aleph_\omega} = 2^{\aleph_{\omega+1}} = \aleph_{\omega+3}$, and there is a family of size $\aleph_{\omega+2}$ of graphs on $\aleph_{\omega+1}$ which is jointly universal for all such graphs.*

7. AFTERWORD

There is some flexibility in the proof of Theorem 6.1, in particular it would be straightforward to modify the construction so that in the final model $2^{\aleph_\omega} = \aleph_{\omega+k}$ for an arbitrary k such that $3 \leq k < \omega$. Larger values can probably be achieved but would require a substantial modification to the construction.

Theorem 6.1 leaves a number of natural questions open:

- Can we have a failure of SCH at \aleph_ω with $u_{\aleph_{\omega+1}} = 1$?
- On a related topic, what is the exact value of $u_{\aleph_{\omega+1}}$ in the model of Theorem 6.1?
- As far as the authors are aware, the only known results on the value of u_{κ^+} for κ singular strong limit and $2^\kappa > \kappa^+$ are consistency results of the kind proved in this paper. In particular, we lack a forcing technique to show that u_{κ^+} can be arbitrarily large.

For κ regular adding Cohen subsets to κ makes u_{κ^+} arbitrarily large, is there an analogous result for κ singular?

- The class of graphs is a very simple class of structures. What can be done in more complex classes?
- In the model of Theorem 6.1, GCH fails cofinally often below \aleph_ω , and in fact $2^{\aleph_n} = \aleph_{n+4}$ for unboundedly many $n < \omega$. Is the conclusion consistent if we demand that GCH holds below \aleph_ω ?

The authors' joint paper with Magidor and Shelah [2] contains some related work, in which the final “Prikry type” forcing is a version of Radin forcing and we obtain models where μ is singular strong limit of uncountable cofinality, SCH fails at μ and $u_{\mu^+} < 2^\mu$.

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