REDUCED PRODUCTS OF METRIC STRUCTURES: A METRIC FEFERMAN-VAUGHT THEOREM

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ABSTRACT. We extend the classical Feferman-Vaught theorem to logic for metric structures. This implies that the reduced powers of elementarily equivalent structures are elementarily equivalent, and therefore they are isomorphic under the Continuum Hypothesis. We also prove the existence of two separable C*-algebras of the form $\bigoplus_i M_{k(i)}(\mathbb{C})$ such that the assertion that their coronas are isomorphic is independent from ZFC, which gives the first example of genuinely non-commutative coronas of separable C*-algebras with this property.

1. INTRODUCTION

In classical model theory S. Feferman and R.L. Vaught ([16] and [5, §6.3]) gave an effective (recursive) way to determine the satisfaction of formulas in the reduced products of models of the same language, over the ideal of all finite sets, *Fin.* They showed the preservation of the elementary equivalence relation \equiv by arbitrary direct products and also by reduced products over *Fin.* Later Frayne, Morel and Scott ([17]) noticed that the results extend to arbitrary reduced products (see also [26]). Even though reduced products have been vastly studied for various metric structures, e.g., Banach spaces, C*-algebras, etc., unlike classical first order logic, their model theory has not been studied until very recently in [20] and [15]. The classical Feferman-Vaught theorem effectively determines the truth value of a formula φ in reduced products of discrete structures $\{\mathcal{A}_{\gamma} : \gamma \in \Omega\}$ over an ideal \mathcal{I} on Ω , by the truth values of certain formulas in the models \mathcal{A}_{γ} and in the Boolean algebra $P(\Omega)/\mathcal{I}$.

In the present paper we prove a metric version of the Feferman-Vaught theorem (Theorem 3.3) for reduced products of metric structures, which also implies the preservation of \equiv by arbitrary direct products, ultraproducts and reduced products of metric structures. This answers a question stated in [20]. We also use this theorem to solve an outstanding problem on coronas of C*-algebras (§5).

In the last few years the model theory for operator algebras has been developed and specialized from the model theory for metric structures. The reader may refer to [13] for a detailed introduction to the model theory of operator algebras, and [10] for an overview of the applications of logic to operator algebras. This has proved to be very fruitful as many properties of C*-algebras and tracial von Neumann algebras have equivalent model theoretic reformulations ([13], [11], ...). In particular for a sequence of C*-algebras $\{\mathcal{A}_n : n \in \mathbb{N}\}$ the asymptotic sequence algebra $\prod_n \mathcal{A}_n / \bigoplus_n \mathcal{A}_n$ is the reduced product over

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the Fréchet ideal and is an important example of corona algebras. If $\mathcal{A}_n = \mathcal{A}$ for all n, we write $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ instead of $\prod_n \mathcal{A}_n / \bigoplus_n \mathcal{A}_n$ In section 2.1 we briefly recall some of the basic concepts regarding corona algebras and the isomorphisms between them.

In model theory saturated structures have many intriguing properties. A transfinite extension of Cantor's back-and-forth method shows that for any uncountable cardinal κ , two elementarily equivalent κ -saturated metric structures (in the same countable language \mathcal{L}) of density character κ , are isomorphic (see e.g., [13] or [2]). Recall that for a topological space X, the density character of X is the smallest cardinality of a dense subset of X. In [15] Farah-Shelah (Theorem 4.3) showed that for any sequence $\{\mathcal{A}_n : n \in \mathbb{N}\}$ of metric structures the reduced product $\prod_n \mathcal{A}_n / \bigoplus_n \mathcal{A}_n$ over the Fréchet ideal is \aleph_1 -saturated (we will usually say "countably saturated" instead of \aleph_1 -saturated). Hence the question whether two such reduced products of the density character \aleph_1 are isomorphic reduces to the weaker question of whether they are elementarily equivalent. For instance if \mathcal{A} is separable C*-algebra, the asymptotic sequence algebra $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ is the corona algebra of the separable C*-algebra $\ell_{\infty}(\mathcal{A})$ and therefore it is non-separable (see [1, Theorem 2.7]). Thus under the Continuum Hypothesis, the asymptotic sequence algebras of two separable C^{*}-algebras \mathcal{A} and \mathcal{B} are isomorphic if and only if they (the asymptotic sequence algebras) are elementarily equivalent. One of the main consequences of our metric Feferman-Vaught theorem (Proposition 3.6) implies that if separable \mathcal{A} and \mathcal{B} are elementarily equivalent, so are their asymptotic sequence algebras and therefore $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ is isomorphic to $\ell_{\infty}(\mathcal{B})/c_0(\mathcal{B})$, under the Continuum Hypothesis.

We say an ideal \mathcal{I} on \mathbb{N} is *atomless* if the Boolean algebra $P(\mathbb{N})/\mathcal{I}$ is atomless. The metric extension of the Feferman-Vaught theorem is used to prove the following theorem.

Theorem 1.1. Suppose \mathcal{A} is a metric \mathcal{L} -structure and ideals \mathcal{I} and \mathcal{J} on \mathbb{N} are atomless, then the reduced powers of \mathcal{A} over \mathcal{I} and \mathcal{J} are elementarily equivalent.

Therefore in particular if \mathcal{A} is a separable C*-algebra then under the Continuum Hypothesis such reduced powers of \mathcal{A} , if they are countably saturated, are all isomorphic to $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$.

For an ultrafilter \mathcal{U} Loś's theorem implies that a metric structure \mathcal{A} is elementarily equivalent to its ultrapower $\mathcal{A}_{\mathcal{U}}$. Therefore Farah-Shelah's result shows that under the Continuum Hypothesis if \mathcal{A} is a separable C*-algebra, $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ is isomorphic to its ultrapower associated with any nonprincipal ultrafilter on \mathbb{N} ([15, Corollary 4.1]). Theorem 1.1 can be used (§4) to show that under the Continuum Hypothesis any reduced power of an asymptotic sequence algebra $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ over a large class of atomless ideals is also isomorphic to $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ itself.

In Section 5 we show there are two reduced products (of matrix algebras) which are isomorphic under the Continuum Hypothesis but there are no "trivial" isomorphisms (see 2.1) between them. Commutative examples of such reduced products are well-known, for example under the Continuum Hypothesis $C(\beta \omega \setminus \omega) \cong C(\beta \omega^2 \setminus \omega^2)$ (note that $\ell_{\infty}/c_0 \cong$ $C(\beta \omega \setminus \omega)$), since by a well-known result of Parovičenko ([22]) under the Continuum Hypothesis $\beta \omega \setminus \omega$ and $\beta \omega^2 \setminus \omega^2$ are homeomorphic. However under the proper forcing axiom they are not isomorphic (see [6] and [7, Chapter 4]). A naive way to obtain nontrivial isomorphisms, under the Continuum Hypothesis, between non-commutative coronas is by tensoring $C(\beta \omega \setminus \omega)$ and $C(\beta \omega^2 \setminus \omega^2)$ with a full matrix algebra. However, such nontrivial isomorphisms are just amplifications of the nontrivial isomorphisms between their corresponding commutative factors (see Section 5 for details). It has been asked by I. Farah to give examples of genuinely non-commutative corona algebras which are nontrivially isomorphic under the Continuum Hypothesis, for non-commutative reasons. Let M_n denote the space of all $n \times n$ matrices over the field of complex numbers. In [18] it has been proved that it is relatively consistent with ZFC that all isomorphisms between reduced products of matrix algebras over analytic P-ideals (e.g., the corona of $\bigoplus M_{k(n)}$) are trivial (cf. Theorem 5.1).

Theorem 1.2. There is an increasing sequence of natural numbers $\{k_{\infty}(i) : i \in \mathbb{N}\}$ such that if $\{g(i)\}$ and $\{h(i)\}$ are two subsequences of $\{k_{\infty}(i)\}$, then under the Continuum Hypothesis, $\mathcal{M}_g = \prod_i M_{g(i)} / \bigoplus M_{g(i)}$ is isomorphic to $\mathcal{M}_h = \prod_i M_{h(i)} / \bigoplus M_{h(i)}$. Moreover, the following are equivalent.

- (1) \mathcal{M}_q and \mathcal{M}_h are isomorphic in ZFC.
- (2) \mathcal{M}_g and \mathcal{M}_h are trivially isomorphic, i.e., $\{g(i) : i \in \mathbb{N}\}$ and $\{h(i) : i \in \mathbb{N}\}$ are equal modulo finite sets.

Thus if $\{g(i) : i \in \mathbb{N}\}$ and $\{h(i) : i \in \mathbb{N}\}$ are almost disjoint, this gives an example of two genuinely non-commutative reduced products for which the question "whether or not they are isomorphic?", is independent from ZFC. We will also show (Theorem 5.4) that there is an abundance of different theories of reduced products of sequences of matrix algebras, by exhibiting 2^{\aleph_0} pairwise non-elementarily equivalent such reduced products.

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2. Some preliminaries

2.1. Corona algebras and trivial isomorphisms. The reader may refer to [3] for a detailed book on C*-algebras. A C*-algebra \mathcal{A} is a Banach *-algebra over the field of complex numbers which satisfies the C*-identity

$$|xx^*|| = ||x||^2$$

for all $x \in \mathcal{A}$. The Gelfand-Naimark theorem states that every commutative C*-algebra is isomorphic to some $C_0(X)$, the algebra of all continuous functions on a locally compact Hausdorff space X which vanish at infinity, with the pointwise multiplication. In general a C*-algebra may not have a unit element. For a non-unital C*-algebra \mathcal{A} there are various ways in which \mathcal{A} can be embedded as an (essential) ideal in a unital C*-algebra. If $\mathcal{A} = C_0(X)$ is commutative this corresponds to the ways in which the locally compact Hausdorff space X can be embedded as an open (dense) set in a compact Hausdorff

space Y. The analogue of the Čech-Stone compactification for (non-commutative) C^{*}algebras is called the *multiplier algebra* of \mathcal{A} . The multiplier algebra $\mathcal{M}(\mathcal{A})$ of \mathcal{A} is the unital C^{*}-algebra containing \mathcal{A} as an essential ideal, which is universal in the sense that whenever \mathcal{A} is an ideal of a unital C^{*}-algebra \mathcal{D} , the identity map on \mathcal{A} extends uniquely to a *-homomorphism from \mathcal{D} into $\mathcal{M}(\mathcal{A})$ (cf. [3, II.7.3]). The quotient C^{*}-algebra $\mathcal{C}(\mathcal{A}) = \mathcal{M}(\mathcal{A})/\mathcal{A}$ is called the *corona* of \mathcal{A} . A few well-known examples are corona algebras are listed below.

(i) If \mathcal{A} is unital, then $\mathcal{M}(\mathcal{A}) = \mathcal{A}$. Therefore the corona of \mathcal{A} is trivial.

(ii) If $\mathcal{A} = \mathcal{K}(H)$, the algebra of all compact operators on a Hilbert space H, then $\mathcal{M}(\mathcal{A}) = \mathcal{B}(H)$, the algebra of all bounded linear operators on H. The corona of \mathcal{A} is the Calkin algebra $\mathcal{C}(H)$.

(iii) If X is a locally compact Hausdorff space, then $\mathcal{M}(C_0(X)) \cong C(\beta X)$, which is isomorphic to the C*-algebra $C_b(X)$ of bounded continuous complex-valued functions on X. The corona of $C_0(X)$ is isomorphic to the C*-algebra $C_b(X)/C_0(X) \cong C(X^*)$, where X^* is the Čech-Stone remainder $\beta X \setminus X$ of X.

(iv) Suppose \mathcal{A}_n is a sequence of unital C*-algebras and let

$$\prod_{n=1}^{\infty} \mathcal{A}_{n} = \{(x_{n}): x_{n} \in \mathcal{A}_{n} \text{ and } \sup_{n} ||x_{n}|| < \infty\}$$
$$\bigoplus_{n=1}^{\infty} \mathcal{A}_{n} = \{(x_{n}) \in \prod_{n=1}^{\infty} \mathcal{A}_{n}: ||x_{n}|| \to 0\}.$$

If $\mathcal{A} = \bigoplus_n \mathcal{A}_n$ then $\mathcal{M}(\mathcal{A}) = \prod_n \mathcal{A}_n$ and the corona of \mathcal{A} is $\prod_n \mathcal{A}_n / \bigoplus_n \mathcal{A}_n$.

(v) If \mathcal{A} is a C*-algebra and X is a locally compact Hausdorff space, let

 $C_b(X, \mathcal{A}) = \{ f : X \to \mathcal{A} : f \text{ is a norm continuous and bounded function} \}$ $C_0(X, \mathcal{A}) = \{ f : X \to \mathcal{A} : f \text{ is continuous and vanishes at } \infty \},$

then the $\mathcal{M}(C_0(X,\mathcal{A})) \cong C_b(X,\mathcal{A})$. Therefore $\mathcal{C}(C_0(X,\mathcal{A})) \cong C_b(X,\mathcal{A})/C_0(X,\mathcal{A}) \cong \mathcal{C}(X^*,\mathcal{C}(\mathcal{A}))$.

Our notion of trivial isomorphisms between corona algebras follows the one in [14], where it is considered more generally for isomorphisms between quotients of algebraic structures.

Definition 2.1. Assume \mathcal{A} and \mathcal{B} are non-unital C*-algebras. A *-homomorphism Φ : $\mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{B})$ is trivial if there is a *-homomorphism $\Phi_* : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{B})$ such that

$$\begin{array}{c} \mathcal{M}(\mathcal{A}) \xrightarrow{\Phi_{*}} \mathcal{M}(\mathcal{B}) \\ \downarrow^{\pi_{\mathcal{A}}} \qquad \qquad \downarrow^{\pi_{\mathcal{B}}} \\ \mathcal{C}(\mathcal{A}) \xrightarrow{\Phi} \mathcal{C}(\mathcal{B}) \end{array}$$

commutes, where $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{B}}$ are the canonical quotient maps.

In the case of automorphisms, this notion of triviality is clearly weaker than the automorphisms being inner, i.e., being implemented by a unitary element. The question of whether the automorphisms, or generally the isomorphisms, between corona algebras are trivial turns out to be very sensitive to the additional set-theoretic assumptions. The most notorious results concerning the triviality of automorphisms of corona of non-commutative C*-algebras concern the Calkin algebra. It has been shown ([23] and [8]) that the statement that all the automorphisms of the Calkin algebra are inner, is independent from ZFC. Refer to [18, \S 1] for a short overview of the some of the important results about triviality of the isomorphisms between corona algebras.

2.2. Logic for metric structures. A main reference for logic for metric structures is [2], see also [19]. A metric structure, in the sense of [2], is a many-sorted structure in which each sort is a complete bounded metric space. A slightly modified version of the this logic was introduced in [13] which does not require the structures to be bounded, and it is more suited while working with operator algebras. A key feature of this logic is that each structure \mathcal{A} is equipped with *domains of quantification*, bounded subsets of \mathcal{A} on which all functions and predicates are uniformly continuous (with a fixed modulus of uniform continuity) which the quantification is allowed. For simplicity in notations we follow [2] in this paper and assume that the metric structures are bounded, but the arguments can be adapted, in the obvious way, to unbounded metric structures as in [13]. For a language \mathcal{L} for metric structures, as usual a metric \mathcal{L} -structure is a complete bounded metric space (for each sort, in the case of many-sorted structures) with appropriate interpretations for predicates, functions and constant symbols in \mathcal{L} . Each predicate and function symbol is equipped with a modulus of uniform continuity. In particular for every \mathcal{L} -formula φ , the set of all evaluations of φ in any \mathcal{L} -structure with the diameter less than a fixed constant, is a bounded subset of the real numbers. Throughout this and the following section we assume formulas are [0, 1]-valued. This is not always an assumption in continuous model theory (e.g., formulas in the model theory for operator algebras). However, since the ranges of (evaluations of) formulas are always bounded in all interpretations allowing $[0,\infty)$ -valued or even negative evaluations results in equivalent logics; see [10, 4.1] for more details on this. We assume that the reader is familiar with the basic definitions of model theory for metric structures, e.g., terms, atomic formulas, formulas, types, etc.

For a topological space X, let $\chi(X)$ denote the density character of X, i.e., the smallest cardinality of a dense subset of X. Suppose κ is an infinite cardinal. A model \mathcal{A} is κ -saturated if every consistent type **t** over $X \subseteq \mathcal{A}$ with $|X| < \kappa$, is realized in \mathcal{A} . If \mathcal{A} is $\chi(X)$ -saturated then \mathcal{A} is called saturated. We say \mathcal{A} is countably saturated if it is \aleph_1 -saturated.

We let \mathbb{F}_n be the set of all formulas whose free variables are included in $\{x_1, \ldots, x_n\}$. For a metric structure \mathcal{A} we usually abbreviate a tuple (a_1, \ldots, a_n) of elements of \mathcal{A} by \bar{a} , when there is no confusion about the length of the tuple, and $\varphi(\bar{a})^{\mathcal{A}}$ is the *interpretation* of φ in \mathcal{A} at \bar{a} .

Define the *theory* of \mathcal{A} to be

$$Th(\mathcal{A}) = \{ \varphi \in \mathbb{F}_0 : \varphi^{\mathcal{A}} = 0 \}.$$

Two metric structures \mathcal{A} and \mathcal{B} are elementarily equivalent, $\mathcal{A} \equiv \mathcal{B}$, if $Th(\mathcal{A}) = Th(\mathcal{B})$. The universal theory $Th_{\forall}(\mathcal{A})$ of \mathcal{A} is the subset of $Th(\mathcal{A})$ consisting of sentences of the form $\sup_{\bar{x}} \varphi(\bar{x})$ where φ is a quantifier-free formula and \bar{x} is a finite tuple.

Lets recall some requirements regarding the connectives from [2, Chapter 6], which will be used throughout this paper. Connectives are continuous functions from $[0,1]^n$ to [0,1] for some $n \ge 1$.

Definition 2.2. A closed system of connectives is a family $\mathcal{F} = (F_n : n \ge 1)$ where each F_n is a set of connectives $f : [0,1]^n \to [0,1]$ satisfying the following conditions.

- (i) For each n, F_n contains the projection onto the j^{th} coordinate for each j = 1, ..., n.
- (ii) For each n and m, if $u \in F_n$, and $v_1, \ldots, v_n \in F_m$, then the function $w : [0,1]^m \to [0,1]$ defined by $w(t) = u(v_1(t), \ldots, v_n(t))$ belongs to F_m .

Definition 2.3. Given a closed system of connectives \mathcal{F} , the collection of \mathcal{F} -restricted formulas is defined by induction.

- (1) Atomic formulas are \mathcal{F} -restricted.
- (2) If $u \in F_n$ and $\varphi_1, \ldots, \varphi_n$ are \mathcal{F} -restricted formulas, then $u(\varphi_1, \ldots, \varphi_n)$ is also an \mathcal{F} -restricted formula.
- (3) If φ is an \mathcal{F} -restricted formula, so are $\sup_x \varphi$ and $\inf_x \varphi$.

Define a binary function $\dot{-}: [0,1]^2 \rightarrow [0,1]$ by

$$x - y = \begin{cases} x - y & x \ge y \\ 0 & \text{otherwise} \end{cases}$$

and let $\mathcal{F}_0 = (F_n : n \ge 1)$ be the closed system of connectives generated from $\{0, 1, x/2, -\}$ by closing it under (i) and (ii) (where 0 and 1 are constant functions with one variable).

Proposition 2.4. [2, Proposition 6.6] The set of all \mathcal{F}_0 -restricted \mathcal{L} -formulas are uniformly dense in the set of all \mathcal{L} -formulas; that is, for any $\epsilon > 0$ and any \mathcal{L} -formula $\varphi(x_1, \ldots, x_n)$, there is an \mathcal{F}_0 -restricted \mathcal{L} -formula $\psi(x_1, \ldots, x_n)$ such that for all \mathcal{L} -structures \mathcal{A} we have

 $|\varphi(a_1,\ldots,a_n) - \psi(a_1,\ldots,a_n)| < \epsilon$

for all $a_1, \ldots, a_n \in \mathcal{A}$. In particular if \mathcal{L} is countable, there is a countable set of \mathcal{L} -formulas which is uniformly dense in the set of all \mathcal{L} -formulas.

2.3. Reduced products of metric structures. Lets recall some definitions and basic theorems regarding reduced products of metric structures from [15] and [20]. Fix a language \mathcal{L} in logic of metric structures. Throughout this paper \mathcal{L} can be many-sorted, but in order to avoid distracting notations we shall assume it is one-sorted. Assume $\{(\mathcal{A}_{\gamma}, d_{\gamma}), \gamma \in \Omega\}$ is a family of metric \mathcal{L} -structures indexed by a set Ω , all having diameter $\leq K$ for some constant K. Consider the direct product

$$\prod_{\Omega} \mathcal{A}_{\gamma} = \{ \langle a(\gamma) \rangle : \ a(\gamma) \in \mathcal{A}_{\gamma} \text{ for all } \gamma \in \Omega \}.$$

Let \mathcal{I} be an ideal on Ω . Define a map $d_{\mathcal{I}}$ on $\prod_{\Omega} \mathcal{A}_{\gamma}$ by

$$d_{\mathcal{I}}(x,y) = \limsup_{\gamma \to \mathcal{I}} d_{\gamma}(x(\gamma), y(\gamma)) = \inf_{S \in \mathcal{I}} \sup_{\gamma \notin S} d_{\gamma}(x(\gamma), y(\gamma))$$

where $x = \langle x(\gamma) : \gamma \in \Omega \rangle$ and $y = \langle y(\gamma) : \gamma \in \Omega \rangle$. Note that the limit notation above corresponds to the notation $\lim_{\gamma \to \mathcal{I}} r_{\gamma} = L$ meaning that for every $\epsilon > 0$ the set $\{\gamma \in \Omega : |r_{\gamma} - L| \ge \epsilon\}$ belongs to the ideal \mathcal{I} .

The map $d_{\mathcal{I}}$ defines a pseudometric metric on $\prod_{\Omega} \mathcal{A}_{\gamma}$. For $x, y \in \prod_{\Omega} \mathcal{A}_{\gamma}$ define $x \sim_{\mathcal{I}} y$ to mean $d_{\mathcal{I}}(x, y) = 0$. Then $\sim_{\mathcal{I}}$ is an equivalence relation and the quotient

$$\prod_{\mathcal{I}} \mathcal{A}_{\gamma} = (\prod_{\Omega} \mathcal{A}_{\gamma}) / \sim_{\mathcal{I}}$$

with the induced metric $d_{\mathcal{I}}$ is a complete bounded metric space. We will use $\pi_{\mathcal{I}}$ to denote the natural quotient map from $\prod_{\Omega} \mathcal{A}_{\gamma}$ onto $\prod_{\mathcal{I}} \mathcal{A}_{\gamma}$. For a tuple $\bar{a} = (a_1, \ldots, a_k)$ of elements of $\prod_{\Omega} \mathcal{A}_{\gamma}$ we write $\pi_{\mathcal{I}}(\bar{a})$ for $(\pi_{\mathcal{I}}(a_1), \ldots, \pi_{\mathcal{I}}(a_k))$ and by $\bar{a}(\gamma)$ we denote the corresponding tuple $(a_1(\gamma), \ldots, a_k(\gamma))$ of elements of \mathcal{A}_{γ} .

Let R be a predicate symbol in \mathcal{L} and \bar{a} be a tuple of elements of $\prod_{\Omega} \mathcal{A}_{\gamma}$ of appropriate size. Define

$$R(\pi_{\mathcal{I}}(\bar{a})) = \limsup_{\mathcal{I}} R(\bar{a}(\gamma)).$$

If f is a function symbol in \mathcal{L} for an appropriate \bar{a} define

$$f(\pi_{\mathcal{I}}(\bar{a})) = \pi_{\mathcal{I}}(\langle f(\bar{a}(\gamma)) \rangle),$$

and if $c \in \mathcal{L}$ is a constant symbol let

$$c^{\prod_{\mathcal{I}} \mathcal{A}_{\gamma}} = \pi_{\tau}(\langle c^{\mathcal{A}_{\gamma}} \rangle).$$

The quotient $\prod_{\mathcal{I}} \mathcal{A}_{\gamma}$ is called the *reduced product* of the family $\{(\mathcal{A}_{\gamma}, d_{\gamma}) : \gamma \in \Omega\}$ over the ideal \mathcal{I} . Note that if \mathcal{I} is a maximal (prime) ideal, then $\prod_{\mathcal{I}} \mathcal{A}_{\gamma}$ is the ultraproduct of the family $\{\mathcal{A}_{\gamma}, \gamma \in \Omega\}$ over the ultrafilter \mathcal{U} consisting of the complements of the elements of \mathcal{I} , usually denoted by $\prod_{\mathcal{U}} \mathcal{A}_{\gamma}$ or $(\prod_{\Omega} \mathcal{A}_{\gamma})_{\mathcal{U}}$ or $\prod_{\Omega} \mathcal{A}_{\gamma}/\mathcal{U}$. Also, in the case when \mathcal{L} includes a distinguished constant symbol for 0 (e.g., the language of C*-algebras) the reduced product of \mathcal{L} -structures $\{\mathcal{A}_{\gamma}, \gamma \in \Omega\}$ over \mathcal{I} is the quotient of $\prod_{\Omega} \mathcal{A}_{\gamma}$ over its closed ideal $\bigoplus_{\mathcal{I}} \mathcal{A}_{\gamma}$ defined by

$$\bigoplus_{\mathcal{I}} \mathcal{A}_{\gamma} = \{ a \in \prod_{\Omega} \mathcal{A}_{\gamma} : \ d_{\mathcal{I}}(a, 0^{\prod_{\Omega} \mathcal{A}_{\gamma}}) = 0 \},\$$

and usually denoted by $\prod_{\Omega} \mathcal{A}_{\gamma} / \bigoplus_{\mathcal{I}} \mathcal{A}_{\gamma}$ (see [15]).

Proposition 2.5. The structure $\langle \prod_{\mathcal{I}} \mathcal{A}_{\gamma}, d_{\mathcal{I}} \rangle$ in the language \mathcal{L} with the interpretations of constants, functions and predicates as above, is a metric \mathcal{L} -structure.

Proof. We only have to check that each function and predicate symbol has the same modulus of uniform continuity. we shall prove this only for a function symbol f of arity k. Let $\Delta : [0,1] \rightarrow [0,1]$ be the modulus of uniform continuity of f, i.e., for $\epsilon > 0$ and $\bar{x} = (x_1, \ldots, x_k), \ \bar{y} = (y_1, \ldots, y_k)$ tuples in each \mathcal{A}_{γ} we have

$$d_{\gamma}(\bar{x}, \bar{y}) < \Delta(\epsilon) \rightarrow d_{\gamma}(f(\bar{x}), f(\bar{y})) \le \epsilon,$$

where $d_{\gamma}(\bar{x}, \bar{y}) < \Delta(\epsilon)$ means $d_{\gamma}(x_i, y_i) < \Delta(\epsilon)$ for ever $i = \{1, \dots, k\}$.

Suppose \bar{a} and \bar{b} in $(\prod_{\Omega} \mathcal{A}_{\gamma})^k$ are such that $d_{\mathcal{I}}(\pi_I(\bar{a}), \pi_{\mathcal{I}}(\bar{b})) < \Delta(\epsilon)$. Then by the definition of $d_{\mathcal{I}}$ there is an \mathcal{I} -positive set $S \subseteq \Omega$ such that for every $\gamma \in S$ we have $d_{\gamma}(\bar{a}(\gamma), \bar{b}(\gamma)) < \Delta(\epsilon)$, and therefore $d_{\gamma}(f(\bar{a}(\gamma)), f(\bar{b}(\gamma))) \leq \epsilon$. This implies that $d_{\mathcal{I}}(\pi_{\mathcal{I}}(f(\bar{a})), \pi_{\mathcal{I}}(f(\bar{y}))) \leq \epsilon$. \Box

Lemma 2.6. Assume \mathcal{I} is an ideal on Ω . If $\varphi(\bar{y})$ is an atomic \mathcal{L} -formula and \bar{a} is a tuple of elements of $\prod_{\Omega} \mathcal{A}_{\gamma}$, then

$$\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\prod_{\mathcal{I}}\mathcal{A}_{\gamma}} = \limsup_{\mathcal{I}} \varphi(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}}$$

Proof. This easily follows from the definition of $d_{\mathcal{I}}$ and the interpretation of atomic formulas.

3. An extension of Feferman-Vaught theorem for reduced products of metric structures

The evaluation of a non-atomic formula in reduced products turns out to be more complicated than the atomic case, see [20]. In this section we give an extension of Feferman-Vaught theorem to reduced products of metric structures, which just like its classical version, gives a powerful tool to prove elementary equivalence of reduced products.

Suppose $\{\mathcal{A}_{\gamma} : \gamma \in \Omega\}$ is a family of metric structures in a fixed language \mathcal{L} and \mathcal{I} is an ideal on Ω . For the purposes of this section let

$$\mathcal{A}_{\Omega} = \prod_{\Omega} \mathcal{A}_{\gamma}, \qquad \qquad \mathcal{A}_{\mathcal{I}} = \prod_{\mathcal{I}} \mathcal{A}_{\gamma}.$$

For an \mathcal{L} -formula $\varphi(\bar{x})$, a tuple \bar{a} of elements of \mathcal{A} and

$$X = \{ \gamma \in \Omega : \varphi(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > r \}$$

for some $r \in \mathbb{R}$, we use \tilde{X} to denote the set

$$\tilde{X} = \{ \gamma \in \Omega : \varphi(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} \ge r \}$$

Definition 3.1. For an \mathcal{F}_0 -restricted \mathcal{L} -formula $\varphi(x_1, \ldots, x_l)$, we say φ is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$ if

(1) Each σ_i is a formula in the language of Boolean algebras with at most $s = m2^n$ many variables, which is monotonic, i.e.,

$$T_{BA} \vdash \forall y_1 \dots, y_s, z_1, \dots, z_s (\sigma_i(y_1, \dots, y_s) \land \bigwedge_{i=1}^s y_i \le z_i)$$
$$\to \sigma_i(z_1, \dots, z_s)).$$

(Here, T_{BA} denotes the theory of Boolean algebras.) (2) Each $\psi_i(x_1, \ldots, x_l)$ is an \mathcal{F}_0 -restricted \mathcal{L} -formula for $j = 0, \ldots, m-1$. (3) For any indexed set Ω , an ideal \mathcal{I} on Ω , a family $\{\mathcal{A}_{\gamma} : \gamma \in \Omega\}$ of metric \mathcal{L} -structures and $a_1, \ldots, a_l \in \mathcal{A}_{\Omega}$ the following hold: for every $\ell = 0, \ldots, 2^n$

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^0]_{\mathcal{I}}, \dots, [X_0^{m-1}]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}) \implies \varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n,$$

and

$$\begin{split} \varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^{n} \\ \implies P(\Omega)/\mathcal{I} \vDash & \sigma_{\ell}([\tilde{X}_{0}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{0}^{m-1}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{m-1}]_{\mathcal{I}}) \\ where \ X_{i}^{j} = \{ \gamma \in \Omega : \psi_{j}(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^{n} \} \ for \ each \ j = 0, \dots, m-1 \ and \\ i = 0, \dots, 2^{n}. \end{split}$$

Using Lemma 2.4 we can generalize this definition to all \mathcal{L} -formulas.

Definition 3.2. We say an \mathcal{L} -formula φ is determined up to 2^{-n} if there is an \mathcal{F}_0 restricted \mathcal{L} -formula $\tilde{\varphi}$ which is uniformly within 2^{-n-1} of φ and $\tilde{\varphi}$ is determined up to 2^{-n} by some $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$.

Theorem 3.3. Every formula is determined up to 2^{-n} for any given $n \in \mathbb{N}$.

Proof. By Definition 3.2 and Lemma 2.4, without loss of generality, we can assume that formulas are \mathcal{F}_0 -restricted. Assume φ is an atomic \mathcal{L} -formula and for each $i \leq 2^n$ define

$$\sigma_i(y_0,\ldots,y_{2^n}) := y_i \neq 0$$

We show that φ is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \varphi)$. Conditions (1) and (2) of Definition 3.1 are clearly satisfied. For an indexed set Ω , an ideal \mathcal{I} on Ω , a family $\{\mathcal{A}_{\gamma}: \gamma \in \Omega\}$ of metric \mathcal{L} -structures and $a_1, \ldots, a_l \in \mathcal{A}_{\Omega}$ let

$$X_i = \{ \gamma \in \Omega : \varphi(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \},\$$

since $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} = \limsup_{\mathcal{I}} \varphi(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}}$ we have

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([X_0]_{\mathcal{I}}, \dots, [X_{2^n}]_{\mathcal{I}}) \iff X_{\ell} \notin \mathcal{I}$$
$$\iff \varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n.$$

Since each $X_i^j \subseteq \tilde{X}_i^j$, by the monotonicity of σ_ℓ , $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n$ also implies that $P(\Omega)/\mathcal{I} \vDash \sigma_\ell([\tilde{X}_0]_{\mathcal{I}}, \ldots, [\tilde{X}_{2^n}]_{\mathcal{I}})$. Thus φ is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \varphi)$.

Assume $\varphi(\bar{x}) = f(\alpha(\bar{x}))$ where $f \in \{0, 1, x/2\}$ and α is some \mathcal{L} -formula determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. The cases where $f \in \{0, 1\}$ are trivial; for example if f = 0 then $\varphi(\bar{x})$ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; 0)$ where $\tau_i := 1 \neq 1$ for each $i = 0, \ldots, 2^n$. If f(x) = x/2 then it is also straightforward to check that φ is determined up to 2^{-n-1} by $(\sigma_0, \ldots, \sigma_{2^n}, \tau_{2^n+1}, \ldots, \tau_{2^{n+1}}; f(\psi_0), \ldots, f(\psi_{m-1}))$, where each τ_i is a false sentence (e.g, $1 \neq 1$).

Let $\varphi(\bar{x}) = \alpha_1(\bar{x}) \div \alpha_2(\bar{x})$ where each α_t $(t \in \{1, 2\})$ is determined up to 2^{-n} by $(\sigma_0^t, \ldots, \sigma_{2^n}^t; \psi_0^t, \ldots, \psi_{m_t-1}^t)$. We claim that φ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; \psi_0^1, \ldots, \psi_{m_t-1}^t)$.

$$\begin{array}{l} ,\psi_{m_{1}-1}^{1},1-\psi_{0}^{2},\ldots,1-\psi_{m_{2}-1}^{2}) \text{ where the Boolean algebra formulas } \tau_{k} \text{ are defined by} \\ \tau_{k}(x_{0}^{0},\ldots,x_{2^{n}}^{0},\ldots,x_{0}^{m_{1}-1},\ldots,x_{2^{n}}^{m_{1}-1},z_{0}^{0},\ldots,z_{2^{n}}^{0},\ldots,z_{0}^{m_{2}-1},\ldots,z_{2^{n}}^{m_{2}-1}):= \\ & \bigvee_{i_{0}=k}^{2^{n}} \left[\sigma_{i_{0}}^{1}(x_{0}^{0},\ldots,x_{2^{n}}^{0},\ldots,x_{0}^{m_{1}-1},\ldots,x_{2^{n}}^{m_{1}-1})\right. \\ & \wedge \neg \sigma_{i_{0}-k}^{2}(-z_{2^{n}}^{0},\ldots,-z_{0}^{0},\ldots,-z_{2^{n}}^{m_{2}-1},\ldots,-z_{0}^{m_{2}-1})\right], \end{array}$$

(here -z is the Boolean algebra complement of z). Conditions (1) and (2) in Definition 3.1 are clearly satisfied. For (3) let $\mathcal{A}_{\mathcal{I}}$ be a reduced product of \mathcal{L} -structures (indexed by Ω and over an ideal \mathcal{I}) and $a_1, \ldots, a_l \in \mathcal{A}_{\Omega}$. Let

$$X_i^j = \{ \gamma \in \Omega : \psi_j^1(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \} \qquad 0 \le j \le m_1 - 1,$$

$$Y_i^j = \{ \gamma \in \Omega : \psi_j^2(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \} \qquad 0 \le j \le m_2 - 1,$$

and

$$Z_i^j = \{ \gamma \in \Omega : 1 - \psi_j^2(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \} \qquad 0 \le j \le m_2 - 1.$$

Note that $\tilde{Y}_{2^n-i}^j = (Z_i^j)^c$ for each *i* and *j*. Assume

$$P(\Omega)/\mathcal{I} \vDash \tau_k([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^{m_1-1}]_{\mathcal{I}}, [Z_0^0]_{\mathcal{I}}, \dots, [Z_{2^n}^{m_2-1}]_{\mathcal{I}}),$$

then for some $i_0 \ge k$,

$$P(\Omega)/\mathcal{I} \vDash \sigma_{i_0}^1([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^{m_1-1}]_{\mathcal{I}}) \land \neg \sigma_{i_0-k}^2([(Z_{2^n}^0)^c]_{\mathcal{I}}, \dots, [(Z_0^{m_1-1})^c]_{\mathcal{I}}),$$

$$\implies P(\Omega)/\mathcal{I} \vDash \sigma_{i_0}^1([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^{m_1-1}]_{\mathcal{I}}) \land \neg \sigma_{i_0-k}^2([\tilde{Y}_0^0]_{\mathcal{I}}, \dots, [\tilde{Y}_{2^n}^{m_1-1}]_{\mathcal{I}}),$$

and therefore

$$\alpha_1(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > i_0/2^n \text{ and } \alpha_2(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} \le (i_0 - k)/2^n.$$

Hence $\varphi(\pi_{\mathfrak{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > k/2^n$. To prove the other direction assume $\varphi(\pi_{\mathfrak{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > k/2^n$. For some $i_0 \geq k$,

$$\alpha_1(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > i_0/2^n \text{ and } \alpha_2(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} \le (i_0 - k)/2^n$$

By the induction assumptions

$$P(\Omega)/\mathcal{I} \vDash \sigma_{i_0}^1([\tilde{X}_0^0]_{\mathcal{I}}, \dots, [\tilde{X}_{2^n}^{m_1-1}]_{\mathcal{I}}) \land \neg \sigma_{i_0-k}^2([\tilde{Y}_0^0]_{\mathcal{I}}, \dots, [\tilde{Y}_{2^n}^{m_1-1}]_{\mathcal{I}}),$$

and note that $(\tilde{Z}^j_{2^n-i})^c \subseteq \tilde{Y}^j_i$ for each i and j, which implies

$$P(\Omega)/\mathcal{I} \vDash \sigma_{i_0}^1([\tilde{X}_0^0]_{\mathcal{I}}, \dots, [\tilde{X}_{2^n}^{m_1-1}]_{\mathcal{I}}) \land \neg \sigma_{i_0-k}^2([(\tilde{Z}_{2^n}^0)^c]_{\mathcal{I}}, \dots, [(\tilde{Z}_0^{m_1-1})^c]_{\mathcal{I}}),$$

 \mathbf{SO}

$$P(\Omega)/\mathcal{I} \vDash \tau_k([\tilde{X}_0^0]_{\mathcal{I}}, \dots, [\tilde{X}_{2^n}^{m_1-1}]_{\mathcal{I}}, [\tilde{Z}_0^0]_{\mathcal{I}}, \dots, [\tilde{Z}_{2^n}^{m_2-1}]_{\mathcal{I}})$$

Therefore φ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; \psi_0^1, \ldots, \psi_{m_1-1}^1, 1-\psi_0^2, \ldots, 1-\psi_{m_2-1}^2)$.

Assume $\varphi(\bar{x}) = \sup_{z} \psi(\bar{x}, z)$ where ψ is determined by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$ up to 2^{-n} . Let $d = 2^{n+m} - 1$ and s_0, \ldots, s_{d-1} be an enumeration of non-empty elements of $\prod_{i=0}^{2^n} P(\{0, \ldots, m-1\})$, i.e., each $s_k = (s_k(0), \ldots, s_k(2^n))$ where $s_k(i) \subseteq \{0, \ldots, m-1\}$ for

each *i*. Also assume that for each $0 \le k \le m-1$ we have $s_k = \{\{k\}, \emptyset, \dots, \emptyset\}$. For any $k \in \{0, \dots, d-1\}$ define an \mathcal{L} -formula θ_k by

$$\theta_k(\bar{x}) = \sup_z \min\left\{\psi_j(\bar{x}, z) : j \in \bigcup_{i=0}^{2^n} s_k(i)\right\}$$

Note that if $0 \le k \le m-1$ then $\theta_k(\bar{x}) = \sup_z \psi_k(\bar{x}, z)$. For each $i \in \{0, \ldots, 2^n\}$ define a Boolean algebra formula τ_i by

$$\begin{aligned} \tau_i(y_0^0, \dots, y_{2^n}^0, \dots, y_0^{d-1}, \dots, y_{2^n}^{d-1}) &= & \exists z_0^0, \dots, z_{2^n}^0, \dots, z_0^{d-1}, \dots, z_{2^n}^{d-1} \\ & [\bigwedge_{j=0}^{d-1} \bigwedge_{i=0}^{2^n} (z_i^j \le y_i^j) \land \bigwedge_{i=0}^{2^n} \bigwedge_{\substack{s_k(t) \cup s_{k'}(t) = s_{k''}(t) \\ \forall t}} (z_i^k. z_i^{k'} = z_i^{k''}) \\ & \land \sigma_i(z_0^0, \dots, z_{2^n}^0, \dots, z_0^{m-1}, \dots, z_{2^n}^{m-1})]. \end{aligned}$$

We claim that φ is determined up to 2^{-n} by $(\tau_0, \ldots, \tau_{2^n}; \theta_0, \ldots, \theta_{d-1})$. Again condition (1) is clearly satisfied. Condition (2) is also satisfied, since $\min\{x, y\} = x - (x - y)$. For (3) assume a reduced product $\mathcal{A}_{\mathcal{I}}$ and $a_1, \ldots, a_l \in \mathcal{A}_{\Omega}$ are given.

(b) assume a reduced product $\mathcal{F}_{\mathcal{I}}$ and $a_1, \ldots, a_l \in \mathcal{F}_{\mathcal{I}}$ and

First assume $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n$ for some ℓ . Let

$$\delta = \frac{\min\{\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} - \ell/2^n, 1/2^n\}}{2}$$

and find $c = (c(\gamma))_{\gamma \in \Omega}$ such that

$$\psi(\pi_{\mathcal{I}}(\bar{a},c))^{\mathcal{A}_{\mathcal{I}}} > \varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} - \delta > \ell/2^{n}.$$

For each $i \leq 2^n$ and $k \leq d-1$ let

$$Y_i^k = \{ \gamma \in \Omega : \theta_k(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \}.$$

and let

$$Z_i^k = \{ \gamma \in \Omega : \min\{\psi_j(\bar{x}(\gamma), c(\gamma)) : j \in \bigcup_{t=0}^{2^n} s_k(t) \}^{\mathcal{A}_{\gamma}} > i/2^n \}.$$

From definition of θ_k it is clear that $\tilde{Z}_i^k \subseteq \tilde{Y}_i^k$, and

$$\bigwedge_{i=0}^{2^{n}} \bigwedge_{\substack{s_k(t) \cup s_{k'}(t) = s_{k''}(t) \\ \forall t}} (\tilde{Z}_i^k \cap \tilde{Z}_i^{k'} = \tilde{Z}_i^{k''}),$$

and by the inductive assumption

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([\tilde{Z}_0^0]_{\mathcal{I}}, \dots, [\tilde{Z}_{2^n}^0]_{\mathcal{I}}, \dots, [\tilde{Z}_0^{m-1}]_{\mathcal{I}}, \dots, [\tilde{Z}_{2^n}^{m-1}]_{\mathcal{I}}).$$

Hence $P(\Omega)/\mathcal{I} \vDash \tau_{\ell}([\tilde{Y}_0^0]_{\mathcal{I}}, \dots, [\tilde{Y}_{2^n}^0]_{\mathcal{I}}, \dots, [\tilde{Y}_0^{d-1}]_{\mathcal{I}}, \dots, [\tilde{Y}_{2^n}^{d-1}]_{\mathcal{I}}).$

For the other direction let

$$Y_i^k = \{ \gamma \in \Omega : \theta_k(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \},\$$

and suppose $P(\Omega)/\mathcal{I} \models \tau_{\ell}([Y_0^0]_{\mathcal{I}}, \dots, [Y_{2^n}^0]_{\mathcal{I}}, \dots, [Y_0^{d-1}]_{\mathcal{I}}, \dots, [Y_{2^n}^{d-1}]_{\mathcal{I}})$. There are sets $Z_0^0, \dots, Z_{2^n}^0, \dots, Z_{2^n}^{d-1}, \dots, Z_{2^n}^{d-1}$ such that the following hold.

$$\begin{split} [Z_{i}^{k}]_{\mathcal{I}} &\subseteq [Y_{i}^{k}]_{\mathcal{I}} & 0 \leq i \leq 2^{n}, 0 \leq k \leq d-1, \\ [Z_{i}^{k}]_{\mathcal{I}} &\cap [Z_{i}^{k'}]_{\mathcal{I}} = [Z_{i}^{k''}]_{\mathcal{I}} & \forall t \ s_{k}(t) \cup s_{k'}(t) = s_{k''}(t), \\ P(\Omega)/\mathcal{I} &\models \sigma_{\ell}([Z_{0}^{0}]_{\mathcal{I}}, \dots, [Z_{2^{n}}^{0}]_{\mathcal{I}}, \dots, [Z_{0}^{m-1}]_{\mathcal{I}}, \dots, [Z_{2^{n}}^{m-1}]_{\mathcal{I}}). \end{split}$$

Since there are only finitely many conditions above, we can find a set $S \in \mathcal{I}$ such that if $D = \Omega \setminus S$ then

(1)
$$Z_i^k \cap D \subseteq Y_i^k \qquad 0 \le i \le 2^n, 0 \le k \le d-1,$$
$$Z_i^k \cap Z_i^{k'} \cap D = Z_i^{k''} \cap D \qquad \forall t \ s_k(t) \cup s_{k'}(t) = s_{k''}(t),$$

Fix $\gamma \in D$, and for each $i \in \{0, \ldots, 2^n\}$ let $u(i) = \{j \in \{0, \ldots, m-1\} : \gamma \in Z_i^j\}$. If $k \in \{0, \ldots, d-1\}$ be such that $s_k = (u(0), \ldots, u(2^n))$, then since $\gamma \in Z_i^j$ for all $j \in u(i)$, using (1) we have $\gamma \in Y_i^k$ (for all i) and hence

$$\theta_k(\bar{a}(\gamma))^{\mathcal{A}_{\mathcal{I}}} = \sup_{z} \min_{j \in \bigcup_{t=0}^{2^n} u(t)} \psi_j(\bar{a}(\gamma), z) > i/2^n.$$

Let

$$\delta = \frac{\min_{i,k} \{\theta_k(\bar{a}(\gamma))^{\mathcal{A}_{\mathcal{I}}} - i/2^n, 1/2^n\}}{2}$$

We can pick $c(\gamma) \in A_{\gamma}$ such that for every i

(2)
$$\min_{j \in u(i)} \psi_j(\bar{a}(\gamma), c(\gamma)) > \theta_k(a(\gamma)) - \delta \ge i/2^n.$$

For $\gamma \notin D$ define $c(\gamma)$ arbitrarily and let $c = (c(\gamma))_{\gamma \in \Omega}$. For each $j \in \{0, \ldots, m-1\}$ and $i \in \{0, \ldots, 2^n\}$ let

$$X_i^j = \{ \gamma \in \Omega : \psi_j(\bar{a}(\gamma), c(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \}.$$

Now (1) and (2) imply that $Z_i^j \cap D \subseteq X_i^j$ for all *i* and *j*. Therefore

$$P(\Omega)/\mathcal{I} \models \bigwedge_{j=0}^{m-1} \bigwedge_{i=0}^{2^n} [Z_i^j]_{\mathcal{I}} \le [X_i^j]_{\mathcal{I}}.$$

Since $P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([Z_0^0]_{\mathcal{I}}, \dots, [Z_{2^n}^0]_{\mathcal{I}}, \dots, [Z_0^{m-1}]_{\mathcal{I}}, \dots, [Z_{2^n}^{m-1}]_{\mathcal{I}})$, by monotonicity of σ_{ℓ} we have

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^0]_{\mathcal{I}}, \dots, [X_0^{m-1}]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}).$$

Therefore by the induction assumption we have

$$\psi(\pi_{\mathcal{I}}(\bar{a},c))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n,$$

which implies that $\varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n$.

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Remark 3.4. From the proof of Theorem 3.3 at each step of the induction, it is straightforward (however lengthy) to check that for each $\ell \in \{1, \ldots, 2^n\}$, the Boolean algebra formulas σ_{ℓ} have the property that

$$T_{BA} \models \sigma_{\ell-1}(z_0^0, \dots, z_{2^n}^{m-1}) \leftrightarrow \sigma_{\ell}(\Omega, z_0^0, z_1^0, \dots, z_{2^n-1}^0, \Omega, z_0^1, z_1^1, \dots, z_{2^n-1}^1, \dots, \Omega, z_0^{m-1}, z_1^{m-1}, \dots, z_{2^n-1}^{m-1}).$$

Lemma 3.5. Assume $\varphi(\bar{x})$ is an \mathcal{F}_0 -restricted \mathcal{L} -formula which is determined up to 2^n by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. Assume $\mathcal{A}_{\mathcal{I}}$ is a reduced product over an ideal \mathcal{I} and $a_1, \ldots, a_l \in \mathcal{A}_{\Omega}$. Then for each $\ell \in \{1, \ldots, 2^n\}$

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([\tilde{X}_{0}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{0}^{m-1}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{m-1}]_{\mathcal{I}})$$
$$\Longrightarrow \varphi(\pi_{\mathcal{I}}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > (\ell-1)/2^{n},$$

where $X_i^j = \{ \gamma \in \Omega : \psi_j(\bar{a}(\gamma))^{\mathcal{A}_{\gamma}} > i/2^n \}, \ j \in \{0, \dots, m-1\} \ and \ i \in \{0, \dots, 2^n\}.$

Proof. Assume

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([\tilde{X}_{0}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{0}^{m-1}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{m-1}]_{\mathcal{I}})$$

and note that $\tilde{X}_0^j = \Omega$ and $\tilde{X}_i^j \subseteq X_{i-1}^j$ for each $j \in \{0, \ldots, m-1\}$ and $i \in \{1, \ldots, 2^n\}$. Thus by the monotonicity of σ_ℓ we have

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([\Omega]_{\mathcal{I}}, [X_0^0]_{\mathcal{I}}, \dots, [X_{2^{n-1}}^0]_{\mathcal{I}}, \dots, [\Omega]_{\mathcal{I}}, [X_0^{m-1}]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}),$$

which implies that (Remark 3.4)

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell-1}([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}).$$

Therefore $\varphi(\pi_{\tau}(\bar{a}))^{\mathcal{A}_{\mathcal{I}}} > (\ell - 1)/2^n$.

Let us give some easy applications of Theorem 3.3. Assume $\{\mathcal{A}_{\gamma} : \gamma \in \Omega\}$ and $\{\mathcal{B}_{\gamma} : \gamma \in \Omega\}$ are families of metric \mathcal{L} -structures indexed by Ω and for an ideal \mathcal{I} on Ω let $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{B}_{\mathcal{I}}$ denote the corresponding reduced products over \mathcal{I} . Next proposition shows that if each $\mathcal{A}_{\gamma} \equiv \mathcal{B}_{\gamma}$ for $\gamma \in \Omega$ then $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{B}_{\mathcal{I}}$ are also elementarily equivalent.

Proposition 3.6. *Reduced products, direct products and ultraproducts preserve elementary equivalence.*

Proof. We only need to show this for reduced products, since the others are special cases of reduced products. Let $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{B}_{\mathcal{I}}$ be two reduced products over ideal \mathcal{I} such that $\mathcal{A}_{\gamma} \equiv \mathcal{B}_{\gamma}$ for every $\gamma \in \Omega$. Let φ be an \mathcal{L} -sentence. By Proposition 2.4 we can assume φ is an \mathcal{F}_0 -restricted \mathcal{L} -sentence. For a given $n \in \mathbb{N}$ suppose φ is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. For each $i \in \{0, \ldots, 2^n\}$ and $j \in \{0, \ldots, m-1\}$ let

$$X_i^j = \{ \gamma \in \Omega : \psi_j^{\mathcal{A}\gamma} > i/2^n \},$$

$$Y_i^j = \{ \gamma \in \Omega : \psi_j^{\mathcal{B}\gamma} > i/2^n \}.$$

By our assumption $X_i^j = Y_i^j$ for all *i* and *j*. Therefore

$$P(\Omega)/\mathcal{I} \vDash \sigma_i([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}) \leftrightarrow \sigma_i([Y_0^0]_{\mathcal{I}}, \dots, [Y_{2^n}^{m-1}]_{\mathcal{I}}),$$

and

$$P(\Omega)/\mathcal{I} \vDash \sigma_i([\tilde{X}_0^0]_{\mathcal{I}}, \dots, [\tilde{X}_{2^n}^{m-1}]_{\mathcal{I}}) \leftrightarrow \sigma_i([\tilde{Y}_0^0]_{\mathcal{I}}, \dots, [\tilde{Y}_{2^n}^{m-1}]_{\mathcal{I}}).$$

If $\varphi^{\mathcal{A}_{\mathcal{I}}} > \ell/2^n$ by Theorem 3.3 and Definition 3.1 (3) we have

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([\tilde{X}_{0}^{0}]_{\mathcal{I}}, \dots, [\tilde{X}_{2^{n}}^{m-1}]_{\mathcal{I}})$$

which is of course

$$P(\Omega)/\mathcal{I} \vDash \sigma_{\ell}([\tilde{Y}_0^0]_{\mathcal{I}}, \dots, [\tilde{Y}_{2^n}^{m-1}]_{\mathcal{I}}).$$

Thus Lemma 3.5 implies that $\varphi^{\mathcal{B}_{\mathcal{I}}} > (\ell - 1)/2^n$. Similarly $\varphi^{\mathcal{B}_{\mathcal{I}}} > \ell/2^n$ implies that $\varphi^{\mathcal{A}_{\mathcal{I}}} > (\ell - 1)/2^n$. This means that $|\varphi^{\mathcal{A}_{\mathcal{I}}} - \varphi^{\mathcal{B}_{\mathcal{I}}}| \leq 1/2^n$. Since *n* was arbitrary by approaching *n* to infinity, we have $\varphi^{\mathcal{A}_{\mathcal{I}}} = \varphi^{\mathcal{B}_{\mathcal{I}}}$. Therefore $\mathcal{A}_{\mathcal{I}} \equiv \mathcal{B}_{\mathcal{I}}$.

Theorem 3.7. Assume \mathcal{A} is an \mathcal{L} -structure and \mathcal{I} and \mathcal{J} are atomless ideals on Ω , then the reduced powers of \mathcal{A} over \mathcal{I} and \mathcal{J} are elementarily equivalent.

Proof. Let $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ denote the reduced powers of \mathcal{A} ($\mathcal{A}_{\gamma} = \mathcal{A}$ for all $\gamma \in \Omega$) over \mathcal{I} and \mathcal{J} , respectively. Let φ be an \mathcal{L} -sentence and for $n \geq 1$ find an \mathcal{F}_0 -restricted \mathcal{L} -sentence $\tilde{\varphi}$ which is uniformly within 2^{-n} of φ and is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. Then

$$X_i^j = \{ \gamma \in \Omega : \psi_j^{\mathcal{A}_\gamma} > i/2^n \}$$

is clearly either Ω or \emptyset , therefore $[X_i^j]_{\mathcal{I}} = [X_i^j]_{\mathcal{J}} = 0$ or 1. Since any two atomless Boolean algebras are elementarily equivalent, for every $i = 0, \ldots, 2^n$

$$P(\Omega)/\mathcal{I} \vDash \sigma_i([X_0^0]_{\mathcal{I}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{I}}) \Longleftrightarrow P(\Omega)/\mathcal{J} \vDash \sigma_i([X_0^0]_{\mathcal{J}}, \dots, [X_{2^n}^{m-1}]_{\mathcal{J}})$$

Thus by Theorem 3.3 and the same argument as the proof of Proposition 3.6 we have $\varphi^{\mathcal{A}_{\mathcal{I}}} = \varphi^{\mathcal{B}_{\mathcal{I}}}$.

4. Isomorphisms of reduced products under the Continuum Hypothesis

In C*-algebra context an important class of corona algebras is the reduced power of a C*-algebra \mathcal{A} over the Fréchet ideal *Fin*. It is called the *asymptotic sequence algebra* of \mathcal{A} and denoted by $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$. The C*-algebra \mathcal{A} can be identified with its diagonal image in $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$. We will also use the same notation $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ for the reduced power of an arbitrary metric structure \mathcal{A} over *Fin*.

As mentioned in the introduction a result of Farah-Shelah (Theorem 4.3) shows that asymptotic sequence algebras are countably saturated and therefore if \mathcal{A} is separable, assuming the Continuum Hypothesis, they have 2^{\aleph_1} automorphisms, hence non-trivial ones. The last statement follows from a folklore theorem (refer to [15, Corollary 4.1] for a proof due to Bradd Hart) which states that any κ -saturated metric structure of density character κ has 2^{κ} automorphisms. Furthermore, κ -saturated structures of the same density character κ which are elementarily equivalent are isomorphic. Assuming the Continuum Hypothesis, for a separable \mathcal{A} and a nonprincipal ultrafilter \mathcal{U} on \mathbb{N} , the asymptotic sequence algebra $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ and its ultrapower $(\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A}))_{\mathcal{U}}$ have the same density character \aleph_1 (see [1, Theorem 2.7]), and therefore they are isomorphic. We will show that

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this is also the case for the reduced powers of separable metric structures over a large family of ideals (Corollary 4.6), e.g., $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ is isomorphic to the asymptotic sequence algebra of $\ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$, under the Continuum Hypothesis.

We briefly recall a few definitions and facts about ideals on \mathbb{N} . The properties of various *definable* (F_{σ} , $F_{\delta\sigma}$, Borel, analytic, ...) ideals and their Boolean algebra quotients have been vastly studied, see e.g., [7], [24].

For any $A \subseteq \mathbb{N} \times \mathbb{N}$ the vertical section of A at m is the set $A_m = \{n \in \mathbb{N} : (m, n) \in A\}$. For two ideals \mathcal{I} and \mathcal{J} on \mathbb{N} , the Fubini product, $\mathcal{I} \times \mathcal{J}$, of \mathcal{I} and \mathcal{J} is the ideal on $\mathbb{N} \times \mathbb{N}$ defined by

$$A \in \mathcal{I} \times \mathcal{J} \iff \{i : A_i \notin \mathcal{J}\} \in \mathcal{I}.$$

Definition 4.1. A map $\mu : \mathcal{P}(\mathbb{N}) \to [0,\infty]$ is a submeasure supported by \mathbb{N} if for $A, B \subseteq \mathbb{N}$

$$\mu(\emptyset) = 0$$

$$\mu(A) \le \mu(A \cup B) \le \mu(A) + \mu(B).$$

It is lower semicontinuous if for all $A \subseteq \mathbb{N}$ we have

$$\mu(A) = \lim_{n \to \infty} \mu(A \cap [1, n]).$$

Layered ideals. An ideal \mathcal{I} is *layered* if there is $f: P(\mathbb{N}) \to [0,\infty]$ such that

(1) $f(A) \leq f(B)$ if $A \subseteq B$, (2) $\mathcal{I} = \{A : f(A) < \infty\}$, (3) $f(A) = \infty$ implies $f(A) = \sup_{B \subset A} f(B)$.

Layered ideals were introduced in [9], where in particular the following is proved.

Lemma 4.2. [9, Proposition 6.6]

- (1) Every F_{σ} ideal is layered.
- (2) If \mathcal{J} is a layered ideal and \mathcal{I} is an arbitrary ideal on \mathbb{N} , then $\mathcal{J} \times \mathcal{I}$ is layered.

Proof. By a theorem of K. Mazur ([21]) for every F_{σ} ideal \mathcal{I} there is a lower semicontinuous submeasure μ such that

$$\mathcal{I} = \{A \subseteq \mathbb{N} \; : \; \mu(A) < \infty\}$$

and $f = \mu$ satisfies all the conditions above. For (2) let $f_{\mathcal{J}}$ be a map witnessing that \mathcal{J} is layered, and define f by

$$f(A) = f_{\mathcal{J}}(\{n : A_n \notin \mathcal{I}\})$$

for $A \subseteq \mathbb{N}^2$. It is not hard to see that f satisfies the conditions (1) - (3) stated above. \Box

Theorem 4.3 (Farah-Shelah). Every reduced product $\prod_n A_n / \bigoplus_{\mathcal{I}} A_n$ is countably saturated if \mathcal{I} is a layered ideal.

Proof. See [15, Theorem 2.7].

Therefore an immediate consequence of Theorem 1.1 and Theorem 4.3 implies the following corollary.

Corollary 4.4. Assume the Continuum Hypothesis. If \mathcal{A} is a separable metric structure, \mathcal{I} and \mathcal{J} are atomless layered ideals, then the reduced powers $\mathcal{A}_{\mathcal{I}}$ and $\mathcal{A}_{\mathcal{J}}$ are isomorphic.

In corollary 4.6 we give an application of this result, but before we need the following lemma.

Lemma 4.5. Suppose \mathcal{I} and \mathcal{J} are ideals on \mathbb{N} and $\mathcal{A}_{\mathcal{I}}$ is the reduced power of \mathcal{A} over the ideal \mathcal{I} . Then

$$\frac{\prod \mathcal{A}_{\mathcal{I}}}{\bigoplus_{\mathcal{J}} \mathcal{A}_{\mathcal{I}}} \cong \frac{\prod_{\mathbb{N}^2} \mathcal{A}}{\bigoplus_{\mathcal{J} \times \mathcal{I}} \mathcal{A}}.$$

Proof. Assume $\langle a_{n,m} \rangle$ is an element of $\prod_{\mathbb{N}^2} \mathcal{A}$. Define the map $\rho : \prod_{\mathbb{N}^2} \mathcal{A} / \bigoplus_{\mathcal{J} \times \mathcal{I}} \mathcal{A} \to \prod(\mathcal{A}_{\mathcal{I}}) / \bigoplus_{\mathcal{J}} (\mathcal{A}_{\mathcal{I}})$ by

$$\rho(\pi_{\mathcal{J}\times\mathcal{I}}(\langle a_{m,n}\rangle)) = \pi_{\mathcal{J}}(\langle b_m\rangle),$$

where $b_m = \pi_{\mathcal{I}}(\langle a_{m,n} \rangle_n)$ for each $m \in \mathbb{N}$. In order to see this map is well-defined assume $\pi_{\mathcal{J} \times \mathcal{I}}(\langle a_{m,n} \rangle) = 0$. If $\pi_{\mathcal{J}}(\langle b_m \rangle) \neq 0$, then there is $\epsilon > 0$ such that for every $S \in \mathcal{J}$ we have

$$\sup_{m \notin S} \|b_m\|_{\mathcal{A}_{\mathcal{I}}} \ge \epsilon.$$

Since $\pi_{\mathcal{J}\times\mathcal{I}}(\langle a_{n,m}\rangle) = 0$, there is $X \in \mathcal{J}\times\mathcal{I}$ such that

$$\sup_{(m,n)\notin X} \|\langle a_{m,n}\rangle)\|_{\mathcal{A}} < \epsilon/4.$$

The set $S = \{m : X_m \notin \mathcal{I}\}$ belongs to \mathcal{J} and hence $\sup_{m\notin S} \|b_m\|_{\mathcal{A}_{\mathcal{I}}} \geq \epsilon$. Pick $m_0 \notin S$ such that $\|b_{m_0}\|_{\mathcal{A}_{\mathcal{I}}} \geq \epsilon/2$ and then pick $n_0 \notin X_{m_0}$ such that $\|a_{m_0,n_0}\|_{\mathcal{A}} \geq \epsilon/4$, which is a contradiction. Therefore $\pi_{\mathcal{I}}(\langle b_m \rangle) = 0$.

To show the injectivity of ρ assume $\pi_{\mathcal{J}}(\langle b_m \rangle) = 0$. Therefore for every $\epsilon > 0$ there is $S \in \mathcal{J}$ such that $\|b_m\|_{\mathcal{A}_{\mathcal{I}}} \leq \epsilon$ for every $m \in \mathbb{N} \setminus S$. So for each $m \in \mathbb{N} \setminus S$ there is $X_m \in \mathcal{I}$ such that

$$\sup_{n \notin X_m} \|a_{(m,n)}\|_{\mathcal{A}} \le 2\epsilon.$$

The set $X = (S \times \mathbb{N}) \cup \{(m, n) : n \in X_m\}$ belongs to the ideal $\mathcal{J} \times \mathcal{I}$ and

$$\sup_{(m,n)\notin X} \|a_{m,n}\|_{\mathcal{A}} \le 2\epsilon.$$

Therefore $\pi_{\mathcal{T}\times\mathcal{I}}(\langle a_{n,m}\rangle) = 0$. It is easy to check that ρ is a surjective *-homomorphism. \Box

The following corollary follows form Lemma 4.2 and Corollary 4.4.

Corollary 4.6. Assume the Continuum Hypothesis. Suppose $\mathfrak{A} = \ell_{\infty}(\mathcal{A})/c_0(\mathcal{A})$ is the asymptotic sequence algebra of \mathcal{A} and \mathcal{I} is an atomless layered ideal on \mathbb{N} , then

$$\frac{\prod\mathfrak{A}}{\bigoplus_{\mathcal{I}}\mathfrak{A}}\cong\mathfrak{A}$$

5. Non-trivially isomorphic reduced products of matrix algebras.

In this section we use Theorem 3.3 in order to prove the existence of two reduced products of matrix algebras which are isomorphic under the Continuum Hypothesis, but not isomorphic in ZFC. Note that in model theory for operator algebras, the ranges of formulas are bounded subsets of reals possibly different from [0, 1] (see for example [13]). Nevertheless Definition 3.1 can be easily adjusted for any formula in the language of C*-algebras \mathcal{L} and Theorem 3.3 can be proved similarly.

As mentioned in the introduction, commutative examples of such reduced products are well-known, for example by a classical result of Parovičenko, under the Continuum Hypothesis $(\ell_{\infty}(\mathbb{N})/c_0(\mathbb{N}) \cong) C(\beta \omega \setminus \omega) \cong C(\beta \omega^2 \setminus \omega^2)$, however under the proper forcing axiom they are not isomorphic, since there are no trivial isomorphisms between them (see [7, Chapter 4]). Other examples of non-trivial isomorphisms between (non-commutative) reduced products can be obtained by tensoring a matrix algebra with these commutative algebras. Recall that ([3]) for a locally compact Hausdorff topological space X and for any C*- algebra \mathcal{A} , $C_0(X, \mathcal{A})$ can be identified with $C_0(X) \otimes \mathcal{A}$, under the map $(f \otimes a)(x) =$ f(x)a. Let M_n denote the space of all $n \times n$ matrices over the field of complex numbers. Assume $\mathfrak{A} = \ell_{\infty}(M_2)/c_0(M_2)$ is the asymptotic sequence algebra of M_2 , we have

$$\mathfrak{A} \cong C(\beta \omega \setminus \omega) \otimes M_2 \cong M_2(C(\beta \omega \setminus \omega)),$$

and Corollary 4.6 implies that

$$\frac{\ell_{\infty}(\mathfrak{A})}{c_0(\mathfrak{A})} \cong \frac{\prod_{\mathbb{N}^2} M_2}{\bigoplus_{Fin \times Fin} M_2} \equiv \mathfrak{A}.$$

Since $\prod_{\mathbb{N}^2} M_2 / \bigoplus_{Fin \times Fin} M_2 \cong M_2(C(\beta \omega^2 \setminus \omega^2))$, for the same reason as the commutative case, under the Continuum Hypothesis \mathfrak{A} and $\ell_{\infty}(\mathfrak{A})/c_0(\mathfrak{A})$ are non-trivially isomorphic.

Recall that an ideal \mathcal{I} on \mathbb{N} is a P-ideal if for every sequence $\{A_n\}_{n=1}^{\infty}$ of sets in \mathcal{I} there is a set $A_{\infty} \in \mathcal{I}$ such that $A_n \setminus A_{\infty}$ is finite, for all n.

Theorem 5.1. [18, Corollary 1.1] Assume there is a measurable cardinal. It is relatively consistent with ZFC that for every analytic P-ideal \mathcal{I} on \mathbb{N} , the reduced products $\prod M_{k(n)} / \bigoplus_{\mathcal{I}} M_{k(n)}$ and $\prod M_{l(n)} / \bigoplus_{\mathcal{I}} M_{l(n)}$ are isomorphic if and only if there are sets $A, B \in \mathcal{I}$ and a bijection $\nu : \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ such that $k(\nu(n)) = l(n)$ for all $n \in \mathbb{N} \setminus A$.

Moreover if Φ is such an isomorphism, there exists a bounded linear operator u: $\prod_{n \in \mathbb{N}} M_{k(n)} \to \prod_{n \in \mathbb{N}} M_{l(n)}$ such that $\pi_{\mathcal{I}}(u)$ is a unitary and $\Phi(\pi_{\mathcal{I}}(a)) = \pi_{\mathcal{I}}(uau^*)$.

Such isomorphisms are trivial in the sense of [14] and in the sense of Definition 2.1 if $\mathcal{I} = \mathcal{J} = Fin.$

Lemma 5.2. There is an increasing sequence of natural numbers $\{k_{\infty}(i) : i \in \mathbb{N}\}$ such that for every \mathcal{F}_0 -restricted \mathcal{L} -sentence ψ

$$\lim_{i \to \infty} \psi^{M_{k_{\infty}(i)}} = r_{\psi}$$

for some real number r_{ψ} .

Proof. Let ψ_1, ψ_2, \ldots be an enumeration of all \mathcal{F}_0 -restricted \mathcal{L} -sentences. Starting with ψ_1 , since the range of it is a bounded set, find a sequence $\{k_1(i)\}$ such that $\psi_1^{M_{k_1(i)}} \to r_{\psi_1}$ for some r_{ψ_1} . Similarly find a subsequence $\{k_2(i)\}$ of $\{k_1(i)\}$ such that $\psi_2^{M_{k_2(i)}} \to r_{\psi_2}$ for some r_{ψ_2} , and so on. If we let

$$k_{\infty}(i) = k_i(i) \qquad i \in \mathbb{N},$$

then $\{k_{\infty}(i)\}$ has the required property.

Proposition 5.3. For any ideal \mathcal{I} on \mathbb{N} containing all finite sets, if $\{g(i)\}$ and $\{h(i)\}$ are two almost disjoint subsequences of $\{k_{\infty}(i)\}$, then

$$\frac{\prod_i M_{g(i)}}{\bigoplus_{\mathcal{I}} M_{g(i)}} \equiv \frac{\prod_i M_{h(i)}}{\bigoplus_{\mathcal{I}} M_{h(i)}},$$

hence if \mathcal{I} is a layered P-ideal, they are isomorphic under the Continuum Hypothesis, with no trivial isomorphisms between them.

This together with Theorem 5.1 implies that these reduced products are not isomorphic in ZFC, and therefore Theorem 1.2 follows.

Proof. Let φ be an \mathcal{L} -sentence and for $n \geq 1$ find an \mathcal{F}_0 -restricted \mathcal{L} -sentence $\tilde{\varphi}$ which is uniformly within 2^{-n} of φ and it is determined up to 2^{-n} by $(\sigma_0, \ldots, \sigma_{2^n}; \psi_0, \ldots, \psi_{m-1})$. Let

$$X_{i}^{j} = \{l \in \mathbb{N} : \psi_{j}^{M_{g(l)}} > i/2^{n}\}$$

and

$$Y_i^j = \{l \in \mathbb{N} : \psi_j^{M_{h(l)}} > i/2^n\}.$$

Since each ψ_j is an \mathcal{F}_0 -restricted formula and $\lim_l \psi_j^{M_{k_\infty(l)}} = r_{\psi_j}$ and \mathcal{I} contains all finite sets, we have $[X_i^j]_{\mathcal{I}} = [Y_i^j]_{\mathcal{I}}$. Hence Theorem 3.3 implies that $\tilde{\varphi}^{\mathcal{A}_{\mathcal{I}}} = \tilde{\varphi}^{\mathcal{B}_{\mathcal{I}}}$. By uniform density of \mathcal{F}_0 -restricted \mathcal{L} -sentences, the result follows.

The following theorem shows the abundance of different theories of reduced products of matrix algebras.

Theorem 5.4. For any ideal \mathcal{I} , there are 2^{\aleph_0} -many reduced products of matrix algebras over \mathcal{I} which are pairwise non-elementarily equivalent.

Proof. Let $E = \{p_1, p_2, ...\} \subset \mathbb{N}$ be an increasing enumeration of prime numbers. Assume $\{A_{\xi} : \xi < 2^{\aleph_0}\}$ is an almost disjoint family of subsets of E. Let $A_{\xi} = \{n_1^{\xi}, n_2^{\xi}, ...\}$ be an increasing enumeration of A_{ξ} . For each $\xi < 2^{\aleph_0}$ define a sequence $\langle k^{\xi}(n) \rangle$ of natural numbers by

$$\langle k^{\xi}(n) \rangle = \langle n_{1}^{\xi}, n_{1}^{\xi} n_{2}^{\xi}, n_{1}^{\xi} n_{2}^{\xi} n_{3}^{\xi}, \dots \rangle.$$

We will show that for any distinct $\xi, \eta < 2^{\aleph_0}$ the reduced products $\prod M_{k^{\xi}(n)} / \bigoplus_{\mathcal{I}} M_{k^{\xi}(n)}$ and $\prod M_{k^{\eta}(n)} / \bigoplus_{\mathcal{I}} M_{k^{\eta}(n)}$ are not elementarily equivalent.

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Fix such ξ and η . Since A_{ξ} and A_{η} are almost disjoint, pick m such that $\{n_m^{\xi}, n_{m+1}^{\xi}, \ldots\} \cap \{n_1^{\eta}, n_2^{\eta}, \ldots\} = \emptyset$. Define a formula $\varphi(\bar{x}, \bar{y})$ by

$$\varphi(x_1, \dots, x_{n_m^{\xi}}, y_2, \dots, y_{n_m^{\xi}}) = \sum_{i=1}^{n_m^{\xi}} (\|x_i^2 - x_i\| + \|x_i^* - x_i\|) + \|\sum_{i=1}^{n_m^{\xi}} x_i - 1\| + \sum_{i \neq j}^{n_m^{\xi}} \|x_i x_j\| + \sum_{i=2}^{n_m^{\xi}} (\|y_i y_i^* - x_1\| + \|y_i^* y_i - x_i\|)$$

For a unital C*-algebra \mathcal{A} and tuples \bar{a} and \bar{v} if $\varphi(\bar{a}, \bar{v})^{\mathcal{A}} = 0$ then $a_1, a_2, \ldots, a_{n_m^{\xi}}$ are orthogonal pairwise Murray-von Neumann equivalent projections of \mathcal{A} , and therefore $M_{n_m^{\xi}}$ can be embedded into \mathcal{A} . Since for every $k \geq m$

$$n_m^{\xi} \mid n_1^{\xi} n_2^{\xi} \dots n_k^{\xi}$$

one can easily find a tuple of projections \bar{a} and a tuple of partial isometries \bar{v} in $\prod M_{k^{\xi}(n)}$ such that

$$\varphi(\pi_{\mathcal{I}}(\bar{a}),\pi_{\mathcal{I}}(\bar{v}))^{\prod M_{k^{\xi}(n)}/\bigoplus_{\mathcal{I}} M_{k^{\xi}(n)}} = 0,$$

but it is not possible to find such projections and partial isometries in $\prod M_{k^{\eta}(n)}$ since

 $n_m^{\xi} \nmid n_1^{\eta} n_2^{\eta} \dots n_k^{\eta}$

for every $k \in \mathbb{N}$. Hence $\prod M_{k^{\xi}(n)} / \bigoplus_{\mathcal{I}} M_{k^{\xi}(n)} \not\equiv \prod M_{k^{\eta}(n)} / \bigoplus_{\mathcal{I}} M_{k^{\eta}(n)}$.

6. FURTHER REMARKS AND QUESTIONS

For a locally compact Hausdorff topological space X and a metric structure \mathcal{A} the continuous reduced products $C_b(X, \mathcal{A})/C_0(X, \mathcal{A})$ are studied as models for metric structures in [15], where in particular it has been shown that certain continuous reduced products, e.g., $C([0, 1)^*)$, are countably saturated. In general $C_b(X, \mathcal{A})$ is a submodel of $\prod_{t \in X} \mathcal{A}$ and one may hope to use a similar approach as in section 4 in order to prove the following preservation (of \equiv) question.

Question (1). Assume \mathcal{A} and \mathcal{B} are elementarily equivalent metric structures and X is a locally compact, non-compact Polish space. Are $C_b(X, \mathcal{A})/C_0(X, \mathcal{A})$ and $C_b(X, \mathcal{B})/C_0(X, \mathcal{B})$ elementarily equivalent?

Note that if X is a discrete space (e.g., \mathbb{N}) this follows from Proposition 3.6 since $C_b(X, \mathcal{A})/C_0(X, \mathcal{A}) \cong \prod_{t \in X} \mathcal{A}/\bigoplus \mathcal{A}$.

In [12] the authors showed the existence of two C*-algebras \mathcal{A} and \mathcal{B} such that $\mathcal{A} \equiv \mathcal{B}$, where $C([0,1]) \otimes \mathcal{A} \not\equiv C([0,1]) \otimes \mathcal{B}$, i.e., tensor products in the category of C*-algebras, do not preserve elementary equivalence. In general it is not clear "how the theory of $C_0(X, \mathcal{A})$ is related to the theory of \mathcal{A} ".

Question (2). Assume \mathcal{A} and \mathcal{B} are elementarily equivalent C*-algebras. For which locally compact, Hausdorff spaces, like $X, C_0(X) \otimes \mathcal{A} \equiv C_0(X) \otimes \mathcal{B}$ is true?

We conclude with an interesting observation which might give some insight to the previous question.

The Cone and Suspension Algebras. Let \mathcal{A} be a C*-algebra. The cone $C\mathcal{A} = C_0((0, 1], \mathcal{A})$ and suspension $S\mathcal{A} = C_0((0, 1), \mathcal{A})$ over \mathcal{A} are the most important examples of contractible and subcontractible C*-algebras ([3]). Since $S\mathcal{A} \subset C\mathcal{A}$ and $C\mathcal{A}$ is homotopic to $\{0\}$ (contractible) by a well-known result of D. Voiculescu ([25]) both $C\mathcal{A}$ and $S\mathcal{A}$ are quasidiagonal C*-algebras. Every quasidiagonal C*-algebra embeds into a reduced product of full matrix algebras over the Fréchet ideal, $\prod M_{k(n)} / \bigoplus_{\mathcal{I}} M_{k(n)}$ for some sequence $\{k(n)\}$ (such C*-algebras are called MF, see e.g., [4] and [3]). In general it is easy to check that if a metric structure \mathcal{A} embeds into \mathcal{B} then the universal theory of \mathcal{A} , $Th_{\forall}(\mathcal{A})$, contains the universal theory of \mathcal{B} , $Th_{\forall}(\mathcal{B})$. Hence for any C*-algebra \mathcal{A}

$$C\mathcal{A} \hookrightarrow \prod M_{k(n)} / \bigoplus M_{k(n)}$$

for some $\{k(n)\}$, which implies that

$$Th_{\forall}(C\mathcal{A}) \supseteq Th_{\forall}(\prod M_{k(n)} / \bigoplus M_{k(n)}).$$

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