# ON DECIDABLE ALGEBRAIC FIELDS 

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#### Abstract

We prove the following propositions. Theorem 1: Let $M$ be a subfield of a fixed algebraic closure $\mathbb{\mathbb { Q }}$ of $\mathbb{Q}$ whose existential elementary theory is decidable (resp. primitively decidable). Then, M is conjugate to a recursive (resp. primitive recursive) subfield $L \subset \tilde{\mathbb{Q}}$.

Theorem 2: For each positive integer $e$ there are infinitely many $e$-tuples $\sigma \in \operatorname{Gal}(\mathbb{Q})^{e}$ such that the field $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ - the fixed field of $\boldsymbol{\sigma}$, is recursive in $\tilde{\mathbb{Q}}$ and its elementary theory is decidable. Moreover, $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ is PAC and $\operatorname{Gal}(\tilde{\mathbb{Q}}(\boldsymbol{\sigma}))$ is isomorphic to the free profinite group on $e$ generators.


## 1. Introduction

The main theme of this work is the interplay between decidability of large algebraic extensions of $\mathbb{Q}$ and their recursiveness in a fixed algebraic closure $\tilde{\mathbb{Q}}$ of $\mathbb{Q}$. One of the main results of [JaK75] gives for each positive integer $e$ a recursive procedure to decide whether a sentence $\theta$ in the language of rings is true in the field $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ for all $\boldsymbol{\sigma} \in \operatorname{Gal}(\mathbb{Q})^{e}$ outside a set of Haar measure zero (see also [FrJ08, p. 442, Thm. 20.6.7]). Here, $\operatorname{Gal}(\mathbb{Q})=\operatorname{Gal}(\tilde{\mathbb{Q}} / \mathbb{Q})$ is the absolute Galois group of $\mathbb{Q}$, and for each $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{e}\right) \in \operatorname{Gal}(\mathbb{Q})^{e}, \tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ is the fixed field of $\sigma_{1}, \ldots, \sigma_{e}$ in $\widetilde{\mathbb{Q}}$. The results of [FHJ84] even give a primitive recursive procedure for the same decision problem (see also [FrJ08, p. 722, Thm. 30.6.1]).

Note that the above procedures give no information about individual fields of the form $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$. Indeed, by Proposition4.1, there are uncountably many elementary equivalence classes of fields $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$. On the other hand, since the language of rings is countable, there are at most countably many decision procedures. Hence, all but at most countably many fields of the form $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ are undecidable.

Another question that one may ask in this context is about the relation between an individual field $\widetilde{\mathbb{Q}}(\boldsymbol{\sigma})$ and $\widetilde{\mathbb{Q}}$. To this end we recall that one may order the elements of $\widetilde{\mathbb{Q}}$ in a primitive recursive sequence and give a primitive recursive procedure to carry out the field theoretic operations among the elements of that sequence. It therefore makes sense to ask about a subfield $M$ of $\tilde{\mathbb{Q}}$ whether $M$ is a recursive subset of $\tilde{\mathbb{Q}}$ (in which case $M$ is also a recursive (or a computable) subfield of $\widetilde{\mathbb{Q}})$.

Usually, this is not the case, because $\tilde{\mathbb{Q}}$ has only countably many recursive subsets. Even if the elementary theory of $M$ is decidable, it may happen that $M$ has uncountably many conjugates (Example 2.9, when $M$ is a real or a $p$-adic closure of $\mathbb{Q}$ ). The elementary theory of each of them is the same as that of $M$, so is also decidable. But only countably many of them are recursive in ©

We shed light on these problems by proving two results:

[^0]Theorem 1. Let $M$ be a subfield of $\tilde{\mathbb{Q}}$ whose existential elementary theory is decidable (resp. primitively decidable). Then, $M$ is conjugate to a recursive (resp. primitive recursive) subfield $L$ of $\widetilde{\mathbb{Q}}$.

In view of this theorem, given a subfield of $\tilde{\mathbb{Q}}$ with undecidable existential (or elementary) theory in the language of rings, one can distinguish between two cases. The theory can be undecidable because the field has no computable conjugate within the given copy of $\tilde{\mathbb{Q}}$ or the theory can be undecidable for a different arithmetic reason. In the first case it is tempting to say that the theory is trivially undecidable. A simple example of a field with a trivially undecidable existential theory is a Galois extension of $\mathbb{Q}$ which is not recursive as a subset of $\tilde{\mathbb{Q}}$.

Theorem 2. For each positive integer e there are infinitely many e-tuples $\boldsymbol{\sigma} \in \operatorname{Gal}(\mathbb{Q})^{e}$ such that the field $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ is recursive in $\tilde{\mathbb{Q}}$ and its elementary theory is decidable. Moreover, $\tilde{\mathbb{Q}}(\boldsymbol{\sigma})$ is PAC and $\operatorname{Gal}(\tilde{\mathbb{Q}}(\boldsymbol{\sigma}))$ is isomorphic to the free profinite group on e generators.

Both theorems make sense, because we can list the elements of $\tilde{\mathbb{Q}}$ in a primitive recursive sequence. The proof of Theorem $\square$ depends on our ability to perform the basic field theoretic operations including the factorization of polynomials over given number fields and even over $\tilde{\mathbb{Q}}$ in a primitive recursive way. The proof of Theorem 2 uses in addition a recursive (but not primitive recursive) version of Hilbert irreducibility theorem (Lemma3.2).

All of these operations can be carried out over each given finitely generated field (over its prime field). In the terminology of [FrJ08, Chap. 19], these fields have the "elimination theory". So, actually we prove Theorems 1 and 2 for fields with elimination theory. In particular, they hold for finitely generated fields.

Theorem 13.3.5 and Proposition 13.2.1 of [FrJ08] state that every infinite finitely generated field $K$ is Hilbertian. An analysis of their proofs seems to show that the procedure to find a point in a given Hilbertian subset of $K^{r}$ is primitive recursive. If this is true, we may strengthen Theorem B and replace "recursive" and "decidable" in that theorem by "primitive recursive" and "primitively decidable", respectively. However, carrying out this check in the present work will take us away from its main topic. So, we don't do it here.

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## 2. Recursive Subfields of $\tilde{K}$

We consider a presented field $K$ with elimination theory in the sense of [FrJ08, p. 410, Def. 19.2.8]. This is a field which is explicitly constructed from the ring $\mathbb{Z}$ of integers, one has "effective recipes" to add and multiply given elements and to "effectively compute" the inverse of each given non-zero element. In particular, $K$ is countable. Most important, there is an "effective algorithm" to factor each given non-zero polynomial $f \in K[Z]$ into a product of irreducible polynomials. Moreover, it is possible to "effectively adjoin" a root $z$ of $f$ to $K$. Each element $z^{\prime}$ of $K(z)$ is then uniquely given as a sum $\sum_{i=0}^{d-1} a_{i} z^{i}$ with $d=\operatorname{deg}(f)$ and $a_{0}, \ldots, a_{d-1} \in K$, and one can effectively compute $\operatorname{irr}\left(z^{\prime}, K\right)$. The field $K(z)$ has again the elimination theory. An effective version of the primitive element theorem is also true, that is if $z_{1}, \ldots, z_{n}$ are roots of given irreducible
separable polynomials $f_{1}, \ldots, f_{n} \in K[Z]$, respectively, then one can "effectively compute" an irreducible separable polynomial $f \in K[Z]$ and a root $z$ of $K$ such that $K\left(z_{1}, \ldots, z_{n}\right)=K(z)$. Moreover, one can "effectively present" $z$ as a linear combination of $z_{1}, \ldots, z_{n}$ with coefficients in $K$ and "effectively present" each $z_{i}$ as a polynomial in $z$ with coefficients in $K$.

All of these notions and algorithms are rigorously defined, explained, and proved in [Fr]08, Sections 19.1 and 19.2]. Moreover, it is proved there that the above algorithms are primitive recursive in the usual sense (e.g. as defined in [FrJ08, Sec. 8.4]). In this case we also say that the above algorithms are effective. It is further proved in [FrJ08, Sec. 19.4] that both the separable closure $K_{s}$ and the algebraic closure $\tilde{K}$ of $K$ can be presented, and then have elimination theory.

Having done so, we say that a subfield $M$ of $\tilde{K}$ is recursive (resp. primitive recursive), if $M$ is a recursive (resp. primitive recursive) subset of $\tilde{K}$ (e.g. [FrJ08, Sections 8.4 and 8.5 , where in each case one examines the characteristic function of the subset]). Since addition, multiplication, and taking inverse of elements of $\tilde{K}$ are primitive recursive, these operations in $M$ are also recursive (resp. primitive recursive).

Independently of the question whether $M$ is a recursive subfield of $\tilde{K}$ or not, we may consider the language $\mathscr{L}$ (ring, $K$ ) of rings with a constant symbol for each element of $K$. An existential sentence in $\mathscr{L}$ (ring, $K$ ) is a sentence which is equivalent to a sentence of the form $\left(\exists X_{1}\right) \cdots\left(\exists X_{n}\right)\left[\bigvee_{i} \wedge_{j} f_{i j}\left(X_{1}, \ldots, X_{n}\right)=0\right]$ [FrJ08, p. 462]. Let $\operatorname{Ex}(M)$ be the set of all existential sentences in $\mathscr{L}$ (ring, $K$ ) which are true in $M$. This is a subset of the elementary theory $\operatorname{Th}(M)$ consisting of all sentences of $\mathscr{L}$ (ring, $K$ ) that hold in $M$. We say that $\operatorname{Ex}(M)$ is decidable (resp. primitively decidable) if there exists an algorithm (resp. primitive recursive algorithm) to decide whether a given existential sentence of $\mathscr{L}$ (ring, $K$ ) holds in $M$ or not.

The procedures we describe below use verbs like "construct", "find", "compute", etc. When these terms are preceded by the adverb "effectively", then the corresponding parts of the procedures are primitive recursive.

By definition, each primitive recursive subset of $\tilde{K}$ is recursive. Similarly, if $\operatorname{Th}(M)(\operatorname{resp} . \operatorname{Ex}(M))$ is primitively decidable, then $\operatorname{Th}(M)$ (resp. $\operatorname{Ex}(M)$ ) is decidable. (See also a comparison in [FrJ08, pp.,159-150, Sec. 8.6] between recursive and primitive recursive decidability procedures.)

Note that, in the cases we consider, $\operatorname{Th}(M)$ involves only constant symbols of $K$ but not of $\tilde{K} \backslash K$. Thus, the question whether $M$ is a recursive (respectively, primitive recursive) subfield of $\tilde{K}$ is independent of the question whether $\operatorname{Th}(M)$ is decidable (resp. primitively recursiv). Indeed, even if $\operatorname{Th}(M)$ is decidable, $M$ may have uncountably many $K$-conjugates in $\tilde{K}$ (Example 2.9). Since $K$ is countable, so is the language $\mathscr{L}$ (ring, $K$ ). Hence, $\tilde{K}$ has only countably many recursive subfields. It follow that at most countably many of the $K$-conjugates of $M$ in $\tilde{K}$ are recursive in $\tilde{K}$. All the others are non-recursive in $\tilde{K}$.

However, we prove in this section that if $M$ is a field extension of $K$ in $\tilde{K}$ and $\operatorname{Ex}(M)$ is decidable (resp. recursively decidable), then $M$ has a $K$-conjugate $L$ which is recursive (resp. primitive recursive) in $\tilde{K}$. In particular, $\operatorname{Th}(L)=\operatorname{Th}(M)$, hence $\operatorname{Ex}(L)=\operatorname{Ex}(M)$, so $\operatorname{Ex}(L)$ is in addition decidable (resp. primitively decidable).
Remark 2.1. Let $K$ be a field, $p \in K[Z]$, and $\phi: L \rightarrow L^{\prime}$ a $K$-isomorphism of subfields of $\tilde{K}$. Denote the set of roots of $p$ in $\tilde{K}$ by $P$. Then, $\phi(P \cap L) \subset P \cap L^{\prime}$ and $\phi^{-1}\left(P \cap L^{\prime}\right) \subset P \cap L$. Therefore, $\phi(P \cap L)=P \cap L^{\prime}$.
Lemma 2.2. Let $M$ be a subfield of $\tilde{K}$ that contains $K$ such that $E x(M)$ is decidable (resp. recursively decidable). Suppose we are given
(a1) a finite separable extension $L$ of $K$ and a $K$-embedding of $L$ into $M$, and
(a2) a monic separable polynomial p in $K[Z]$.
Let P be the set of roots of $p$ in $K_{s}$. Then,
(b) we can determine (resp. effectively determine) whether there exists a $K$-embedding of $L(z)$ into $M$ with $z \in P \backslash L$;

Proof. We effectively decompose $p(Z)$ into a product of monic irreducible factors over $L$,

$$
p(Z)=\left(Z-a_{1}\right) \cdots\left(Z-a_{l}\right) h_{1}(Z) \cdots h_{m}(Z) .
$$

such that $a_{1}, \ldots, a_{l} \in L$ and $\operatorname{deg}\left(h_{i}\right) \geq 2$ for $i=1, \ldots, m$ [FrJ08, p. 407, Lemma 19.2.2]. If $m=0$, then $P \subset L$, so there is no embedding of $L(z)$ into $M$ with $z \in P \backslash L$.

Otherwise, we effectively construct a primitive element $y$ for $L / K$, effectively compute $f=$ $\operatorname{irr}(y, K)$, and set $d=\operatorname{deg}(f)$. For each $1 \leq i \leq m$ we set $d_{i}=\operatorname{deg}\left(h_{i}\right)$. Then, we effectively compute for each $0 \leq j \leq d_{i}$ the unique polynomial $g_{i j}$ in $K[Y]$ of degree at most $d-1$ such that $h_{i}(Z)=\sum_{j=0}^{d_{i}} g_{i j}(y) Z^{j}$. We set $g_{i}(Y, Z)=\sum_{j=0}^{d_{i}} g_{i j}(Y) Z^{j}$ and observe that $g_{i} \in K[Y, Z]$ and $g_{i}(y, Z)=h_{i}(Z)$. Then, we denote the existential sentence

$$
(\exists Y)(\exists Z)\left[f(Y)=0 \wedge g_{i}(Y, Z)=0\right]
$$

of $\mathscr{L}$ (ring, $K$ ) by $\theta_{i}$.
Since the existential theory of $M$ in the language $\mathscr{L}$ (ring, $K$ ) is decidable (resp. primitively decidable), we may check (resp. effectively check) the truth of each $\theta_{i}$ in $M$. If none of the sentences $\theta_{1}, \ldots, \theta_{m}$ is true in $M$, then there exists no $K$-embedding $\phi^{\prime}$ of $L(z)$ into $M$ with $z \in$ $P \backslash L$.

Indeed, if such $z$ and $\phi^{\prime}$ exist, we write $y^{\prime}=\phi^{\prime}(y)$ and $z^{\prime}=\phi^{\prime}(z)$. Then, $z \in P \backslash\left\{a_{1}, \ldots, a_{l}\right\}$, so there exists $1 \leq i \leq m$ with $h_{i}(z)=0$. Hence, $g_{i}(y, z)=0$. Applying $\phi^{\prime}$, we see that $f\left(y^{\prime}\right)=0$ and $g_{i}\left(y^{\prime}, z^{\prime}\right)=0$, with $y^{\prime}, z^{\prime} \in M$. Thus, $\theta_{i}$ holds in $M$, in contrast to our assumption.

Finally suppose that one of the sentences $\theta_{1}, \ldots, \theta_{m}$, say $\theta_{1}$, is true in $M$. Thus, there exist $y^{\prime}, z^{\prime} \in M$ with $f\left(y^{\prime}\right)=0$ and $g_{1}\left(y^{\prime}, z^{\prime}\right)=0$. Since $f$ is irreducible over $K$, we may extend (resp. effectively extend) the map $y \rightarrow y^{\prime}$ to a $K$-isomorphism $\phi_{1}^{\prime}$ of $L=K(y)$ onto $K\left(y^{\prime}\right)$. Since $g_{i}(y, Z)=h_{i}(Z)$ is irreducible over $K(y)=L$, the polynomial $g_{i}\left(y^{\prime}, Z\right)$ is irreducible over $K\left(y^{\prime}\right)$. Since $z^{\prime}$ is a root of the latter polynomial, we may find (resp. effectively find) a root $z$ of $g_{i}(y, Z)$ in $K_{s}$ and conclude that the isomorphism $\left(\phi_{1}^{\prime}\right)^{-1}: K\left(y^{\prime}\right) \rightarrow K(y)$ extends to an isomorphism $K\left(y^{\prime}, z^{\prime}\right) \rightarrow K(y, z)$ that maps $z^{\prime}$ onto $z$. In particular, $h_{1}(z)=g_{1}(y, z)=0$, so $z \in P \backslash L$. Then, the inverse isomorphism $\phi^{\prime}: K(y, z) \rightarrow K\left(y^{\prime}, z^{\prime}\right)$ of the latter isomorphism is the desired one.

Remark 2.3. Note that the $K$-embedding $\phi^{\prime}: L(z) \rightarrow M$ constructed in the last paragraph of the latter proof does not necessarily extend the $K$-embedding $\phi: L \rightarrow M$.
Lemma 2.4. Let $M$ be a subfield of $\tilde{K}$ that contains $K$ such that $E x(M)$ is decidable (resp. recursively decidable). Suppose we are given
(a1) a finite separable extension $L$ of $K$ and a $K$-embedding $\phi: L \rightarrow M$ and
(a2) a monic separable polynomial $p$ in $K[Z]$.
Let $P$ be the set of roots of $p$ in $K_{s}$. Then, we can find (resp. effectively find) a subset I of P such that there exists a $K$-embedding $\psi: L(I) \rightarrow M$ with the property that $I=P \cap L(I)$ and $\psi(I)=P \cap M$.

Proof. The assumptions of our lemma coincide with the assumptions of Lemma 2.2, so the conclusion of that lemma holds in our situation. We set $L^{\prime}=\phi(L)$ and consider the two possible cases.

Case A: There is no $K$-embedding $L(z) \rightarrow M$ with $z \in P \backslash L$ Then, we set $I=P \cap L$ and $\psi=\phi$. Hence, $L(I)=L$, so $I=P \cap L(I)$. By Remark 2.1, $\psi(P \cap L)=P \cap L^{\prime}$.

Note that $\psi(I)=\psi(P \cap L) \subset P \cap M$. If there exists $z^{\prime} \in P \cap M \backslash \psi(I)$, then by the preceding paragraph, $z^{\prime} \in M \backslash L^{\prime}$. Thus, there exists $z \in K_{s}$ and an extension of $\psi$ to an isomorphism $\psi^{\prime}: L(z) \rightarrow L^{\prime}\left(z^{\prime}\right)$ such that $\psi^{\prime}(z)=z^{\prime}$. Hence, $z \in P \backslash L$, in contrast to our assumption.

It follows from this contradiction that $\psi(I)=P \cap M$.
Case B: Case A does not occur Using Lemma 2.2, we find (resp. effectively find) $z \in P \backslash L$ and construct (effectively construct) a $K$-embedding $\phi^{\prime}$ of $L(z)$ into $M$. Then, $|P \backslash L(z)|<\mid P \backslash$ $L \mid$. By induction, we may find (resp. effectively find) a subset $I$ of $P$ and a $K$-embedding $\psi$ of $L(z, I)$ into $M$ such that $I=P \cap L(z, I)$ and $\psi(I)=P \cap M$. In particular, since $z \in P$, we have $z \in I$. Thus, $L(I, z)=L(I)$ and $I=P \cap L(I)$.

Lemma 2.5. Let $M$ be a subfield of $\tilde{K}$ that contains $K$, let $p \in K[Z]$ be a monic separable polynomial, and let $P$ be the set of all roots of $p$ in $K_{s}$. Consider an extension $L$ of $K$ in $K_{s}$ and let $\phi$ and $\phi^{\prime}$ be $K$-embeddings of $L$ into $M$. Suppose that $\phi(P \cap L)=P \cap M$. Then, $\phi^{\prime}(P \cap L)=P \cap M$.

Proof. We set $M_{0}=\phi(L)$ and $M_{0}^{\prime}=\phi^{\prime}(L)$ and let $\tau: M_{0} \rightarrow M_{0}^{\prime}$ be the $K$-isomorphism that satisfies $\tau \circ \phi=\phi^{\prime}$. By Observation 2.1] $\phi(P \cap L)=P \cap M_{0}, \phi^{\prime}(P \cap L)=P \cap M_{0}^{\prime}$, and $\tau\left(P \cap M_{0}\right)=P \cap M_{0}^{\prime}$. It follows from $\phi(P \cap L)=P \cap M$ that $P \cap M=P \cap M_{0}$. By the relation $\tau\left(P \cap M_{0}\right)=P \cap M_{0}^{\prime}$ and the injectivity of $\tau$, we have $\left|P \cap M_{0}^{\prime}\right|=\left|P \cap M_{0}\right|=|P \cap M|$. Since $P \cap M_{0}^{\prime} \subset P \cap M$, we deduce that $P \cap M_{0}^{\prime}=P \cap M$. Finally, $\phi^{\prime}(P \cap L)=\tau(\phi(P \cap L))=\tau\left(P \cap M_{0}\right)=P \cap M_{0}^{\prime}=P \cap M$, as asserted.

We denote the maximal purely inseparable extension of a field $F$ by $F_{\text {ins }}$.
Lemma 2.6. Let $F$ be a recursive (resp. primitive recursive) subfield of $K_{s}$ that contains $K$. Suppose that we can decide (resp. primitively decide) for each monic separable polynomial $f \in K[X]$ whether $f$ has a root in $F$. Then, $F_{\text {ins }}$ is also a recursive (resp. primitive recursive) subfield of $\tilde{K}$ and we can decide (resp. primitively decide) for each monic polynomial $f \in K[X]$ whether $f$ has a root in $F_{\text {ins }}$.

Proof. It suffices to consider the case where $p=\operatorname{chr}(K)>0$. By our assumptions on $K$, we are able to effectively factor each monic $f \in K[X]$ into irreducible polynomials over $K$. Hence, in order to decide whether $f$ has a root in $F_{\text {ins }}$, we may assume that $f$ is irreducible. In this case we write $f(X)=g\left(X^{q}\right)$, where $g$ is an irreducible separable polynomial in $K[X]$ and $q$ is a power of $p$. If $y$ is a root of $g$ in $F$ and $z^{q}=y$ with $z \in \tilde{K}$, then $f(z)=0$ and $z \in F_{\text {ins. }}$. On the other hand, if $z \in F_{\text {ins }}$ and $f(z)=0$, we get for $y=z^{q}$ that $g(y)=f(z)=0$ and $y \in K_{s} \cap F_{\text {ins }}=F$. By assumption, we may decide (resp. effectively decide) whether $g$ has a root in $F$. Hence, we may decide (resp. effectively decide) whether $f$ has a root in $F_{\text {ins }}$.

Consider $x \in \tilde{K}$ and compute $f(X)=\operatorname{irr}(x, K)$. Then, we write $f(X)=g\left(X^{q}\right)$, where $g \in K[X]$ is separable and $q$ is a power of $p$. Then, $x^{q} \in K_{s}$ and we can check (resp. effectively check) whether $x^{q} \in F$, because $F$ is a recursive (resp. primitive recursive) subfield of $K_{s}$. This is the case if and only if $x \in F_{\text {ins }}$.

Theorem 2.7. Let $M$ be a perfect subfield of $\tilde{K}$ that contains K. Suppose that the existential theory of $M$ in $\mathscr{L}$ (ring, $K$ ) is decidable (resp. primitively decidable). Then, $\tilde{K}$ has a recursive (resp. primitive recursive) subfield $L$ that contains $K$ and is $K$-isomorphic to $M$.

Proof. We make a primitive recursive list, $p_{1}, p_{2}, p_{3}, \ldots$, of all monic separable irreducible polynomials in $K[Z]$. For each positive integer $j$ we compute the set $P_{j}$ of all roots of $p_{j}$ in $K_{s}$. Since $p_{1}, p_{2}, p_{3}, \ldots$ are distinct and irreducible polynomials,
(1) the sets $P_{1}, P_{2}, P_{3}, \ldots$ are disjoint.

Using Lemma 2.4, we inductively construct a recursive (resp. primitive recursive) ascending tower $L_{0} \subset L_{1} \subset L_{2} \subset \cdots$ of finite extensions of $K$ in $K_{s}$ with $L_{0}=K$. Moreover, for each positive integer $j$ there is a $K$-embedding $\psi_{j}: L_{j} \rightarrow M$ such that
(2) $\psi_{j}\left(P_{j} \cap L_{j}\right)=P_{j} \cap M$.

We consider the subfield $L_{\infty}=\cup_{j=1}^{\infty} L_{j}$ of $K_{s}$ (hence, also of $\tilde{K}$ ) and set $M_{\infty}=M \cap K_{s}$.
For each positive integer $j$ we let $E_{j}$ be the set of all $K$-embeddings $\psi: L_{j} \rightarrow M$ such that $\psi\left(P_{j} \cap L_{j}\right)=P_{j} \cap M$. The set $E_{j}$ is non-empty (because $\psi_{j} \in E_{j}$ ) and finite (because $\left.\left[L_{j}: K\right]<\infty\right)$. Moreover, if $\phi \in E_{j+1}$, then by Lemma 2.5, $\left.\phi\right|_{L_{j}} \in E_{j}$. Thus, $E_{0}, E_{1}, E_{2}, \ldots$ form an inverse system of finite sets. Hence, by [FrJ08, p. 3, Cor. 1.1.4], there exists a $K$-embedding $\phi: L_{\infty} \rightarrow M$ such that $\left.\phi\right|_{L_{j}} \in E_{j}$, i.e.
(3) $\phi\left(P_{j} \cap L_{j}\right)=P_{j} \cap M$
for each positive integer $j$.
Claim A: $\phi$ maps $L_{\infty}$ isomorphically onto $M_{\infty}$ Indeed, since $\phi\left(L_{\infty}\right) \subset M$ and $L_{\infty} \subset K_{s}$, we have $\phi\left(L_{\infty}\right) \subset M_{\infty}$. Conversely, let $y \in M_{\infty}$. By definition, $K_{s}=\cup_{j=1}^{\infty} P_{j}$. Hence, there exists a positive integer $j$ such that $y \in P_{j}$. By (3) there exists $x \in P_{j} \cap L_{j}$ (hence, $x \in L_{\infty}$ ) such that $\phi(x)=y$. Thus, $\phi\left(L_{\infty}\right)=M_{\infty}$. Since $\phi$ is injective, $\phi$ maps $L_{\infty}$ isomorphically onto $M_{\infty}$.
Claim B: $L_{\infty}=\cup_{j=1}^{\infty} P_{j} \cap L_{j}$ Indeed, by definition, the right hand side is contained in the left hand side. Conversely, let $x \in L_{\infty}$. Then, there exists a positive integer $j$ with $x \in P_{j}$. Thus, $\phi(x) \in P_{j} \cap M$. By (3), $x \in P_{j} \cap L_{j}$, as claimed.
Claim C: The field $L_{\infty}$ is recursive (resp. primitive recursive) in $K_{s}$. Indeed, given $x \in K_{s}$ we can effectively find a positive integer $j$ with $p_{j}(x)=0$. This means that $x \in P_{j}$. Then, we check (resp. effectively check) if $x \in L_{j}$. If this is the case, then $x \in L_{\infty}$. Otherwise, $x \notin L_{\infty}$. Indeed, if $x \in L_{\infty}$, then $\phi(x) \in M_{\infty}=\cup_{j^{\prime}=1}^{\infty} P_{j^{\prime}} \cap M_{\infty}$. Thus, there exists a positive integer $j^{\prime}$ with $\phi(x) \in P_{j^{\prime}}$. Since $\phi\left(P_{j}\right)=P_{j}$, it follows from (1) that $j^{\prime}=j$, Therefore, by (3), $x \in L_{j}$, in contrast to our assumption.
Conclusion of the proof: Since $M$ is perfect and $M_{\infty}=M \cap K_{s}$, the field $M$ is the maximal purely inseparable extension of $M_{\infty}$. Let $L$ be the maximal purely inseparable extension of $L_{\infty}$ and extend $\phi$, using Claim A, in the unique possible way to an isomorphism $\phi: L \rightarrow M$. By Claim C and Lemma 2.6, $L$ is a recursive (resp. primitive recursive) subfield of $\tilde{K}$.

Remark 2.8. Note that we do not claim nor we do not prove that the $K$-isomorphism $L \rightarrow M$ mentioned in Theorem 2.7is recursive. Indeed, let $\mathscr{M}$ be a $K$-conjugacy class of fields with an existential decidable theory. Only countably many of them are recursive subfields of $\tilde{K}$. For each of them there are only countably many recursive $K$-embeddings into $\tilde{K}$. Thus, all but countably many fields in $\mathscr{M}$ are not images of those embeddings.

Example 2.9. Let $R$ be a real closure of $\mathbb{Q}$. Then, $R$ is elementarily equivalent to the field $\mathbb{R}$ of real numbers [Pre81, p. 51, Cor. 5.3]. By Tarski [Tar48, p. 42, Thm. 37], Th $(\mathbb{R})$ is primitively decidable, hence so are $\operatorname{Th}(R)$ and $\operatorname{Ex}(R)$. Similarly, for each positive integer $p$, we choose a Henselization $\mathbb{Q}_{p}$ of $\mathbb{Q}$ with respect to the $p$-adic valuation of $\mathbb{Q}$. Then, $\mathbb{Q}_{p}$ is elementary equivalent to the field $\widehat{\mathbb{Q}}_{p}$ of all $p$-adic numbers [PrR84, p. 86, Thm. 5.1]. We know that $\operatorname{Th}\left(\hat{\mathbb{Q}}_{p}\right)$ is decidable [Mar02,
p. 97, Cor. 3.3.16], and even primitively decidable [Wei84, p. 84, Cor. 3.11(iii)]. Hence, so are $\operatorname{Th}\left(\mathbb{Q}_{p}\right)$ and $\operatorname{Ex}\left(\mathbb{Q}_{p}\right)$.

By Emil $\operatorname{Artin}, \operatorname{Aut}(R)$ is trivial [Lan93, p. 455, Thm. XI.2.9]. By F. K. Schmidt [Sch33], the same is true for $\operatorname{Aut}\left(\mathbb{Q}_{p}\right)$ [Jar91, Prop. 14.5]. Since $\operatorname{Gal}(\mathbb{Q})$ is uncountable, the fields $R$ and $\mathbb{Q}_{p}$ have uncountably many $\mathbb{Q}$-conjugates. It follows from Theorem 2.7that there exists a primitive recursive subfield $L$ of $\tilde{\mathbb{Q}}$ which is isomorphic to $R$ (resp. to $\mathbb{Q}_{p}$ ). However, by Remark[2.8, there exists a conjugate $R^{\prime}$ of $R\left(\right.$ resp. $\mathbb{Q}_{p}^{\prime}$ of $\mathbb{Q}_{p}$ ) which is not the image of a recursive subfield $L$ of $\tilde{\mathbb{Q}}$ by a recursive isomorphism.

## 3. Decidable Large Fields

We consider a presented field $K$ with elimination theory as in Section 2, In addition to the field operations discussed in that section, we note that all of the standard operations on Galois extensions of $K$ and of given finite extensions of $K$ and with their Galois groups can be carried out in a primitive recursive way [FrJ08, pp. 411-412, Sec. 19.3]. In addition, using [FrJ08, p. 413, Lemma 19.4.1], we effectively construct a sequence $z_{1}, z_{2}, z_{3}, \ldots$ of elements that presents $K_{s}$ over $K$. We can also effectively construct a sequence $\tilde{z}_{1}, \tilde{z}_{2}, \tilde{z}_{3}, \ldots$ of elements that presents $\tilde{K}$ over $K$. Thus, $K_{s}=K\left(z_{1}, z_{2}, z_{3}, \ldots\right)$ and $\tilde{K}=K\left(\tilde{z}_{1}, \tilde{z}_{2}, \tilde{z}_{3}, \ldots\right)$. Again, we recall that by the above mentioned lemma, both fields have the splitting algorithm.

Notation 3.1. We consider variables $T_{1}, \ldots, T_{r}, X$ over $K$ and abbreviate $T_{1}, \ldots, T_{r}$ by T. Let $f_{1}, \ldots, f_{m}$ be polynomials in $K(\mathbf{T})[X]$ which are irreducible and separable in the ring $K(\mathbf{T})[X]$ and let $g$ be a non-zero polynomial in $K[\mathbf{T}]$. Following [FrJ08, Sec. 12.1], we write $H_{K}\left(f_{1}, \ldots, f_{m} ; g\right)$ for the set of all $\mathbf{a} \in K^{r}$ such that $f_{1}(\mathbf{a}, X), \ldots, f_{m}(\mathbf{a}, X)$ are defined, irreducible, and separable in $K[X]$. In addition, $g(\mathbf{a})$ is defined and non-zero. Then, we call $H_{K}\left(f_{1}, \ldots, f_{r} ; g\right)$ a separable Hilbert subset of $K^{r}$. A separable Hilbert set of $K$ is a separable Hilbert subset of $K^{r}$ for some positive integer $r$. We say that $K$ is Hilbertian if each separable Hilbert set of $K$ is non-empty.
Lemma 3.2. Suppose $K$ is Hilbertian. Given a separable Hilbertsubset $H$ of $K^{r}$, we can recursively find $\left(a_{1}, \ldots, a_{r}\right) \in H$.

Proof. Using Cantor's first diagonal method, we can effectively write down a list ( $\mathbf{a}_{1}, \mathbf{a}_{2}, \mathbf{a}_{3}, \ldots$ ), with $\mathbf{a}_{i}=\left(a_{i 1}, \ldots, a_{i r}\right)$, of all elements of $K^{r}$. Let $H=H_{K}\left(f_{1}, \ldots, f_{m} ; g\right)$ as in Notation3.1,

Since $K$ has the splitting algorithm, we may effectively check the irreducibility of the polynomials $f_{1}\left(\mathbf{a}_{i}, X\right), \ldots, f_{m}\left(\mathbf{a}_{i}, X\right)$ over $K$ and their separability for $i=1,2,3, \ldots$, and also the condition $g\left(\mathbf{a}_{i}\right) \neq 0$. Since $K$ is Hilbertian, we will certainly hit an $i$ such that $f_{1}\left(\mathbf{a}_{i}, X\right), \ldots, \mathbf{f}_{m}\left(\mathbf{a}_{i}, X\right)$ are defined, irreducible, and separable over $K$, and $g\left(\mathbf{a}_{i}\right) \neq 0$, as needed.

This gives us a recursive procedure (but not a primitive recursive procedure) to find $\mathbf{a}$ in $H$.

Lemma 3.3. Suppose that $K$ is Hilbertian, $L$ is a given finite separable extension of $K$, and $H$ is a given separable Hilbert subset of $L^{r}$. Then, we can effectively find a separable Hilbert subset $H_{K}$ of $K^{r}$ which is contained in $H$.

Proof. Let $H=H_{L}\left(f_{1}, \ldots, f_{m} ; g\right)$, where $f_{1}, \ldots, f_{m}$ are irreducible separable polynomials in $L(\mathbf{T})[X]$ and $g \in L[\mathbf{T}]$ non-zero. Without loss we may assume that the coefficients of $f_{1}, \ldots, f_{m}$ are in $L[\mathbf{T}]$. The proof of [FrJ08, p. 224, Lemma 12.2.2] uses the proof of [FrJ08, p. 223, Lemma 12.2.1] with $L\left(T_{1}, \ldots, T_{r}\right)$ replacing $L$ in the latter lemma to effectively produce a non-zero polynomial $h \in L\left[T_{1}, \ldots, T_{r}\right]$ and irreducible separable polynomials $p_{1}, \ldots, p_{m}$ in $K\left(T_{1}, \ldots, T_{r}\right)[X]$ with the
following property: If $\mathbf{a} \in K^{r}, h(\mathbf{a}) \neq 0$, and the $p_{i}(\mathbf{a}, X)$ 's are defined, irreducible, and separable in $K[X]$, then $f_{1}(\mathbf{a}, X), \ldots, f_{m}(\mathbf{a}, X)$ are defined, irreducible, and separable in $L[X]$, and $g(\mathbf{a}) \neq 0$. Replacing $h$ by the product of all its $K$-conjugates (an effective operation), $H_{K}\left(p_{1}, \ldots, p_{m} ; h\right.$ ) is a separable Hilbert subset of $K^{r}$ which is contained in $H$.

Recall that a field $M$ is PAC if every absolutely integral variety over $M$ has an $M$-rational point. We denote the free profinite group on $e$ generators by $h a t F_{e}$ [FrJ08, p. 349, first paragraph]. We also denote the absolute Galois group of a field $F$ by $\operatorname{Gal}(F)$.
Lemma 3.4. Let $M$ be an extension of $K$ in $\tilde{K}$. Suppose $M$ is perfect and $P A C, \operatorname{Gal}(M) \simeq \hat{F}_{e}$, and one may check whether a given monic polynomial in $K[X]$ has a root in $M$. Then, $\operatorname{Th}(M)$ is decidable.

Proof. Let $\operatorname{Root}(M)$ be the recursive set of all monic polynomials in $K[X]$ that have a root in $M$. Let $\operatorname{Ax}(K, e)$ be the set of axioms in the language $\mathscr{L}($ ring, $K$ ) given in [FrJ08, p. 437, Prop. 20.4.4]. Thus, a field extension $F$ of $K$ satisfies $\operatorname{Ax}(K, e)$ if and only if $F$ is perfect, PAC, and $\operatorname{Gal}(F) \simeq$ $\hat{F}_{e}$. Let $\operatorname{Ax}(K, M)$ be the union of $\operatorname{Ax}(K, e)$ and the axioms that say that each $f \in \operatorname{Root}(M)$ has a root in $M$ and each monic $f \in K[X] \backslash \operatorname{Root}(M)$ does not have a root in $M$. In particular, $M \vDash$ $\operatorname{Ax}(K, M)$. Thus, if a field extension $F$ of $K$ is equivalent to $M$ as a structure of $\mathscr{L}$ (ring, $K$ ), then $F \vDash \operatorname{Ax}(K, M)$.

Conversely, if $F \vDash \operatorname{Ax}(K, M)$, then a monic polynomial $f \in K[X]$ has a root in $F$ if and only if $f$ has a root in $M$. By [FrJ08, p. 441, Lemma 20.6.3(b)], $F \cap \tilde{K} \simeq_{K} M$. Hence, by [FrJ08, p. 436, Cor. 20.4.2], $F$ is elementarily equivalent to $M$ as a structure of $\mathscr{L}$ (ring, $K$ ).

It follows from Gödel completeness theorem [FrJ08, p. 154, Cor. 8.2.6] that a sentence $\theta$ of $\mathscr{L}$ (ring, $K$ ) is true in $M$ if and only if $\theta$ has a formal proof in $\mathscr{L}$ (ring, $K$ ) from the axioms $\operatorname{Ax}(K, M)$ [FrJ08, Sec. 8.1, p. 150]. Thus, to check whether $\theta$ is true in $M$, one makes a list of all formal proofs in $\mathscr{L}$ (ring, $K$ ) from the axioms $\operatorname{Ax}(K, M)$ and check them one by one. After finitely many steps, one finds a proof of $\theta$ or of $\neg \theta$. In the first case $\theta$ is true in $M$ in the latter case $\theta$ is false in $M$. Consequently, $\operatorname{Th}(M)$ is decidable.

Proposition 3.5. Let $K$ be a presented field with elimination theory. Suppose that $K$ is Hilbertian. Then, we can for every positive integer e recursively construct a decidable perfect PAC algebraic extension $M$ of $K$ with $\operatorname{Gal}(M) \simeq \hat{F}_{e}$ which is recursive in $\tilde{K}$.

Proof. We construct a primitive recursive list ( $G_{1}, G_{2}, G_{3}, \ldots$ ) of all finite non-trivial groups that are generated by $e$ elements. As mentioned in the first paragraph of this section, $\tilde{K}$ has the splitting algorithm. Hence, by [FrJ08, p. 405, Lemma 19.1.3(c)] applied to $\tilde{K}$ rather than to $K$, we may find out whether a given polynomial $f \in K[T, X]$ is irreducible over $\tilde{K}$. We use this test to build a recursive list ( $f_{1}, f_{2}, f_{3}, \ldots$ ) of all absolutely irreducible polynomials in $K[T, X]$ that are monic and separable in $X$. The rest of the proof breaks up into several parts.
Part A: The induction plan By induction we effectively construct an ascending sequence ( $N_{0}, N_{1}, N_{2}, \ldots$ ) of presented finite Galois extensions of $K$ in $K_{s}$ and for each $n$ a presented subfield $M_{n}$ of $N_{n}$ that contains $K$, such that the following conditions hold for each $n \in \mathbb{N}$ :
(1a) $z_{n} \in N_{n}$ (where, as in the beginning of this section, $z_{1}, z_{2}, z_{3}, \ldots$ present $K_{s}$ over $K$ ).
(1b) There exist $a \in K$ and $b \in M_{n}$ such that $f_{n}(a, b)=0$.
(1c) The group $\operatorname{Gal}\left(N_{n} / M_{n}\right)$ is generated by $e$ elements, it has $G_{n}$ as a quotient, and $N_{n-1} \cap$ $M_{n}=M_{n-1}$.

Part B: The field $N_{n}^{\prime}$ We start the induction by setting $M_{0}=N_{0}=K$. Next we consider $n \geq 1$ and assume that $N_{0}, \ldots, N_{n}$ and $M_{0}, \ldots, M_{n}$ have already been effectively constructed such that (I) holds with $n$ replaced by $m$ for $m=0, \ldots, n$.

Since $f_{n+1}$ is absolutely irreducible, $f_{n+1}$ is irreducible over $N_{n}$. Hence, we can use Lemma 3.3 to construct a separable Hilbert subset $H$ of $K$ such that $f_{n+1}(a, X)$ is irreducible over $N_{n}$ for each $a \in H$. Then, we use Lemma 3.2 to effectively choose $a \in K$ such that $a \in H$. In the next step we choose $b \in K_{s}$ with $f_{n+1}(a, b)=0$. Hence, $N_{n}$ and $K(b)$ are linearly disjoint over $K$, so $N_{n} \cap M_{n}(b)=M_{n}$. Therefore, res: $\operatorname{Gal}\left(N_{n}(b) / M_{n}(b)\right) \rightarrow \operatorname{Gal}\left(N_{n} / M_{n}\right)$ is an epimorphism. We use [FrJ08, p. 412, Lemma 19.3.2] to effectively construct the Galois closure $N_{n}^{\prime}$ of $N_{n}\left(b, z_{n+1}\right) / K$.


Part C: Construction of $N_{n+1}$. We compute the order $r$ of $G_{n+1}$ and embed $G_{n+1}$ into the symmetric group $S_{r}$. For every field $F$, the Galois group of the general polynomial $X^{r}+T_{1} X^{r-1}+$ $\cdots+T_{r}$ over $F\left(T_{1}, \ldots, T_{r}\right)$ is the symmetric group $S_{r}$ [Lan93, p. 272, Example VI.2.2]. The proof of [FrJ08, p. 231, Lemma 13.3.1] gives a separable Hilbert subset $H$ of $F^{r}$ such that $\operatorname{Gal}\left(X^{r}+\right.$ $\left.a_{1} X^{r-1}+\cdots+a_{r}, F\right) \simeq S_{r}$ for each $\mathbf{a} \in F^{r}$.

Next we compute the number $s$ of subfields of $N_{n}^{\prime}$ that properly contain $K$, and use the preceding paragraph and Lemmas 3.2 and 3.3 to construct $s+1$ linearly disjoint Galois extensions $L_{1}, \ldots, L_{s+1}$ of $K$ with Galois group $S_{r}$. The intersection of at least one of these fields with $N_{n}^{\prime}$ is $K$. Computing the intersections $N_{n}^{\prime} \cap L_{1}, \ldots, N_{n}^{\prime} \cap L_{s+1}$, we find an $i$ between 1 and $s+1$ such that $N_{n}^{\prime} \cap L_{i}=K$. In other words, $N_{n}^{\prime}$ and $L_{i}$ are linearly disjoint over $K$. Hence, $N_{n}^{\prime}$ and $M_{n}(b) L_{i}$ are linearly disjoint over $M_{n}(b)$. We set $N_{n+1}=N_{n}^{\prime} L_{i}$.
Part D: Construction of $M_{n+1}$. By the preceding paragraph,

$$
\begin{equation*}
\operatorname{Gal}\left(N_{n+1} / M_{n}(b)\right) \simeq \operatorname{Gal}\left(N_{n}^{\prime} / M_{n}(b)\right) \times \operatorname{Gal}\left(M_{n}(b) L_{i} / M_{n}(b)\right) . \tag{2}
\end{equation*}
$$

In addition, $M_{n}(b)$ is linearly disjoint from $L_{i}$ over $K$. We effectively find $\tau_{1}, \ldots, \tau_{e}$ in $\operatorname{Gal}\left(L_{i} / K\right)$ that generate a subgroup which is isomorphic to $G_{n+1}\left[F r J 08, ~ p . ~ 412\right.$, Lemma 19.3.2] and set $K_{i}$
to be the fixed field of $\tau_{1}, \ldots, \tau_{e}$ in $L_{i}$. Then, $\operatorname{Gal}\left(M_{n}(b) L_{i} / M_{n}(b) K_{i}\right) \simeq \operatorname{Gal}\left(L_{i} / K_{i}\right) \simeq G_{n+1}$.


By (1c), we find $\sigma_{n, 1}, \ldots, \sigma_{n, e}$ of $\operatorname{Gal}\left(N_{n}^{\prime} / M_{n}(b)\right)$ whose restriction to $N_{n}(b)$ generate $\operatorname{Gal}\left(N_{n}(b) / M_{n}(b)\right.$ ), so their restrictions to $N_{n}$ generate $\operatorname{Gal}\left(N_{n} / M_{n}\right)$. By (2), we can effectively find $\sigma_{n+1,1}, \ldots, \sigma_{n+1, e}$ in $\operatorname{Gal}\left(N_{n+1} / M_{n}(b)\right)$ whose restrictions to $N_{n}^{\prime}$ are $\sigma_{n, 1}, \ldots, \sigma_{n, e}$, respectively, and whose restrictions to $L_{i}$ are $\tau_{1}, \ldots, \tau_{e}$, respectively. Thus, by (1c), both $G_{n}$ and $G_{n+1}$ are quotients of the subgroup $H=\left\langle\sigma_{n+1,1}, \ldots, \sigma_{n+1, e}\right\rangle$ of $\operatorname{Gal}\left(N_{n+1} / M_{n}(b)\right)$. Then, the restriction of $H$ to $L_{i}$ is $G_{n+1}$ and the restriction of $H$ to $N_{n}$ is $\operatorname{Gal}\left(N_{n} / M_{n}\right)$. Let $M_{n+1}$ be the fixed field of $H$ in $N_{n+1}$. Then, $G_{n+1}$ is a quotient of $\operatorname{Gal}\left(N_{n+1} / M_{n+1}\right)$, and $N_{n} \cap M_{n+1}=M_{n}$. This concludes the ( $n+1$ )th step of the induction.

We put all of the fields mentioned above appears in the following diagram of fields:


Part E: The field $M_{\infty}$ By the defining property of $z_{1}, z_{2}, z_{3}, \ldots$ and by (1a), $\cup_{n=1}^{\infty} N_{n}=K_{s}$. By Part A, $M_{\infty}=\cup_{n=1}^{\infty} M_{n}$ is a presented recursive subfield of $K_{s}$. Moreover, for $n^{\prime}>n$, (1c) and induction on $n^{\prime}-n$ imply that $M_{n^{\prime}} \cap N_{n}=M_{n}$., Hence, $M_{\infty} \cap N_{n}=M_{n}$ for each positive integer $n$. Also, $\operatorname{Gal}\left(M_{\infty}\right)$ is the inverse limit of the groups $\operatorname{Gal}\left(N_{n} / M_{n}\right)$. Since each of these groups is generated by $e$ elements, so is $\operatorname{Gal}\left(M_{\infty}\right)$ (as a profinite group). In addition, since $G_{n}$ is a quotient of $\operatorname{Gal}\left(N_{n} / M_{n}\right)$, each finite group which is generated by $e$ elements is a quotient of $\operatorname{Gal}\left(M_{\infty}\right)$. Hence, $\operatorname{Gal}\left(M_{\infty}\right) \simeq \hat{F}_{e}$ [FrJ08, p. 360, Lemma 17.7.1]. Finally, by (1b), each absolutely irreducible polynomial in two variables with coefficients in $K$ has a zero in $M_{\infty}$. Therefore, by [FrJ08, p. 195, Thm. 11.2.3], $M_{\infty}$ is PAC.
Part F:The field $M_{\infty}$ is recursive in $K_{s}$ Next we show how to decide whether a given monic separable polynomial $f \in K[X]$ has a root in $M_{\infty}$. Since $K$ has the splitting algorithm, we may
assume that $f$ is irreducible. Moreover, since $K_{s}$ has the splitting algorithm, we may find a root $z$ of $f$ in $K_{s}$ and identify $z$ as $z_{n}$ for some positive integer $n$. By (3), $z \in N_{n}$ and so $f$ totally splits in $N_{n}$. We check whether $G_{n}$ fixes any of the roots of $f$ (by [FrJ08, p. 412, Lemma 19.3.2]). This will be the case if and only if $f$ has a root in $M_{n}$. Since, by Part E, $N_{n} \cap M_{\infty}=M_{n}$, this will be the case if and only if $f$ has a root in $M_{\infty}$.

In the situation of the preceding paragraph, we may check whether $G_{n}$ fixes $z$, hence whether $z \in M_{\infty}$. This proves that $M_{\infty}$ is recursive in $K_{s}$.
Part G: Conclusion of the proof. Finally, let $M$ be the maximal purely inseparable extension of $M_{\infty}$ in $\tilde{K}$. Then, $M$ is recursive in $\tilde{K}$ (Lemma 2.6), $M$ is PAC [FrJ08, p. 196, Cor. 11.2.5], and $\operatorname{Gal}(M) \simeq \operatorname{Gal}\left(M_{\infty}\right) \simeq$
$h a t F_{e}$. It follows from Lemma[3.4 that $M$ is decidable.
We are now in a position to prove a stronger version of Theorem B of the introduction.
Theorem 3.6. Let $K$ be a presented field with elimination theory. Suppose that $K$ is Hilbertian. Then, we can for every positive integer e construct an infinite sequence of decidable PAC perfect fields which are recursive in $\tilde{K}$, each with absolute Galois group isomorphic to $h a t F_{e}$.

Proof. Let $n$ be a non-negative number and assume that we have already constructed $n$ distinct decidable PAC perfect fields $M^{(1)}, \ldots, M^{(n)}$ with absolute Galois groups isomorphic to $h a t F_{e}$ and which are recursive in $\tilde{K}$. In particular, $M^{(1)}, \ldots, M^{(n)}$ are proper $K$-vector-subspaces of $\tilde{K}$. Hence, $\bigcup_{i=1}^{n} M^{(i)}$ is a proper subset of $\tilde{K}$ [Hup90, p. 11, A 1.1.c]. Since each $M^{(i)}$ is a recursive subset of $\tilde{K}$, we may find $z \in \tilde{K} \backslash \bigcup_{i=1}^{n} M^{(i)}$.

Since $K$ has elimination theory, so has $K(z)$ [FrJ08, p. 410, Def. 19.2.8]. By [FrJ08, p. 224, Cor. 12.2.3], $K(z)$ is Hilbertian. Hence, by Proposition 3.5, $K(z)$ has an extension $M^{(n+1)}$ which is perfect, PAC, decidable, $\operatorname{Gal}\left(M^{(n+1)}\right) \simeq \hat{F}_{e}$, and is a recursive subfield of $\tilde{K}$. Since $z \in M^{(n+1)}$, we have $M^{(n+1)} \neq M^{(i)}$ for $i=1, \ldots, n$. This concludes the induction.

Remark 3.7. If $N$ is a Galois extension of $K$ and $\operatorname{Ex}(N)$ is decidable (resp. primitively decidable), then $N$ is also a recursive (resp. primitive recursive) subfield of $K_{s}$, hence also of $\tilde{K}$. Indeed, if $z \in K_{s}$, we construct $\operatorname{irr}(z, K)$ and check whether $\operatorname{irr}(z, K)$ has a root in $N$. This is the case if and only if all of the roots of $\operatorname{irr}(z, K)$ belong to $N$, hence if and only if $z \in N$.

For example, the field $\mathbb{Q}_{\text {tr }}$ of all totally real algebraic numbers is a Galois extension of $\mathbb{Q}$ and $\mathrm{Th}\left(\mathbb{Q}_{\mathrm{tr}}\right)$ is primitively decidable. [FHV94, p. 90, Thm. 10.1]. If $S$ is a finite number of prime numbers, then the field $\mathbb{Q}_{\text {tot }}$ of all totally $S$-adic numbers is decidable [Ers96]. By the preceding paragraph, $\mathbb{Q}_{\text {tr }}$ is primitive recursive in $\tilde{\mathbb{Q}}$ and $\mathbb{Q}_{\text {totS }}$ is recursive in $\tilde{\mathbb{Q}}$.

Remark 3.8. Primitive recursive procedures All of the procedures we use to construct the field $M$ in Proposition3.5are primitive recursive except the procedure we use in Lemma3.2, Thus, if we can effectively find a point in every separable Hilbert subset of $K$ (in which case we say that is $K$ effectively Hilbertian), then we can effectively construct $M$ in Proposition 3.5 and then also effectively construct the infinite sequence of fields $M^{(1)}, M^{(2)}, M^{(3)}, \ldots$ that appears in Theorem 3.6. In addition, we can prove that the theories of those fields are primitive recursive.

We can prove the above mentioned effectiveness results at least in the case where $K$ is an infinite finitely generated field (over its prime field). Unfortunately, these improvements would need a lot of space and so would go beyond the scope of this work. So, we restrict ourselves to some hints for the proofs.

In order to prove that a presented finitely generated infinite field $K$ is effectively Hilbertian, it suffices to prove it only in the cases where $K$ is either $K_{0}(t)$, where $K_{0}$ is an infinite finitely generated field and $t$ is indeterminate, or $K=\mathbb{Q}$, or $K=\mathbb{F}_{p}(t)$.

In the first case, we notice that the proof of [FrJ08, p. 236, Prop. 13.2.1] reduces the effective Hilbertianity to the primitive recursiveness of $\operatorname{Th}\left(\tilde{K}_{0}\right)$, which is proved in $[\operatorname{FrJ} 08, \mathrm{p} .170$, Prop. 9.4.3].

The case where $K=\mathbb{Q}$ or $K=\mathbb{F}_{p}(t)$ involves effective operations in $\mathbb{Z}$ like factoring positive integers into products of prime numbers and effective Chebotarev density theorem. All of this appear in the proof of Theorem 13.3.5 of [FrJ08] and the lemmata that proceed it in Chapter 13 of [FrJ08].

Finally, we have to improve Lemma 3.4 and prove that, under the conditions of that lemma, $\operatorname{Th}(M)$ is primitive recursive. This depends on "elimination of quantifiers in the language of Galois stratification" given by [FrJ08, p. 721, Prop. 30.5.3].

## 4. Appendix

We prove in this appendix a statement made in the introduction.
For each field $K$, every positive integer, and every $\boldsymbol{\sigma} \in \operatorname{Gal}(K)^{e}$ we denote the maximal purely inseparable extension of $K_{s}(\boldsymbol{\sigma})$ by $\tilde{K}(\boldsymbol{\sigma})$.

Proposition 4.1. Let $K$ be a Hilbertian field and let e be a positive integer. Then, there are uncountably many elementary equivalence classes (in the language $\mathscr{L}$ (ring, $K$ )) of fields of the form $\tilde{K}(\boldsymbol{\sigma})$ with $\boldsymbol{\sigma} \in \operatorname{Gal}(K)^{e}$.

Moreover, the Haar measure of the set of pairs $\left(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{\prime}\right) \in \operatorname{Gal}(K)^{2 e}$ such that $K_{s}(\boldsymbol{\sigma})$ and $K_{s}\left(\boldsymbol{\sigma}^{\prime}\right)$ are not equivalent as structures of the language $\mathscr{L}$ (ring, $K$ ) is 1 .

Proof. Using the assumption that $K$ is Hilbertian, we construct, by induction, a linearly disjoint sequence $K_{1}, K_{2}, K_{3}, \ldots$ of quadratic separable extensions of $K$ [FrJ08, p. 297, Cor. 16.2.7(b)] (if $\operatorname{chr}(K)=2$, one has to use [FrJ08, p. 296, Example 16.2.5(c) and p. 297, Lemma 16.2.6]). Let $L$ be the compositum of all these extensions. Then, $\operatorname{Gal}(L / K)$ is an infinite profinite group of exponent 2. In particular, the closed subgroup generated by every finite subset is finite, hence has Haar measure 0 in $\operatorname{Gal}(L / K)$.

We denote the normalized Haar measure of $\operatorname{Gal}(L / K)$ by $\mu$. For each $\boldsymbol{\sigma}=\left(\sigma_{1}, \ldots, \sigma_{e}\right) \in \operatorname{Gal}(K)^{e}$ we denote the fixed field of $\sigma_{1}, \ldots, \sigma_{e}$ in $L$ by $L(\boldsymbol{\sigma})$ and let $\langle\boldsymbol{\sigma}\rangle=$ $\operatorname{Gal}(L / L(\boldsymbol{\sigma}))$ be the closed subgroup of $\operatorname{Gal}(L / K)$ generated by $\sigma_{1}, \ldots, \sigma_{e}$.

If there are only countably many fields $L\left(\boldsymbol{\sigma}^{(1)}\right), L\left(\boldsymbol{\sigma}^{(2)}\right), L\left(\boldsymbol{\sigma}^{(3)}\right), \ldots$ with $\boldsymbol{\sigma}^{(i)} \in \operatorname{Gal}(L / K)^{e}$, then $\operatorname{Gal}(L / K)=\bigcup_{i=1}^{\infty}\left\langle\boldsymbol{\sigma}^{(i)}\right\rangle$, so $1=\mu(\operatorname{Gal}(L / K)) \leq \sum_{i=1}^{\infty} \mu\left(\left\langle\boldsymbol{\sigma}^{(i)}\right\rangle\right)=0$, which is a contradiction. Hence, there is an uncountable subset $S$ of $\operatorname{Gal}(L / K)^{e}$ such that $L(\boldsymbol{\sigma}) \neq L\left(\boldsymbol{\sigma}^{\prime}\right)$ for every distinct elements $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}^{\prime}$ of $S$.

We extend each $\boldsymbol{\sigma} \in S$ to an element $\tilde{\boldsymbol{\sigma}}$ of $\operatorname{Gal}(K)^{e}$. If $\boldsymbol{\sigma}^{\prime} \in S$ and $\boldsymbol{\sigma}^{\prime} \neq \boldsymbol{\sigma}$, then $K_{S}(\tilde{\boldsymbol{\sigma}})$ is not $K-$ conjugate to $K_{s}\left(\widetilde{\boldsymbol{\sigma}^{\prime}}\right)$, otherwise $L(\boldsymbol{\sigma})$ and $L\left(\boldsymbol{\sigma}^{\prime}\right)$ are $K$-conjugate, hence equal, because $\operatorname{Gal}(L / K)$ is abelian. It follows from [FrJ08, p. 441, Lemma 20.6.3(b)] that $\tilde{K}(\tilde{\boldsymbol{\sigma}})$ and $\tilde{K}\left(\widetilde{\boldsymbol{\sigma}^{\prime}}\right)$ are not elementarily equivalent as structures of $\mathscr{L}$ (ring, $K$ ). This proves the first statement of the proposition.

The proof of the second statement of the proposition is based on the observation that the diagonal $D$ of $\operatorname{Gal}(L / K)^{e}$ has Haar measure 0 in $\operatorname{Gal}(L / K)^{2 e}$. This is so, because $\operatorname{Gal}(L / K)^{e}$ is an infinite profinite group and for every finite group $G$, the proportion of the diagonal $\left\{(g, g) \in G^{2} \mid g \in\right.$ $G\}$ in $G^{2}$ is $\frac{1}{|G|}$. It follows from [Hal68, p. 279, Thm. C] that the set $\tilde{D}^{\prime}=\left\{\left(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{\prime}\right) \in \operatorname{Gal}(K)^{2 e}|\boldsymbol{\sigma}|_{L} \neq\right.$
$\left.\left.\boldsymbol{\sigma}^{\prime}\right|_{L}\right\}$ has Haar measure 1. By the preceding paragraph, $K_{S}(\boldsymbol{\sigma})$ is not elementarily equivalent to $K_{S}\left(\boldsymbol{\sigma}^{\prime}\right)$ for all $\left(\boldsymbol{\sigma}, \boldsymbol{\sigma}^{\prime}\right) \in \tilde{D}$.

## References

[1] Yu. L. Ershov, Nice local-global fields I, Algebra and Logic 35 (1996), 229-235.
[2] M. Fried, D. Haran, and M. Jarden, Galois stratification over Frobenius fields, Advances of Mathematics, 51 (1984), 1-35.
[3] M. D. Fried, D. Haran, H. Völklein, Real hilbertianity and the field of totally real numbers, Contemporary Mathematics 174 (1994), 1-34.
[4] M. D. Fried and M. Jarden, Field Arithmetic, Third Edition, revised by Moshe Jarden, Ergebnisse der Mathematik (3) 11, Springer, Heidelberg, 2008.
[5] P. R. Halmos, Measure Theory, D. Van Nostrand Company, Princeton 1968.
[6] B. Huppert, Angewandte Linear Algebra, de Gruyter, Berlin 1990.
[7] M. Jarden, Intersection of local algebraic extensions of a Hilbertian field, in "Generators and Relations in Groups and Geometries" (A. Barlotti et al., eds), NATO ASI Series C 333, 343-405, Kluwer, Dordrecht, 1991.
[8] M. Jarden and U. Kiehne, The elementary theory of algebraic fields of finite corank, Inventiones Mathematicae 30 (1975), 275-294.
[9] S. Lang, Algebra, third edition, Addison-Wesley, Reading, 1993.
[10] D. Marker, Model Theory: an Introduction, Springer 2002.
[11] A. Prestel, Pseudo real closed fields, Set Theory and Model Theory, Lecture Notes 872, (1981), 127-156, Springer.
[12] A. Prestel and P. Roquette, Formally p-adic Fields, Lecture Notes in Mathematics 1050, Springer, Berlin, 1984.
[13] F.K. Schmidt , Mehrfach perfekte Körper, Mathematische Annalen 108 (1933), 1-25.
[14] A. Tarski, A decision method for elementary algebra and geometry, The RAND corporation, Santa Monica, 1948,
http:<br>www.rand.org $\backslash$ content \dam\rand \pubs $\backslash$ reports $\backslash 2008 \backslash R 109 . p d f$.
[15] Weispfenning, Quantifier elimination and decision procedures for valued fields, in Models and Sets, Lecture Notes in Mathematics 1103, 419-472, Springer 1984.

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