CONSTRUCTING TYPES IN DIFFERENTIALLY CLOSED FIELDS THAT ARE ANALYSABLE IN THE CONSTANTS

RUIZHANG JIN

ABSTRACT. Analysability of finite U-rank types are explored both in general and in the theory DCF₀. The well-known fact that the equation $\delta(\log \delta x) = 0$ is analysable in but not almost internal to the constants is generalized to show that $\log \delta \ldots \log \delta x = 0$ is not analysable in the constants in (n - 1)-steps. The

CONTENTS

| 1. | Introduction | 1 |
|------------|---|----|
| 2. | Analysability | 2 |
| 3. | Iterated Logarithmic Derivative | 5 |
| 4. | Analyses by reductions and coreductions | 7 |
| 5. | A Construction in DCF_0 | 12 |
| References | | 20 |

1. INTRODUCTION

That differential-algebraic geometry is an expansion of algebraic geometry is reflected in model theory by viewing the theory of algebraically closed fields as a reduct of the theory of differentially closed fields. The locus of that reduct is the field of constants. The smallest intermediate reduct that properly expands algebraic geometry is that of differential varieties that are *almost internal* to the constants: differential varieties that over possibly additional parameters become definable finite covers of algebraic varieties in the constants. Here already one observes new and interesting geometric and model theoretic phenomena. A further step would be to consider those differential varieties that are built up through a finite sequence of fibrations whose fibres are almost internal to the constants; these are the differential varieties that are *analysable in the constants*, and they are the focus of this paper. In particular, we give some constructions that exhibit the richness of this category.

notion of a canonical analysis is introduced – namely an analysis that is of minimal length and interalgebraic with every other analysis of that length. Not every analysable type admits a canonical analysis. Using properties of reductions and coreductions in theories with the canonical base property, it is constructed, for any sequence of positive integers $(n_1, ..., n_\ell)$, a type in DCF₀ that admits a canonical analysis with the property that the *i*th step has *U*-rank n_i .

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Differential varieties analysable in the constants have come up recently in applications; it is shown in [1] that they give rise to a new class of associative algebras satisfying the classical Dixmier-Moeglin equivalence.

Probably the best known example of an analysable but not internal to the constants differential variety is the one defined by the equation $\delta\left(\frac{\delta x}{x}\right) = 0$. It decomposes as an extension of the additive group of constants by the multiplicative group of constants, without itself being almost internal to the constants. Our first observation is to generalize this construction by iterating the logarithmic derivative. Writing $\log \delta x := \frac{\delta x}{x}$ and $\log \delta^{(m)} = \underbrace{\log \delta \dots \log \delta}_{m}$ we consider the equation

 $\log \delta^{(m)} x = 0$, and show in Section 3 that while it is analysable in the constants in m steps, it is not analysable in m-1 steps. This is done in Section 3 by essentially reducing to the m = 2 case.

Note that each step in the analysis of $\log \delta^{(m)} x = 0$ is of *U*-rank one. It is not hard to produce from this example, using methods that work generally in stable theories satisfying the canonical base property (CBP), including reductions and coreductions, other examples of types analysable in the constants in *m*-steps but not in m-1-steps. We may even require this type to satisfy the property that the *i*th step of the analysis by reductions of this type is of *U*-rank n_i , for any given increasing sequence $(n_i)_{i=1}^m$, or that the *i*th step of the analysis by coreductions of this type is of *U*-rank n_i , for any given decreasing sequence $(n_i)_{i=1}^m$. This is done in Section 4.

But we look for more; we want analyses of a type p that are *canonical* in the sense that up to interalgebraicity there is no other analyses of p in the constants of the same (minimal) length. Not every finite rank type, even in DCF₀, admits a canonical analysis (see Example 4.1). However, we show in Section 5 that given any sequence of positive integers (n_1, \ldots, n_m) there exists in DCF₀ a type that has a canonical analysis in the constants with *i*th step having *U*-rank n_i . Unlike in the logarithmic derivative case, these examples are not differential algebraic groups, and hence that theory is not directly available to us. Our proofs involve a careful algebraic analysis of the equations that arise. Note that the situation is very different for differential algebraic groups; in [1] it is shown that every differential algebraic group over the constants is analysable in at most 3 steps.

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2. Analysability

As a general setting, we work in a saturated model \mathcal{U} of a complete stable theory T that eliminates imaginaries. We review in this section some classical notions around finite rank types. As a general reference we suggest [7]. We have provided proofs where explicit references were not possible.

Let \mathcal{P} be a set of partial types (over different parameter sets) which is invariant under automorphisms over \emptyset , and q be a stationary type over a parameter set A.

Recall that a stationary type q over A is \mathcal{P} -internal (or almost \mathcal{P} -internal) if for some (equivalently any) realization a of q, there exists $B \supseteq A$ which is independent from a over A, and c_1, \ldots, c_k realizations of types in \mathcal{P} whose parameter sets are contained in B, such that $a \in dcl(Bc_1...c_k)$ (or $a \in acl(Bc_1...c_k)$). The type q over A is \mathcal{P} -analysable if for some (equivalently any) realization a of q, there are $a_1, ..., a_k$ such that $\operatorname{stp}(a_1/A)$ is almost \mathcal{P} -internal, $a_{i-1} \in \operatorname{dcl}(Aa_i)$, $\operatorname{stp}(a_i/Aa_{i-1})$ is almost \mathcal{P} -internal for i = 2, 3, ..., k, and $\operatorname{acl}(Aa) = \operatorname{acl}(Aa_k)$. The sequence $(a_i)_{i=1}^k$ mentioned above is called a \mathcal{P} -analysis of q and a \mathcal{P} -analysis of a over A. For notational convenience, for any analysis $(a_i)_{i=1}^k$ we use a_0 to denote the empty tuple. We call k the length of the analysis. Note that an algebraic type has a \mathcal{P} -analysis of length zero, and an almost \mathcal{P} -internal type has a \mathcal{P} -analysis of length 1.

The *U*-type of the analysis is the sequence $(U(a_i/Aa_{i-1}))_{i=1}^k$. We say the analysis is nondegenerated if each entry of the *U*-type is nonzero.

Note that the definition of analysable here is in fact the definition of *almost* analysable in the literature (for example, section 1 of [6]), and we may instead say that a type is *strictly* \mathcal{P} -analysable if $\operatorname{stp}(a_i/a_{i-1})$ is internal (rather than almost internal) to \mathcal{P} . The following proposition proves that these two definitions are in fact equivalent.

Proposition 2.1. A stationary type q over A is \mathcal{P} -analysable iff it is strictly \mathcal{P} -analysable.

We need the following lemma.

Lemma 2.2. If a stationary type q over A is almost \mathcal{P} -internal, then for any $a \models q$, there exists a tuple a_0 such that $\operatorname{tp}(a_0/A)$ is \mathcal{P} -internal and $\operatorname{acl}(Aa) = \operatorname{acl}(Aa_0)$.

Proof. Given any realization $a \vDash q$, let n be the least number such that there exists an L_A -formula $\varphi(x, y, z)$, a tuple b independent from a over A and a tuple c realizing types in \mathcal{P} such that $\vDash \varphi(a, b, c)$ and $\varphi(\mathcal{U}, b, c)$ is of size n. We fix these b, c, and φ that satisfy $|\varphi(\mathcal{U}, b, c)| = n$.

Step 1. We prove that $\varphi(\mathcal{U}, b, c) \subseteq \operatorname{acl}(Aa)$.

Let $a = a_1, a_2, ..., a_n$ be the elements of $\varphi(\mathcal{U}, b, c)$. Towards a contradiction, suppose without loss of generality that $a_2 \notin \operatorname{acl}(Aa)$. Then there are a'_2, b' and c'such that $\operatorname{tp}(a'_2b'c'/Aa) = \operatorname{tp}(a_2bc/Aa)$ and $a'_2b' \downarrow_{Aa} a_2...a_nb$. Since $a'_2 \notin \operatorname{acl}(Aa)$ and $a'_2 \downarrow_{Aa} a_2...a_n b, a'_2 \notin \operatorname{acl}(Aaa_2...a_nb)$. In particular, $a'_2 \neq a_i$ for i = 1, 2, ..., n. Also, since $a \downarrow_A b$ and $b \downarrow_{Aa} b'$, we have $b \downarrow_A ab'$, and therefore $b \downarrow_{Ab'} a$. As $\operatorname{tp}(b'/Aa) = \operatorname{tp}(b/Aa)$ and $b \downarrow_A a$, we have $b' \downarrow_A a$, which, together with $b \downarrow_{Ab'} a$, yields $bb' \downarrow_A a$. Now the fact that q is almost \mathcal{P} -internal is witnessed by $a \models \varphi(x, b, c) \land \varphi(x, b', c')$, and the size of $\varphi(\mathcal{U}, b, c) \land \varphi(\mathcal{U}, b', c')$ is smaller then n (notice that $|\varphi(\mathcal{U}, b, c)| = |\varphi(\mathcal{U}, b', c')| = n$, but the two sets are not the same), contradicting minimality of n.

Step 2. Let d be the code of the set $\varphi(\mathcal{U}, b, c)$. Then $\operatorname{tp}(d/A)$ is \mathcal{P} -internal and $\operatorname{acl}(Aa) = \operatorname{acl}(Ad)$.

We have $a \in \operatorname{acl}(d) \subseteq \operatorname{acl}(Ad)$ by the definition of a code, and $d \in \operatorname{dcl}(aa_2...a_n) \subseteq \operatorname{acl}(Aa)$. Moreover, as $a \, \bigcup_A b$, we have $d \, \bigcup_A b$. Since d is the code of $\varphi(\mathcal{U}, b, c)$ where φ is an L_A -formula, $d \in \operatorname{dcl}(Abc)$. Therefore $\operatorname{tp}(d/A)$ is \mathcal{P} -internal.

Proof of Proposition 2.1. The nontrivial direction is from left to right. Suppose $(b_1, ..., b_k)$ is an analysis of a over A. For convenience, let a_0 be the empty tuple. We now construct the sequence $(a_1, ..., a_k)$.

Suppose we already have $(a_1, ..., a_{i-1})$ for $1 \le i \le k$ such that $\operatorname{stp}(a_j/Aa_{j-1})$ is \mathcal{P} -internal, $a_{j-1} \in \operatorname{dcl}(Aa_j)$, and $\operatorname{acl}(Aa_j) = \operatorname{acl}(Ab_j)$ for j = 1, 2, ..., i-1. Then as $\operatorname{stp}(b_i/Ab_{i-1})$ is almost \mathcal{P} -internal and $\operatorname{acl}(Aa_{i-1}) = \operatorname{acl}(Ab_{i-1})$, we have that

 $\operatorname{stp}(b_i/Aa_{i-1})$ is almost \mathcal{P} -internal, so by Lemma 2.2, there exists a^* such that $\operatorname{acl}(Aa_{i-1}a^*) = \operatorname{acl}(Aa_{i-1}b_i)$ and $\operatorname{tp}(a^*/Aa_{i-1})$ is \mathcal{P} -internal. Let $a_i = a_{i-1}a^*$. Then we have $a_{i-1} \in \operatorname{dcl}(Aa_i)$, $\operatorname{acl}(Aa_i) = \operatorname{acl}(Aa_{i-1}b_i) = \operatorname{acl}(Ab_{i-1}b_i) = \operatorname{acl}(Ab_i)$, and $\operatorname{tp}(a_i/Aa_{i-1})$ is \mathcal{P} -internal.

The sequence $(a_1, ..., a_k)$ then witnesses that tp(a/A) is strictly analysable. \Box

We use the following definitions in order to better describe analysable types and their analyses. We say that the type q is k-step \mathcal{P} -analysable, or \mathcal{P} -analysable in ksteps, if the analysability of q is witnessed by a \mathcal{P} -analysis of length k. A \mathcal{P} -analysis $(a_i)_{i=1}^k$ is said to be *incompressible* if $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is not almost \mathcal{P} -internal for all i = 1, 2, ..., k - 1. A \mathcal{P} -analysis of q is *minimal* if there is no \mathcal{P} -analysis of q of strictly shorter length.

The following lemma shows that incompressibility implies minimality if the U-type of an analysis is (1, 1, ..., 1).

Lemma 2.3. Let $(a_1, ..., a_n)$ be an incompressible \mathcal{P} -analysis of a over A of U-type (1, 1, ..., 1). Then the analysis is minimal, i.e., tp(a/A) is not \mathcal{P} -analysable in n-1

Proof. For n = 2, the only possibility that the analysis is not minimal is that stp(a/A) is 1-step \mathcal{P} -analysable, i.e., almost \mathcal{P} -internal, which contradicts the fact that (a_1, a_2) is an incompressible analysis.

Assume we have proved the conclusion for n < k. Suppose towards a contradiction that $(a_1, ..., a_k)$ is an incompressible \mathcal{P} -analysis of a over A of U-type (1, 1, ..., 1) which is not minimal. Let $(c_1, ..., c_{k-1})$ be another \mathcal{P} -analysis of a over

A. Note that $(a_1c_1, a_2c_2, ..., a_{k-1}c_{k-1})$ is also a \mathcal{P} -analysis of a over A. Let $b_1, ..., b_\ell$ be a subsequence of $(a_ic_i)_{i=1}^{k-1}$ such that $(b_j)_{j=1}^{\ell}$ is a nondegenerated \mathcal{P} -analysis of p. This can be done by taking away all elements a_ic_i in $(a_ic_i)_{i=1}^{k-1}$ such that $U(a_ic_i/Aa_{i-1}c_{i-1}) = 0$. Let $b_j = a$ for $\ell + 1 \leq j \leq k-1$. Then the only zero entries of the U-type of $(b_j)_{j=1}^{k-1}$ (if any) are at the end of the sequence.

If $U(b_1/A) = 1$, then $\operatorname{acl}(Ab_1) = \operatorname{acl}(Aa_1)$, and $\operatorname{stp}(a/Aa_1) = \operatorname{stp}(a/Ab_1)$. But then $(a_2, ..., a_k)$ is a k - 1-step incompressible \mathcal{P} -analysis of a over Aa_1 of U-type $\underbrace{(1, 1, ..., 1)}_{k-1}$, while $(b_2, ..., b_{k-1})$ is a k - 2-step \mathcal{P} -analysis of the same type with

shorter length, contradicting our induction hypothesis.

Now suppose $U(b_1/A) \geq 2$. If the *U*-type of $(b_j)_{j=1}^{k-1}$ is degenerated, then $U(b_{k-1}/b_{k-2}) = 0$, and we have $U(b_{k-2}/A) = U(a/A) = k$. If $(b_j)_{j=1}^{k-1}$ is nondegenerated, then $U(b_j/Ab_{j-1}) \geq 1$ for any j = 1, ..., k-2 which gives us $U(b_j/A) \geq j+1$ for any j = 1, ..., k-2. In both cases $U(b_{k-2}/A) \geq k-1$. By the induction hypothesis, $\operatorname{acl}(Ab_{k-2}) \neq \operatorname{acl}(Aa_{k-1})$: otherwise, $(a_i)_{i=1}^{k-1}$ is a k-1-step incompressible \mathcal{P} -analysis of a_{k-1} over A of U-type $\underbrace{(1,1,...,1)}_{k-1}$, while $(b_i)_{i=1}^{k-2}$ is a k-2-

step \mathcal{P} -analysis of the same type, contradicting our induction hypothesis. Similarly, $\operatorname{acl}(Ab_{k-2}) \supseteq \operatorname{acl}(Aa_{k-1})$ does not hold: otherwise $U(b_{k-2}/Aa_{k-1}) \ge 1$, and since $b_{k-2} \in \operatorname{acl}(Aa)$ and $U(a/Aa_{k-1}) = 1$, we have $\operatorname{acl}(Ab_{k-2}) = \operatorname{acl}(Aa)$; therefore $(a_2, ..., a_k)$ is a k - 1-step incompressible \mathcal{P} -analysis of $\operatorname{stp}(a/Aa_1)$ of U-type $\underbrace{(1, 1, ..., 1)}_{k-1}$, while $(b_1, ..., b_{k-2})$ is a k - 2-step \mathcal{P} -analysis of the same type,

contradicting our induction hypothesis. Hence $\operatorname{acl}(Ab_{k-2}) \supseteq \operatorname{acl}(Aa_{k-1})$ does not hold, i.e., $a_{k-1} \notin \operatorname{acl}(Ab_{k-2})$. We have $k = U(a/A) \ge U(a_{k-1}b_{k-2}/A) =$ $U(b_{k-2}/A) + U(a_{k-1}b_{k-2}/Ab_{k-2}) \ge (k-1) + 1 = k$, so $\operatorname{acl}(Ab_{k-2}a_{k-1}) = \operatorname{acl}(Aa)$. But then since $\operatorname{stp}(b_{k-2}/Aa_1)$ and $\operatorname{stp}(a_{k-1}/Aa_1)$ are k-2-step \mathcal{P} -analysable, so is $\operatorname{stp}(a/Aa_1)$, while (a_2, \dots, a_k) is a k-1-step incompressible \mathcal{P} -analysis of a over Aa_1 of U-type $\underbrace{(1, 1, \dots, 1)}_{k-2}$, contradicting our induction hypothesis. \Box

$$k - 1$$

3. Iterated Logarithmic Derivative

Our primary interest is in DCF₀, the theory of differential closed field of characteristic 0. The theory DCF₀ is complete, stable, and eliminates both quantifiers and imaginaries. We assume some familiarity of this theory. The language used is $(0, 1, +, \times, \delta)$, and $\mathcal{U} = (U, 0, 1, +, \times, \delta)$ is the saturated model, where δ is the derivative on the field. We often omit $0, 1, +, \times$ and write $\mathcal{U} = (U, \delta)$.

We focus on types which are almost C-internal or C-analysable in DCF₀, where $C = \{x : \delta x = 0\}$ is the field of constants.

We often use the term "generic type" in DCF₀. The generic type of an irreducible Kolchin closed set D over a δ -field k is the type which says that x is in D but not in any k-definable Kolchin closed subset of D. A definable set is *irreducible* if its Kolchin closure is. By the generic type of an irreducible definable set, we mean the generic type of its Kolchin closure. Note that this does not always coincide with the type of greatest U-rank.

Recall that in DCF₀, the logarithmic derivative of x is defined as $\log \delta x = \frac{\delta x}{x}$. The logarithmic derivative is used extensively in this section. Note that $\log \delta : \mathbb{G}_m \to \mathbb{G}_a$ is a definable group homomorphism between algebraic groups, and the kernel of the map is $\mathbb{G}_m(\mathcal{C})$. Here \mathbb{G}_m is the universe (take away 0) viewed as a multiplicative group, \mathbb{G}_a is the universe viewed as an additive group, and $\mathbb{G}_m(\mathcal{C})$ is the constant points of \mathbb{G}_m .

Fact 3.1 (see, for example, Fact 4.2 of [2]). Let G be the differential algebraic subgroup of \mathbb{G}_m defined by $\{x : \delta(\log \delta x) = 0\}$. The generic type of G is 2-step C-analysable but not almost C-internal.

It follows that any C-analysis of this type is of U-type (1, 1).

We will be considering iterated logarithmic derivatives. For any $n \ge 1$ we set $\log \delta^{(n)}(x) := \underbrace{\log \delta \log \delta \dots \log \delta(x)}_{n \text{ times}}$. Note that $\log \delta^{(n)}(x)$ is only defined at x if

 $\log \delta^{(i)}(x) \neq 0$ for i = 0, 1, ..., n-1 where $\log \delta^{(0)}(x) = x$. Whenever we write $\log \delta^{(n)}(x)$ it is always assumed that x is in this domain of definition. Note that for any $h \in \mathcal{U}$, the equation $\log \delta^{(n)}(x) = h$ defines an irreducible Kolchin constructible subset B of \mathcal{U} . Indeed, B is isomorphic to

$$B^* = \{ (x, \log\delta(x), ..., \log\delta^{(n-1)}(x)) : x \in B \}$$
$$= \{ (x_1, ..., x_n) : x_i \neq 0; \frac{\delta x_i}{x_i} = x_{i+1}, i = 1, 2, ..., n-1; \frac{\delta x_n}{x_n} = h \}$$

whose Kolchin closure is $\{(x_1, ..., x_n) : \delta x_i = x_i x_{i+1}, i = 1, 2, ..., n-1; \delta x_n = h x_n\},\$ which is irreducible since it is the set of *D*-points (or "sharp" set) corresponding to the irreducible *D*-variety (\mathbb{A}^n , *s*) where $s(x_1, ..., x_{n-1}, x_n) = (x_1 x_2, ..., x_{n-1} x_n, hx_n)$. (For details on D-varieties see [4].)

In particular, $\{x : \log \delta^{(2)}(x) = h\}$ is irreducible. Note also that the generic type of $\log \delta^{(2)}(x) = 0$ is the same as that of G defined in Fact 3.1. So the following proposition is a generalisation of Fact 3.1.

Proposition 3.2. Let $h \in U$ and consider $B = \{x : \log \delta^{(2)}(x) = h\}$. Let k be a δ -field containing h, and p be the generic type of B over k. Then p is not almost C-internal.

Proof. We may assume that k contains an element of the form $a = \log \delta q_0$ where $q_0 \in B$. Indeed, this follows from the fact that for any $q_0 \in B$, p is almost C-internal iff the non-forking extension of p to $k\langle g_0 \rangle$ is, and $p|k\langle g_0 \rangle$ is the generic type of B over $k\langle q_0\rangle$.

We now construct a new model $\mathcal{V} = (U, D)$ of DCF₀ as follows. The set U and the interpretation of 0, 1, + and \times remain the same, while $Dg := \frac{\delta g}{a}$ for all $g \in \mathcal{U}$. Notice that \mathcal{V} is also a model of DCF₀ with the same field of constants as \mathcal{U} , and any definable set in one model is definable in the other, with the same set of parameters, as long as the parameter set contains a. Now let q be a type in the model \mathcal{V} over k so that q and p have the same set of realizations in U. This can be done by replacing each occurrence of δ in formulas in p by aD.

Assume towards a contradiction that p is almost C-internal. Hence, for any $g \models p$, there is $B \supset k$ such that $g \downarrow_k B$ and $g \in \operatorname{acl}(BC)$, in the model \mathcal{U} . Replacing δ by aD in the formulas witnessing this fact, we have that $g \in \operatorname{acl}(BC)$ in \mathcal{V} as well. Moreover, $g igsquarepsilon_k B$ holds in \mathcal{V} because U-ranks of types are the same in \mathcal{U} and \mathcal{V} if the parameter set contains a. We get that q is almost \mathcal{C} -internal in \mathcal{V} .

However, q is the generic type of B, since Kolchin closed sets definable over k (which contains a) are the same in \mathcal{U} and \mathcal{V} . The set B is defined in \mathcal{U} by the formula $\log \delta(\log \delta x) = h$, which is just $a \log D(\log Dx) = h$, which is equivalent to $\log D(\log Df) = 0$. So q is the generic type of $B = \{x : \log D(\log Dx) = 0\}$, which is not almost C-internal in \mathcal{V} by Fact 3.1, a contradiction. \square

We can now show that the iterated logarithmic derivatives give rise to *n*-step C-analysable types that are not n - 1-step C-analysable.

Corollary 3.3. In DCF₀, let $D = \{x \in U : \underbrace{\log \delta \log \delta \dots \log \delta}_{n} x = 0\}$. Then the generic type p of D is n-step C-analysable but not n - 1-step C-analysable.

Proof. Let $a \in D$ be generic. Let $a_n = a$, $a_k = \log \delta a_{k+1}$ for k = 0, 1, ..., n-1. Note that $a_0 = 0$, $a_k \in dcl(a_{k+1})$ for k = 0, 1, ..., n-1, and a is interdefinable with $(a_1, ..., a_n).$

As a is generic in D, $a_{i+1} \notin \operatorname{acl}(a_i)$ for each i = 0, 1, ..., n-1. By additivity of U-rank, for each $i = 0, 1, ..., n - 1, U(a_{i+1}/a_i) = 1$. Hence, $stp(a_{i+1}/a_i)$ is the generic type over a_i of $\log \delta(x) = a_i$. The latter equation defines a multiplicative translation of $\mathbb{G}_m(\mathcal{C}) = \ker(\log \delta)$, so $\operatorname{stp}(a_{i+1}/a_i)$ is almost \mathcal{C} -internal of U-rank 1. That is, $(a_1, a_2, ..., a_n = a)$ is a C-analysis of p of U-type (1, 1, ..., 1).

 $\mathbf{6}$

For each i = 1, 2, ..., n - 1, $stp(a_{i+1}/a_{i-1})$ is the generic type of $log\delta^{(2)}x = a_{i-1}$ over a_{i-1} . Proposition 3.2 tells us that this type is not almost \mathcal{C} -internal. That is, (a_1, a_2, \dots, a_n) is an incompressible *C*-analysis.

Hence, by Lemma 2.3, p is not C-analysable in n-1 steps.

4. Analyses by reductions and coreductions

In this section we return to the general setting of Section 2: so T is a complete stable theory that eliminates imaginaries, \mathcal{U} is a sufficiently saturated model of T, and \mathcal{P} is a set of partial types invariant over automorphisms of the universe.

Note that Lemma 2.3 does not hold if the entries of the U-type are not all 1.

Example 4.1. Let stp(a) be 2-step \mathcal{P} -analysable with an incompressible \mathcal{P} -analysis (a_1, a) . Now let (b_1, b) be such that $bb_1 \perp aa_1$ and $stp(bb_1) = stp(aa_1)$. Let c = ab. Then c is 3-step \mathcal{P} -analysable, with an analysis $(a_1, ab_1, c = ab)$. This analysis is incompressible: $stp(ab_1)$ is not almost \mathcal{P} -internal because stp(a) is not almost \mathcal{P} -internal and $\operatorname{stp}(ab/a_1)$ is not almost \mathcal{P} -internal because $\operatorname{stp}(b)$ is not almost \mathcal{P} -internal, and $\operatorname{stp}(b/a_1)$ is its non-forking extension. But c is 2-step \mathcal{P} -analysable by $(a_1b_1, c = ab)$, so the \mathcal{P} -analysis $(a_1, ab_1, c = ab)$ is not minimal despite being incompressible.

To generalize Lemma 2.3 to higher U-rank cases, we need each step to satisfy some maximality or minimality property. We will use the notions of \mathcal{P} -reduction and \mathcal{P} -coreduction.

Definition 4.2 (See, for example, Section 4 of [5]). Suppose a is a tuple and A is a parameter set. We say a tuple b is a \mathcal{P} -reduction of a over A if b is maximally almost \mathcal{P} -internal over A in acl(Aa), i.e., stp(b/A) is almost \mathcal{P} -internal, $b \in acl(Aa)$, and if $b' \in \operatorname{acl}(Aa)$ and $\operatorname{stp}(b'/A)$ is almost \mathcal{P} -internal then $b' \in \operatorname{acl}(Ab)$. We say a nondegenerated \mathcal{P} -analysis (a_1, \dots, a_n) of a over A is a \mathcal{P} -analysis by reductions of a over A if a_k is the \mathcal{P} -reduction of a over Aa_{k-1} for k = 1, 2, ..., n.

Note that by definition \mathcal{P} -reductions are unique up to interalgebraicity over the parameter set, i.e., if b and c are both \mathcal{P} -reductions of a over A, then $\operatorname{acl}(Ab) =$ $\operatorname{acl}(Ac)$. We may therefore call b the \mathcal{P} -reduction of a over A.

Remark 4.3. It is clear that if $U(a/A) < \omega$, then a \mathcal{P} -reduction of a over A always exists. In fact, let b be a tuple that has maximal U-rank over A satisfying the condition that stp(b/A) is almost \mathcal{P} -internal and $b \in acl(Aa)$. Then b is a \mathcal{P} reduction of a over A: if c also satisfies this condition, then stp(bc/A) is almost \mathcal{P} -internal and $bc \in \operatorname{acl}(Aa)$, so U(bc/A) = U(b/A), which means $c \in \operatorname{acl}(Ab)$. Hence, if tp(a/A) is \mathcal{P} -analysable of finite U-rank then a \mathcal{P} -analysis by reductions always exists.

Definition 4.4 (See, for example, Definition 4.1 of [5]). Suppose a is a tuple and A is a parameter set. We say a tuple b is a \mathcal{P} -coreduction of a over A if b is minimal in $\operatorname{acl}(Aa)$ such that a is almost \mathcal{P} -internal over Ab, i.e., $\operatorname{stp}(a/Ab)$ is almost \mathcal{P} internal, $b \in \operatorname{acl}(Aa)$, and if $b' \in \operatorname{acl}(aA)$ and b' satisfies that $\operatorname{stp}(a/Ab')$ is almost \mathcal{P} -internal then $b \in \operatorname{acl}(Ab')$. We say a nondegenerated \mathcal{P} -analysis $(a_1, ..., a_n)$ of a over A is a \mathcal{P} -analysis by coreductions of a over A if a_{k-1} is a \mathcal{P} -coreduction of a_k over A for k = 2, ..., n.

Note similarly that by definition \mathcal{P} -coreductions are unique up to interalgebraicity over the parameter set. We may therefore call b the \mathcal{P} -coreduction of a over A.

Recall that T has the canonical base property (CBP) if whenever $U(a/b) < \omega$ and $\operatorname{acl}(b) = \operatorname{acl}(\operatorname{Cb}(a/b))$, then $\operatorname{stp}(b/a)$ is almost \mathbb{P} -internal, where \mathbb{P} is the set of all nonmodular minimal types. See, for example, Section 1 of [6]. It is a fact that if T has CBP then \mathbb{P} -coreductions exist for any finite-rank type (see Theorem 2.4 of [3]). Hence, assuming T has CBP, if $\operatorname{stp}(a/A)$ is \mathbb{P} -analysable of finite U-rank then a \mathbb{P} -analysis by coreductions always exists.

The following lemma shows that in $DCF_0 C$ -coreductions of any finite-rank type always exist. This is because any nonmodular minimal type in DCF_0 is almost C-internal.

Lemma 4.5. We work in DCF_0 in this lemma. If U(a/A) is finite, then the C-coreduction of a over A exists.

Proof. Let \mathbb{P} be the set of all nonmodular minimal types in $\mathcal{U} \models \text{DCF}_0$. By Theorem 1.1 of [8], DCF₀ has CBP. Therefore, there exists b which is the \mathbb{P} -coreduction of a over A.

We want to show that b is the C-coreduction of a over A. In fact, we only need to show that if a type is almost \mathbb{P} -internal then it is almost C-internal. Suppose $\operatorname{tp}(e/D)$ is \mathbb{P} -internal. Then for some $B \supset D$ such that $B \bigcup_{D} e$ and a tuple c consists of realizations of types in \mathbb{P} with bases in B, $e \in \operatorname{acl}(Bc)$. Since every minimal nonmodular type in DCF₀ is almost C-internal, there exist $F \supset B$ such that $F \bigcup_{B} ec$ and $c \in \operatorname{acl}(FC)$. Now $e \in \operatorname{acl}(Bc) \subseteq \operatorname{acl}(FC)$, and since $e \bigcup_{B} F$ and $e \bigcup_{D} B$, we have $e \bigcup_{D} F$. This shows that $\operatorname{tp}(e/D)$ is almost C-internal. \Box

It is not hard to see that analyses by reductions or coreductions are incompressible. If $(a_1, ..., a_n)$ is a \mathcal{P} -analysis by reductions of $\operatorname{tp}(a/A)$ and $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is almost \mathcal{P} -internal for some i = 1, 2, ..., n - 1, then since a_i is the \mathcal{P} -reduction of a over $Aa_{i-1}, a_{i+1} \in \operatorname{acl}(Aa_i)$ which implies $\operatorname{acl}(Aa_i) = \operatorname{acl}(Aa_{i+1})$. Now for any j > i, assume that $\operatorname{acl}(Aa_j) = \operatorname{acl}(Aa_i)$. Then since a_{j+1} is the \mathcal{P} -reduction of a over Aa_j and $\operatorname{acl}(Aa_j) = \operatorname{acl}(Aa_i)$. Then since a_{j+1} is the \mathcal{P} -reduction of a over Aa_j and $\operatorname{acl}(Aa_j) = \operatorname{acl}(Aa_i)$. Thus $a_i, ..., a_n$ are all the same up to interalgebraicity over A, and this is possible only if i = n, contradicting the fact that $i \leq n - 1$. Similarly, if $(a_1, ..., a_n)$ is a \mathcal{P} -analysis by coreductions of $\operatorname{tp}(a/A)$ and $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is almost \mathcal{P} -internal for some i = 1, 2, ..., n - 1, then since a_i is the \mathcal{P} -coreduction of a_{i+1} over $a_{i-1}, a_i \in \operatorname{acl}(Aa_{i-1})$ which implies a_i and a_{i-1} are interalgebraic over A. An inductive argument similar to the reduction case shows that $a_0, ..., a_i$ are all the same up to interalgebraicity over A, and this is possible only if i = 0, contradicting the fact that $i \geq 1$.

However, more is true: they are actually minimal.

Proposition 4.6. Analysis by reductions and coreductions are minimal.

Proof. Let $(a_1, ..., a_n)$ and $(c_1, ..., c_\ell)$ be \mathcal{P} -analyses of a over A with $(a_1, ..., a_n)$ being by reductions. We shall prove that $n \leq \ell$. We show that $c_i \in \operatorname{acl}(Aa_i)$ for $i = 1, 2, ..., \min(n, \ell)$. For i = 1, since $\operatorname{stp}(c_1/A)$ is almost \mathcal{P} -internal and a_1 is the \mathcal{P} -reduction of a over A, $c_1 \in \operatorname{acl}(Aa_1)$. Now if $c_{i-1} \in \operatorname{acl}(Aa_{i-1})$, then

 $\operatorname{stp}(c_i/a_{i-1})$ is almost \mathcal{P} -internal, and as a_i is the \mathcal{P} -reduction of a over Aa_{i-1} , $c_i \in \operatorname{acl}(Aa_i)$ as desired. Suppose $\ell < n$. Then $\operatorname{acl}(Aa_\ell) \subsetneq \operatorname{acl}(Aa_n)$ since (a_1, \dots, a_n) is incompressible, so $\operatorname{acl}(Aa) = \operatorname{acl}(Ac_\ell) \subseteq \operatorname{acl}(Aa_\ell) \subsetneq \operatorname{acl}(Aa_n) = \operatorname{acl}(Aa)$, a contradiction.

Now suppose $(b_1, ..., b_m)$ is a \mathcal{P} -analysis by coreductions of a over A. We shall prove that $m \leq \ell$. We show that $b_{m-j} \in \operatorname{acl}(Ac_{\ell-j})$ for $j = 0, 1, ..., \min(m, \ell) - 1$. For j = 0, notice that b_m, c_ℓ are both interalgebraic over A with a. Now if $b_{m-j+1} \in$ $\operatorname{acl}(Ac_{\ell-j+1})$, then $\operatorname{stp}(b_{m-j+1}/c_{\ell-j})$ is almost \mathcal{P} -internal, and as b_{m-j} is the \mathcal{P} coreduction of b_{m-j+1} over $A, b_{m-j} \in \operatorname{acl}(Ac_{\ell-j})$ as desired. Assume towards a contradiction that $\ell < m$. Then $\operatorname{acl}(Ab_{m-\ell+1}) \subseteq \operatorname{acl}(Ac_1)$. Since $m - \ell + 1 \geq 2$, $\operatorname{stp}(b_{m-\ell+1}/A)$ is not almost \mathcal{P} -internal because $(b_1, ..., b_m)$ is incompressible, but $\operatorname{stp}(c_1/A)$ is almost \mathcal{P} -internal, a contradiction. \Box

So analyses by reductions and coreductions are of the same length. However, analyses by reductions and coreductions do not always have to agree (even up to interalgebraicity).

Definition 4.7. We say that two \mathcal{P} -analyses $(a_1, ..., a_n)$ and $(b_1, ..., b_m)$ of a over A are *interalgebraic over* A if n = m and $\operatorname{acl}(Aa_i) = \operatorname{acl}(Ab_i)$ for i = 1, 2, ..., n. We call an analysis *canonical* if it is minimal and interalgebraic with every other minimal analysis.

Example 4.8. Using the notation of Example 4.1, the \mathcal{P} -analysis by reductions of ab_1 over \emptyset is (a_1b_1, ab_1) , while the \mathcal{P} -analysis by coreductions of ab_1 is (a_1, ab_1) . But (a_1b_1, ab_1) and (a_1, ab_1) are not interalgebraic. In particular, $stp(ab_1)$ does not have a canonical \mathcal{P} -analysis.

The next proposition points out, however, that if an analysis by reductions has the same U-type as one by coreductions, then they are interalgebraic and are in fact the unique minimal analysis up to interalgebraicity.

Proposition 4.9. Let $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ be \mathcal{P} -analyses by reductions and coreductions of a over A, respectively. If the U-types of $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ are the same, then $(a_1, ..., a_n)$ is interalgebraic with $(b_1, ..., b_n)$ over A. Moreover, if $(c_1, ..., c_n)$ is another \mathcal{P} -analysis of a over A, then $(c_1, ..., c_n)$ is also interalgebraic with both $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ over A.

In particular, if p has an analysis by reductions and an analysis by coreductions of the same U-type, then these analyses are canonical. Conversely, any canonical analysis is an analysis by both reductions and coreductions.

Proof. Having the same U-type implies that $U(a_i/A) = U(b_i/A)$ for i = 1, 2, ..., n. Let $(c_1, ..., c_n)$ be another \mathcal{P} -analysis of a over A, We have seen in the proof of 4.6 that $c_i \in \operatorname{acl}(Aa_i)$ and $b_i \in \operatorname{acl}(Ac_i)$ for i = 1, 2, ..., n. Therefore $U(a_i/A) = U(b_i/A) = U(c_i/A)$ and $\operatorname{acl}(Aa_i) = \operatorname{acl}(Ab_i) = \operatorname{acl}(Ac_i)$ for i = 1, 2, ..., n, as desired.

The "in particular" clause now follows by Proposition 4.6. For the converse, let $(a_i)_{i=1}^n, (b_i)_{i=1}^n, (c_i)_{i=1}^n$ be \mathcal{P} -analyses of a over A, which are an analysis by reductions, an analysis by coreductions, and a canonical analysis, respectively. We have that a_i is the \mathcal{P} -reduction of a over Aa_{i-1} , $\operatorname{acl}(Aa_i) = \operatorname{acl}(Ac_i)$, and $\operatorname{acl}(Aa_{i-1}) = \operatorname{acl}(Ac_{i-1})$, so c_i is the \mathcal{P} -reduction of a over Ac_{i-1} . Thus $(c_i)_{i=1}^n$ is a \mathcal{P} -analysis by reductions. Similarly, we have that b_i is the \mathcal{P} -coreduction of b_{i+1} over A, $\operatorname{acl}(Ab_i) = \operatorname{acl}(Ac_i)$, and $\operatorname{acl}(Ab_{i+1}) = \operatorname{acl}(Ac_{i+1})$, so c_i is the \mathcal{P} -coreduction of a over Ac_{i-1} . Thus $(c_i)_{i=1}^n$ is a \mathcal{P} -analysis by coreductions. \Box

Here is a local criterion to determine whether an analysis is an analysis by reductions.

Lemma 4.10. Let $(a_1, ..., a_n)$ be a \mathcal{P} -analysis of a over A. Then it is a \mathcal{P} -analysis by reductions iff a_i is a \mathcal{P} -reduction of a_{i+1} over Aa_{i-1} for i = 1, ..., n-1.

Proof. Suppose $(a_1, ..., a_n)$ is a \mathcal{P} -analysis by reductions of a over A. For any k = 1, 2, ..., n - 1, a_k is a \mathcal{P} -reduction of a over Aa_{k-1} , i.e., for any $a'_k \in \operatorname{acl}(Aa)$, if $\operatorname{stp}(a'_k/Aa_{k-1})$ is almost \mathcal{P} -internal, then $a'_k \in \operatorname{acl}(a_k)$. In particular, for any $a'_k \in \operatorname{acl}(Aa_{k+1})$, if $\operatorname{stp}(a'_k/Aa_{k-1})$ is almost \mathcal{P} -internal, then $a'_k \in \operatorname{acl}(a_k)$. Note that $a_k \in \operatorname{acl}(Aa_{k+1})$, so a_k is a \mathcal{P} -reduction of a_{k+1} over Aa_{k-1} .

Now suppose $(a_1, ..., a_n)$ is a \mathcal{P} -analysis of a over A such that a_i is a \mathcal{P} -reduction of a_{i+1} over Aa_{i-1} for i = 1, ..., n-1. We need to check that a_k is the \mathcal{P} -reduction of a over Aa_{k-1} . In fact, let a'_k be the \mathcal{P} -reduction of a over Aa_{k-1} , then we only need to show that $a'_k \in \operatorname{acl}(Aa_k)$.

We know $a'_k \in \operatorname{acl}(Aa_n)$. Suppose $a'_k \in \operatorname{acl}(Aa_i)$ for some i such that $k < i \leq n$. Since a'_k is almost \mathcal{P} -internal over Aa_{k-1} and k-1 < i-1, a'_k is \mathcal{P} -internal over Aa_{i-2} . Now a_{i-1} is a \mathcal{P} reduction of a_i over Aa_{i-2} , $a'_k \in \operatorname{acl}(Aa_i)$, and a'_k is almost \mathcal{P} -internal over Aa_{i-2} , so $a'_k \in \operatorname{acl}(Aa_{i-1})$. By induction we get $a'_k \in \operatorname{acl}(Aa_k)$. \Box

We have a similar criterion for analyses by coreductions.

Lemma 4.11. A \mathcal{P} -analysis $(a_1, ..., a_n)$ of a over A is a \mathcal{P} -analysis by coreductions iff a_i is a \mathcal{P} -coreduction of a_{i+1} over Aa_{i-1} for i = 1, ..., n-1.

Proof. Suppose $(a_1, ..., a_n)$ is a \mathcal{P} -analysis by coreductions of a over A. For any $k = 1, 2, ..., n-1, a_k$ is a \mathcal{P} -coreduction of a_{k+1} over A, i.e., for any $a'_k \in \operatorname{acl}(Aa_{k+1})$, if $\operatorname{stp}(a_{k+1}/Aa'_k)$ is \mathcal{P} -internal, then $a_k \in \operatorname{acl}(Aa'_k)$. In particular, for any $a'_k \in \operatorname{acl}(Aa_{k+1})$, if $\operatorname{stp}(a_{k+1}/Aa_{k-1}a'_k)$ is \mathcal{P} -internal, then $a_k \in \operatorname{acl}(Aa_{k-1}a'_k)$. So we have that a_k is a reduction of a_{k+1} over Aa_{k-1} .

Now suppose $(a_1, ..., a_n)$ is a \mathcal{P} -analysis of a over A such that a_i is a \mathcal{P} -coreduction of a_{i+1} over Aa_{i-1} for i = 1, ..., n-1. Fixing a $k \in \{1, 2, ..., n-1\}$, we need to check that a_k is the \mathcal{P} -coreduction of a_{k+1} over A. In fact, let a' be be such that $\operatorname{stp}(a_{k+1}/Aa')$ is almost \mathcal{P} -internal. We need to prove that $a_k \in \operatorname{acl}(Aa')$.

We know that $a_1 \in \operatorname{acl}(Aa')$. This is because a_1 is the \mathcal{P} -coreduction of a_2 over A, and $\operatorname{stp}(a_2/Aa')$ is almost \mathcal{P} -internal (since $a_2 \in \operatorname{dcl}(Aa_{k+1})$).

Suppose $a_{i-1} \in \operatorname{acl}(Aa')$ for some *i* such that $1 < i \leq k$. Since a_{i+1} is almost \mathcal{P} internal over Aa' (as $i+1 \leq k+1$, $a_{i+1} \in \operatorname{acl}(Aa_{k+1})$), and a_i is the \mathcal{P} -coreduction of a_{i+1} over Aa_{i-1} , we have that $a_i \in \operatorname{acl}(Aa')$. By induction we get $a_k \in \operatorname{acl}(Aa')$. \Box

It follows from the above lemma that an incompressible analysis of U-type (1, 1, ..., 1) is canonical. Indeed, for such an analysis $(a_1, ..., a_n)$ of a over A, as $stp(a_{i+1}/Aa_{i-1})$ is not almost \mathcal{P} -internal, by rank consideration, a_i must be both the \mathcal{P} -reduction and the \mathcal{P} -coreduction of a_{i+1} over Aa_{i-1} for i = 1, 2, ..., n-1.

We end this section by pointing out that once we have a type with an incompressible analysis of U-type (1, 1, ..., 1) – as for example we do in DCF₀ by Corollary

3.3 – then every decreasing sequence of positive integers of length n appears as the \mathcal{U} -type of the \mathcal{P} -analysis by reductions of some other type in this theory. A similar statement holds for increasing sequences and \mathcal{P} -analyses by coreductions provided that every finite U-rank type has a \mathcal{P} -coreduction. For convenience we work over the empty set.

Proposition 4.12. Suppose $(a_1, ..., a_n)$ is a \mathcal{P} -analysis of a of U-type (1, 1, ..., 1).

- (a) Given positive integers $s_1 \ge ... \ge s_n$, there exists a tuple whose \mathcal{P} -analysis by reductions is of U-type $(s_1, ..., s_n)$.
- (b) Suppose every type of finite U-rank has a \mathcal{P} -coreduction. Given positive integers $s_1 \leq ... \leq s_n$, there exists a tuple whose \mathcal{P} -analysis by coreductions is of U-type $(s_1, ..., s_n)$.

Proof. (a) Let $\bar{a}^{(j)} = (a_1^{(j)}, ..., a_n^{(j)})$, j = 1, 2, ... be tuples such that $(\bar{a}^{(1)}, \bar{a}^{(2)}, ...)$ is a Morley sequence of $\operatorname{tp}(a_1, ..., a_n)$. In particular, $a_i^{(j)}$ is the \mathcal{P} -reduction and the \mathcal{P} -coreduction of $a_{i+1}^{(j)}$. Let $\alpha_i = (a_i^{(1)}, ..., a_i^{(s_i)})$ and $\beta_i = (\alpha_1, ..., \alpha_i)$. Note that $a_i^{(j)} \in \beta_i$ for $j = 1, 2, ..., s_i$. We claim the tuple β_n is \mathcal{P} -analysable and its \mathcal{P} -analysis by reductions is of U-type $(s_1, ..., s_n)$. To show this, since $(\bar{a}^{(j)})_j$ is a Morley sequence, we have

$$U(\beta_i/\beta_{i-1}) = U(\alpha_i/\beta_{i-1}) = U(a_i^{(1)}...a_i^{(s_i)}/\beta_{i-1}) = U(a_i^{(1)}...a_i^{(s_i)}/a_{i-1}^{(1)}a_{i-1}^{(s_i)}) = s_i,$$

l

so we only need to prove that the \mathcal{P} -analysis by reductions of β is $(\beta_1, \beta_2, ..., \beta_n)$.

Let b_i be the reduction of β_n over β_{i-1} . We claim that b_i is interalgebraic with β_i . Since $a_{i-1}^{(j)} \in \operatorname{dcl}(\beta_{i-1})$ for $j = 1, 2, ..., s_i$ (since $s_{i-1} \geq s_i$), $\operatorname{stp}(a_i^{(j)}/\beta_{i-1})$ is almost \mathcal{P} -internal for $j = 1, 2, ..., s_i$, so $\operatorname{stp}(\alpha_i/\beta_{i-1})$ is almost \mathcal{P} -internal. Since $\beta_i \in \operatorname{dcl}(\alpha_i, \beta_{i-1})$, $\operatorname{stp}(\beta_i/\beta_{i-1})$ is almost \mathcal{P} -internal, so $\beta_i \in b_i$. We now need to show that $U(b_i/\beta_i) = 0$. Toward a contradiction, suppose $U(b_i/\beta_i) > 0$.

Set $B = \beta_i$, which is the collection of elements of the form $a_p^{(q)}$ where $1 \le p \le i$ and $1 \le q \le s_i$. Now we add elements of the form $a_p^{(q)}$ one by one into B according to dictionary order of (p,q) where $i+1 \le p \le n$ and $1 \le q \le s_i$ as long as $U(b_i/B)$ remains unchanged. Since $b_i \in \beta_n$, $U(b_i/\beta_n) = 0$, so this process will terminate for some $a_p^{(q)}$ where $U(b_i/Ba_p^{(q)}) < U(b_i/B)$.

Now *B* contains elements of the form $a_{p'}^{(q')}$ where (p',q') < (p,q) by dictionary order. We have $a_p^{(q)} \not\perp_B b_i$. As $a_{p-1}^{(q)} \in B$ and $a_p^{(q)} \not\downarrow_{a_{p-1}^{(q)}} B$, $U(a_p^{(q)}/B) = 1$, so

 $a_p^{(q)} \in \operatorname{acl}(Bb_i)$. However, Let $C = \{a_i^{(j)} : a_{i+1}^{(j)} \in \operatorname{dcl}(B)\}$. Then $\operatorname{stp}(B/C)$ is almost \mathcal{P} -internal as $\operatorname{stp}(a_{i+1}^{(j)}/a_i^{(j)})$ is almost internal for any i, j, and $\operatorname{stp}(b_i/C)$ is almost \mathcal{P} -internal because $\beta_{i-1} \in \operatorname{dcl}(C)$. But $\operatorname{stp}(a_p^{(q)}/C)$ is not almost \mathcal{P} -internal: since $a_{p-1}^{(q)} \notin \operatorname{acl}(a_{p-2}^{(q)})$ and $a_{p-1}^{(q)} \underset{a_{p-2}^{(q)}}{\cup} C$, we have $a_{p-1}^{(q)} \notin \operatorname{acl}(C)$.

(b) Let $\bar{a}^{(j)} = (a_1^{(j)}, ..., a_n^{(j)}), \ j = 1, 2, ...$ be tuples such that $(\bar{a}^{(1)}, \bar{a}^{(2)}, ...)$ is a Morley sequence of $\operatorname{tp}(a_1, ..., a_n/A)$. Let $\beta_i = (a_1^{(1)} ... a_1^{(s_{n-i+1})}, ..., a_i^{(1)} ... a_i^{(s_1)})$. Let $f(j) = \min\{k : j \le s_k\}$, and let f(j) be infinity if it is not defined. Then $a_k^{(j)} \in \beta_i$ iff $k \le i - f(j) + 1$ and $\beta_i = \bigcup_{j=1}^{s_i} a_{i+1-f(j)}^{(j)}$. We claim the tuple β_n is \mathcal{P} -analysable

and its \mathcal{P} -analysis by coreductions is of U-type $(s_1, ..., s_n)$. Since $\beta_i = \bigcup_{j=1}^{s_i} a_{i+1-f(j)}^{(j)}$

and $\beta_{i-1} = \bigcup_{j=1}^{s_i} a_{i-f(j)}^{(j)}$ (as i - f(j) = 0 for $s_{i-1} < j \le s_i$, we may set the upper bound as s_i), we have

$$U(\beta_i/\beta_{i-1}) = U(\bigcup_{j=1}^{s_i} a_{i+1-f(j)}^{(j)}/\beta_{i-1})$$
$$= \sum_{j=1}^{s_i} U(a_{i+1-f(j)}^{(j)}/a_{i-f(j)}^{(j)})$$
$$= s_i$$

as $(\bar{a}^{(j)})_j$ is a Morley sequence. Thus we only need to prove that the \mathcal{P} -analysis by coreductions of β is $(\beta_1, \beta_2, ..., \beta_n)$.

Suppose b is the \mathcal{P} -coreduction of β_{i+1} over the empty set. We claim that $\operatorname{acl}(b) = \operatorname{acl}(\beta_i)$. Note that $\operatorname{stp}(\beta_{i+1}/\beta_i)$ is almost \mathcal{P} -internal, so $b \in \operatorname{acl}(\beta_i)$. Take any $a_j^{(k)} \in \beta_i$. Since $a_{j+1}^{(k)} \in \beta_{i+1}$ and β_{i+1} is almost \mathcal{P} -internal over b, $a_{j+1}^{(k)}$ is almost \mathcal{P} -internal over b, so $a_j^{(k)} \in b$ since $a_j^{(k)}$ is the \mathcal{P} -coreduction of $a_{j+1}^{(k)}$. We therefore have that $\beta_i \in \operatorname{acl}(b)$.

5. A Construction in DCF_0

In this section we show that in DCF_0 we can do better than the conclusions of Proposition 4.12. Given any sequence of positive integers we provide a type which has a canonical C-analysis with that U-type. Throughout we use the fact proven in Lemma 4.5 that any finite rank type has a C-coreduction.

Suppose $n_1, ..., n_\ell$ are positive integers. We want to construct a type admitting a C-analysis in ℓ steps where the *i*th step has U-rank n_i , and such that the analysis is canonical. Here is our construction.

For convenience, we name everything in \mathbb{Q}^{alg} in the language. Let $c_{ij} \in \mathbb{Q}^{\text{alg}}$ be algebraic numbers for $i = 1, 2, ..., \ell$ and $1 \leq j \leq n_i$ such that $\{c_{ij}\}_{j=1}^{n_i}$ is \mathbb{Q} -linearly independent for $i = 1, 2, ..., \ell$.

We inductively define (D_i, e_i) for $i = 1, 2, ..., \ell$ as follows:

Set $D_1 := \delta$ and let e_1 be a generic solution over \emptyset to

(E₁)
$$(D_1 - c_{11})(D_1 - c_{12})...(D_1 - c_{1n_1})x = 0.$$

For i > 1 set $D_i := \frac{\delta}{\prod_{j=1}^{i-1} e_j}$ and let e_i be a generic solution over $\{e_1, \dots, e_{i-1}\}$ to

(E_i)
$$(D_i - c_{i1})(D_i - c_{i2})...(D_i - c_{in_i})x = 0.$$

The notation $D_i - c_{ij}$ here represents a linear operator which sends y to $D_i y - c_{ij} y$, so equation (E_i) is a linear differential equation over $\{e_1, \dots e_{i-1}\}$ of order n_i .

Now let $a_i = (e_1, ..., e_i)$ for i = 1, 2, ..., n, and $a_0 = \emptyset$. We will show that $(a_1...a_\ell)$ is a canonical \mathcal{C} -analysis of a_ℓ of U-type $(n_1, ..., n_\ell)$.

Since e_i is a generic solution of (E_i) , an order n_i linear differential equation over a_{i-1} , we have $U(a_i/a_{i-1}) = n_i$, and $\operatorname{stp}(a_i/a_{i-1})$ is almost \mathcal{C} -internal. So this is

a C-analysis of the correct U-type. We need to show it is by C-reductions and C-coreduction.

Fixing $i \in \{1, 2, ..., \ell\}$, the following coordinatisation of solutions of (E_i) is a useful tool that we will apply often.

Lemma 5.1. If f is any solution to (E_i) then we can decompose $f = \sum_{j=1}^{n_i} f_j$ such that each f_j is a solution to $D_i x - c_{ij} x = 0$ and f is interdefinable with $(f_1, ..., f_{n_i})$ over a_{i-1} .

Proof. Indeed, let g_j be a generic solution of $D_i x - c_{ij} x = 0$. The set $\{g_j : j = 1, 2, ..., n_i\}$ is *C*-linearly independent because g_j 's are nonzero eigenvectors of different eigenvalues under the *C*-linear operator D_i . Note that since $(D_i - c_{ij})$ commutes with $(D_i - c_{ij'})$ for any j, j', each g_j is a solution to (E_i) . Since (E_i) is an order n_i linear differential equation and $\{g_j : j = 1, 2, ..., n_i\}$ is a set of *C*-linearly independent solutions of (E_i) , any solution of (E_i) is a *C*-linear combination of g_j 's. In

particular, f is of the form $\sum_{j=1}^{n} u_j g_j$ where $u_j \in \mathcal{C}$ for $j = 1, ..., n_i$. Let $f_j = u_j g_j$,

so
$$f = \sum_{j=1}^{n_i} f_j$$
, and $f \in \operatorname{dcl}(f_1, ..., f_{n_i})$. Also,
 $D_i f_j - c_{ij} f_j = u_j (D_i g_j - c_{ij} g_j) = 0$,

so f_i is a solution to $D_i x - c_{ij} x = 0$.

We still need to verify that $(f_1, ..., f_{n_i}) \in \operatorname{dcl}(a_{i-1}f)$. Indeed, suppose $(f_j^*)_{j=1}^{n_i}$ and $(f_j)_{j=1}^{n_i}$ have the same type over $a_{i-1}f$. Then in particular f_j^* is a solution to $D_i x - c_{ij} x = 0$, and

$$\sum_{j=1}^{n_i} f_j = f = \sum_{j=1}^{n_i} f_j^*$$

which gives us $\sum_{j=1}^{n_i} (f_j - f_j^*) = 0$. As $\{f_j - f_j^* : j = 1, 2, ..., n_i\}$ is a set of eigenvectors

of different eigenvalues under the C-linear operator D_i , we then have $f_j - f_j^* = 0$ for all $j = 1, 2, ..., n_i$, so $(f_j^*)_{j=1}^{n_i} = (f_j)_{j=1}^{n_i}$.

Lemma 5.2. If f is a generic solution to (E_i) over a_{i-1} , then $\{f_1, ..., f_{n_i}\}$ obtained in Lemma 5.1 is independent over a_{i-1} and each f_j is a generic solution to $D_i x - c_{ij} x = 0$.

Proof. Since f is a generic solution over a_{i-1} to (E_i) , which is a linear differential equation of order n_i , we have $U(f/a_{i-1}) = n_i$ Since f_j is a solution for $D_i x - c_{ij} x = 0$, $U(f_{ij}/a_{i-1}) \leq 1$. But

$$n_{i} = U(f/a_{i-1})$$

= $U(f_{1}f_{2}...f_{n_{i}}/a_{i-1})$
= $U(f_{1}/a_{i-1}) + U(f_{2}/a_{i-1}f_{1}) + ... + U(f_{n_{i}}/a_{i-1}f_{1}f_{2}...f_{n_{i}-1})$
 $\leq U(f_{1}/a_{i-1}) + U(f_{2}/a_{i-1}) + ... + U(f_{n_{i}}/a_{i-1})$
 $\leq n_{i}.$

So $U(f_j/a_{i-1}) = 1$ and $U(f_j/a_{i-1}f_1f_2...f_{j-1}) = 1$ for $j = 1, 2, ..., n_i$. This means that $\{f_1, ..., f_{n_i}\}$ is independent over a_{i-1} and each f_j is a generic solution to $D_i x - c_{ij} x = 0.$

Lemma 5.3. Let f be a generic solution over \mathbb{Q}^{alg} to (E_1) . Then $\operatorname{acl}(f) \cap \mathcal{C} = \mathbb{Q}^{\text{alg}}$.

Proof. Let $m = n_1$. Let $(f_1, ..., f_m)$ be the decomposition of f by Lemma 5.1 with respect to (E₁). Since f is generic, $f_j \neq 0$ for j = 1, 2, ..., m. Suppose the conclusion is false and there exists some c such that $c \in (\operatorname{acl}(f) \cap \mathcal{C}) \setminus \mathbb{Q}^{\operatorname{alg}}$. Note that $\operatorname{acl}(f) = \mathbb{Q}(f_1, \dots, f_m)^{\operatorname{alg}}$ since $\delta f_j = c_{1j} f_j \in \mathbb{Q}^{\operatorname{alg}}(f_j)$.

For simplicity, let $\overline{f} = (f_1, ..., f_m)$, and $\overline{y} = (y_1, ..., y_m)$. Let $F(x, \overline{y})$ be a polynomial with coefficients in \mathbb{Q}^{alg} such that $F(c, \bar{f}) = 0$ and $F(x, \bar{f}) \neq 0$. Since $c \notin \mathbb{Q}^{alg}, F(c,\bar{y}) \neq 0$. Let $G(\bar{y})$ be a nonzero polynomial over \mathcal{C} with minimal number of terms such that $G(\bar{f}) = 0$. Since $F(c, \bar{y}) \neq 0$ and $F(c, \bar{f}) = 0$, $F(c, \bar{y})$ satisfies all conditions of G except for the minimality, so such a G exists.

Let

$$G(\bar{y}) = \sum_{\bar{r}\in I} s_{\bar{r}} \bar{y}^{\bar{r}},$$

where I is a set of m-tuples of nonnegative integers, $\bar{y}^{\bar{r}} = y_1^{r_1} \dots y_m^{r_m}$, and $s_{\bar{r}} \in \mathcal{C}$. Let $\bar{c} = (c_{11}, ..., c_{1m})$, and set $\bar{f}\bar{c} := \sum_{i=1}^{m} f_j c_{1j}$.

We claim that

$$\bar{r}^{(1)}\bar{c} = \bar{r}^{(2)}\bar{c}$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Indeed, otherwise, fixing any $\bar{r}^* \in I$, we have $G^*(\bar{y}) := \bar{r}^* \bar{c} G(\bar{y}) - \delta(G(\bar{y}))$

$$= \sum_{\bar{r}\in I} (\bar{r}^*\bar{c}) s_{\bar{r}} \bar{y}^{\bar{r}} - \sum_{\bar{r}\in I} s_{\bar{r}} \delta \bar{y}^i$$
$$= \sum_{\bar{r}\in I} (\bar{r}^*\bar{c} - \bar{r}\bar{c}) s_{\bar{r}} \bar{y}^{\bar{r}}$$

is a polynomial with fewer terms than G (since the term with index \bar{r}^* is cancelled) such that its coefficients are in \mathcal{C} , $G^*(f) = 0$ since $G(f) = \delta(G(f)) = 0$, and $G^*(\bar{y}) \neq 0$ as there exist $\bar{r} \in I$ such that $\bar{r}\bar{c} \neq \bar{r}^{(*)}\bar{c}$. This contradicts the minimality of G.

We now have $\bar{r}^{(1)}\bar{c} = \bar{r}^{(2)}\bar{c}$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$, i.e., $(\bar{r}^{(1)} - \bar{r}^{(2)})\bar{c} = 0$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. But $\{c_{11}, \dots, c_{1m}\}$ is Q-linearly independent, so in fact $\bar{r}^{(1)} = \bar{r}^{(2)}$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Therefore there is only one element \bar{r} in I, and $G(\bar{f}) = s_{\bar{r}} \bar{f}^{\bar{r}}$. Since all f_j 's are nonzero, $s_{\bar{r}} = 0$, so G is the zero polynomial, a contradiction. \square

The following lemma is the technical heart of the construction.

Lemma 5.4. Fix $i \in \{1, 2, ..., \ell - 1\}$, and for notational convenience, let $m := n_i$ and $L := \operatorname{acl}(a_{i-1})$. Then the following are true:

- (i) Suppose f is a solution of (E_i) and $(f_1, ..., f_m)$ is the decomposition of f by Lemma 5.1. Then f is generic over L iff all the f_j are nonzero.
- (ii) Suppose f is a generic solution to (E_i) over L, $\alpha \in \mathbb{Q}^{alg}$ is nonzero, and h is a nonzero solution of $D_i x - \alpha f x = 0$. Then f is the C-coreduction of h over L.
- (iii) The C-coreduction of a_{i+1} over a_{i-1} is a_i .

(iv) The C-reduction of a_{i+1} over a_{i-1} is a_i .

Proof. We use induction on i.

(i) Suppose the conclusion is true for i - 1.

By Lemma 5.2, if f is a generic solution to (E_i) over L, then for any $j \in \{1, 2, ..., m\}$, f_j is a generic solution to $D_i x - c_{ij} x = 0$. In particular, $f_j \neq 0$.

Now suppose $f_j \neq 0$ for all j = 1, 2, ..., m, but f is not generic, i.e., U(f/L) < m. Since

$$U(f/a_{i-1}) = U(f_1 f_2 \dots f_m/a_{i-1})$$

= $U(f_1/a_{i-1}) + U(f_2/a_{i-1}f_1) + \dots + U(f_m/a_{i-1}f_1f_2 \dots f_{m-1}),$

 $U(f_j/a_{i-1}f_1f_2...f_{j-1}) < 1$ for some j, and hence $f_j \in L\langle \bigcup_{k \neq j} f_k \rangle$ for that j. Note

that

$$\delta f_k = (D_i f_k) \prod_{j=1}^{i-1} e_j = c_{ik} f_k \prod_{j=1}^{i-1} e_j \in L(f_k),$$

so $f_j \in L \langle \bigcup_{k \neq j} f_k \rangle = L(\bigcup_{k \neq j} f_k)$, which means that $\{f_1, ..., f_m\}$ is algebraically dependent over L in the field theoretic sense.

Let $f = (f_1, ..., f_m)$, $\bar{y} = (y_1, ..., y_m)$, and $\bar{c} = (c_{i1}, ..., c_{im})$. Let $G(\bar{y})$ be a nonzero polynomial with minimal number of terms such that its coefficients are in L and $G(\bar{f}) = 0$. We will use a minimality argument similar to that in the proof of Lemma 5.3. Suppose

$$G(y_1, \dots, y_m) = \sum_{\bar{r} \in I} s_{\bar{r}} \bar{y}^{\bar{r}},$$

where I is a set of m-tuples of nonnegative integers, and $s_{\bar{r}} \in L$ for $\bar{r} \in I$. Now

$$D_i(G(\bar{f})) = D_i \sum_{\bar{r} \in I} s_{\bar{r}} \bar{f}^{\bar{r}}$$
$$= \sum_{\bar{r} \in I} (\bar{f}^{\bar{r}} D_i s_{\bar{r}} + s_{\bar{r}} D_i \bar{f}^{\bar{r}})$$
$$= \sum_{\bar{r} \in I} (\log D_i s_{\bar{r}} + \bar{r}\bar{c}) s_{\bar{r}} \bar{f}^{\bar{r}}.$$

We claim that

$$\log D_i s_{\bar{r}^{(1)}} + \bar{r}^{(1)} \bar{c} = \log D_i s_{\bar{r}^{(2)}} + \bar{r}^{(2)} \bar{c}$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Indeed, otherwise, fixing any $\bar{r}^* \in I$, we have

$$\begin{aligned} G^*(\bar{y}) &:= (\log D_i s_{\bar{r}^*} + \bar{r}^* \bar{c}) G(\bar{y}) - D_i(G(\bar{y})) \\ &= \sum_{\bar{r} \in I} (\log D_i s_{\bar{r}^*} + \bar{r}^* \bar{c} - \log D_i s_{\bar{r}} - \bar{r} \bar{c}) s_{\bar{r}} \bar{y}^{\bar{r}} \end{aligned}$$

is a polynomial with fewer terms than G (since the term with index \bar{r}^* is cancelled) such that its coefficients are in L, $G^*(\bar{f}) = 0$ as $G(\bar{f}) = D_i(G(\bar{f})) = 0$, and $G^*(\bar{y}) \neq 0$ as there exist \bar{r} in I such that $\log D_i s_{\bar{r}} + \bar{r}\bar{c} \neq \log D_i s_{\bar{r}^*} + \bar{r}^*\bar{c}$. This contradicts the minimality of G.

There are at least two terms in $G(\bar{y})$. Indeed, if there is only one term in G, then $G(\bar{y}) = s_{\bar{r}}\bar{y}^{\bar{r}}$ for the unique $\bar{r} \in I$. Since $f_j \neq 0$ for j = 1, 2, ..., m and $G(\bar{f}) = 0$, we have $s_{\bar{r}} = 0$, so $G(\bar{y}) = 0$, contradicting the fact that G is nonzero.

We now have

$$\log D_i s_{\bar{r}^{(1)}} + \bar{r}^{(1)} \bar{c} = \log D_i s_{\bar{r}^{(2)}} + \bar{r}^{(2)} \bar{c}$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Note that $\log D_i s_{\bar{r}} + \bar{r}\bar{c} = \log D_i(s_{\bar{r}}\bar{f}^{\bar{r}})$ for any $\bar{r} \in I$. Therefore, fixing $\bar{r}^{(1)} \neq \bar{r}^{(2)}$ in I, we get $s_{\bar{r}^{(1)}}\bar{f}^{\bar{r}^{(1)}} = cs_{\bar{r}^{(2)}}\bar{f}^{\bar{r}^{(2)}}$ for some nonzero $c \in \mathcal{C}$. This means that

(*)
$$c\bar{f}^{\bar{r}^{(2)}-\bar{r}^{(1)}} = s_{\bar{r}^{(1)}}s_{\bar{r}^{(2)}}^{-1}$$

Note that as all $f_j \neq 0$, $\bar{f}^{\bar{r}^{(2)}-\bar{r}^{(1)}}$ makes sense and is nonzero. Let $h = c\bar{f}^{\bar{r}^{(2)}-\bar{r}^{(1)}}$. Then h is a nonzero solution to

(**)
$$\log D_i x = (\bar{r}^{(2)} - \bar{r}^{(1)})\bar{c}$$

When i = 1, right side of (*) is in $\operatorname{acl}(a_0) = \mathbb{Q}^{\operatorname{alg}} \subset \mathcal{C}$, so h is also a constant, but then it is not a solution for (**). When i > 1, we apply part (ii) of the lemma for i-1 with e_{i-1} a generic solution of (\mathbb{E}_{i-1}) over a_{i-2} , $\alpha = (\bar{r}^{(2)} - \bar{r}^{(1)})\bar{c}^* \neq 0$, and h a nonzero solution of $D_{i-1}x - dx = 0$. We get that e_{i-1} is the coreduction of h over a_{i-2} . In particular, since $e_{i-1} \notin \operatorname{acl}(a_{i-2})$, we have that $\operatorname{stp}(h/a_{i-2})$ is not almost \mathcal{C} -internal. But the right side of (*) is in L which is almost \mathcal{C} -internal over a_{i-2} , a contradiction.

(ii) Suppose the conclusion is true for i - 1, and (i) is true for i.

We use induction on m, the order of the differential equation (E_i).

If $m = n_i = 1$, we have that $\log D_i h = \alpha f$ and $\log D_i(\alpha f) = c_{i1}$. Let h^* be a generic solution of $\log D_i x = \alpha f$ over Lf. Since f is a generic solution of $\log D_i(x) = c_{i1}$ over L, αf is also a generic solution of $\log D_i(x) = c_{i1}$ over L, and therefore h^* is a generic solution of $\log D_i^{(2)} x = c_{i1}$ over a_{i-1} . Thus $\operatorname{stp}(h^*/L)$ is not almost \mathcal{C} -internal by Proposition 3.2. Since h^* is a constant multiple of h, $\operatorname{stp}(h/L)$ is also not almost \mathcal{C} -internal. Note that (f, h) is a \mathcal{C} -analysis of h over L, and as it is incompressible of U-type (1, 1), we have that f is the \mathcal{C} -coreduction of h over L.

Now suppose the conclusion of (ii) is proven if the order of the equation (E_i) is less than or equal to m - 1.

Let β be the C-coreduction of h over L. Since $\operatorname{stp}(h/Lf)$ is almost C-internal, we only need to show that $f \in \operatorname{acl}(L\beta)$. Let $(f_1, ..., f_m)$ be the decomposition of f by Lemma 5.1. By Lemma 5.2, f_j is a generic solution of $D_i x - c_{ij} x = 0$ for j = 1, 2, ..., m. Suppose towards a contradiction that $f \notin \operatorname{acl}(L\beta)$. We may, without loss of generality, suppose $f_1, ..., f_s \notin \operatorname{acl}(L\beta)$ and $f_{s+1}, ..., f_m \in \operatorname{acl}(L\beta)$ for some $1 \leq s \leq m$.

In the rest of the proof we seek a contradiction to the above assumption.

We prove first that s = m. Suppose not, so $f_m \in \operatorname{acl}(L\beta)$. Let h_m be a nonzero solution to $D_i x - \alpha f_m x = 0$. We have that $\operatorname{stp}(h_m/Lf_m)$ is almost \mathcal{C} -internal. Since $f_m \in \operatorname{acl}(L\beta)$, $\operatorname{stp}(h_m/L\beta)$ is almost \mathcal{C} -internal. Let $h^* = hh_m^{-1}$. Then

$$\log D_i(h^*) = \log D_i(h) - \log D_i(h_m) = \alpha (f_1 + ... + f_{m-1} + f_m) - \alpha f_m = \alpha (f_1 + ... + f_{m-1}).$$

Let $f^* = f_1 + \dots + f_{m-1}$. Then h^* is a nonzero solution to $D_i x - \alpha f^* x = 0$. From (i), since f_1, \dots, f_{m-1} are all nonzero, f^* is a generic solution over L to

$$(D_i - c_{i1})...(D_i - c_{i,m-1})x = 0.$$

By the induction hypothesis, we conclude that the C-coreduction of h^* over L is f^* . Since h and h_m are almost C-internal over $L\beta$ and $h^* = hh_m^{-1}$, we get that

 $f^* \in \operatorname{acl}(L\beta)$. As f^* is interdefinable with (f_1, \dots, f_{m-1}) over $L, f_1 \in \operatorname{acl}(L\beta)$, contradicting our assumption.

Let $g_{t1} = tf_1$ for $t = 1, 2, \dots$ We show that $stp(g_{t1}/L\beta) = stp(f_1/L\beta)$. Since

(5.1)
$$D_i g_{t1} - c_{i1} g_{t1} = t D_i f_1 - t c_{i1} f_1 = 0,$$

we have that $g_{t1} \in \{x : D_i x - c_{i1} x = 0\}$, a strongly minimal set. Thus in order to prove $\operatorname{stp}(g_{t1}/L\beta) = \operatorname{stp}(f_1/L\beta)$ we only need to show that $g_{t1} \notin \operatorname{acl}(L\beta)$, which follows from $f_1 \notin \operatorname{acl}(L\beta)$.

For each integer $t \ge 1$, let η_t be an automorphism fixing $\operatorname{acl}(L\beta)$ and taking f_1 to g_{t1} . Set $g_{tj} := \eta_t(f_j)$ for all $j = 1, 2, ..., m, g_t := \eta_t(f)$, and $h_t := \eta_t(h)$. So $\operatorname{stp}(h_t, g_t, g_{t1}, ..., g_{tm}/L\beta) = \operatorname{stp}(h, f, f_1, ..., f_m/L\beta)$ for all $t \ge 1$. In particular, g_t is a generic solution to (\mathbf{E}_i) over L, h_t is a nonzero solution to $D_i x - \alpha g_t x = 0$,

 $g_t = \sum_{j=1}^{n} g_{tj}$ is the decomposition by Lemma 5.1, and $\operatorname{stp}(h_t/\beta)$ is almost \mathcal{C} -internal.

We next show that $g_{tj} = tf_j$ for all $t \ge 1$ and all j.

Towards a contradiction, suppose that $g_{tj} \neq tf_j$ for some t and j. Fix this t. We argue first that $g_{tj} - tf_j \in \operatorname{acl}(L\beta)$. Let $H = h_t h^{-t}$, and let $I = \{j : 2 \leq j \leq j \leq j \leq j \}$ $m, g_{tj} - tf_j \neq 0$ (note that $g_{t1} = tf_1$, so we only need $j \ge 2$; also note that I is nonempty since $g_{tj} \neq tf_j$ for some j by assumption). We have that

$$D_i H = (\log D_i H) H$$

= $(\log D_i h_t - t \log D_i h) H$
= $(\alpha g_t - t\alpha f) H$
= $(\alpha \sum_{j=1}^m (g_{tj} - tf_j)) H$,
= $(\alpha \sum_{j \in I} (g_{tj} - tf_j)) H$.

So *H* is a nonzero solution of $D_i x - (\alpha \sum_{j \in I} (g_{tj} - tf_j))x = 0$. Note that $\sum_{j \in I} (g_{tj} - tf_j)$ is a solution to

(5.2)
$$\left(\prod_{j\in I} (D_i - c_{ij})\right)(x) = 0.$$

This is because (5.2) is linear, and for each $j \in I$,

$$(D_i - c_{ij})(g_{tj} - tf_j) = (D_i - c_{ij})g_{tj} - (D_i - c_{ij})tf_j = 0.$$

The decomposition of $\sum_{j \in I} (g_{tj} - tf_j)$ by Lemma 5.1 with respect to (5.2) is $(g_{tj} - tf_j)_{j \in I}$, and $g_{tj} - tf_j \neq 0$ for every $j \in I$. Therefore, applying part (i) where we replace (E_i) with (5.2), we get that $\sum_{j \in I} (g_{tj} - tf_j)$ is a generic solution to (5.2) over L.

Now, since (5.2) is of order less than m and H is a nonzero solution of $D_i x$ – $(\alpha \sum_{j \in I} (g_{tj} - tf_j))x = 0$, by the induction hypothesis, the coreduction of H over L is $\sum_{j \in I} (g_{tj} - tf_j)$. Since $H = h_t h^{-t}$ and both h and h_t are almost \mathcal{C} -internal over $L\beta$, we have $\operatorname{stp}(H/L\beta)$ is almost \mathcal{C} -internal. Therefore, for any $j \in I$, $g_{tj} - tf_j \in$ $\operatorname{acl}(L\beta)$. We now fix some $j \in I$.

Let $\gamma = \frac{g_{tj}}{f_j} - t = \frac{g_{tj} - tf_j}{f_j} \neq 0$. Then γ is a constant in $\operatorname{acl}(LF)\backslash\operatorname{acl}(L\beta)$. Indeed, γ is a constant because g_{tj} and f_j are both solutions to $D_i x - c_{ij} x = 0$, and hence $\frac{g_{tj}}{f_j} \in \mathcal{C}$. We get $\gamma \in \operatorname{acl}(Lf)$ by the fact that $g_{tj} - tf_j \in \operatorname{acl}(L\beta) \subseteq \operatorname{acl}(Lf)$.

And $\gamma \notin \operatorname{acl}(L\beta)$ because if it were, then so would $f_j = \frac{g_{tj} - tf_j}{\gamma}$, but we know that is not the case.

When i = 1 this is impossible, since $\operatorname{acl}(Lf) = \operatorname{acl}(f)$, and Lemma 5.3 tells us that $\operatorname{acl}(f) \cap \mathcal{C} = \mathbb{Q}^{\operatorname{alg}}$.

Suppose i > 1. We apply part (iv) of the lemma for i - 1 and get that the C-reduction of a_i over a_{i-2} is a_{i-1} . As f is a generic solution of (E_i) over L, $\operatorname{stp}(f/L) = \operatorname{stp}(e_i/L)$, so the C-reduction of f over a_{i-2} is a_{i-1} . Since $\gamma \in \operatorname{acl}(Lf) \setminus \operatorname{acl}(L\beta)$, $\gamma \notin L = \operatorname{acl}(a_{i-1})$. So $\operatorname{stp}(\gamma/a_{i-2})$ is not almost C-internal. On the other hand, γ is a constant, a contradiction.

What we have actually shown is that for any $t \ge 1$, $\operatorname{stp}(tf_1/L\beta) = \operatorname{stp}(f_1/L\beta)$, and if $\operatorname{stp}(\tilde{f}_1, \tilde{f}_2, ..., \tilde{f}_m/L\beta) = \operatorname{stp}(f_1, ..., f_m/L\beta)$ and $\tilde{f}_1 = tf_1$, then $\tilde{f}_j = tf_j$ for j = 2, 3, ..., m. In particular, $\operatorname{stp}(tf_1, ..., tf_m/L\beta) = \operatorname{stp}(f_1, ..., f_m/L\beta)$ holds for all t. In addition, the case of t = 1 tells us that $f_j \in \operatorname{dcl}(f_1 \operatorname{acl}(L\beta))$ for j = 2, 3, ..., m.

We now show that $\frac{f_j}{f_1} \in \operatorname{acl}(L\beta)$ for j = 2, 3, ..., m. Fix some j. Since $f_j \in \operatorname{dcl}(f_1 \operatorname{acl}(L\beta))$, there exists a formula $\varphi_1(x, y)$ over $\operatorname{acl}(L\beta)$ such that $\varphi_1(\mathcal{U}, f_1) = \{f_j\}$. Since $\operatorname{stp}(tf_1, tf_j/L\beta) = \operatorname{stp}(f_1, f_j/L\beta)$, we have $\varphi_1(\mathcal{U}, tf_1) = \{tf_j\}$ for all t. Now set $\varphi_2(x, y) := \forall z(\varphi_1(z, y) \to x = \frac{z}{y})$. Then $\varphi_2(\mathcal{U}, tf_1) = \left\{\frac{f_j}{f_1}\right\}$ for all t. So we have

$$\{tf_1: t \ge 1\} \subseteq \left\{b \in \mathcal{U}: \log D_i b = c_{i1} \text{ and } \varphi_2(\mathcal{U}, b) = \left\{\frac{f_j}{f_1}\right\}\right\}.$$

Since $\log D_i x = c_{i1}$ is strongly minimal, it must be that for all but finitely many solutions to $\log D_i x = c_{i1}$, $\varphi_2(\mathcal{U}, b) = \left\{\frac{f_j}{f_1}\right\}$. It follows that $\frac{f_j}{f_1} \in \operatorname{acl}(L\beta)$.

Let g_{01} be a generic solution over Lh to $D_i x - c_{i1} x = 0$, and $g_{0j} = g_{01} \frac{f_j}{f_1}$ for j = 2, 3, ..., m. We have shown that each $\frac{f_j}{f_1}$ is in $\operatorname{acl}(L\beta)$, so $(g_{01}, ..., g_{0m}) \in \operatorname{acl}(L\beta g_{01})$. Let $c_{01} = \frac{f_1}{g_{01}} \in \mathcal{C}$. Now,

$$\log D_i^{(2)}(h) = \log D_i(\alpha f) = \log D_i(\alpha (f_1 + ... + f_m)) = \log D_i(\alpha c_{01}(g_{01} + ... + g_{0m})) = \log D_i(g_{01} + ... + g_{0m}) =: \epsilon.$$

Hence h is a solution to $\log D_i^{(2)}(x) = \epsilon$ which is over $\operatorname{acl}(L\beta g_{01})$, so $U(h/L\beta g_{01}) \leq 2$. 2. Note that $U(h/L\beta) \geq 2$ since h is a generic solution to $\log D_i x = \alpha f$ and $U(f/L\beta) \geq 1$. But we also have $h \downarrow_{L\beta} g_{01}$ (recall that $\beta \in \operatorname{acl}(Lh)$), so $U(h/L\beta g_{01}) = U(h/L\beta) \geq 2$. Thus $U(h/L\beta g_{01}) = 2$, and h is a generic solution to $\log D_i^{(2)}(x) = \epsilon$

over $\operatorname{acl}(L\beta g_{01})$. Hence $\operatorname{stp}(h/L\beta g_{01})$ is not almost \mathcal{C} -internal by Proposition 3.2, and therefore $\operatorname{stp}(h/L\beta)$ is not almost \mathcal{C} -internal, contradicting the definition of β .

(iii) Assume part (ii) of the lemma is true for i.

Let $e_{i+1} = \sum_{j=1}^{n_{i+1}} b_{i+1,j}$ be the decomposition by Lemma 5.1 with respect to (E_{i+1}) . We have that $\operatorname{stp}(a_{i+1}/a_i)$ is almost \mathcal{C} -internal. Also, by part (ii) applied to $f = e_i$ and $h = b_{i+1,1}$, the \mathcal{C} -coreduction of $b_{i+1,1}$ over a_{i-1} is e_i , which is interdefinable over a_{i-1} with a_i . Since $b_{i+1,1} \in \operatorname{dcl}(a_i e_{i+1}) = \operatorname{dcl}(a_{i+1})$, the \mathcal{C} -coreduction of a_{i+1} over a_{i-1} is a_i .

(iv) Assume parts (i) and (ii) of the lemma are true for *i*. For simplicity, we use *n* to denote n_{i+1} . Let *K* be the algebraically closed field generated by a_i . Let $\bar{b}_{i+1} = (b_{i+1,1}, \dots, b_{i+1,n})$.

We already know that $\operatorname{stp}(a_i/a_{i-1})$ is \mathcal{C} -internal. Suppose $\beta \in \operatorname{acl}(a_{i+1})$ is almost \mathcal{C} -internal over a_{i-1} and $\beta \notin \operatorname{acl}(a_i)$. Since e_{i+1} is interalgebraic with \overline{b}_{i+1} over $a_i, \beta \in \operatorname{acl}(a_i\overline{b}_{i+1})$, which means $\beta \in K\langle \overline{b}_{i+1}\rangle^{\operatorname{alg}}$. Since $\delta b_{i+1,j} = c_{i+1,j}b_{i+1,j}\prod_{k=1}^{i}e_k \in K(b_{i+1,j})$ for j = 1, 2, ..., n, we have $K\langle \overline{b}_{i+1}\rangle = K(\overline{b}_{i+1})$, so $\beta \in K(\overline{b}_{i+1})^{\operatorname{alg}}$. Thus there exist a polynomial $F(x, y_1, ..., y_n)$ with coefficients in K such that $F(\beta, b_{i+1,1}, ..., b_{i+1,n}) = 0$ and $F(x, b_{i+1,1}, ..., b_{i+1,n}) \neq 0$. Also, $F(\beta, y_1, ..., y_n) \neq 0$ since $\beta \notin K$.

Suppose $G(y_1, ..., y_n)$ is a nonzero polynomial with minimal number of terms such that the coefficients of G are almost C-internal over a_{i-1} and $G(\bar{b}_{i+1}) = 0$. Note that this is well-defined because $F(\beta, y_1, ..., y_n)$ satisfies all the conditions except for the minimality, as K and β are both almost C-internal over a_{i-1} .

Let

$$G(y_1, ..., y_n) = \sum_{\bar{r} \in I} s_{\bar{r}} \bar{y}^{\bar{r}},$$

where I is a set of *n*-tuples of nonnegative integers, and $\operatorname{stp}(s_{\bar{r}}/a_{i-1})$ is almost C-internal. Let $\bar{c}_{i+1} = (c_{i+1,1}, \dots, c_{i+1,n})$. Arguing exactly as in the proof of part (i) of the lemma, we get by minimality of G that

(5.3)
$$\log D_i s_{\bar{r}^{(1)}} + \bar{r}^{(1)} \bar{c}_{i+1} e_i = \log D_i s_{\bar{r}^{(2)}} + \bar{r}^{(2)} \bar{c}_{i+1} e_i$$

for any $r^{(1)}, r^{(2)} \in I$. Indeed,

$$D_{i}(G(\bar{b}_{i+1})) = \sum_{\bar{\in}I} (\bar{b}_{i+1}^{\bar{r}} D_{i} s_{\bar{r}} + s_{\bar{r}} D_{i} \bar{b}_{i+1}^{\bar{r}})$$

$$= \sum_{\bar{\in}I} (\bar{b}_{i+1}^{\bar{r}} D_{i} s_{\bar{r}} + s_{\bar{r}} \bar{r} \bar{c}_{i+1} e_{i} \bar{b}_{i+1}^{\bar{r}})$$

$$= \sum_{\bar{\in}I} (\log D_{i} s_{\bar{r}} + \bar{r} \bar{c}_{i+1} e_{i}) s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}},$$

where the second equality is by the fact that

$$D_{i}\bar{b}_{i+1}^{\bar{r}} = \bar{r}\bar{b}_{i+1}^{\bar{r}-\bar{1}}D_{i}\bar{b}_{i+1}$$
$$= \bar{r}\bar{b}_{i+1}^{\bar{r}-\bar{1}}e_{i}D_{i+1}\bar{b}_{i+1}$$
$$= \bar{r}\bar{b}_{i+1}^{\bar{r}-\bar{1}}e_{i}\bar{c}_{i+1}\bar{b}_{i+1}$$
$$= \bar{r}e_{i}\bar{c}_{i+1}\bar{b}_{i+1}^{\bar{r}}.$$

Now if (5.3) failed, then fixing any $\bar{r}^* \in I$ we see that

$$\begin{aligned} G^*(\bar{y}) &:= (\log D_i s_{\bar{r}^*} + \bar{r}^* \bar{c}_{i+1} e_i) G(\bar{y}) - D_i G(\bar{y}) \\ &= \sum_{\bar{r} \in I} (\log D_i s_{\bar{r}^*} + \bar{r}^* \bar{c}_{i+1} e_i - \log D_i s_{\bar{r}} - \bar{r} \bar{c}_{i+1} e_i) s_{\bar{r}} \bar{y}^{\bar{r}} \end{aligned}$$

whose coefficients are again almost C-internal over a_{i-1} , would contradict the minimal choice of G.

If G has only one term, then for the only $\bar{r} \in I$, $G(\bar{b}_{i+1}) = s_{\bar{r}}\bar{b}_{i+1}^{\bar{r}}$. Since $b_{i+1,j} \neq 0$ for $j = 1, 2, ..., n, s_{\bar{r}} = 0$, which means $G(\bar{y}) = 0$, a contradiction. Now fix $r^{(1)} \neq r^{(2)}$ in *I*. Since $\log D_i s_{\bar{r}} + \bar{r} \bar{c}_{i+1} e_i = \log D_i (s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}})$ for any $\bar{r} \in I$, we have

In $r^{(1)} \neq r^{(2)}$ in *I*. Since $\log D_i s_{\bar{r}} + \bar{r} \bar{c}_{i+1} e_i = \log D_i (s_{\bar{r}} b_{i+1}^r)$ for any $\bar{r} \in I$, we have $s_{\bar{r}^{(1)}} \bar{b}_{i+1}^{\bar{r}^{(1)}} = cs_{\bar{r}^{(2)}} \bar{b}_{i+1}^{\bar{r}^{(2)}}$ for some $c \in C$. This means that $\bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}} = cs_{\bar{r}^{(2)}} s_{\bar{r}^{(1)}}^{-1}$. So $\bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}}$ is almost C-internal over a_{i-1} . On the other hand, as $D_{i+1} \bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}} = (\bar{r}^{(1)} - \bar{r}^{(2)}) \bar{c}_{i+1} \bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}}$, $\bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}}$ is a solution of $(D_{i+1} - (\bar{r}^{(1)} - \bar{r}^{(2)}) \bar{c}_{i+1}) x = 0$, with $(\bar{r} - \bar{r}^*) \bar{c}_{i+1} \neq 0$ since $\{c_{i+1,j} : j = 1, 2, ..., n\}$ is \mathbb{Q} -linearly independent. By part (ii) of the lemma with $f = e_i$, $h = \bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}}$, and $\alpha = (\bar{r}^{(1)} - \bar{r}^{(2)}) \bar{c}_{i+1}$, e_i is a C-coreduction of $\bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}}$ over a_{i-1} . In particular, $\bar{b}_{i+1}^{\bar{r}^{(1)} - \bar{r}^{(2)}}$ is not almost C-internal over a_{i-1} . This contradiction proves part (iy) of the lemma. part (iv) of the lemma.

We have accomplished the desired construction:

Theorem 5.5. Given positive integers $n_1, ..., n_\ell$, there exists in DCF₀ a type over \mathbb{Q}^{alg} that admits a canonical \mathcal{C} -analysis of U-type (n_1, \ldots, n_ℓ) .

Proof. Let $(a_1, ..., a_\ell)$ be as in the above construction. We have seen that $(a_1, ..., a_\ell)$ is a C-analysis of $p = \operatorname{stp}(a_{\ell})$ of U-type (n_1, \dots, n_{ℓ}) . By Lemmas 4.10 and 4.11, parts (iii) and (iv) of Lemma 5.4 prove that it is a C-analysis by reductions and coreductions, as desired. \square

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