# CONSTRUCTING TYPES IN DIFFERENTIALLY CLOSED FIELDS THAT ARE ANALYSABLE IN THE CONSTANTS 

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#### Abstract

Analysability of finite $U$-rank types are explored both in general and in the theory $\mathrm{DCF}_{0}$. The well-known fact that the equation $\delta(\log \delta x)=0$ is analysable in but not almost internal to the constants is generalized to show   notion of a canonical analysis is introduced - namely an analysis that is of minimal length and interalgebraic with every other analysis of that length. Not every analysable type admits a canonical analysis. Using properties of reductions and coreductions in theories with the canonical base property, it is constructed, for any sequence of positive integers $\left(n_{1}, \ldots, n_{\ell}\right)$, a type in $\mathrm{DCF}_{0}$ that admits a canonical analysis with the property that the $i$ th step has $U$-rank $n_{i}$.


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## 1. Introduction

That differential-algebraic geometry is an expansion of algebraic geometry is reflected in model theory by viewing the theory of algebraically closed fields as a reduct of the theory of differentially closed fields. The locus of that reduct is the field of constants. The smallest intermediate reduct that properly expands algebraic geometry is that of differential varieties that are almost internal to the constants: differential varieties that over possibly additional parameters become definable finite covers of algebraic varieties in the constants. Here already one observes new and interesting geometric and model theoretic phenomena. A further step would be to consider those differential varieties that are built up through a finite sequence of fibrations whose fibres are almost internal to the constants; these are the differential varieties that are analysable in the constants, and they are the focus of this paper. In particular, we give some constructions that exhibit the richness of this category.

[^0]Differential varieties analysable in the constants have come up recently in applications; it is shown in [1] that they give rise to a new class of associative algebras satisfying the classical Dixmier-Moeglin equivalence.

Probably the best known example of an analysable but not internal to the constants differential variety is the one defined by the equation $\delta\left(\frac{\delta x}{x}\right)=0$. It decomposes as an extension of the additive group of constants by the multiplicative group of constants, without itself being almost internal to the constants. Our first observation is to generalize this construction by iterating the logarithmic derivative. Writing $\log \delta x:=\frac{\delta x}{x}$ and $\log \delta^{(m)}=\underbrace{\log \delta \ldots \log \delta}_{m}$ we consider the equation $\log \delta^{(m)} x=0$, and show in Section 3 that while it is analysable in the constants in $m$ steps, it is not analysable in $m-1$ steps. This is done in Section 3 by essentially reducing to the $m=2$ case.

Note that each step in the analysis of $\log \delta^{(m)} x=0$ is of $U$-rank one. It is not hard to produce from this example, using methods that work generally in stable theories satisfying the canonical base property (CBP), including reductions and coreductions, other examples of types analysable in the constants in $m$-steps but not in $m-1$-steps. We may even require this type to satisfy the property that the $i$ th step of the analysis by reductions of this type is of $U$-rank $n_{i}$, for any given increasing sequence $\left(n_{i}\right)_{i=1}^{m}$, or that the $i$ th step of the analysis by coreductions of this type is of $U$-rank $n_{i}$, for any given decreasing sequence $\left(n_{i}\right)_{i=1}^{m}$. This is done in Section 4

But we look for more; we want analyses of a type $p$ that are canonical in the sense that up to interalgebraicity there is no other analyses of $p$ in the constants of the same (minimal) length. Not every finite rank type, even in $\mathrm{DCF}_{0}$, admits a canonical analysis (see Example 4.1). However, we show in Section 5 that given any sequence of positive integers $\left(n_{1}, \ldots, n_{m}\right)$ there exists in $\mathrm{DCF}_{0}$ a type that has a canonical analysis in the constants with $i$ th step having $U$-rank $n_{i}$. Unlike in the logarithmic derivative case, these examples are not differential algebraic groups, and hence that theory is not directly available to us. Our proofs involve a careful algebraic analysis of the equations that arise. Note that the situation is very different for differential algebraic groups; in [1] it is shown that every differential algebraic group over the constants is analysable in at most 3 steps.

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## 2. Analysability

As a general setting, we work in a saturated model $\mathcal{U}$ of a complete stable theory $T$ that eliminates imaginaries. We review in this section some classical notions around finite rank types. As a general reference we suggest [7]. We have provided proofs where explicit references were not possible.

Let $\mathcal{P}$ be a set of partial types (over different parameter sets) which is invariant under automorphisms over $\varnothing$, and $q$ be a stationary type over a parameter set $A$.

Recall that a stationary type $q$ over $A$ is $\mathcal{P}$-internal (or almost $\mathcal{P}$-internal) if for some (equivalently any) realization $a$ of $q$, there exists $B \supseteq A$ which is independent from $a$ over $A$, and $c_{1}, \ldots, c_{k}$ realizations of types in $\mathcal{P}$ whose parameter sets are contained in $B$, such that $a \in \operatorname{dcl}\left(B c_{1} \ldots c_{k}\right)$ (or $\left.a \in \operatorname{acl}\left(B c_{1} \ldots c_{k}\right)\right)$.

The type $q$ over $A$ is $\mathcal{P}$-analysable if for some (equivalently any) realization $a$ of $q$, there are $a_{1}, \ldots, a_{k}$ such that $\operatorname{stp}\left(a_{1} / A\right)$ is almost $\mathcal{P}$-internal, $a_{i-1} \in \operatorname{dcl}\left(A a_{i}\right)$, $\operatorname{stp}\left(a_{i} / A a_{i-1}\right)$ is almost $\mathcal{P}$-internal for $i=2,3, \ldots, k$, and $\operatorname{acl}(A a)=\operatorname{acl}\left(A a_{k}\right)$. The sequence $\left(a_{i}\right)_{i=1}^{k}$ mentioned above is called a $\mathcal{P}$-analysis of $q$ and a $\mathcal{P}$-analysis of $a$ over $A$. For notational convenience, for any analysis $\left(a_{i}\right)_{i=1}^{k}$ we use $a_{0}$ to denote the empty tuple. We call $k$ the length of the analysis. Note that an algebraic type has a $\mathcal{P}$-analysis of length zero, and an almost $\mathcal{P}$-internal type has a $\mathcal{P}$-analysis of length 1.

The $U$-type of the analysis is the sequence $\left(U\left(a_{i} / A a_{i-1}\right)\right)_{i=1}^{k}$. We say the analysis is nondegenerated if each entry of the $U$-type is nonzero.

Note that the definition of analysable here is in fact the definition of almost analysable in the literature (for example, section 1 of [6]), and we may instead say that a type is strictly $\mathcal{P}$-analysable if $\operatorname{stp}\left(a_{i} / a_{i-1}\right)$ is internal (rather than almost internal) to $\mathcal{P}$. The following proposition proves that these two definitions are in fact equivalent.
Proposition 2.1. A stationary type $q$ over $A$ is $\mathcal{P}$-analysable iff it is strictly $\mathcal{P}$ analysable.

We need the following lemma.
Lemma 2.2. If a stationary type $q$ over $A$ is almost $\mathcal{P}$-internal, then for any $a \vDash q$, there exists a tuple $a_{0}$ such that $\operatorname{tp}\left(a_{0} / A\right)$ is $\mathcal{P}$-internal and $\operatorname{acl}(A a)=\operatorname{acl}\left(A a_{0}\right)$.

Proof. Given any realization $a \vDash q$, let $n$ be the least number such that there exists an $L_{A}$-formula $\varphi(x, y, z)$, a tuple $b$ independent from $a$ over $A$ and a tuple $c$ realizing types in $\mathcal{P}$ such that $\vDash \varphi(a, b, c)$ and $\varphi(\mathcal{U}, b, c)$ is of size $n$. We fix these $b, c$, and $\varphi$ that satisfy $|\varphi(\mathcal{U}, b, c)|=n$.

Step 1. We prove that $\varphi(\mathcal{U}, b, c) \subseteq \operatorname{acl}(A a)$.
Let $a=a_{1}, a_{2}, \ldots, a_{n}$ be the elements of $\varphi(\mathcal{U}, b, c)$. Towards a contradiction, suppose without loss of generality that $a_{2} \notin \operatorname{acl}(A a)$. Then there are $a_{2}^{\prime}, b^{\prime}$ and $c^{\prime}$ such that $\operatorname{tp}\left(a_{2}^{\prime} b^{\prime} c^{\prime} / A a\right)=\operatorname{tp}\left(a_{2} b c / A a\right)$ and $a_{2}^{\prime} b^{\prime} \downarrow_{A a} a_{2} \ldots a_{n} b$. Since $a_{2}^{\prime} \notin \operatorname{acl}(A a)$ and $a_{2}^{\prime} \downarrow_{A a} a_{2} \ldots a_{n} b, a_{2}^{\prime} \notin \operatorname{acl}\left(A a a_{2} \ldots a_{n} b\right)$. In particular, $a_{2}^{\prime} \neq a_{i}$ for $i=1,2, \ldots, n$. Also, since $a \downarrow_{A} b$ and $b \downarrow_{A a} b^{\prime}$, we have $b \downarrow_{A} a b^{\prime}$, and therefore $b \downarrow_{A b^{\prime}} a$. As $\operatorname{tp}\left(b^{\prime} / A a\right)=\operatorname{tp}(b / A a)$ and $b \downarrow_{A} a$, we have $b^{\prime} \downarrow_{A} a$, which, together with $b \downarrow_{A b^{\prime}} a$, yields $b b^{\prime} \downarrow_{A} a$. Now the fact that $q$ is almost $\mathcal{P}$-internal is witnessed by $a \vDash \varphi(x, b, c) \wedge \varphi\left(x, b^{\prime}, c^{\prime}\right)$, and the size of $\varphi(\mathcal{U}, b, c) \wedge \varphi\left(\mathcal{U}, b^{\prime}, c^{\prime}\right)$ is smaller then $n$ (notice that $|\varphi(\mathcal{U}, b, c)|=\left|\varphi\left(\mathcal{U}, b^{\prime}, c^{\prime}\right)\right|=n$, but the two sets are not the same), contradicting minimality of $n$.

Step 2. Let $d$ be the code of the set $\varphi(\mathcal{U}, b, c)$. Then $\operatorname{tp}(d / A)$ is $\mathcal{P}$-internal and $\operatorname{acl}(A a)=\operatorname{acl}(A d)$.

We have $a \in \operatorname{acl}(d) \subseteq \operatorname{acl}(A d)$ by the definition of a code, and $d \in \operatorname{dcl}\left(a a_{2} \ldots a_{n}\right) \subseteq$ $\operatorname{acl}(A a)$. Moreover, as $a \downarrow_{A} b$, we have $d \downarrow_{A} b$. Since $d$ is the code of $\varphi(\mathcal{U}, b, c)$ where $\varphi$ is an $L_{A}$-formula, $d \in \operatorname{dcl}(A b c)$. Therefore $\operatorname{tp}(d / A)$ is $\mathcal{P}$-internal.

Proof of Proposition 2.1. The nontrivial direction is from left to right. Suppose $\left(b_{1}, \ldots, b_{k}\right)$ is an analysis of $a$ over $A$. For convenience, let $a_{0}$ be the empty tuple. We now construct the sequence $\left(a_{1}, \ldots, a_{k}\right)$.

Suppose we already have $\left(a_{1}, \ldots, a_{i-1}\right)$ for $1 \leq i \leq k$ such that $\operatorname{stp}\left(a_{j} / A a_{j-1}\right)$ is $\mathcal{P}$-internal, $a_{j-1} \in \operatorname{dcl}\left(A a_{j}\right)$, and $\operatorname{acl}\left(A a_{j}\right)=\operatorname{acl}\left(A b_{j}\right)$ for $j=1,2, \ldots, i-1$. Then as $\operatorname{stp}\left(b_{i} / A b_{i-1}\right)$ is almost $\mathcal{P}$-internal and $\operatorname{acl}\left(A a_{i-1}\right)=\operatorname{acl}\left(A b_{i-1}\right)$, we have that
$\operatorname{stp}\left(b_{i} / A a_{i-1}\right)$ is almost $\mathcal{P}$-internal, so by Lemma 2.2, there exists $a^{*}$ such that $\operatorname{acl}\left(A a_{i-1} a^{*}\right)=\operatorname{acl}\left(A a_{i-1} b_{i}\right)$ and $\operatorname{tp}\left(a^{*} / A a_{i-1}\right)$ is $\mathcal{P}$-internal. Let $a_{i}=a_{i-1} a^{*}$. Then we have $a_{i-1} \in \operatorname{dcl}\left(A a_{i}\right), \operatorname{acl}\left(A a_{i}\right)=\operatorname{acl}\left(A a_{i-1} b_{i}\right)=\operatorname{acl}\left(A b_{i-1} b_{i}\right)=\operatorname{acl}\left(A b_{i}\right)$, and $\operatorname{tp}\left(a_{i} / A a_{i-1}\right)$ is $\mathcal{P}$-internal.

The sequence $\left(a_{1}, \ldots, a_{k}\right)$ then witnesses that $\operatorname{tp}(a / A)$ is strictly analysable.

We use the following definitions in order to better describe analysable types and their analyses. We say that the type $q$ is $k$-step $\mathcal{P}$-analysable, or $\mathcal{P}$-analysable in $k$ steps, if the analysability of $q$ is witnessed by a $\mathcal{P}$-analysis of length $k$. A $\mathcal{P}$-analysis $\left(a_{i}\right)_{i=1}^{k}$ is said to be incompressible if $\operatorname{stp}\left(a_{i+1} / A a_{i-1}\right)$ is not almost $\mathcal{P}$-internal for all $i=1,2, \ldots, k-1$. A $\mathcal{P}$-analysis of $q$ is minimal if there is no $\mathcal{P}$-analysis of $q$ of strictly shorter length.

The following lemma shows that incompressibility implies minimality if the $U$ type of an analysis is $(1,1, \ldots, 1)$.

Lemma 2.3. Let $\left(a_{1}, \ldots, a_{n}\right)$ be an incompressible $\mathcal{P}$-analysis of a over $A$ of $U$-type $\underbrace{(1,1, \ldots, 1)}_{n}$. Then the analysis is minimal, i.e., $\operatorname{tp}(a / A)$ is not $\mathcal{P}$-analysable in $n-1$

Proof. For $n=2$, the only possibility that the analysis is not minimal is that $\operatorname{stp}(a / A)$ is 1 -step $\mathcal{P}$-analysable, i.e., almost $\mathcal{P}$-internal, which contradicts the fact that $\left(a_{1}, a_{2}\right)$ is an incompressible analysis.

Assume we have proved the conclusion for $n<k$. Suppose towards a contradiction that $\left(a_{1}, \ldots, a_{k}\right)$ is an incompressible $\mathcal{P}$-analysis of $a$ over $A$ of $U$-type $\underbrace{(1,1, \ldots, 1)}_{k}$ which is not minimal. Let $\left(c_{1}, \ldots, c_{k-1}\right)$ be another $\mathcal{P}$-analysis of $a$ over
$A$. Note that $\left(a_{1} c_{1}, a_{2} c_{2}, \ldots, a_{k-1} c_{k-1}\right)$ is also a $\mathcal{P}$-analysis of $a$ over $A$. Let $b_{1}, \ldots, b_{\ell}$ be a subsequence of $\left(a_{i} c_{i}\right)_{i=1}^{k-1}$ such that $\left(b_{j}\right)_{j=1}^{\ell}$ is a nondegenerated $\mathcal{P}$-analysis of $p$. This can be done by taking away all elements $a_{i} c_{i}$ in $\left(a_{i} c_{i}\right)_{i=1}^{k-1}$ such that $U\left(a_{i} c_{i} / A a_{i-1} c_{i-1}\right)=0$. Let $b_{j}=a$ for $\ell+1 \leq j \leq k-1$. Then the only zero entries of the $U$-type of $\left(b_{j}\right)_{j=1}^{k-1}$ (if any) are at the end of the sequence.

If $U\left(b_{1} / A\right)=1$, then $\operatorname{acl}\left(A b_{1}\right)=\operatorname{acl}\left(A a_{1}\right)$, and $\operatorname{stp}\left(a / A a_{1}\right)=\operatorname{stp}\left(a / A b_{1}\right)$. But then $\left(a_{2}, \ldots, a_{k}\right)$ is a $k-1$-step incompressible $\mathcal{P}$-analysis of $a$ over $A a_{1}$ of $U$-type $\underbrace{(1,1, \ldots, 1)}_{k-1}$, while $\left(b_{2}, \ldots, b_{k-1}\right)$ is a $k-2$-step $\mathcal{P}$-analysis of the same type with shorter length, contradicting our induction hypothesis.

Now suppose $U\left(b_{1} / A\right) \geq 2$. If the $U$-type of $\left(b_{j}\right)_{j=1}^{k-1}$ is degenerated, then $U\left(b_{k-1} / b_{k-2}\right)=0$, and we have $U\left(b_{k-2} / A\right)=U(a / A)=k$. If $\left(b_{j}\right)_{j=1}^{k-1}$ is nondegenerated, then $U\left(b_{j} / A b_{j-1}\right) \geq 1$ for any $j=1, \ldots, k-2$ which gives us $U\left(b_{j} / A\right) \geq j+1$ for any $j=1, \ldots, k-2$. In both cases $U\left(b_{k-2} / A\right) \geq k-1$. By the induction hypothesis, $\operatorname{acl}\left(A b_{k-2}\right) \neq \operatorname{acl}\left(A a_{k-1}\right)$ : otherwise, $\left(a_{i}\right)_{i=1}^{k-1}$ is a $k-1$-step incompressible $\mathcal{P}$-analysis of $a_{k-1}$ over $A$ of $U$-type $\underbrace{(1,1, \ldots, 1)}_{k-1}$, while $\left(b_{i}\right)_{i=1}^{k-2}$ is a $k-2$ step $\mathcal{P}$-analysis of the same type, contradicting our induction hypothesis. Similarly, $\operatorname{acl}\left(A b_{k-2}\right) \supsetneq \operatorname{acl}\left(A a_{k-1}\right)$ does not hold: otherwise $U\left(b_{k-2} / A a_{k-1}\right) \geq 1$, and since $b_{k-2} \in \operatorname{acl}(A a)$ and $U\left(a / A a_{k-1}\right)=1$, we have $\operatorname{acl}\left(A b_{k-2}\right)=\operatorname{acl}(A a)$;
therefore $\left(a_{2}, \ldots, a_{k}\right)$ is a $k-1$-step incompressible $\mathcal{P}$-analysis of $\operatorname{stp}\left(a / A a_{1}\right)$ of $U$ type $\underbrace{(1,1, \ldots, 1)}_{k-1}$, while $\left(b_{1}, \ldots, b_{k-2}\right)$ is a $k-2$-step $\mathcal{P}$-analysis of the same type,
contradicting our induction hypothesis. Hence $\operatorname{acl}\left(A b_{k-2}\right) \supseteq \operatorname{acl}\left(A a_{k-1}\right)$ does not hold, i.e., $a_{k-1} \notin \operatorname{acl}\left(A b_{k-2}\right)$. We have $k=U(a / A) \geq U\left(a_{k-1} b_{k-2} / A\right)=$ $U\left(b_{k-2} / A\right)+U\left(a_{k-1} b_{k-2} / A b_{k-2}\right) \geq(k-1)+1=k, \operatorname{so} \operatorname{acl}\left(A b_{k-2} a_{k-1}\right)=\operatorname{acl}(A a)$. But then since $\operatorname{stp}\left(b_{k-2} / A a_{1}\right)$ and $\operatorname{stp}\left(a_{k-1} / A a_{1}\right)$ are $k-2$-step $\mathcal{P}$-analysable, so is $\operatorname{stp}\left(a / A a_{1}\right)$, while $\left(a_{2}, \ldots, a_{k}\right)$ is a $k-1$-step incompressible $\mathcal{P}$-analysis of $a$ over $A a_{1}$ of $U$-type $\underbrace{(1,1, \ldots, 1)}_{k-1}$, contradicting our induction hypothesis.

## 3. Iterated Logarithmic Derivative

Our primary interest is in $\mathrm{DCF}_{0}$, the theory of differential closed field of characteristic 0 . The theory $\mathrm{DCF}_{0}$ is complete, stable, and eliminates both quantifiers and imaginaries. We assume some familiarity of this theory. The language used is $(0,1,+, \times, \delta)$, and $\mathcal{U}=(U, 0,1,+, \times, \delta)$ is the saturated model, where $\delta$ is the derivative on the field. We often omit $0,1,+, \times$ and write $\mathcal{U}=(U, \delta)$.

We focus on types which are almost $\mathcal{C}$-internal or $\mathcal{C}$-analysable in $\mathrm{DCF}_{0}$, where $\mathcal{C}=\{x: \delta x=0\}$ is the field of constants.

We often use the term "generic type" in $\mathrm{DCF}_{0}$. The generic type of an irreducible Kolchin closed set $D$ over a $\delta$-field $k$ is the type which says that $x$ is in $D$ but not in any $k$-definable Kolchin closed subset of $D$. A definable set is irreducible if its Kolchin closure is. By the generic type of an irreducible definable set, we mean the generic type of its Kolchin closure. Note that this does not always coincide with the type of greatest $U$-rank.

Recall that in $\mathrm{DCF}_{0}$, the logarithmic derivative of $x$ is defined as $\log \delta x=\frac{\delta x}{x}$. The logarithmic derivative is used extensively in this section. Note that $\log \delta$ : $\mathbb{G}_{m} \rightarrow \mathbb{G}_{a}$ is a definable group homomorphism between algebraic groups, and the kernel of the map is $\mathbb{G}_{m}(\mathcal{C})$. Here $\mathbb{G}_{m}$ is the universe (take away 0 ) viewed as a multiplicative group, $\mathbb{G}_{a}$ is the universe viewed as an additive group, and $\mathbb{G}_{m}(\mathcal{C})$ is the constant points of $\mathbb{G}_{m}$.
Fact 3.1 (see, for example, Fact 4.2 of [2]). Let $G$ be the differential algebraic subgroup of $\mathbb{G}_{m}$ defined by $\{x: \delta(\log \delta x)=0\}$. The generic type of $G$ is 2-step $\mathcal{C}$-analysable but not almost $\mathcal{C}$-internal.

It follows that any $\mathcal{C}$-analysis of this type is of $U$-type $(1,1)$.
We will be considering iterated logarithmic derivatives. For any $n \geq 1$ we set $\log \delta^{(n)}(x):=\underbrace{\log \delta \log \delta \ldots \log \delta(x)}$. Note that $\log \delta^{(n)}(x)$ is only defined at $x$ if $\log \delta^{(i)}(x) \neq 0$ for $i=0,1, \ldots, n-1$ where $\log \delta^{(0)}(x)=x$. Whenever we write $\log \delta^{(n)}(x)$ it is always assumed that $x$ is in this domain of definition. Note that for any $h \in \mathcal{U}$, the equation $\log \delta^{(n)}(x)=h$ defines an irreducible Kolchin constructible subset $B$ of $\mathcal{U}$. Indeed, $B$ is isomorphic to

$$
\begin{aligned}
B^{*} & =\left\{\left(x, \log \delta(x), \ldots, \log \delta^{(n-1)}(x)\right): x \in B\right\} \\
& =\left\{\left(x_{1}, \ldots, x_{n}\right): x_{i} \neq 0 ; \frac{\delta x_{i}}{x_{i}}=x_{i+1}, i=1,2, \ldots, n-1 ; \frac{\delta x_{n}}{x_{n}}=h\right\}
\end{aligned}
$$

whose Kolchin closure is $\left\{\left(x_{1}, \ldots, x_{n}\right): \delta x_{i}=x_{i} x_{i+1}, i=1,2, \ldots, n-1 ; \delta x_{n}=h x_{n}\right\}$, which is irreducible since it is the set of $D$-points (or "sharp" set) corresponding to the irreducible $D$-variety $\left(\mathbb{A}^{n}, s\right)$ where $s\left(x_{1}, \ldots, x_{n-1}, x_{n}\right)=\left(x_{1} x_{2}, \ldots, x_{n-1} x_{n}, h x_{n}\right)$. (For details on $D$-varieties see [4].)

In particular, $\left\{x: \log \delta^{(2)}(x)=h\right\}$ is irreducible. Note also that the generic type of $\log \delta^{(2)}(x)=0$ is the same as that of $G$ defined in Fact 3.1. So the following proposition is a generalisation of Fact 3.1.
Proposition 3.2. Let $h \in U$ and consider $B=\left\{x: \log \delta^{(2)}(x)=h\right\}$. Let $k$ be $a$ $\delta$-field containing $h$, and $p$ be the generic type of $B$ over $k$. Then $p$ is not almost $\mathcal{C}$-internal.

Proof. We may assume that $k$ contains an element of the form $a=\log \delta g_{0}$ where $g_{0} \in B$. Indeed, this follows from the fact that for any $g_{0} \in B, p$ is almost $\mathcal{C}$-internal iff the non-forking extension of $p$ to $k\left\langle g_{0}\right\rangle$ is, and $p \mid k\left\langle g_{0}\right\rangle$ is the generic type of $B$ over $k\left\langle g_{0}\right\rangle$.

We now construct a new model $\mathcal{V}=(U, D)$ of $\mathrm{DCF}_{0}$ as follows. The set $U$ and the interpretation of $0,1,+$ and $\times$ remain the same, while $D g:=\frac{\delta g}{a}$ for all $g \in \mathcal{U}$. Notice that $\mathcal{V}$ is also a model of $\mathrm{DCF}_{0}$ with the same field of constants as $\mathcal{U}$, and any definable set in one model is definable in the other, with the same set of parameters, as long as the parameter set contains $a$. Now let $q$ be a type in the model $\mathcal{V}$ over $k$ so that $q$ and $p$ have the same set of realizations in $U$. This can be done by replacing each occurrence of $\delta$ in formulas in $p$ by $a D$.

Assume towards a contradiction that $p$ is almost $\mathcal{C}$-internal. Hence, for any $g \vDash p$, there is $B \supset k$ such that $g \downarrow_{k} B$ and $g \in \operatorname{acl}(B C)$, in the model $\mathcal{U}$. Replacing $\delta$ by $a D$ in the formulas witnessing this fact, we have that $g \in \operatorname{acl}(B C)$ in $\mathcal{V}$ as well. Moreover, $g \downarrow_{k} B$ holds in $\mathcal{V}$ because $U$-ranks of types are the same in $\mathcal{U}$ and $\mathcal{V}$ if the parameter set contains $a$. We get that $q$ is almost $\mathcal{C}$-internal in $\mathcal{V}$.

However, $q$ is the generic type of $B$, since Kolchin closed sets definable over $k$ (which contains $a$ ) are the same in $\mathcal{U}$ and $\mathcal{V}$. The set $B$ is defined in $\mathcal{U}$ by the formula $\log \delta(\log \delta x)=h$, which is just $a \log D(\log D x)=h$, which is equivalent to $\log D(\log D f)=0$. So $q$ is the generic type of $B=\{x: \log D(\log D x)=0\}$, which is not almost $\mathcal{C}$-internal in $\mathcal{V}$ by Fact 3.1, a contradiction.

We can now show that the iterated logarithmic derivatives give rise to $n$-step $\mathcal{C}$-analysable types that are not $n-1$-step $\mathcal{C}$-analysable.
Corollary 3.3. In $\mathrm{DCF}_{0}$, let $D=\{x \in U: \log \delta \log \delta \ldots \log \delta x=0\}$. Then the generic type $p$ of $D$ is $n$-step $\mathcal{C}$-analysable but not $n-{ }_{1}^{n}$-step $\mathcal{C}$-analysable.
Proof. Let $a \in D$ be generic. Let $a_{n}=a, a_{k}=\log \delta a_{k+1}$ for $k=0,1, \ldots, n-1$. Note that $a_{0}=0, a_{k} \in \operatorname{dcl}\left(a_{k+1}\right)$ for $k=0,1, \ldots, n-1$, and $a$ is interdefinable with $\left(a_{1}, \ldots, a_{n}\right)$.

As $a$ is generic in $D, a_{i+1} \notin \operatorname{acl}\left(a_{i}\right)$ for each $i=0,1, \ldots, n-1$. By additivity of $U$-rank, for each $i=0,1, \ldots, n-1, U\left(a_{i+1} / a_{i}\right)=1$. Hence, $\operatorname{stp}\left(a_{i+1} / a_{i}\right)$ is the generic type over $a_{i}$ of $\log \delta(x)=a_{i}$. The latter equation defines a multiplicative translation of $\mathbb{G}_{m}(\mathcal{C})=\operatorname{ker}(\log \delta)$, so $\operatorname{stp}\left(a_{i+1} / a_{i}\right)$ is almost $\mathcal{C}$-internal of $U$-rank 1 . That is, $\left(a_{1}, a_{2}, \ldots, a_{n}=a\right)$ is a $\mathcal{C}$-analysis of $p$ of $U$-type $\underbrace{(1,1, \ldots, 1)}_{n}$.

For each $i=1,2, \ldots, n-1, \operatorname{stp}\left(a_{i+1} / a_{i-1}\right)$ is the generic type of $\log \delta^{(2)} x=a_{i-1}$ over $a_{i-1}$. Proposition 3.2 tells us that this type is not almost $\mathcal{C}$-internal. That is, $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ is an incompressible $\mathcal{C}$-analysis.

Hence, by Lemma 2.3, $p$ is not $\mathcal{C}$-analysable in $n-1$ steps.

## 4. Analyses by reductions and coreductions

In this section we return to the general setting of Section 2 so $T$ is a complete stable theory that eliminates imaginaries, $\mathcal{U}$ is a sufficiently saturated model of $T$, and $\mathcal{P}$ is a set of partial types invariant over automorphisms of the universe.

Note that Lemma 2.3 does not hold if the entries of the $U$-type are not all 1.
Example 4.1. Let $\operatorname{stp}(a)$ be 2 -step $\mathcal{P}$-analysable with an incompressible $\mathcal{P}$-analysis $\left(a_{1}, a\right)$. Now let $\left(b_{1}, b\right)$ be such that $b b_{1} \downarrow a a_{1}$ and $\operatorname{stp}\left(b b_{1}\right)=\operatorname{stp}\left(a a_{1}\right)$. Let $c=a b$. Then $c$ is 3-step $\mathcal{P}$-analysable, with an analysis ( $a_{1}, a b_{1}, c=a b$ ). This analysis is incompressible: $\operatorname{stp}\left(a b_{1}\right)$ is not almost $\mathcal{P}$-internal because $\operatorname{stp}(a)$ is not almost $\mathcal{P}$-internal and $\operatorname{stp}\left(a b / a_{1}\right)$ is not almost $\mathcal{P}$-internal because $\operatorname{stp}(b)$ is not almost $\mathcal{P}$-internal, and $\operatorname{stp}\left(b / a_{1}\right)$ is its non-forking extension. But $c$ is 2 -step $\mathcal{P}$-analysable by $\left(a_{1} b_{1}, c=a b\right)$, so the $\mathcal{P}$-analysis $\left(a_{1}, a b_{1}, c=a b\right)$ is not minimal despite being incompressible.

To generalize Lemma 2.3 to higher $U$-rank cases, we need each step to satisfy some maximality or minimality property. We will use the notions of $\mathcal{P}$-reduction and $\mathcal{P}$-coreduction.

Definition 4.2 (See, for example, Section 4 of [5]). Suppose $a$ is a tuple and $A$ is a parameter set. We say a tuple $b$ is a $\mathcal{P}$-reduction of $a$ over $A$ if $b$ is maximally almost $\mathcal{P}$-internal over $A$ in $\operatorname{acl}(A a)$, i.e., $\operatorname{stp}(b / A)$ is almost $\mathcal{P}$-internal, $b \in \operatorname{acl}(A a)$, and if $b^{\prime} \in \operatorname{acl}(A a)$ and $\operatorname{stp}\left(b^{\prime} / A\right)$ is almost $\mathcal{P}$-internal then $b^{\prime} \in \operatorname{acl}(A b)$. We say a nondegenerated $\mathcal{P}$-analysis $\left(a_{1}, \ldots, a_{n}\right)$ of $a$ over $A$ is a $\mathcal{P}$-analysis by reductions of $a$ over $A$ if $a_{k}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{k-1}$ for $k=1,2, \ldots, n$.

Note that by definition $\mathcal{P}$-reductions are unique up to interalgebraicity over the parameter set, i.e., if $b$ and $c$ are both $\mathcal{P}$-reductions of $a$ over $A$, then $\operatorname{acl}(A b)=$ $\operatorname{acl}(A c)$. We may therefore call $b$ the $\mathcal{P}$-reduction of $a$ over $A$.

Remark 4.3. It is clear that if $U(a / A)<\omega$, then a $\mathcal{P}$-reduction of $a$ over $A$ always exists. In fact, let be a tuple that has maximal $U$-rank over $A$ satisfying the condition that $\operatorname{stp}(b / A)$ is almost $\mathcal{P}$-internal and $b \in \operatorname{acl}(A a)$. Then $b$ is a $\mathcal{P}$ reduction of $a$ over $A$ : if $c$ also satisfies this condition, then $\operatorname{stp}(b c / A)$ is almost $\mathcal{P}$-internal and $b c \in \operatorname{acl}(A a)$, so $U(b c / A)=U(b / A)$, which means $c \in \operatorname{acl}(A b)$. Hence, if $\operatorname{tp}(a / A)$ is $\mathcal{P}$-analysable of finite $U$-rank then a $\mathcal{P}$-analysis by reductions always exists.

Definition 4.4 (See, for example, Definition 4.1 of [5]). Suppose $a$ is a tuple and $A$ is a parameter set. We say a tuple $b$ is a $\mathcal{P}$-coreduction of $a$ over $A$ if $b$ is minimal in $\operatorname{acl}(A a)$ such that $a$ is almost $\mathcal{P}$-internal over $A b$, i.e., $\operatorname{stp}(a / A b)$ is almost $\mathcal{P}$ internal, $b \in \operatorname{acl}(A a)$, and if $b^{\prime} \in \operatorname{acl}(a A)$ and $b^{\prime}$ satisfies that $\operatorname{stp}\left(a / A b^{\prime}\right)$ is almost $\mathcal{P}$-internal then $b \in \operatorname{acl}\left(A b^{\prime}\right)$. We say a nondegenerated $\mathcal{P}$-analysis $\left(a_{1}, \ldots, a_{n}\right)$ of $a$ over $A$ is a $\mathcal{P}$-analysis by coreductions of $a$ over $A$ if $a_{k-1}$ is a $\mathcal{P}$-coreduction of $a_{k}$ over $A$ for $k=2, \ldots, n$.

Note similarly that by definition $\mathcal{P}$-coreductions are unique up to interalgebraicity over the parameter set. We may therefore call $b$ the $\mathcal{P}$-coreduction of $a$ over A.

Recall that $T$ has the canonical base property (CBP) if whenever $U(a / b)<\omega$ and $\operatorname{acl}(b)=\operatorname{acl}(\operatorname{Cb}(a / b))$, then $\operatorname{stp}(b / a)$ is almost $\mathbb{P}$-internal, where $\mathbb{P}$ is the set of all nonmodular minimal types. See, for example, Section 1 of [6. It is a fact that if $T$ has CBP then $\mathbb{P}$-coreductions exist for any finite-rank type (see Theorem 2.4 of [3]). Hence, assuming $T$ has CBP, if $\operatorname{stp}(a / A)$ is $\mathbb{P}$-analysable of finite $U$-rank then a $\mathbb{P}$-analysis by coreductions always exists.

The following lemma shows that in $\mathrm{DCF}_{0} \mathcal{C}$-coreductions of any finite-rank type always exist. This is because any nonmodular minimal type in $\mathrm{DCF}_{0}$ is almost $\mathcal{C}$-internal.

Lemma 4.5. We work in $\mathrm{DCF}_{0}$ in this lemma. If $U(a / A)$ is finite, then the $\mathcal{C}$-coreduction of a over $A$ exists.

Proof. Let $\mathbb{P}$ be the set of all nonmodular minimal types in $\mathcal{U} \models \mathrm{DCF}_{0}$. By Theorem 1.1 of [8], $\mathrm{DCF}_{0}$ has CBP. Therefore, there exists $b$ which is the $\mathbb{P}$-coreduction of $a$ over $A$.

We want to show that $b$ is the $\mathcal{C}$-coreduction of $a$ over $A$. In fact, we only need to show that if a type is almost $\mathbb{P}$-internal then it is almost $\mathcal{C}$-internal. Suppose $\operatorname{tp}(e / D)$ is $\mathbb{P}$-internal. Then for some $B \supset D$ such that $B \underset{D}{\downarrow} e$ and a tuple $c$ consists of realizations of types in $\mathbb{P}$ with bases in $B, e \in \operatorname{acl}(B c)$. Since every minimal nonmodular type in $\mathrm{DCF}_{0}$ is almost $\mathcal{C}$-internal, there exist $F \supset B$ such that $F \underset{B}{\downarrow} e c$ and $c \in \operatorname{acl}(F \mathcal{C})$. Now $e \in \operatorname{acl}(B c) \subseteq \operatorname{acl}(F \mathcal{C})$, and since $e \underset{B}{\downarrow} F$ and $e \underset{D}{\downarrow} B$, we have $e \underset{D}{\downarrow} F$. This shows that $\operatorname{tp}(e / D)$ is almost $\mathcal{C}$-internal.

It is not hard to see that analyses by reductions or coreductions are incompressible. If $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis by reductions of $\operatorname{tp}(a / A)$ and $\operatorname{stp}\left(a_{i+1} / A a_{i-1}\right)$ is almost $\mathcal{P}$-internal for some $i=1,2, \ldots, n-1$, then since $a_{i}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{i-1}, a_{i+1} \in \operatorname{acl}\left(A a_{i}\right)$ which implies $\operatorname{acl}\left(A a_{i}\right)=\operatorname{acl}\left(A a_{i+1}\right)$. Now for any $j>i$, assume that $\operatorname{acl}\left(A a_{j}\right)=\operatorname{acl}\left(A a_{i}\right)$. Then since $a_{j+1}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{j}$ and $\operatorname{acl}\left(A a_{j}\right)=\operatorname{acl}\left(A a_{i}\right), a_{j+1}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{i}$, so $\operatorname{acl}\left(A a_{j+1}\right)=\operatorname{acl}\left(A a_{i+1}\right)=\operatorname{acl}\left(A a_{i}\right)$. Thus $a_{i}, \ldots, a_{n}$ are all the same up to interalgebraicity over $A$, and this is possible only if $i=n$, contradicting the fact that $i \leq n-1$. Similarly, if $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis by coreductions of $\operatorname{tp}(a / A)$ and $\operatorname{stp}\left(a_{i+1} / A a_{i-1}\right)$ is almost $\mathcal{P}$-internal for some $i=1,2, \ldots, n-1$, then since $a_{i}$ is the $\mathcal{P}$-coreduction of $a_{i+1}$ over $a_{i-1}, a_{i} \in \operatorname{acl}\left(A a_{i-1}\right)$ which implies $a_{i}$ and $a_{i-1}$ are interalgebraic over $A$. An inductive argument similar to the reduction case shows that $a_{0}, \ldots, a_{i}$ are all the same up to interalgebraicity over $A$, and this is possible only if $i=0$, contradicting the fact that $i \geq 1$.

However, more is true: they are actually minimal.
Proposition 4.6. Analysis by reductions and coreductions are minimal.
Proof. Let $\left(a_{1}, \ldots, a_{n}\right)$ and $\left(c_{1}, \ldots, c_{\ell}\right)$ be $\mathcal{P}$-analyses of $a$ over $A$ with $\left(a_{1}, \ldots, a_{n}\right)$ being by reductions. We shall prove that $n \leq \ell$. We show that $c_{i} \in \operatorname{acl}\left(A a_{i}\right)$ for $i=1,2, \ldots, \min (n, \ell)$. For $i=1$, since $\operatorname{stp}\left(c_{1} / A\right)$ is almost $\mathcal{P}$-internal and $a_{1}$ is the $\mathcal{P}$-reduction of $a$ over $A, c_{1} \in \operatorname{acl}\left(A a_{1}\right)$. Now if $c_{i-1} \in \operatorname{acl}\left(A a_{i-1}\right)$, then
$\operatorname{stp}\left(c_{i} / a_{i-1}\right)$ is almost $\mathcal{P}$-internal, and as $a_{i}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{i-1}$, $c_{i} \in \operatorname{acl}\left(A a_{i}\right)$ as desired. Suppose $\ell<n$. Then $\operatorname{acl}\left(A a_{\ell}\right) \subsetneq \operatorname{acl}\left(A a_{n}\right)$ since $\left(a_{1}, \ldots, a_{n}\right)$ is incompressible, so $\operatorname{acl}(A a)=\operatorname{acl}\left(A c_{\ell}\right) \subseteq \operatorname{acl}\left(A a_{\ell}\right) \subsetneq \operatorname{acl}\left(A a_{n}\right)=\operatorname{acl}(A a)$, a contradiction.

Now suppose $\left(b_{1}, \ldots, b_{m}\right)$ is a $\mathcal{P}$-analysis by coreductions of $a$ over $A$. We shall prove that $m \leq \ell$. We show that $b_{m-j} \in \operatorname{acl}\left(A c_{\ell-j}\right)$ for $j=0,1, \ldots, \min (m, \ell)-1$. For $j=0$, notice that $b_{m}, c_{\ell}$ are both interalgebraic over $A$ with $a$. Now if $b_{m-j+1} \in$ $\operatorname{acl}\left(A c_{\ell-j+1}\right)$, then $\operatorname{stp}\left(b_{m-j+1} / c_{\ell-j}\right)$ is almost $\mathcal{P}$-internal, and as $b_{m-j}$ is the $\mathcal{P}$ coreduction of $b_{m-j+1}$ over $A, b_{m-j} \in \operatorname{acl}\left(A c_{\ell-j}\right)$ as desired. Assume towards a contradiction that $\ell<m$. Then $\operatorname{acl}\left(A b_{m-\ell+1}\right) \subseteq \operatorname{acl}\left(A c_{1}\right)$. Since $m-\ell+1 \geq 2$, $\operatorname{stp}\left(b_{m-\ell+1} / A\right)$ is not almost $\mathcal{P}$-internal because $\left(b_{1}, \ldots, b_{m}\right)$ is incompressible, but $\operatorname{stp}\left(c_{1} / A\right)$ is almost $\mathcal{P}$-internal, a contradiction.

So analyses by reductions and coreductions are of the same length. However, analyses by reductions and coreductions do not always have to agree (even up to interalgebraicity).
Definition 4.7. We say that two $\mathcal{P}$-analyses $\left(a_{1}, \ldots, a_{n}\right)$ and $\left(b_{1}, \ldots, b_{m}\right)$ of $a$ over $A$ are interalgebraic over $A$ if $n=m$ and $\operatorname{acl}\left(A a_{i}\right)=\operatorname{acl}\left(A b_{i}\right)$ for $i=1,2, \ldots, n$. We call an analysis canonical if it is minimal and interalgebraic with every other minimal analysis.

Example 4.8. Using the notation of Example 4.1, the $\mathcal{P}$-analysis by reductions of $a b_{1}$ over $\varnothing$ is $\left(a_{1} b_{1}, a b_{1}\right)$, while the $\mathcal{P}$-analysis by coreductions of $a b_{1}$ is $\left(a_{1}, a b_{1}\right)$. But $\left(a_{1} b_{1}, a b_{1}\right)$ and $\left(a_{1}, a b_{1}\right)$ are not interalgebraic. In particular, $\operatorname{stp}\left(a b_{1}\right)$ does not have a canonical $\mathcal{P}$-analysis.

The next proposition points out, however, that if an analysis by reductions has the same $U$-type as one by coreductions, then they are interalgebraic and are in fact the unique minimal analysis up to interalgebraicity.

Proposition 4.9. Let $\left(a_{1}, \ldots, a_{n}\right)$ and $\left(b_{1}, \ldots, b_{n}\right)$ be $\mathcal{P}$-analyses by reductions and coreductions of a over $A$, respectively. If the $U$-types of $\left(a_{1}, \ldots, a_{n}\right)$ and $\left(b_{1}, \ldots, b_{n}\right)$ are the same, then $\left(a_{1}, \ldots, a_{n}\right)$ is interalgebraic with $\left(b_{1}, \ldots, b_{n}\right)$ over $A$. Moreover, if $\left(c_{1}, \ldots, c_{n}\right)$ is another $\mathcal{P}$-analysis of a over $A$, then $\left(c_{1}, \ldots, c_{n}\right)$ is also interalgebraic with both $\left(a_{1}, \ldots, a_{n}\right)$ and $\left(b_{1}, \ldots, b_{n}\right)$ over $A$.

In particular, if $p$ has an analysis by reductions and an analysis by coreductions of the same $U$-type, then these analyses are canonical. Conversely, any canonical analysis is an analysis by both reductions and coreductions.
Proof. Having the same $U$-type implies that $U\left(a_{i} / A\right)=U\left(b_{i} / A\right)$ for $i=1,2, \ldots, n$. Let $\left(c_{1}, \ldots, c_{n}\right)$ be another $\mathcal{P}$-analysis of $a$ over $A$, We have seen in the proof of 4.6 that $c_{i} \in \operatorname{acl}\left(A a_{i}\right)$ and $b_{i} \in \operatorname{acl}\left(A c_{i}\right)$ for $i=1,2, \ldots, n$. Therefore $U\left(a_{i} / A\right)=$ $U\left(b_{i} / A\right)=U\left(c_{i} / A\right)$ and $\operatorname{acl}\left(A a_{i}\right)=\operatorname{acl}\left(A b_{i}\right)=\operatorname{acl}\left(A c_{i}\right)$ for $i=1,2, \ldots, n$, as desired.

The "in particular" clause now follows by Proposition 4.6. For the converse, let $\left(a_{i}\right)_{i=1}^{n},\left(b_{i}\right)_{i=1}^{n},\left(c_{i}\right)_{i=1}^{n}$ be $\mathcal{P}$-analyses of $a$ over $A$, which are an analysis by reductions, an analysis by coreductions, and a canonical analysis, respectively. We have that $a_{i}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{i-1}, \operatorname{acl}\left(A a_{i}\right)=\operatorname{acl}\left(A c_{i}\right)$, and $\operatorname{acl}\left(A a_{i-1}\right)=\operatorname{acl}\left(A c_{i-1}\right)$, so $c_{i}$ is the $\mathcal{P}$-reduction of $a$ over $A c_{i-1}$. Thus $\left(c_{i}\right)_{i=1}^{n}$ is a $\mathcal{P}$-analysis by reductions. Similarly, we have that $b_{i}$ is the $\mathcal{P}$-coreduction of $b_{i+1}$ over $A, \operatorname{acl}\left(A b_{i}\right)=\operatorname{acl}\left(A c_{i}\right)$, and $\operatorname{acl}\left(A b_{i+1}\right)=\operatorname{acl}\left(A c_{i+1}\right)$, so $c_{i}$ is the $\mathcal{P}$-coreduction of $a$ over $A c_{i-1}$. Thus $\left(c_{i}\right)_{i=1}^{n}$ is a $\mathcal{P}$-analysis by coreductions.

Here is a local criterion to determine whether an analysis is an analysis by reductions.

Lemma 4.10. Let $\left(a_{1}, \ldots, a_{n}\right)$ be a $\mathcal{P}$-analysis of a over $A$. Then it is a $\mathcal{P}$-analysis by reductions iff $a_{i}$ is a $\mathcal{P}$-reduction of $a_{i+1}$ over $A a_{i-1}$ for $i=1, \ldots, n-1$.

Proof. Suppose $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis by reductions of $a$ over $A$. For any $k=1,2, \ldots, n-1, a_{k}$ is a $\mathcal{P}$-reduction of $a$ over $A a_{k-1}$, i.e., for any $a_{k}^{\prime} \in \operatorname{acl}(A a)$, if $\operatorname{stp}\left(a_{k}^{\prime} / A a_{k-1}\right)$ is almost $\mathcal{P}$-internal, then $a_{k}^{\prime} \in \operatorname{acl}\left(a_{k}\right)$. In particular, for any $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{k+1}\right)$, if $\operatorname{stp}\left(a_{k}^{\prime} / A a_{k-1}\right)$ is almost $\mathcal{P}$-internal, then $a_{k}^{\prime} \in \operatorname{acl}\left(a_{k}\right)$. Note that $a_{k} \in \operatorname{acl}\left(A a_{k+1}\right)$, so $a_{k}$ is a $\mathcal{P}$-reduction of $a_{k+1}$ over $A a_{k-1}$.

Now suppose $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis of $a$ over $A$ such that $a_{i}$ is a $\mathcal{P}$-reduction of $a_{i+1}$ over $A a_{i-1}$ for $i=1, \ldots, n-1$. We need to check that $a_{k}$ is the $\mathcal{P}$-reduction of $a$ over $A a_{k-1}$. In fact, let $a_{k}^{\prime}$ be the $\mathcal{P}$-reduction of $a$ over $A a_{k-1}$, then we only need to show that $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{k}\right)$.

We know $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{n}\right)$. Suppose $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{i}\right)$ for some $i$ such that $k<i \leq n$. Since $a_{k}^{\prime}$ is almost $\mathcal{P}$-internal over $A a_{k-1}$ and $k-1<i-1, a_{k}^{\prime}$ is $\mathcal{P}$-internal over $A a_{i-2}$. Now $a_{i-1}$ is a $\mathcal{P}$ reduction of $a_{i}$ over $A a_{i-2}, a_{k}^{\prime} \in \operatorname{acl}\left(A a_{i}\right)$, and $a_{k}^{\prime}$ is almost $\mathcal{P}$-internal over $A a_{i-2}$, so $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{i-1}\right)$. By induction we get $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{k}\right)$.

We have a similar criterion for analyses by coreductions.
Lemma 4.11. A $\mathcal{P}$-analysis $\left(a_{1}, \ldots, a_{n}\right)$ of a over $A$ is a $\mathcal{P}$-analysis by coreductions iff $a_{i}$ is a $\mathcal{P}$-coreduction of $a_{i+1}$ over $A a_{i-1}$ for $i=1, \ldots, n-1$.
Proof. Suppose $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis by coreductions of $a$ over $A$. For any $k=1,2, \ldots, n-1, a_{k}$ is a $\mathcal{P}$-coreduction of $a_{k+1}$ over $A$, i.e., for any $a_{k}^{\prime} \in \operatorname{acl}\left(A a_{k+1}\right)$, if $\operatorname{stp}\left(a_{k+1} / A a_{k}^{\prime}\right)$ is $\mathcal{P}$-internal, then $a_{k} \in \operatorname{acl}\left(A a_{k}^{\prime}\right)$. In particular, for any $a_{k}^{\prime} \in$ $\operatorname{acl}\left(A a_{k+1}\right)$, if $\operatorname{stp}\left(a_{k+1} / A a_{k-1} a_{k}^{\prime}\right)$ is $\mathcal{P}$-internal, then $a_{k} \in \operatorname{acl}\left(A a_{k-1} a_{k}^{\prime}\right)$. So we have that $a_{k}$ is a reduction of $a_{k+1}$ over $A a_{k-1}$.

Now suppose $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis of $a$ over $A$ such that $a_{i}$ is a $\mathcal{P}$-coreduction of $a_{i+1}$ over $A a_{i-1}$ for $i=1, \ldots, n-1$. Fixing a $k \in\{1,2, \ldots, n-1\}$, we need to check that $a_{k}$ is the $\mathcal{P}$-coreduction of $a_{k+1}$ over $A$. In fact, let $a^{\prime}$ be be such that $\operatorname{stp}\left(a_{k+1} / A a^{\prime}\right)$ is almost $\mathcal{P}$-internal. We need to prove that $a_{k} \in \operatorname{acl}\left(A a^{\prime}\right)$.

We know that $a_{1} \in \operatorname{acl}\left(A a^{\prime}\right)$. This is because $a_{1}$ is the $\mathcal{P}$-coreduction of $a_{2}$ over $A$, and $\operatorname{stp}\left(a_{2} / A a^{\prime}\right)$ is almost $\mathcal{P}$-internal (since $\left.a_{2} \in \operatorname{dcl}\left(A a_{k+1}\right)\right)$.

Suppose $a_{i-1} \in \operatorname{acl}\left(A a^{\prime}\right)$ for some $i$ such that $1<i \leq k$. Since $a_{i+1}$ is almost $\mathcal{P}$ internal over $A a^{\prime}\left(\right.$ as $\left.i+1 \leq k+1, a_{i+1} \in \operatorname{acl}\left(A a_{k+1}\right)\right)$, and $a_{i}$ is the $\mathcal{P}$-coreduction of $a_{i+1}$ over $A a_{i-1}$, we have that $a_{i} \in \operatorname{acl}\left(A a^{\prime}\right)$. By induction we get $a_{k} \in \operatorname{acl}\left(A a^{\prime}\right)$.

It follows from the above lemma that an incompressible analysis of $U$-type $(1,1, \ldots, 1)$ is canonical. Indeed, for such an analysis $\left(a_{1}, \ldots, a_{n}\right)$ of $a$ over $A$, as $\operatorname{stp}\left(a_{i+1} / A a_{i-1}\right)$ is not almost $\mathcal{P}$-internal, by rank consideration, $a_{i}$ must be both the $\mathcal{P}$-reduction and the $\mathcal{P}$-coreduction of $a_{i+1}$ over $A a_{i-1}$ for $i=1,2, \ldots, n-1$.

We end this section by pointing out that once we have a type with an incompressible analysis of $U$-type $\underbrace{(1,1, \ldots, 1)}_{n}$ - as for example we do in $\mathrm{DCF}_{0}$ by Corollary 3.3 - then every decreasing sequence of positive integers of length $n$ appears as the $\mathcal{U}$-type of the $\mathcal{P}$-analysis by reductions of some other type in this theory. A similar statement holds for increasing sequences and $\mathcal{P}$-analyses by coreductions provided that every finite $U$-rank type has a $\mathcal{P}$-coreduction. For convenience we work over the empty set.

Proposition 4.12. Suppose $\left(a_{1}, \ldots, a_{n}\right)$ is a $\mathcal{P}$-analysis of a of $U$-type $(1,1, \ldots, 1)$.
(a) Given positive integers $s_{1} \geq \ldots \geq s_{n}$, there exists a tuple whose $\mathcal{P}$-analysis by reductions is of $U$-type $\left(s_{1}, \ldots, s_{n}\right)$.
(b) Suppose every type of finite $U$-rank has a $\mathcal{P}$-coreduction. Given positive integers $s_{1} \leq \ldots \leq s_{n}$, there exists a tuple whose $\mathcal{P}$-analysis by coreductions is of $U$-type $\left(s_{1}, \ldots, s_{n}\right)$.

Proof. (a) Let $\bar{a}^{(j)}=\left(a_{1}^{(j)}, \ldots, a_{n}^{(j)}\right), j=1,2, \ldots$ be tuples such that $\left(\bar{a}^{(1)}, \bar{a}^{(2)}, \ldots\right)$ is a Morley sequence of $\operatorname{tp}\left(a_{1}, \ldots, a_{n}\right)$. In particular, $a_{i}^{(j)}$ is the $\mathcal{P}$-reduction and the $\mathcal{P}$-coreduction of $a_{i+1}^{(j)}$. Let $\alpha_{i}=\left(a_{i}^{(1)}, \ldots, a_{i}^{\left(s_{i}\right)}\right)$ and $\beta_{i}=\left(\alpha_{1}, \ldots, \alpha_{i}\right)$. Note that $a_{i}^{(j)} \in \beta_{i}$ for $j=1,2, \ldots, s_{i}$. We claim the tuple $\beta_{n}$ is $\mathcal{P}$-analysable and its $\mathcal{P}$-analysis by reductions is of $U$-type $\left(s_{1}, \ldots, s_{n}\right)$. To show this, since $\left(\bar{a}^{(j)}\right)_{j}$ is a Morley sequence, we have

$$
\begin{aligned}
U\left(\beta_{i} / \beta_{i-1}\right) & =U\left(\alpha_{i} / \beta_{i-1}\right) \\
& =U\left(a_{i}^{(1)} \ldots a_{i}^{\left(s_{i}\right)} / \beta_{i-1}\right) \\
& =U\left(a_{i}^{(1)} \ldots a_{i}^{\left(s_{i}\right)} / a_{i-1}^{(1)} a_{i-1}^{\left(s_{i}\right)}\right) \\
& =s_{i},
\end{aligned}
$$

so we only need to prove that the $\mathcal{P}$-analysis by reductions of $\beta$ is $\left(\beta_{1}, \beta_{2}, \ldots, \beta_{n}\right)$.
Let $b_{i}$ be the reduction of $\beta_{n}$ over $\beta_{i-1}$. We claim that $b_{i}$ is interalgebraic with $\beta_{i}$. Since $a_{i-1}^{(j)} \in \operatorname{dcl}\left(\beta_{i-1}\right)$ for $j=1,2, \ldots, s_{i}$ (since $\left.s_{i-1} \geq s_{i}\right), \operatorname{stp}\left(a_{i}^{(j)} / \beta_{i-1}\right)$ is almost $\mathcal{P}$-internal for $j=1,2, \ldots, s_{i}$, so $\operatorname{stp}\left(\alpha_{i} / \beta_{i-1}\right)$ is almost $\mathcal{P}$-internal. Since $\beta_{i} \in \operatorname{dcl}\left(\alpha_{i}, \beta_{i-1}\right), \operatorname{stp}\left(\beta_{i} / \beta_{i-1}\right)$ is almost $\mathcal{P}$-internal, so $\beta_{i} \in b_{i}$. We now need to show that $U\left(b_{i} / \beta_{i}\right)=0$. Toward a contradiction, suppose $U\left(b_{i} / \beta_{i}\right)>0$.

Set $B=\beta_{i}$, which is the collection of elements of the form $a_{p}^{(q)}$ where $1 \leq p \leq i$ and $1 \leq q \leq s_{i}$. Now we add elements of the form $a_{p}^{(q)}$ one by one into $B$ according to dictionary order of $(p, q)$ where $i+1 \leq p \leq n$ and $1 \leq q \leq s_{i}$ as long as $U\left(b_{i} / B\right)$ remains unchanged. Since $b_{i} \in \beta_{n}, U\left(b_{i} / \beta_{n}\right)=0$, so this process will terminate for some $a_{p}^{(q)}$ where $U\left(b_{i} / B a_{p}^{(q)}\right)<U\left(b_{i} / B\right)$.

Now $B$ contains elements of the form $a_{p^{\prime}}^{\left(q^{\prime}\right)}$ where $\left(p^{\prime}, q^{\prime}\right)<(p, q)$ by dictionary order. We have $a_{p}^{(q)} \underset{B}{\perp} b_{i}$. As $a_{p-1}^{(q)} \in B$ and $a_{p}^{(q)} \underset{a_{p-1}^{(q)}}{\perp} B, U\left(a_{p}^{(q)} / B\right)=1$, so $a_{p}^{(q)} \in \operatorname{acl}\left(B b_{i}\right)$. However, Let $C=\left\{a_{i}^{(j)}: a_{i+1}^{(j)} \in \operatorname{dcl}(B)\right\}$. Then $\operatorname{stp}(B / C)$ is almost $\mathcal{P}$-internal as $\operatorname{stp}\left(a_{i+1}^{(j)} / a_{i}^{(j)}\right)$ is almost internal for any $i, j$, and $\operatorname{stp}\left(b_{i} / C\right)$ is almost $\mathcal{P}$-internal because $\beta_{i-1} \in \operatorname{dcl}(C)$. But $\operatorname{stp}\left(a_{p}^{(q)} / C\right)$ is not almost $\mathcal{P}$-internal: since $a_{p-1}^{(q)} \notin \operatorname{acl}\left(a_{p-2}^{(q)}\right)$ and $a_{p-1}^{(q)} \underset{a_{p-2}^{(q)}}{\perp} C$, we have $a_{p-1}^{(q)} \notin \operatorname{acl}(C)$.
(b) Let $\bar{a}^{(j)}=\left(a_{1}^{(j)}, \ldots, a_{n}^{(j)}\right), j=1,2, \ldots$ be tuples such that $\left(\bar{a}^{(1)}, \bar{a}^{(2)}, \ldots\right)$ is a Morley sequence of $\operatorname{tp}\left(a_{1}, \ldots, a_{n} / A\right)$. Let $\beta_{i}=\left(a_{1}^{(1)} \ldots a_{1}^{\left(s_{n-i+1}\right)}, \ldots, a_{i}^{(1)} \ldots a_{i}^{\left(s_{1}\right)}\right)$. Let $f(j)=\min \left\{k: j \leq s_{k}\right\}$, and let $f(j)$ be infinity if it is not defined. Then $a_{k}^{(j)} \in \beta_{i}$ iff $k \leq i-f(j)+1$ and $\beta_{i}=\bigcup_{j=1}^{s_{i}} a_{i+1-f(j)}^{(j)}$. We claim the tuple $\beta_{n}$ is $\mathcal{P}$-analysable
and its $\mathcal{P}$-analysis by coreductions is of $U$-type $\left(s_{1}, \ldots, s_{n}\right)$. Since $\beta_{i}=\bigcup_{j=1}^{s_{i}} a_{i+1-f(j)}^{(j)}$ and $\beta_{i-1}=\bigcup_{j=1}^{s_{i}} a_{i-f(j)}^{(j)}$ (as $i-f(j)=0$ for $s_{i-1}<j \leq s_{i}$, we may set the upper bound as $s_{i}$ ), we have

$$
\begin{aligned}
U\left(\beta_{i} / \beta_{i-1}\right) & =U\left(\bigcup_{j=1}^{s_{i}} a_{i+1-f(j)}^{(j)} / \beta_{i-1}\right) \\
& =\sum_{j=1}^{s_{i}} U\left(a_{i+1-f(j)}^{(j)} / a_{i-f(j)}^{(j)}\right) \\
& =s_{i}
\end{aligned}
$$

as $\left(\bar{a}^{(j)}\right)_{j}$ is a Morley sequence. Thus we only need to prove that the $\mathcal{P}$-analysis by coreductions of $\beta$ is $\left(\beta_{1}, \beta_{2}, \ldots, \beta_{n}\right)$.

Suppose $b$ is the $\mathcal{P}$-coreduction of $\beta_{i+1}$ over the empty set. We claim that $\operatorname{acl}(b)=\operatorname{acl}\left(\beta_{i}\right)$. Note that $\operatorname{stp}\left(\beta_{i+1} / \beta_{i}\right)$ is almost $\mathcal{P}$-internal, so $b \in \operatorname{acl}\left(\beta_{i}\right)$. Take any $a_{j}^{(k)} \in \beta_{i}$. Since $a_{j+1}^{(k)} \in \beta_{i+1}$ and $\beta_{i+1}$ is almost $\mathcal{P}$-internal over $b, a_{j+1}^{(k)}$ is almost $\mathcal{P}$-internal over $b$, so $a_{j}^{(k)} \in b$ since $a_{j}^{(k)}$ is the $\mathcal{P}$-coreduction of $a_{j+1}^{(k)}$. We therefore have that $\beta_{i} \in \operatorname{acl}(b)$.

## 5. A Construction in $\mathrm{DCF}_{0}$

In this section we show that in $\mathrm{DCF}_{0}$ we can do better than the conclusions of Proposition 4.12, Given any sequence of positive integers we provide a type which has a canonical $\mathcal{C}$-analysis with that $U$-type. Throughout we use the fact proven in Lemma 4.5 that any finite rank type has a $\mathcal{C}$-coreduction.

Suppose $n_{1}, \ldots, n_{\ell}$ are positive integers. We want to construct a type admitting a $\mathcal{C}$-analysis in $\ell$ steps where the $i$ th step has $U$-rank $n_{i}$, and such that the analysis is canonical. Here is our construction.

For convenience, we name everything in $\mathbb{Q}^{\text {alg }}$ in the language. Let $c_{i j} \in \mathbb{Q}^{\text {alg }}$ be algebraic numbers for $i=1,2, \ldots, \ell$ and $1 \leq j \leq n_{i}$ such that $\left\{c_{i j}\right\}_{j=1}^{n_{i}}$ is $\mathbb{Q}$-linearly independent for $i=1,2, \ldots, \ell$.

We inductively define $\left(D_{i}, e_{i}\right)$ for $i=1,2, \ldots, \ell$ as follows:
Set $D_{1}:=\delta$ and let $e_{1}$ be a generic solution over $\varnothing$ to
( $\mathrm{E}_{1}$ )

$$
\left(D_{1}-c_{11}\right)\left(D_{1}-c_{12}\right) \ldots\left(D_{1}-c_{1 n_{1}}\right) x=0 .
$$

For $i>1$ set $D_{i}:=\frac{\delta}{\prod_{j=1}^{i-1} e_{j}}$ and let $e_{i}$ be a generic solution over $\left\{e_{1}, \ldots e_{i-1}\right\}$ to ( $\mathrm{E}_{i}$ )

$$
\left(D_{i}-c_{i 1}\right)\left(D_{i}-c_{i 2}\right) \ldots\left(D_{i}-c_{i n_{i}}\right) x=0
$$

The notation $D_{i}-c_{i j}$ here represents a linear operator which sends $y$ to $D_{i} y-c_{i j} y$, so equation $\left(\mathrm{E}_{i}\right)$ is a linear differential equation over $\left\{e_{1}, \ldots e_{i-1}\right\}$ of order $n_{i}$.

Now let $a_{i}=\left(e_{1}, \ldots, e_{i}\right)$ for $i=1,2, \ldots, n$, and $a_{0}=\varnothing$. We will show that $\left(a_{1} \ldots a_{\ell}\right)$ is a canonical $\mathcal{C}$-analysis of $a_{\ell}$ of $U$-type $\left(n_{1}, \ldots, n_{\ell}\right)$.

Since $e_{i}$ is a generic solution of ( $\mathrm{E}_{i}$ ), an order $n_{i}$ linear differential equation over $a_{i-1}$, we have $U\left(a_{i} / a_{i-1}\right)=n_{i}$, and $\operatorname{stp}\left(a_{i} / a_{i-1}\right)$ is almost $\mathcal{C}$-internal. So this is
a $\mathcal{C}$-analysis of the correct $U$-type. We need to show it is by $\mathcal{C}$-reductions and $\mathcal{C}$-coreduction.

Fixing $i \in\{1,2, \ldots, \ell\}$, the following coordinatisation of solutions of $\left(\mathrm{E}_{i}\right)$ is a useful tool that we will apply often.
Lemma 5.1. If $f$ is any solution to $\left(E_{i}\right)$ then we can decompose $f=\sum_{j=1}^{n_{i}} f_{j}$ such that each $f_{j}$ is a solution to $D_{i} x-c_{i j} x=0$ and $f$ is interdefinable with $\left(f_{1}, \ldots, f_{n_{i}}\right)$ over $a_{i-1}$.

Proof. Indeed, let $g_{j}$ be a generic solution of $D_{i} x-c_{i j} x=0$. The set $\left\{g_{j}: j=\right.$ $\left.1,2, \ldots, n_{i}\right\}$ is $\mathcal{C}$-linearly independent because $g_{j}$ 's are nonzero eigenvectors of different eigenvalues under the $\mathcal{C}$-linear operator $D_{i}$. Note that since $\left(D_{i}-c_{i j}\right)$ commutes with $\left(D_{i}-c_{i j^{\prime}}\right)$ for any $j, j^{\prime}$, each $g_{j}$ is a solution to $\left(\mathrm{E}_{i}\right)$. Since $\left(\mathrm{E}_{i}\right)$ is an order $n_{i}$ linear differential equation and $\left\{g_{j}: j=1,2, \ldots, n_{i}\right\}$ is a set of $\mathcal{C}$-linearly independent solutions of $\left(\mathrm{E}_{i}\right)$, any solution of $\left(\mathrm{E}_{i}\right)$ is a $\mathcal{C}$-linear combination of $g_{j}$ 's. In particular, $f$ is of the form $\sum_{j=1}^{n_{i}} u_{j} g_{j}$ where $u_{j} \in \mathcal{C}$ for $j=1, \ldots, n_{i}$. Let $f_{j}=u_{j} g_{j}$, so $f=\sum_{j=1}^{n_{i}} f_{j}$, and $f \in \operatorname{dcl}\left(f_{1}, \ldots, f_{n_{i}}\right)$. Also,

$$
D_{i} f_{j}-c_{i j} f_{j}=u_{j}\left(D_{i} g_{j}-c_{i j} g_{j}\right)=0
$$

so $f_{j}$ is a solution to $D_{i} x-c_{i j} x=0$.
We still need to verify that $\left(f_{1}, \ldots, f_{n_{i}}\right) \in \operatorname{dcl}\left(a_{i-1} f\right)$. Indeed, suppose $\left(f_{j}^{*}\right)_{j=1}^{n_{i}}$ and $\left(f_{j}\right)_{j=1}^{n_{i}}$ have the same type over $a_{i-1} f$. Then in particular $f_{j}^{*}$ is a solution to $D_{i} x-c_{i j} x=0$, and

$$
\sum_{j=1}^{n_{i}} f_{j}=f=\sum_{j=1}^{n_{i}} f_{j}^{*}
$$

which gives us $\sum_{j=1}^{n_{i}}\left(f_{j}-f_{j}^{*}\right)=0$. As $\left\{f_{j}-f_{j}^{*}: j=1,2, \ldots, n_{i}\right\}$ is a set of eigenvectors of different eigenvalues under the $\mathcal{C}$-linear operator $D_{i}$, we then have $f_{j}-f_{j}^{*}=0$ for all $j=1,2, \ldots, n_{i}$, so $\left(f_{j}^{*}\right)_{j=1}^{n_{i}}=\left(f_{j}\right)_{j=1}^{n_{i}}$.

Lemma 5.2. If $f$ is a generic solution to ( $E_{i}$ ) over $a_{i-1}$, then $\left\{f_{1}, \ldots, f_{n_{i}}\right\}$ obtained in Lemma 5.1 is independent over $a_{i-1}$ and each $f_{j}$ is a generic solution to $D_{i} x$ $c_{i j} x=0$.

Proof. Since $f$ is a generic solution over $a_{i-1}$ to $\left(\mathrm{E}_{i}\right)$, which is a linear differential equation of order $n_{i}$, we have $U\left(f / a_{i-1}\right)=n_{i}$ Since $f_{j}$ is a solution for $D_{i} x-c_{i j} x=$ $0, U\left(f_{i j} / a_{i-1}\right) \leq 1$. But

$$
\begin{aligned}
n_{i} & =U\left(f / a_{i-1}\right) \\
& =U\left(f_{1} f_{2} \ldots f_{n_{i}} / a_{i-1}\right) \\
& =U\left(f_{1} / a_{i-1}\right)+U\left(f_{2} / a_{i-1} f_{1}\right)+\ldots+U\left(f_{n_{i}} / a_{i-1} f_{1} f_{2} \ldots f_{n_{i}-1}\right) \\
& \leq U\left(f_{1} / a_{i-1}\right)+U\left(f_{2} / a_{i-1}\right)+\ldots+U\left(f_{n_{i}} / a_{i-1}\right) \\
& \leq n_{i} .
\end{aligned}
$$

So $U\left(f_{j} / a_{i-1}\right)=1$ and $U\left(f_{j} / a_{i-1} f_{1} f_{2} \ldots f_{j-1}\right)=1$ for $j=1,2, \ldots, n_{i}$. This means that $\left\{f_{1}, \ldots, f_{n_{i}}\right\}$ is independent over $a_{i-1}$ and each $f_{j}$ is a generic solution to $D_{i} x-c_{i j} x=0$.

Lemma 5.3. Let $f$ be a generic solution over $\mathbb{Q}^{\text {alg }}$ to $\left(E_{1}\right)$. Then $\operatorname{acl}(f) \cap \mathcal{C}=\mathbb{Q}^{\text {alg }}$.
Proof. Let $m=n_{1}$. Let $\left(f_{1}, \ldots, f_{m}\right)$ be the decomposition of $f$ by Lemma 5.1 with respect to $\left(\mathrm{E}_{1}\right)$. Since $f$ is generic, $f_{j} \neq 0$ for $j=1,2, \ldots, m$. Suppose the conclusion is false and there exists some $c$ such that $c \in(\operatorname{acl}(f) \cap \mathcal{C}) \backslash \mathbb{Q}^{\text {alg }}$. Note that $\operatorname{acl}(f)=\mathbb{Q}\left(f_{1}, \ldots, f_{m}\right)^{\text {alg }}$ since $\delta f_{j}=c_{1 j} f_{j} \in \mathbb{Q}^{\text {alg }}\left(f_{j}\right)$.

For simplicity, let $\bar{f}=\left(f_{1}, \ldots, f_{m}\right)$, and $\bar{y}=\left(y_{1}, \ldots, y_{m}\right)$. Let $F(x, \bar{y})$ be a polynomial with coefficients in $\mathbb{Q}^{\text {alg }}$ such that $F(c, \bar{f})=0$ and $F(x, \bar{f}) \neq 0$. Since $c \notin \mathbb{Q}^{\text {alg }}, F(c, \bar{y}) \neq 0$. Let $G(\bar{y})$ be a nonzero polynomial over $\mathcal{C}$ with minimal number of terms such that $G(\bar{f})=0$. Since $F(c, \bar{y}) \neq 0$ and $F(c, \bar{f})=0, F(c, \bar{y})$ satisfies all conditions of $G$ except for the minimality, so such a $G$ exists.

Let

$$
G(\bar{y})=\sum_{\bar{r} \in I} s_{\bar{r}} \bar{y}^{\bar{r}},
$$

where $I$ is a set of $m$-tuples of nonnegative integers, $\bar{y}^{\bar{r}}=y_{1}^{r_{1}} \ldots y_{m}^{r_{m}}$, and $s_{\bar{r}} \in \mathcal{C}$. Let $\bar{c}=\left(c_{11}, \ldots, c_{1 m}\right)$, and set $\bar{f} \bar{c}:=\sum_{j=1}^{m} f_{j} c_{1 j}$.

We claim that

$$
\bar{r}^{(1)} \bar{c}=\bar{r}^{(2)} \bar{c}
$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Indeed, otherwise, fixing any $\bar{r}^{*} \in I$, we have

$$
\begin{aligned}
G^{*}(\bar{y}) & :=\bar{r}^{*} \bar{c} G(\bar{y})-\delta(G(\bar{y})) \\
& =\sum_{\bar{r} \in I}\left(\bar{r}^{*} \bar{c}\right) s_{\bar{r}} \bar{y}^{\bar{r}}-\sum_{\bar{r} \in I} s_{\bar{r}} \delta \bar{y}^{\bar{r}} \\
& =\sum_{\bar{r} \in I}\left(\bar{r}^{*} \bar{c}-\bar{r} \bar{c}\right) s_{\bar{r}} \bar{y}^{\bar{r}}
\end{aligned}
$$

is a polynomial with fewer terms than $G$ (since the term with index $\bar{r}^{*}$ is cancelled) such that its coefficients are in $\mathcal{C}, G^{*}(\bar{f})=0$ since $G(\bar{f})=\delta(G(\bar{f}))=0$, and $G^{*}(\bar{y}) \neq 0$ as there exist $\bar{r} \in I$ such that $\bar{r} \bar{c} \neq \bar{r}^{(*)} \bar{c}$. This contradicts the minimality of $G$.

We now have $\bar{r}^{(1)} \bar{c}=\bar{r}^{(2)} \bar{c}$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$, i.e., $\left(\bar{r}^{(1)}-\bar{r}^{(2)}\right) \bar{c}=0$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. But $\left\{c_{11} \ldots, c_{1 m}\right\}$ is $\mathbb{Q}$-linearly independent, so in fact $\bar{r}^{(1)}=\bar{r}^{(2)}$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Therefore there is only one element $\bar{r}$ in $I$, and $G(\bar{f})=s_{\bar{r}} \bar{f}^{\bar{r}}$. Since all $f_{j}$ 's are nonzero, $s_{\bar{r}}=0$, so $G$ is the zero polynomial, a contradiction.

The following lemma is the technical heart of the construction.
Lemma 5.4. Fix $i \in\{1,2, \ldots, \ell-1\}$, and for notational convenience, let $m:=n_{i}$ and $L:=\operatorname{acl}\left(a_{i-1}\right)$. Then the following are true:
(i) Suppose $f$ is a solution of $\left(E_{i}\right)$ and $\left(f_{1}, \ldots, f_{m}\right)$ is the decomposition of $f$ by Lemma 5.1. Then $f$ is generic over $L$ iff all the $f_{j}$ are nonzero.
(ii) Suppose $f$ is a generic solution to $\left(E_{i}\right)$ over $L, \alpha \in \mathbb{Q}^{\text {alg }}$ is nonzero, and $h$ is a nonzero solution of $D_{i} x-\alpha f x=0$. Then $f$ is the $\mathcal{C}$-coreduction of $h$ over $L$.
(iii) The $\mathcal{C}$-coreduction of $a_{i+1}$ over $a_{i-1}$ is $a_{i}$.
(iv) The $\mathcal{C}$-reduction of $a_{i+1}$ over $a_{i-1}$ is $a_{i}$.

Proof. We use induction on $i$.
(i) Suppose the conclusion is true for $i-1$.

By Lemma 5.2, if $f$ is a generic solution to $\left(\mathrm{E}_{i}\right)$ over $L$, then for any $j \in$ $\{1,2, \ldots, m\}, f_{j}$ is a generic solution to $D_{i} x-c_{i j} x=0$. In particular, $f_{j} \neq 0$.

Now suppose $f_{j} \neq 0$ for all $j=1,2, \ldots, m$, but $f$ is not generic, i.e., $U(f / L)<m$. Since

$$
\begin{aligned}
U\left(f / a_{i-1}\right) & =U\left(f_{1} f_{2} \ldots f_{m} / a_{i-1}\right) \\
& =U\left(f_{1} / a_{i-1}\right)+U\left(f_{2} / a_{i-1} f_{1}\right)+\ldots+U\left(f_{m} / a_{i-1} f_{1} f_{2} \ldots f_{m-1}\right)
\end{aligned}
$$

$U\left(f_{j} / a_{i-1} f_{1} f_{2} \ldots f_{j-1}\right)<1$ for some $j$, and hence $f_{j} \in L\left\langle\bigcup_{k \neq j} f_{k}\right\rangle$ for that $j$. Note that

$$
\delta f_{k}=\left(D_{i} f_{k}\right) \prod_{j=1}^{i-1} e_{j}=c_{i k} f_{k} \prod_{j=1}^{i-1} e_{j} \in L\left(f_{k}\right)
$$

so $f_{j} \in L\left\langle\bigcup_{k \neq j} f_{k}\right\rangle=L\left(\bigcup_{k \neq j} f_{k}\right)$, which means that $\left\{f_{1}, \ldots, f_{m}\right\}$ is algebraically dependent over $L$ in the field theoretic sense.

Let $\bar{f}=\left(f_{1}, \ldots, f_{m}\right), \bar{y}=\left(y_{1}, \ldots, y_{m}\right)$, and $\bar{c}=\left(c_{i 1}, \ldots, c_{i m}\right)$. Let $G(\bar{y})$ be a nonzero polynomial with minimal number of terms such that its coefficients are in $L$ and $G(\bar{f})=0$. We will use a minimality argument similar to that in the proof of Lemma 5.3. Suppose

$$
G\left(y_{1}, \ldots, y_{m}\right)=\sum_{\bar{r} \in I} s_{\bar{r}} \bar{y}^{\bar{r}}
$$

where $I$ is a set of $m$-tuples of nonnegative integers, and $s_{\bar{r}} \in L$ for $\bar{r} \in I$. Now

$$
\begin{aligned}
D_{i}(G(\bar{f})) & =D_{i} \sum_{\bar{r} \in I} s_{\bar{r}} \bar{f}^{\bar{r}} \\
& =\sum_{\bar{r} \in I}\left(\bar{f}^{\bar{r}} D_{i} s_{\bar{r}}+s_{\bar{r}} D_{i} \bar{f}^{\bar{r}}\right) \\
& =\sum_{\bar{r} \in I}\left(\log D_{i} s_{\bar{r}}+\bar{r} \bar{c}\right) s_{\bar{r}} \bar{f}^{\bar{r}}
\end{aligned}
$$

We claim that

$$
\log D_{i} s_{\bar{r}^{(1)}}+\bar{r}^{(1)} \bar{c}=\log D_{i} s_{\bar{r}^{(2)}}+\bar{r}^{(2)} \bar{c}
$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Indeed, otherwise, fixing any $\bar{r}^{*} \in I$, we have

$$
\begin{aligned}
G^{*}(\bar{y}) & :=\left(\log D_{i} s_{\bar{r}^{*}}+\bar{r}^{*} \bar{c}\right) G(\bar{y})-D_{i}(G(\bar{y})) \\
& =\sum_{\bar{r} \in I}\left(\log D_{i} s_{s_{\bar{r}}}+\bar{r}^{*} \bar{c}-\log D_{i} s_{\bar{r}}-\bar{r} \bar{c}\right) s_{\bar{r}} \bar{y}^{\bar{r}}
\end{aligned}
$$

is a polynomial with fewer terms than $G$ (since the term with index $\bar{r}^{*}$ is cancelled) such that its coefficients are in $L, G^{*}(\bar{f})=0$ as $G(\bar{f})=D_{i}(G(\bar{f}))=0$, and $G^{*}(\bar{y}) \neq 0$ as there exist $\bar{r}$ in $I$ such that $\log D_{i} s_{\bar{r}}+\bar{r} \bar{c} \neq \log D_{i} s_{\bar{r}^{*}}+\bar{r}^{*} \bar{c}$. This contradicts the minimality of $G$.

There are at least two terms in $G(\bar{y})$. Indeed, if there is only one term in $G$, then $G(\bar{y})=s_{\bar{r}} \bar{y}^{\bar{r}}$ for the unique $\bar{r} \in I$. Since $f_{j} \neq 0$ for $j=1,2, \ldots, m$ and $G(\bar{f})=0$, we have $s_{\bar{r}}=0$, so $G(\bar{y})=0$, contradicting the fact that $G$ is nonzero.

We now have

$$
\log D_{i} s_{\bar{r}^{(1)}}+\bar{r}^{(1)} \bar{c}=\log D_{i} s_{\bar{r}^{(2)}}+\bar{r}^{(2)} \bar{c}
$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Note that $\log D_{i} s_{\bar{r}}+\bar{r} \bar{c}=\log D_{i}\left(s_{\bar{r}} \bar{f}^{\bar{r}}\right)$ for any $\bar{r} \in I$. Therefore, fixing $\bar{r}^{(1)} \neq \bar{r}^{(2)}$ in $I$, we get $s_{\bar{r}^{(1)}} \bar{f}^{(1)}=c s_{\bar{r}^{(2)}} \bar{f}^{\bar{r}^{(2)}}$ for some nonzero $c \in \mathcal{C}$. This means that

$$
\begin{equation*}
c \bar{f}^{\bar{r}^{(2)}-\bar{r}^{(1)}}=s_{\bar{r}^{(1)}} s_{\bar{r}^{(2)}}^{-1} . \tag{*}
\end{equation*}
$$

Note that as all $f_{j} \neq 0, \bar{f}^{r^{(2)}-\bar{r}^{(1)}}$ makes sense and is nonzero. Let $h=c \bar{f}^{\bar{r}^{(2)}-\bar{r}^{(1)}}$. Then $h$ is a nonzero solution to

$$
\begin{equation*}
\log D_{i} x=\left(\bar{r}^{(2)}-\bar{r}^{(1)}\right) \bar{c} \tag{**}
\end{equation*}
$$

When $i=1$, right side of $\left(^{*}\right)$ is in $\operatorname{acl}\left(a_{0}\right)=\mathbb{Q}^{\text {alg }} \subset \mathcal{C}$, so $h$ is also a constant, but then it is not a solution for $\left({ }^{* *}\right)$. When $i>1$, we apply part (ii) of the lemma for $i-1$ with $e_{i-1}$ a generic solution of $\left(\mathrm{E}_{i-1}\right)$ over $a_{i-2}, \alpha=\left(\bar{r}^{(2)}-\bar{r}^{(1)}\right) \bar{c}^{*} \neq 0$, and $h$ a nonzero solution of $D_{i-1} x-d x=0$. We get that $e_{i-1}$ is the coreduction of $h$ over $a_{i-2}$. In particular, since $e_{i-1} \notin \operatorname{acl}\left(a_{i-2}\right)$, we have that $\operatorname{stp}\left(h / a_{i-2}\right)$ is not almost $\mathcal{C}$-internal. But the right side of $\left(^{*}\right)$ is in $L$ which is almost $\mathcal{C}$-internal over $a_{i-2}$, a contradiction.
(ii) Suppose the conclusion is true for $i-1$, and (i) is true for $i$.

We use induction on $m$, the order of the differential equation $\left(\mathrm{E}_{i}\right)$.
If $m=n_{i}=1$, we have that $\log D_{i} h=\alpha f$ and $\log D_{i}(\alpha f)=c_{i 1}$. Let $h^{*}$ be a generic solution of $\log D_{i} x=\alpha f$ over $L f$. Since $f$ is a generic solution of $\log D_{i}(x)=c_{i 1}$ over $L, \alpha f$ is also a generic solution of $\log D_{i}(x)=c_{i 1}$ over $L$, and therefore $h^{*}$ is a generic solution of $\log D_{i}^{(2)} x=c_{i 1}$ over $a_{i-1}$. Thus $\operatorname{stp}\left(h^{*} / L\right)$ is not almost $\mathcal{C}$-internal by Proposition 3.2. Since $h^{*}$ is a constant multiple of $h, \operatorname{stp}(h / L)$ is also not almost $\mathcal{C}$-internal. Note that $(f, h)$ is a $\mathcal{C}$-analysis of $h$ over $L$, and as it is incompressible of $U$-type $(1,1)$, we have that $f$ is the $\mathcal{C}$-coreduction of $h$ over $L$.

Now suppose the conclusion of (ii) is proven if the order of the equation $\left(\mathrm{E}_{i}\right)$ is less than or equal to $m-1$.

Let $\beta$ be the $\mathcal{C}$-coreduction of $h$ over $L$. Since $\operatorname{stp}(h / L f)$ is almost $\mathcal{C}$-internal, we only need to show that $f \in \operatorname{acl}(L \beta)$. Let $\left(f_{1}, \ldots, f_{m}\right)$ be the decomposition of $f$ by Lemma [5.1. By Lemma 5.2, $f_{j}$ is a generic solution of $D_{i} x-c_{i j} x=0$ for $j=1,2, \ldots, m$. Suppose towards a contradiction that $f \notin \operatorname{acl}(L \beta)$. We may, without loss of generality, suppose $f_{1}, \ldots, f_{s} \notin \operatorname{acl}(L \beta)$ and $f_{s+1}, \ldots, f_{m} \in \operatorname{acl}(L \beta)$ for some $1 \leq s \leq m$.

In the rest of the proof we seek a contradiction to the above assumption.
We prove first that $s=m$. Suppose not, so $f_{m} \in \operatorname{acl}(L \beta)$. Let $h_{m}$ be a nonzero solution to $D_{i} x-\alpha f_{m} x=0$. We have that $\operatorname{stp}\left(h_{m} / L f_{m}\right)$ is almost $\mathcal{C}$-internal. Since $f_{m} \in \operatorname{acl}(L \beta), \operatorname{stp}\left(h_{m} / L \beta\right)$ is almost $\mathcal{C}$-internal. Let $h^{*}=h h_{m}^{-1}$. Then

$$
\begin{aligned}
\log D_{i}\left(h^{*}\right) & =\log D_{i}(h)-\log D_{i}\left(h_{m}\right) \\
& =\alpha\left(f_{1}+\ldots+f_{m-1}+f_{m}\right)-\alpha f_{m} \\
& =\alpha\left(f_{1}+\ldots+f_{m-1}\right)
\end{aligned}
$$

Let $f^{*}=f_{1}+\ldots+f_{m-1}$. Then $h^{*}$ is a nonzero solution to $D_{i} x-\alpha f^{*} x=0$. From (i), since $f_{1}, \ldots, f_{m-1}$ are all nonzero, $f^{*}$ is a generic solution over $L$ to

$$
\left(D_{i}-c_{i 1}\right) \ldots\left(D_{i}-c_{i, m-1}\right) x=0
$$

By the induction hypothesis, we conclude that the $\mathcal{C}$-coreduction of $h^{*}$ over $L$ is $f^{*}$. Since $h$ and $h_{m}$ are almost $\mathcal{C}$-internal over $L \beta$ and $h^{*}=h h_{m}^{-1}$, we get that
$f^{*} \in \operatorname{acl}(L \beta)$. As $f^{*}$ is interdefinable with $\left(f_{1}, \ldots, f_{m-1}\right)$ over $L, f_{1} \in \operatorname{acl}(L \beta)$, contradicting our assumption.

Let $g_{t 1}=t f_{1}$ for $t=1,2, \ldots$. We show that $\operatorname{stp}\left(g_{t 1} / L \beta\right)=\operatorname{stp}\left(f_{1} / L \beta\right)$. Since

$$
\begin{equation*}
D_{i} g_{t 1}-c_{i 1} g_{t 1}=t D_{i} f_{1}-t c_{i 1} f_{1}=0 \tag{5.1}
\end{equation*}
$$

we have that $g_{t 1} \in\left\{x: D_{i} x-c_{i 1} x=0\right\}$, a strongly minimal set. Thus in order to prove $\operatorname{stp}\left(g_{t 1} / L \beta\right)=\operatorname{stp}\left(f_{1} / L \beta\right)$ we only need to show that $g_{t 1} \notin \operatorname{acl}(L \beta)$, which follows from $f_{1} \notin \operatorname{acl}(L \beta)$.

For each integer $t \geq 1$, let $\eta_{t}$ be an automorphism fixing acl $(L \beta)$ and taking $f_{1}$ to $g_{t 1}$. Set $g_{t j}:=\eta_{t}\left(f_{j}\right)$ for all $j=1,2, \ldots, m, g_{t}:=\eta_{t}(f)$, and $h_{t}:=\eta_{t}(h)$. So $\operatorname{stp}\left(h_{t}, g_{t}, g_{t 1}, \ldots, g_{t m} / L \beta\right)=\operatorname{stp}\left(h, f, f_{1}, \ldots, f_{m} / L \beta\right)$ for all $t \geq 1$. In particular, $g_{t}$ is a generic solution to $\left(\mathrm{E}_{i}\right)$ over $L, h_{t}$ is a nonzero solution to $D_{i} x-\alpha g_{t} x=0$, $g_{t}=\sum_{j=1}^{m} g_{t j}$ is the decomposition by Lemma 5.1, and $\operatorname{stp}\left(h_{t} / \beta\right)$ is almost $\mathcal{C}$-internal.

We next show that $g_{t j}=t f_{j}$ for all $t \geq 1$ and all $j$.
Towards a contradiction, suppose that $g_{t j} \neq t f_{j}$ for some $t$ and $j$. Fix this $t$. We argue first that $g_{t j}-t f_{j} \in \operatorname{acl}(L \beta)$. Let $H=h_{t} h^{-t}$, and let $I=\{j: 2 \leq j \leq$ $\left.m, g_{t j}-t f_{j} \neq 0\right\}$ (note that $g_{t 1}=t f_{1}$, so we only need $j \geq 2$; also note that $I$ is nonempty since $g_{t j} \neq t f_{j}$ for some $j$ by assumption). We have that

$$
\begin{aligned}
D_{i} H & =\left(\log D_{i} H\right) H \\
& =\left(\log D_{i} h_{t}-t \log D_{i} h\right) H \\
& =\left(\alpha g_{t}-t \alpha f\right) H \\
& =\left(\alpha \sum_{j=1}^{m}\left(g_{t j}-t f_{j}\right)\right) H, \\
& =\left(\alpha \sum_{j \in I}\left(g_{t j}-t f_{j}\right)\right) H .
\end{aligned}
$$

So $H$ is a nonzero solution of $D_{i} x-\left(\alpha \sum_{j \in I}\left(g_{t j}-t f_{j}\right)\right) x=0$.
Note that $\sum_{j \in I}\left(g_{t j}-t f_{j}\right)$ is a solution to

$$
\begin{equation*}
\left(\prod_{j \in I}\left(D_{i}-c_{i j}\right)\right)(x)=0 \tag{5.2}
\end{equation*}
$$

This is because (5.2) is linear, and for each $j \in I$,

$$
\left(D_{i}-c_{i j}\right)\left(g_{t j}-t f_{j}\right)=\left(D_{i}-c_{i j}\right) g_{t j}-\left(D_{i}-c_{i j}\right) t f_{j}=0
$$

The decomposition of $\sum_{j \in I}\left(g_{t j}-t f_{j}\right)$ by Lemma 5.1] with respect to (5.2) is $\left(g_{t j}-\right.$ $\left.t f_{j}\right)_{j \in I}$, and $g_{t j}-t f_{j} \neq 0$ for every $j \in I$. Therefore, applying part (i) where we replace $\left(\mathrm{E}_{i}\right)$ with (5.2), we get that $\sum_{j \in I}\left(g_{t j}-t f_{j}\right)$ is a generic solution to (5.2) over $L$.

Now, since (5.2) is of order less than $m$ and $H$ is a nonzero solution of $D_{i} x-$ $\left(\alpha \sum_{j \in I}\left(g_{t j}-t f_{j}\right)\right) x=0$, by the induction hypothesis, the coreduction of $H$ over $L$ is $\sum_{j \in I}\left(g_{t j}-t f_{j}\right)$. Since $H=h_{t} h^{-t}$ and both $h$ and $h_{t}$ are almost $\mathcal{C}$-internal over $L \beta$, we have $\operatorname{stp}(H / L \beta)$ is almost $\mathcal{C}$-internal. Therefore, for any $j \in I, g_{t j}-t f_{j} \in$ $\operatorname{acl}(L \beta)$. We now fix some $j \in I$.

Let $\gamma=\frac{g_{t j}}{f_{j}}-t=\frac{g_{t j}-t f_{j}}{f_{j}} \neq 0$. Then $\gamma$ is a constant in $\operatorname{acl}(L F) \backslash \operatorname{acl}(L \beta)$. Indeed, $\gamma$ is a constant because $g_{t j}$ and $f_{j}$ are both solutions to $D_{i} x-c_{i j} x=0$, and hence $\frac{g_{t j}}{f_{j}} \in \mathcal{C}$. We get $\gamma \in \operatorname{acl}(L f)$ by the fact that $g_{t j}-t f_{j} \in \operatorname{acl}(L \beta) \subseteq \operatorname{acl}(L f)$. And $\gamma \notin \operatorname{acl}(L \beta)$ because if it were, then so would $f_{j}=\frac{g_{t j}-t f_{j}}{\gamma}$, but we know that is not the case.

When $i=1$ this is impossible, since $\operatorname{acl}(L f)=\operatorname{acl}(f)$, and Lemma 5.3 tells us that $\operatorname{acl}(f) \cap \mathcal{C}=\mathbb{Q}^{\text {alg }}$.

Suppose $i>1$. We apply part (iv) of the lemma for $i-1$ and get that the $\mathcal{C}$-reduction of $a_{i}$ over $a_{i-2}$ is $a_{i-1}$. As $f$ is a generic solution of $\left(\mathrm{E}_{i}\right)$ over $L$, $\operatorname{stp}(f / L)=\operatorname{stp}\left(e_{i} / L\right)$, so the $\mathcal{C}$-reduction of $f$ over $a_{i-2}$ is $a_{i-1}$. Since $\gamma \in$ $\operatorname{acl}(L f) \backslash \operatorname{acl}(L \beta), \gamma \notin L=\operatorname{acl}\left(a_{i-1}\right)$. So $\operatorname{stp}\left(\gamma / a_{i-2}\right)$ is not almost $\mathcal{C}$-internal. On the other hand, $\gamma$ is a constant, a contradiction.

What we have actually shown is that for any $t \geq 1, \operatorname{stp}\left(t f_{1} / L \beta\right)=\operatorname{stp}\left(f_{1} / L \beta\right)$, and if $\operatorname{stp}\left(\tilde{f}_{1}, \tilde{f}_{2}, \ldots, \tilde{f}_{m} / L \beta\right)=\operatorname{stp}\left(f_{1}, \ldots, f_{m} / L \beta\right)$ and $\tilde{f}_{1}=t f_{1}$, then $\tilde{f}_{j}=t f_{j}$ for $j=2,3, \ldots, m$. In particular, $\operatorname{stp}\left(t f_{1}, \ldots, t f_{m} / L \beta\right)=\operatorname{stp}\left(f_{1}, \ldots, f_{m} / L \beta\right)$ holds for all $t$. In addition, the case of $t=1$ tells us that $f_{j} \in \operatorname{dcl}\left(f_{1} \operatorname{acl}(L \beta)\right)$ for $j=2,3, \ldots, m$.

We now show that $\frac{f_{j}}{f_{1}} \in \operatorname{acl}(L \beta)$ for $j=2,3, \ldots, m$. Fix some $j$. Since $f_{j} \in$ $\operatorname{dcl}\left(f_{1} \operatorname{acl}(L \beta)\right)$, there exists a formula $\varphi_{1}(x, y)$ over $\operatorname{acl}(L \beta)$ such that $\varphi_{1}\left(\mathcal{U}, f_{1}\right)=$ $\left\{f_{j}\right\}$. Since $\operatorname{stp}\left(t f_{1}, t f_{j} / L \beta\right)=\operatorname{stp}\left(f_{1}, f_{j} / L \beta\right)$, we have $\varphi_{1}\left(\mathcal{U}, t f_{1}\right)=\left\{t f_{j}\right\}$ for all $t$. Now set $\varphi_{2}(x, y):=\forall z\left(\varphi_{1}(z, y) \rightarrow x=\frac{z}{y}\right)$. Then $\varphi_{2}\left(\mathcal{U}, t f_{1}\right)=\left\{\frac{f_{j}}{f_{1}}\right\}$ for all $t$. So we have

$$
\left\{t f_{1}: t \geq 1\right\} \subseteq\left\{b \in \mathcal{U}: \log D_{i} b=c_{i 1} \text { and } \varphi_{2}(\mathcal{U}, b)=\left\{\frac{f_{j}}{f_{1}}\right\}\right\}
$$

Since $\log D_{i} x=c_{i 1}$ is strongly minimal, it must be that for all but finitely many solutions to $\log D_{i} x=c_{i 1}, \varphi_{2}(\mathcal{U}, b)=\left\{\frac{f_{j}}{f_{1}}\right\}$. It follows that $\frac{f_{j}}{f_{1}} \in \operatorname{acl}(L \beta)$.

Let $g_{01}$ be a generic solution over $L h$ to $D_{i} x-c_{i 1} x=0$, and $g_{0 j}=g_{01} \frac{f_{j}}{f_{1}}$ for $j=$ $2,3, \ldots, m$. We have shown that each $\frac{f_{j}}{f_{1}}$ is in $\operatorname{acl}(L \beta)$, so $\left(g_{01}, \ldots, g_{0 m}\right) \in \operatorname{acl}\left(L \beta g_{01}\right)$. Let $c_{01}=\frac{f_{1}}{g_{01}} \in \mathcal{C}$. Now,

$$
\begin{aligned}
\log D_{i}^{(2)}(h) & =\log D_{i}(\alpha f) \\
& =\log D_{i}\left(\alpha\left(f_{1}+\ldots+f_{m}\right)\right) \\
& =\log D_{i}\left(\alpha c_{01}\left(g_{01}+\ldots+g_{0 m}\right)\right) \\
& =\log D_{i}\left(g_{01}+\ldots+g_{0 m}\right)=: \epsilon
\end{aligned}
$$

Hence $h$ is a solution to $\log D_{i}^{(2)}(x)=\epsilon$ which is over acl $\left(L \beta g_{01}\right)$, so $U\left(h / L \beta g_{01}\right) \leq$ 2. Note that $U(h / L \beta) \geq 2$ since $h$ is a generic solution to $\log D_{i} x=\alpha f$ and $U(f / L \beta) \geq 1$. But we also have $h \underset{L \beta}{\downarrow} g_{01}$ (recall that $\left.\beta \in \operatorname{acl}(L h)\right)$, so $U\left(h / L \beta g_{01}\right)=$ $U(h / L \beta) \geq 2$. Thus $U\left(h / L \beta g_{01}\right)=2$, and $h$ is a generic solution to $\log D_{i}^{(2)}(x)=\epsilon$
over $\operatorname{acl}\left(L \beta g_{01}\right)$. Hence $\operatorname{stp}\left(h / L \beta g_{01}\right)$ is not almost $\mathcal{C}$-internal by Proposition 3.2, and therefore $\operatorname{stp}(h / L \beta)$ is not almost $\mathcal{C}$-internal, contradicting the definition of $\beta$.
(iii) Assume part (ii) of the lemma is true for $i$.

Let $e_{i+1}=\sum_{j=1}^{n_{i+1}} b_{i+1, j}$ be the decomposition by Lemma 5.1 with respect to $\left(\mathrm{E}_{i+1}\right)$. We have that $\operatorname{stp}\left(a_{i+1} / a_{i}\right)$ is almost $\mathcal{C}$-internal. Also, by part (ii) applied to $f=e_{i}$ and $h=b_{i+1,1}$, the $\mathcal{C}$-coreduction of $b_{i+1,1}$ over $a_{i-1}$ is $e_{i}$, which is interdefinable over $a_{i-1}$ with $a_{i}$. Since $b_{i+1,1} \in \operatorname{dcl}\left(a_{i} e_{i+1}\right)=\operatorname{dcl}\left(a_{i+1}\right)$, the $\mathcal{C}$ coreduction of $a_{i+1}$ over $a_{i-1}$ is $a_{i}$.
(iv) Assume parts (i) and (ii) of the lemma are true for $i$. For simplicity, we use $n$ to denote $n_{i+1}$. Let $K$ be the algebraically closed field generated by $a_{i}$. Let $\bar{b}_{i+1}=\left(b_{i+1,1}, \ldots, b_{i+1, n}\right)$.

We already know that $\operatorname{stp}\left(a_{i} / a_{i-1}\right)$ is $\mathcal{C}$-internal. Suppose $\beta \in \operatorname{acl}\left(a_{i+1}\right)$ is almost $\mathcal{C}$-internal over $a_{i-1}$ and $\beta \notin \operatorname{acl}\left(a_{i}\right)$. Since $e_{i+1}$ is interalgebraic with $\bar{b}_{i+1}$ over $a_{i}, \beta \in \operatorname{acl}\left(a_{i} \bar{b}_{i+1}\right)$, which means $\beta \in K\left\langle\bar{b}_{i+1}\right\rangle^{\text {alg }}$. Since $\delta b_{i+1, j}=$ $c_{i+1, j} b_{i+1, j} \prod_{k=1}^{i} e_{k} \in K\left(b_{i+1, j}\right)$ for $j=1,2, \ldots, n$, we have $K\left\langle\bar{b}_{i+1}\right\rangle=K\left(\bar{b}_{i+1}\right)$, so $\beta \in K\left(\bar{b}_{i+1}\right)^{\text {alg }}$. Thus there exist a polynomial $F\left(x, y_{1}, \ldots, y_{n}\right)$ with coefficients in $K$ such that $F\left(\beta, b_{i+1,1}, \ldots, b_{i+1, n}\right)=0$ and $F\left(x, b_{i+1,1}, \ldots, b_{i+1, n}\right) \neq 0$. Also, $F\left(\beta, y_{1}, \ldots, y_{n}\right) \neq 0$ since $\beta \notin K$.

Suppose $G\left(y_{1}, \ldots, y_{n}\right)$ is a nonzero polynomial with minimal number of terms such that the coefficients of $G$ are almost $\mathcal{C}$-internal over $a_{i-1}$ and $G\left(\bar{b}_{i+1}\right)=0$. Note that this is well-defined because $F\left(\beta, y_{1}, \ldots, y_{n}\right)$ satisfies all the conditions except for the minimality, as $K$ and $\beta$ are both almost $\mathcal{C}$-internal over $a_{i-1}$.

Let

$$
G\left(y_{1}, \ldots, y_{n}\right)=\sum_{\bar{r} \in I} s_{\bar{r}} \bar{y}^{\bar{r}}
$$

where $I$ is a set of $n$-tuples of nonnegative integers, and $\operatorname{stp}\left(s_{\bar{r}} / a_{i-1}\right)$ is almost $\mathcal{C}$-internal. Let $\bar{c}_{i+1}=\left(c_{i+1,1}, \ldots, c_{i+1, n}\right)$. Arguing exactly as in the proof of part (i) of the lemma, we get by minimality of $G$ that

$$
\begin{equation*}
\log D_{i} s_{\bar{r}^{(1)}}+\bar{r}^{(1)} \bar{c}_{i+1} e_{i}=\log D_{i} s_{\bar{r}^{(2)}}+\bar{r}^{(2)} \bar{c}_{i+1} e_{i} \tag{5.3}
\end{equation*}
$$

for any $r^{(1)}, r^{(2)} \in I$. Indeed,

$$
\begin{aligned}
D_{i}\left(G\left(\bar{b}_{i+1}\right)\right) & =\sum_{\bar{\epsilon} I}\left(\bar{b}_{i+1}^{\bar{r}} D_{i} s_{\bar{r}}+s_{\bar{r}} D_{i} \bar{b}_{i+1}^{\bar{r}}\right) \\
& =\sum_{\bar{\epsilon} I}\left(\bar{b}_{i+1}^{\bar{r}} D_{i} s_{\bar{r}}+s_{\bar{r}} \bar{r} \bar{c}_{i+1} e_{i} \bar{b}_{i+1}^{\bar{r}}\right) \\
& =\sum_{\bar{\epsilon} I}\left(\log D_{i} s_{\bar{r}}+\bar{r} \bar{c}_{i+1} e_{i}\right) s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}}
\end{aligned}
$$

where the second equality is by the fact that

$$
\begin{aligned}
D_{i} \bar{b}_{i+1}^{\bar{r}} & =\bar{r} \bar{b}_{i+1}^{\bar{r}-\overline{1}} D_{i} \bar{b}_{i+1} \\
& =\bar{r} \bar{b} \bar{b}_{i+1}^{\bar{r}}-\overline{1} e_{i} D_{i+1} \bar{b}_{i+1} \\
& =\bar{r} \bar{b} \bar{b}-\overline{1}-\overline{1} e_{i} \bar{c}_{i+1} \bar{b}_{i+1} \\
& =\bar{r} e_{i} \bar{c}_{i+1} \bar{b}_{i+1}^{\bar{r}} .
\end{aligned}
$$

Now if (5.3) failed, then fixing any $\bar{r}^{*} \in I$ we see that

$$
\begin{aligned}
G^{*}(\bar{y}): & :=\left(\log D_{i} s_{\bar{r}^{*}}+\bar{r}^{*} \bar{c}_{i+1} e_{i}\right) G(\bar{y})-D_{i} G(\bar{y}) \\
& =\sum_{\bar{r} \in I}\left(\log D_{i} s_{\bar{r}^{*}}+\bar{r}^{*} \bar{c}_{i+1} e_{i}-\log D_{i} s_{\bar{r}}-\bar{r} \bar{c}_{i+1} e_{i}\right) s_{\bar{r}} \bar{y}^{\bar{r}}
\end{aligned}
$$

whose coefficients are again almost $\mathcal{C}$-internal over $a_{i-1}$, would contradict the minimal choice of $G$.

If $G$ has only one term, then for the only $\bar{r} \in I, G\left(\bar{b}_{i+1}\right)=s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}}$. Since $b_{i+1, j} \neq 0$ for $j=1,2, \ldots, n, s_{\bar{r}}=0$, which means $G(\bar{y})=0$, a contradiction. Now fix $r^{(1)} \neq r^{(2)}$ in $I$. Since $\log D_{i} s_{\bar{r}}+\bar{r} \bar{c}_{i+1} e_{i}=\log D_{i}\left(s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}}\right)$ for any $\bar{r} \in I$, we have $s_{\bar{r}^{(1)}} \bar{b}_{i+1}^{\bar{r}^{(1)}}=c s_{\bar{r}^{(2)}} \bar{b}_{i+1}^{\bar{r}^{(2)}}$ for some $c \in \mathcal{C}$. This means that $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}=c s_{\bar{r}^{(2)}} s_{\bar{r}^{(1)}}^{-1}$. So $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ is almost $\mathcal{C}$-internal over $a_{i-1}$.

On the other hand, as $D_{i+1} \bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}=\left(\bar{r}^{(1)}-\bar{r}^{(2)}\right) \bar{c}_{i+1} \bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}, \bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ is a solution of $\left(D_{i+1}-\left(\bar{r}^{(1)}-\bar{r}^{(2)}\right) \bar{c}_{i+1}\right) x=0$, with $\left(\bar{r}-\bar{r}^{*}\right) \bar{c}_{i+1} \neq 0$ since $\left\{c_{i+1, j}\right.$ : $j=1,2, \ldots, n\}$ is $\mathbb{Q}$-linearly independent. By part (ii) of the lemma with $f=e_{i}$, $h=\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$, and $\alpha=\left(\bar{r}^{(1)}-\bar{r}^{(2)}\right) \bar{c}_{i+1}, e_{i}$ is a $\mathcal{C}$-coreduction of $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ over $a_{i-1}$. In particular, $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ is not almost $\mathcal{C}$-internal over $a_{i-1}$. This contradiction proves part (iv) of the lemma.

We have accomplished the desired construction:
Theorem 5.5. Given positive integers $n_{1}, \ldots, n_{\ell}$, there exists in $\mathrm{DCF}_{0}$ a type over $\mathbb{Q}^{\text {alg }}$ that admits a canonical $\mathcal{C}$-analysis of $U$-type $\left(n_{1}, \ldots, n_{\ell}\right)$.

Proof. Let $\left(a_{1}, \ldots, a_{\ell}\right)$ be as in the above construction. We have seen that $\left(a_{1}, \ldots, a_{\ell}\right)$ is a $\mathcal{C}$-analysis of $p=\operatorname{stp}\left(a_{\ell}\right)$ of $\mathcal{U}$-type $\left(n_{1}, \ldots, n_{\ell}\right)$. By Lemmas 4.10 and 4.11 parts (iii) and (iv) of Lemma 5.4 prove that it is a $\mathcal{C}$-analysis by reductions and coreductions, as desired.

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