THE JOURNAL OF SYMBOLIC LOGIC
Volume 00, Number 0, XXX 0000

# TRUNCATION AND SEMI-DECIDABILITY NOTIONS IN APPLICATIVE THEORIES

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**Abstract.** $\mathsf{BON}^+$ is an applicative theory and closely related to the first order parts of the standard systems of explicit mathematics. As such it is also a natural framework for abstract computations. In this article we analyze this aspect of $\mathsf{BON}^+$ more closely. First a point is made for introducing a new operation $\boldsymbol{\tau}_{\mathsf{N}}$, called truncation, to obtain a natural formalization of partial recursive functions in our applicative framework. Then we introduce the operational versions of a series of notions that are all equivalent to semi-decidability in ordinary recursion theory on the natural numbers, and study their mutual relationships over $\mathsf{BON}^+$ with $\boldsymbol{\tau}_{\mathsf{N}}$.

**§1. Introduction.** Starting point of the following considerations is the applicative theory $\mathsf{BON}^+$ whose universe consists of so-called operations; self-application is possible though not necessarily defined. This basic theory of operations and numbers $\mathsf{BON}^+$ comprises the axioms of partial combinatory algebra, some natural axioms for the data type of the natural numbers, and the schema of induction on the natural numbers for all formulae (hence the symbol "+" in its name).

Moreover, $\mathsf{BON}^+$ is closely related to the first order parts of the standard systems of explicit mathematics introduced in Feferman [3, 4]. Since the notion of a partial combinatory algebra is an interesting generalization of and an abstract framework for computations, this applicative part of explicit mathematics is sometimes called its "computational engine".

In this article we analyze this aspect of $\mathsf{BON}^+$ more closely. First a point is made for introducing a new operation $\boldsymbol{\tau}_{\mathsf{N}}$, called truncation, to obtain a natural formalization of partial recursive functions in our applicative framework. Then we introduce the operational versions of a series of notions that are all equivalent to semi-decidability in ordinary recursion theory on the natural numbers, and study their mutual relationships over $\mathsf{BON}^+$ with $\boldsymbol{\tau}_{\mathsf{N}}$. As it turns out, not all these equivalences can be transferred to their operational variants, and interesting mutual relationships can be discovered.

1991 *Mathematics Subject Classification.* 03B40, 03D25, 03D75, 03E15, 03F99.
*Key words and phrases.* applicative theories, explicit mathematics, abstract computability, (semi-) decidability, Kleene's second model, graph model, Hausdorff–Kuratowski difference hierarchy.

This article is organized as follows. In the next section we present the basic theory $\mathsf{BON}^+$ as well as two notions of representing partial number-theoretic functions and state some central properties of $\mathsf{BON}^+$. In Section 3 we first discuss a few shortcomings of $\mathsf{BON}^+$ with respect to a natural treatment of partial recursive functions and then introduce the truncation operator $\boldsymbol{\tau}_\mathsf{N}$ as a possibility to compensate for these deficits. Section 4 gives two proofs of the undefinability of $\boldsymbol{\tau}_\mathsf{N}$ in $\mathsf{BON}^+$. Section 5 is about models of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$. These models serve several purposes: (i) they underline that the operator $\boldsymbol{\tau}_\mathsf{N}$ reflects a very natural principle in our operational context, (ii) they give us the consistency of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ with the assertion that application is total and a kind of existence property, and (iii) they provide some complexity results that we use in Section 6, where the relationships between our operational versions of classical semi-decidability notions are studied.

**§2. The theory $\mathsf{BON}^+$.** In this section we introduce the basic theory $\mathsf{BON}^+$ of operations and numbers, which is the point of departure for our considerations. $\mathsf{BON}^+$ axiomatizes the basic operational behavior of the first order objects of explicit mathematics. It is closely related to the theory $\mathsf{BON}$ introduced in, for example, Feferman and Jäger [5] and Feferman, Jäger, and Strahm [6], to the theory $\mathsf{EON}$ of Beeson [2, VI.2.4], and to the theory $\mathsf{APP}$ of Troelstra and van Dalen [19, 9.3.3]. In Section 3 we extend $\mathsf{BON}^+$ to the theory $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

The language $L$ of $\mathsf{BON}^+$ and $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ is a first order language with countably many individual variables $a$, $b$, $c$, $u$, $v$, $w$, $x$, $y$, $z$, $f$, $g$, $h$, ... (possibly with subscripts) and the individual constants $0$, $\mathbf{k}$, $\mathbf{s}$, $\mathbf{p}$, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{s}_\mathsf{N}$, $\mathbf{p}_\mathsf{N}$, $\mathbf{d}_\mathsf{N}$, $\boldsymbol{\tau}_\mathsf{N}$, the meaning of which will be explained later. In addition, there is a binary function symbol $\cdot$ for application. The relation symbols are countably many unary relation variables $U$, $V$, $W$, ... plus the specific unary relation symbols $\downarrow$, $\mathsf{N}$, and the binary relation symbol $=$.

The term formation operation is term application and thus the *terms* ($r$, $s$, $t$, $r_1$, $s_1$, $t_1$, ... ) are generated as follows:

(1) Each individual variable is a term.
(2) Each individual constant is a term.
(3) If $s$ and $t$ are terms, then so also is $(s \cdot t)$.

We write $(s \cdot t)$ often just as $(st)$ or $st$. In this simplified form we adopt the convention of association to the left such that, for example, $s_1 s_2 \dots s_n$ stands for $(\dots(s_1 \cdot s_2) \dots \cdot s_n)$. We also use the notation $s(t_1, \dots, t_n)$ for $st_1 \dots t_n$. If $n$ is a natural number, we write $\overline{n}$ for the corresponding numeral, i.e., for the closed term given recursively by $\overline{0} := 0$ and $\overline{n+1} := \mathbf{s}_\mathsf{N}\overline{n}$.

The *formulae* ($\varphi$, $\chi$, $\psi$, $\varphi_1$, $\chi_1$, $\psi_1$, ... ) of $L$ are generated from the atomic formulae $t\downarrow$, $(s = t)$, $\mathsf{N}(t)$, and $U(t)$ by closing them under the usual propositional connectives and quantification over individuals. We will often omit parentheses if there is no danger of confusion.

The logic of $\mathsf{BON}^+$ is the classical version of Beeson's *logic of partial terms* (see Beeson [2, VI.1]). It corresponds to the $E^+$-logic with equality and strictness of Troelstra and van Dalen [18, 2.2.4], where $E(t)$ is written instead of $t\downarrow$. Here $t\downarrow$

is read "$t$ is defined" or "$t$ has a value". The partial equality $\simeq$ is introduced by

$$(s \simeq t) \;:=\; ((s\downarrow \vee\; t\downarrow) \;\rightarrow\; s = t).$$

Furthermore, we write $t \in Z$ instead of $Z(t)$ in case $Z$ is a relation variable or the relation constant $\mathsf{N}$. As usual, $t \notin Z$ and $s \neq t$ stand for $\neg(t \in Z)$ and $\neg(s = t)$, respectively. As additional abbreviations we will use:

$$\begin{aligned}
t \in (\mathsf{N} \rightarrow \mathsf{N}) &\;:=\; (\forall x \in \mathsf{N})(tx \in \mathsf{N}),\\
t \in (\mathsf{N}^m \rightarrow \mathsf{N}) &\;:=\; (\forall x_0, \ldots, x_{m-1} \in \mathsf{N})(t(x_0, \ldots, x_{m-1}) \in \mathsf{N}),\\
f \in \mathit{Char} &\;:=\; (\forall x \in \mathsf{N})\big(fx = \overline{0} \,\vee\, fx = \overline{1}\big),\\
f \in \mathit{Char}_2 &\;:=\; (\forall x, y \in \mathsf{N})\big(f(x,y) = \overline{0} \,\vee\, f(x,y) = \overline{1}\big),\\
t \in \mathsf{N}\backslash U &\;:=\; t \in \mathsf{N} \,\wedge\, t \notin U,\\
U = \emptyset &\;:=\; \neg\exists x\ (x \in U).
\end{aligned}$$

The so-called *strictness axioms* of the logic of partial terms are all formulae of the following form where $\varphi[u]$ is an atomic formula with an occurrence of $u$:

$$\varphi[s] \;\rightarrow\; s\downarrow.$$

Keep in mind that in general $t \notin Z$ does not imply $t\downarrow$ and that we cannot deduce $s\downarrow$ or $t\downarrow$ from $s \neq t$.

The non-logical axioms of $\mathsf{BON}^+$ can be divided into the following four groups.

I. Partial combinatory algebra
(1) $\mathbf{k}(x,y) = x$,
(2) $\mathbf{s}(x,y)\downarrow \,\wedge\, \mathbf{s}(x,y,z) \simeq x(z, yz)$.

II. Pairing and projection
(3) $\mathbf{p}_0(\mathbf{p}(x,y)) = x \,\wedge\, \mathbf{p}_1(\mathbf{p}(x,y)) = y$.

III. Natural numbers
(4) $0 \in \mathsf{N} \,\wedge\, \mathbf{s}_{\mathsf{N}} \in (\mathsf{N} \rightarrow \mathsf{N})$,
(5) $\mathbf{s}_{\mathsf{N}}x \neq 0 \,\wedge\, \mathbf{p}_{\mathsf{N}}0 = 0 \,\wedge\, (\forall x \in \mathsf{N})(\mathbf{p}_{\mathsf{N}}(\mathbf{s}_{\mathsf{N}}x) = x)$,
(6) $x \in U \;\rightarrow\; x \in \mathsf{N}$,
(7) $\varphi[0] \,\wedge\, (\forall x \in \mathsf{N})(\varphi[x] \;\rightarrow\; \varphi[\mathbf{s}_{\mathsf{N}}x]) \;\rightarrow\; (\forall x \in \mathsf{N})\varphi[x]$
for all $L$ formulae $\varphi[x]$.

IV. Definition by cases on $\mathsf{N}$
(8) $x \in \mathsf{N} \,\wedge\, y \in \mathsf{N} \,\wedge\, x = y \;\rightarrow\; \mathbf{d}_{\mathsf{N}}(a,b,x,y) = a$,
(9) $x \in \mathsf{N} \,\wedge\, y \in \mathsf{N} \,\wedge\, x \neq y \;\rightarrow\; \mathbf{d}_{\mathsf{N}}(a,b,x,y) = b$.

$\mathbf{k}$ and $\mathbf{s}$ are the partial versions of the well-known combinators of Curry's combinatory logic. $\mathbf{p}$ provides an injective pairing of the universe with the inverse operations $\mathbf{p}_0$ and $\mathbf{p}_1$. $\mathbf{s}_{\mathsf{N}}$ represents the successor function on the natural numbers and $\mathbf{p}_{\mathsf{N}}$ the predecessor function. Axioms (4) and (5) formulate some basic properties of the natural numbers, axiom (6) simply states that the relation variables range over subsets of the natural numbers, and (7) is the schema of induction. $\mathbf{d}_{\mathsf{N}}$ gives definition by integer cases. Since $\mathsf{BON}^+$ comprises the axioms (1)–(2) of a partial combinatory algebra, we clearly have $\lambda$ abstraction and the usual fixed point theorem; this is mentioned already in Feferman [3] and proved in detail in, e.g., Beeson [2, VI.2.2], Troelstra and van Dalen [19, 9.3.5], and Feferman, Jäger, and Strahm [6].

LEMMA 2.1 ($\lambda$ abstraction). *For each variable $x$ and term $t$ we can construct a term $\lambda x.t$ whose free variables are those of $t$, excluding $x$, such that* $\mathsf{BON}^+$ *proves*

$$\lambda x.t\downarrow \wedge (\lambda x.t)x \simeq t.$$

The generalization of $\lambda$ abstraction to several variables is by simply iterating abstraction for one argument, and we usually write $\lambda x_1 \dots x_n.t$ for the corresponding term.

LEMMA 2.2 (Fixed point). *There exists a closed term* $\mathbf{fix}$ *such that* $\mathsf{BON}^+$ *proves*

$$\mathbf{fix}(f)\downarrow \wedge (g = \mathbf{fix}(f) \to \forall x \, (gx \simeq f(g, x))).$$

COROLLARY 2.3. *Let* $\mathbf{nt}_{\mathsf{N}} := \mathbf{fix}(\lambda xy.\mathbf{s}_{\mathsf{N}}(x(y)))$. *Then* $\mathsf{BON}^+$ *proves*

$$\mathbf{nt}_{\mathsf{N}}\downarrow \wedge \forall x \, (\mathbf{nt}_{\mathsf{N}}(x) \simeq \mathbf{s}_{\mathsf{N}}(\mathbf{nt}_{\mathsf{N}}(x)))$$

*and hence* $\mathbf{nt}_{\mathsf{N}}(0) \notin \mathsf{N}$.

$\mathsf{BON}^+$ is proof-theoretically equivalent to the theory $\mathsf{BON}$ (see Feferman and Jäger [5] and Feferman, Jäger, and Strahm [6]) extended by the schema of induction for arbitrary formulae. It can be shown that all primitive recursive functions can be represented in $\mathsf{BON}^+$ as explained below.

We write $\omega$ for the set of natural numbers. Given a (possibly partial) function $\mathcal{F}$ from $\omega^k$ to $\omega$ we say that a closed term $t$ *numeralwise represents* $\mathcal{F}$ in an $L$ theory $T$ iff

$$\mathcal{F}(m_1, \dots, m_k) = n \iff T \vdash t(\overline{m_1}, \dots, \overline{m_k}) = \overline{n}$$

for all $m_1, \dots, m_k, n \in \omega$. However, this does not guarantee the expected behavior of $t$ on nonstandard natural numbers. In order to impose such a condition, we have to assume that it is described by a formula, e.g., by equations. For example, let us consider a unary function $\mathcal{G}$ that is defined by primitive recursion from a natural number $n$ and a binary function $\mathcal{F}$ as

$$\mathcal{G}(0) = n \text{ and } \mathcal{G}(m+1) = \mathcal{F}(m, \mathcal{G}(m))$$

for all natural numbers $m$. Then, if the terms $s$ and $t$ represent the functions $\mathcal{F}$ and $\mathcal{G}$, respectively, we want the conditional equations

$$t0 = \overline{n} \text{ and } (\forall x \in \mathsf{N})(t(\mathbf{s}_{\mathsf{N}}x) = s(x, tx))$$

to be provable in $T$. If the defining formula of a function $\mathcal{F}$ is provable for a term $t$ in $T$, we say that $t$ *definitionally represents* $\mathcal{F}$ *in* $T$. The following is immediate from Troelstra and van Dalen [19, 9.3].

THEOREM 2.4 (Prim. rec. func.). *For any (definition of a) $k$-ary primitive recursive function $\mathcal{F}$, there exists a closed term* $\mathbf{prim}_{\mathcal{F}}$ *that numeralwise and definitionally represents $\mathcal{F}$ in* $\mathsf{BON}^+$ *and for which* $\mathsf{BON}^+$ *proves* $\mathbf{prim}_{\mathcal{F}} \in (\mathsf{N}^k \to \mathsf{N})$.

Observe, however, that this theorem does not imply that $\mathsf{BON}^+$ proves

$$\mathbf{prim}_{\mathcal{F}}(a_1, \ldots, a_k) \in \mathsf{N} \rightarrow a_1, \ldots, a_k \in \mathsf{N}$$

for the representation $\mathbf{prim}_{\mathcal{F}}$ of a $k$-ary primitive recursive $\mathcal{F}$; this implication is not provable in $\mathsf{BON}^+$ in general.

According to the last theorem there are closed terms **pair**, $\mathbf{proj}_0$, $\mathbf{proj}_1$, numeralwise and definitionally representing a primitive recursive bijective pairing function with its corresponding projections, respectively, such that $\mathsf{BON}^+$ proves the following:

(1) $s, t \in \mathsf{N} \rightarrow (\mathbf{pair}(s, t) \in \mathsf{N} \wedge \mathbf{proj}_0(s) \in \mathsf{N} \wedge \mathbf{proj}_1(s) \in \mathsf{N})$,
(2) $s, t \in \mathsf{N} \rightarrow (\mathbf{proj}_0(\mathbf{pair}(s, t)) = s \wedge \mathbf{proj}_1(\mathbf{pair}(s, t)) = t)$,
(3) $s \in \mathsf{N} \rightarrow \mathbf{pair}(\mathbf{proj}_0(s), \mathbf{proj}_1(s)) = s$.

There is also a closed term **less** for the characteristic function of the primitive recursive less relation, and we often write $a < b$ for $(a, b \in \mathsf{N} \wedge \mathbf{less}(a, b) = 0)$.

In Troelstra and van Dalen [19, 9.3.10], a specific minimum operator is considered. Later, we need the following part of this result.

THEOREM 2.5 ($\mathbf{min}_0$). *There exists a closed term* $\mathbf{min}_0$ *such that* $\mathsf{BON}^+$ *proves*

$$\begin{aligned} &(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N})) \\ &\quad \rightarrow \mathbf{min}_0(f) \in \mathsf{N} \wedge f(\mathbf{min}_0(f)) = 0 \wedge (\forall y < \mathbf{min}_0(f))(0 < fy). \end{aligned}$$

PROOF. Let $t := \lambda fhx.\mathbf{d}_{\mathsf{N}}(\lambda u.x, \lambda u.h(\mathbf{s}_{\mathsf{N}}x), fx, 0)0$. Then, as far as $fx \in \mathsf{N}$,

$$\begin{aligned} \mathbf{fix}(tf, x) \simeq t(f, \mathbf{fix}(tf), x) &\simeq \mathbf{d}_{\mathsf{N}}(\lambda u.x, (\lambda u.h(\mathbf{s}_{\mathsf{N}}x))[\mathbf{fix}(tf)/h], fx, 0)0 \\ &\simeq \begin{cases} x & \text{if } fx = 0, \\ \mathbf{fix}(tf, \mathbf{s}_{\mathsf{N}}x) & \text{otherwise.} \end{cases} \end{aligned}$$

Define $\mathbf{min}_0 := \lambda f.\mathbf{fix}(tf, 0)$. Now we assume

$$(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N})).$$

By induction there exists $a \in \mathsf{N}$ with $fa = 0$ and $(\forall y < a)(0 < fy)$. If $a = 0$, then $\mathbf{min}_0(f) \simeq \mathbf{fix}(tf, a) = a$ is provable. If $0 < a$, a further induction yields

$$y < a \rightarrow \mathbf{min}_0(f) \simeq \mathbf{fix}(tf, \mathbf{s}_{\mathsf{N}}y).$$

Therefore, we have $\mathbf{min}_0(f) \simeq \mathbf{fix}(tf, a) = a$ as well. This proves our claim. $\dashv$

Making use of this minimum operator and following [19] it is routine work to show that every total recursive function can be represented numeralwise (but not definitionally in general) in $\mathsf{BON}^+$ by a closed term. Having primitive recursion and $\mathbf{min}_0$, it is easy to see that even Kleene's enumeration $\{e\}$ of the partial recursive number-theoretic functions can be obtained in $\mathsf{BON}^+$.

**§3. Truncation to $\mathsf{N}$.** In this section we discuss some deficiencies of $\mathsf{BON}^+$ with respect to a "natural treatment" of partial recursive number-theoretic functions within $\mathsf{BON}^+$ and propose the introduction of a new truncation operator to compensate for them.

There are two interesting additional axioms, the totality assertion (TOT-AP) and the assertion (TOT-$\mathsf{N}$) that every object is a natural number, (Tot-Ap) $\forall x\, \forall y\, (xy\downarrow)$,

(Tot-N) $\forall x\, (x \in \mathsf{N})$.

$\mathsf{BON}^+$ is consistent with (Tot-Ap) (as will be shown by the term model in Section 5.1) and with (Tot-N) (as seen by Kleene's first model $[\mathbb{N}, \omega]$ in the notation of Section 4). However, $\mathsf{BON}^+ +$ (Tot-Ap) $+$ (Tot-N) is inconsistent by Corollary 2.3. Thus, both (Tot-Ap) and (Tot-N) are unprovable in $\mathsf{BON}^+$, respectively. Hence, if we want to be compatible with both possible extensions of $\mathsf{BON}^+$, the only way to formally express the non-termination of a partial number-theoretic function $\mathcal{F}$ at input $x$ is to state $t_{\mathcal{F}}(x) \notin \mathsf{N}$ for the associated term $t_{\mathcal{F}}$. In particular, in the presence of (Tot-Ap), the non-termination of $\mathcal{F}(n)$ is represented by having a value outside of $\mathsf{N}$; while in the presence of (Tot-N), it is represented by non-definedness.

Now suppose that a unary partial number-theoretic function $\mathcal{F}$ is the composition of unary partial number-theoretic functions $\mathcal{G}$ and $\mathcal{H}$, i.e.,

$$\mathcal{F}(n) \simeq \mathcal{H}(\mathcal{G}(n))$$

for all natural numbers $n$. Also, if $\mathcal{G}(n)$ does not terminate, then neither does $\mathcal{F}(n)$. If the terms $s$ and $t$ represent $\mathcal{G}$ and $\mathcal{H}$, respectively, we would expect that $r := \lambda x.t(s(x))$ represents $\mathcal{F}$ and

$$a \in \mathsf{N} \wedge sa \notin \mathsf{N} \rightarrow ra \notin \mathsf{N}$$

within $\mathsf{BON}^+$ according to the way of representing the non-termination of partial functions mentioned in the previous paragraph. However, if $\mathcal{H}$ is the function constant 0 and $t := \lambda x.0$ its canonical representation, then

$$a \in \mathsf{N} \wedge sa \notin \mathsf{N} \wedge ra = 0$$

is possible in $\mathsf{BON}^+$. Simply assume that $sa$ has a value outside $\mathsf{N}$.

In ordinary computation theory on the natural numbers and many of its generalizations there exist

(i) a closed term $r$ such that

$$(\forall x \in \mathsf{N})(rx = 0) \wedge \forall x\, (rx \in \mathsf{N} \rightarrow x \in \mathsf{N}),$$

(ii) an operator **op** that maps any partial computable function $f$ to a partial computable function $g = \mathbf{op}(f)$ such that

$$(\forall x \in \mathsf{N})(fx \in \mathsf{N} \leftrightarrow gx = 0).$$

In the following section we will show that both such terms do not exist in our present environment $\mathsf{BON}^+$.

To overcome these problems and similar difficulties, we now make use of the constant $\boldsymbol{\tau}_{\mathsf{N}}$, which did not play a role thus far. Consider the following two $\boldsymbol{\tau}_{\mathsf{N}}$-axioms.

VI. Truncation to $\mathsf{N}$

($\boldsymbol{\tau}_{\mathsf{N}}$.1) $x \in \mathsf{N} \rightarrow \boldsymbol{\tau}_{\mathsf{N}}(f, x) \simeq fx$,

($\boldsymbol{\tau}_{\mathsf{N}}$.2) $\boldsymbol{\tau}_{\mathsf{N}}(f, x) \in \mathsf{N} \rightarrow x \in \mathsf{N}$.

The first axiom states that on $\mathsf{N}$ any operation $f$ behaves exactly as its truncation $\boldsymbol{\tau}_{\mathsf{N}} f$. Moreover, the second axiom states that $\boldsymbol{\tau}_{\mathsf{N}}(f, x)$ can belong to $\mathsf{N}$ only when so does $x$ as well. In this sense, $\boldsymbol{\tau}_{\mathsf{N}}$ truncates every operation $f$ to the natural numbers $\mathsf{N}$.

$\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ is defined to be the extension of $\mathsf{BON}^+$ by the axioms ($\boldsymbol{\tau}_{\mathsf{N}}$.1) and ($\boldsymbol{\tau}_{\mathsf{N}}$.2). In Section 4, we will show that $\boldsymbol{\tau}_{\mathsf{N}}$ cannot be defined in $\mathsf{BON}^+$. Hence $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ is a proper extension of $\mathsf{BON}^+$. And it is easy to check that by means of $\boldsymbol{\tau}_{\mathsf{N}}$ the problems described above can be healed. There is a close relationship between our truncation operator $\boldsymbol{\tau}_{\mathsf{N}}$ and Kahle's notion of $\mathsf{N}$-strictness, introduced in Kahle [8, 9]; for details see Rosebrock [17].

Before turning to the undefinability of $\boldsymbol{\tau}_{\mathsf{N}}$ in $\mathsf{BON}^+$ we want to illustrate that $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ is a natural framework for explicitly dealing with partial recursive functions and their defining equations. We leave it to the readers to convince themselves that without $\boldsymbol{\tau}_{\mathsf{N}}$ and the $\boldsymbol{\tau}_{\mathsf{N}}$-axioms this approach would not have been possible.

It turns out to be important to have a minimum operator that is stronger than $\mathbf{min}_0$ of Theorem 2.5. To establish its existence we start with a preparatory lemma that asserts the existence of a term for deciding admissibility in the sense of Troelstra and van Dalen [19, 9.3.9] up to a natural number.

LEMMA 3.1. *There exists a closed term* $\mathbf{adm}$ *such that* $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ *proves the following:*

(1) $(\forall x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N}) \rightarrow \mathbf{adm}(f, x) = 0)$,
(2) $(\forall x \in \mathsf{N})(\mathbf{adm}(f, x) \in \mathsf{N} \rightarrow \mathbf{adm}(f, x) = fx \wedge (\forall y < x)(fy \in \mathsf{N}))$.

PROOF. We work within $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ and define

$$\mathbf{adm} := \lambda f.\mathbf{fix}(\lambda hx.\mathbf{d}_{\mathsf{N}}(f, \lambda u.\boldsymbol{\tau}_{\mathsf{N}}(\lambda z.fx, h(\mathbf{p}_{\mathsf{N}} x)), x, 0)0).$$

Then we have for all $y \in \mathsf{N}$,

$$\begin{aligned} \mathbf{adm}(f, y) &\simeq \mathbf{d}_{\mathsf{N}}(f, (\lambda u.\boldsymbol{\tau}_{\mathsf{N}}(\lambda z.fy, h(\mathbf{p}_{\mathsf{N}} y)))[\mathbf{adm} f/h], y, 0)0 \\ &\simeq \begin{cases} f0 & \text{if } y = 0, \\ \boldsymbol{\tau}_{\mathsf{N}}(\lambda z.fy, \mathbf{adm}(f, \mathbf{p}_{\mathsf{N}} y)) & \text{otherwise.} \end{cases} \end{aligned}$$

To show (1), pick $x \in \mathsf{N}$ with $fx = 0$ and $(\forall y < x)(fy \in \mathsf{N})$. We prove $y < x \rightarrow \mathbf{adm}(f, y) \in \mathsf{N}$ by induction on $y$ and continue with

$$\mathbf{adm}(f, x) \simeq \left. \begin{cases} f0 & \text{if } x = 0 \\ \boldsymbol{\tau}_{\mathsf{N}}(\lambda z.fx, \mathbf{adm}(f, \mathbf{p}_{\mathsf{N}} x)) & \text{otherwise} \end{cases} \right\} \simeq fx = 0.$$

For establishing (2), we prove

$$\mathbf{adm}(f, x) \in \mathsf{N} \rightarrow \mathbf{adm}(f, x) = fx \wedge (\forall y < x)(fy \in \mathsf{N})$$

by induction on $x$. This is obvious for $x = 0$. Assume $\mathbf{adm}(f, \mathbf{s}_{\mathsf{N}} x) \in \mathsf{N}$. This means $\boldsymbol{\tau}_{\mathsf{N}}(\lambda z.f(\mathbf{s}_{\mathsf{N}} x), \mathbf{adm}(f, x)) \in \mathsf{N}$. Hence ($\boldsymbol{\tau}_{\mathsf{N}}$.2) implies $\mathbf{adm}(f, x) \in \mathsf{N}$, and so ($\boldsymbol{\tau}_{\mathsf{N}}$.1) yields $\mathbf{adm}(f, \mathbf{s}_{\mathsf{N}} x) = f(\mathbf{s}_{\mathsf{N}} x)$. By induction hypothesis we also have $fx = \mathbf{adm}(f, x) \in \mathsf{N}$ and $(\forall y < x)(fy \in \mathsf{N})$. Therefore, we can finally conclude $(\forall y < \mathbf{s}_{\mathsf{N}} x)(fy \in \mathsf{N})$. $\dashv$

THEOREM 3.2 (**min**). *There exists a closed term* $\mathbf{min}$ *such that* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ *proves the following:*

(1) $(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N})) \rightarrow \mathbf{min}(f) \in \mathsf{N}$,
(2) $\mathbf{min}(f) \in \mathsf{N} \rightarrow f(\mathbf{min}(f)) = 0 \wedge (\forall y < \mathbf{min}(f))(0 < fy)$.

PROOF. We define

$$\mathbf{min} := \lambda f.\boldsymbol{\tau}_\mathsf{N}(\lambda u.\mathbf{d}_\mathsf{N}(\lambda v.\mathbf{min}_0(f), \mathbf{nt}_\mathsf{N}, u, 0)0, \mathbf{adm}(f, \mathbf{min}_0(f))),$$

where $\mathbf{nt}_\mathsf{N}$ is defined in Corollary 2.3.

In order to prove (1), assume $(\exists x \in \mathsf{N})(fx = 0 \wedge (\forall y < x)(fy \in \mathsf{N}))$. Theorem 2.5 implies $\mathbf{min}_0(f) \in \mathsf{N}$, $f(\mathbf{min}_0(f)) = 0$, and $(\forall y < \mathbf{min}_0(f))(fy \in \mathsf{N})$. Therefore, $\mathbf{adm}(f, \mathbf{min}_0(f)) = 0$ in view of the previous lemma. By $(\boldsymbol{\tau}_\mathsf{N}.1)$ we have

$$\mathbf{min}(f) \simeq \mathbf{d}_\mathsf{N}(\lambda v.\mathbf{min}_0(f), \mathbf{nt}_\mathsf{N}, 0, 0)0 = \mathbf{min}_0(f).$$

Now we turn to (2) and assume $\mathbf{min}(f) \in \mathsf{N}$. Then $\mathbf{adm}(f, \mathbf{min}_0(f)) \in \mathsf{N}$ by $(\boldsymbol{\tau}_\mathsf{N}.2)$ and the definition of $\mathbf{min}$. Hence,

$$\begin{aligned}\mathbf{min}(f) &\simeq \mathbf{d}_\mathsf{N}(\lambda v.\mathbf{min}_0(f), \mathbf{nt}_\mathsf{N}, \mathbf{adm}(f, \mathbf{min}_0(f)), 0)0 \\ &\simeq \begin{cases}\mathbf{min}_0(f) & \text{if } \mathbf{adm}(f, \mathbf{min}_0(f)) = 0, \\ \mathbf{nt}_\mathsf{N}(0) & \text{otherwise.}\end{cases}\end{aligned}$$

By $\mathbf{min}(f) \in \mathsf{N}$ and $\mathbf{nt}_\mathsf{N}(0) \notin \mathsf{N}$, the second case is ruled out. Therefore $\mathbf{adm}(f, \mathbf{min}_0(f)) = 0$ and $\mathbf{min}(f) = \mathbf{min}_0(f)$. According to the previous lemma, we thus have

$$(\forall y < \mathbf{min}(f))(fy \in \mathsf{N}) \wedge f(\mathbf{min}(f)) = \mathbf{adm}(f, \mathbf{min}(f)) = 0.$$

Now we apply Theorem 2.5 and obtain $(\forall y < \mathbf{min}(f))(0 < fy)$. ⊣

Now we are ready to turn to the definitional representation of all partial recursive (number-theoretic) functions. We start off from the definition of the partial recursive functions as the least class of number-theoretic functions that (i) contains the constant-zero function, the successor function, the projections and (ii) is closed under compositions and minimizations.

THEOREM 3.3 (Part. rec. func.: definit. repr.). *For any (definition of a) partial recursive number-theoretic function* $\mathcal{F}$*, there is a closed term* $\mathbf{g}_\mathcal{F}$ *such that* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ *proves the defining formulae for both the domain and the values of* $\mathbf{g}_\mathcal{F}$.

PROOF. We prove this by induction on the definition of the class of the partial recursive functions.

(i) Initial functions. Clearly, the term $\mathbf{s}_\mathsf{N}$ represents the unary successor function and the corresponding defining equations are provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$. The term $\mathbf{zero}^k := \lambda x_0 \dots x_{k-1}.0$ and the term $\mathbf{proj}^k_i := \lambda x_0 \dots x_{k-1}.x_i$ represent the $k$-ary zero function and the $k$-ary $i$-th projection function (for $0 \leq i < k$), respectively, with the equations
   (1) $\mathbf{zero}^k(x_0, \dots, x_{k-1}) \in \mathsf{N} \leftrightarrow \top$,
   (2) $\mathbf{zero}^k(x_0, \dots, x_{k-1}) = 0$,

(3) $\mathbf{proj}^k_i(x_0,\dots,x_{k-1}) \in \mathsf{N} \leftrightarrow \top$,

(4) $\mathbf{proj}^k_i(x_0,\dots,x_{k-1}) = x_i$

being provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ for all $x_0,\dots,x_{k-1} \in \mathsf{N}$.

(ii) Composition. For notational simplicity we restrict ourselves to the case of the composition of a binary with two unary functions,

$$\mathcal{F}(n) \simeq \mathcal{I}(\mathcal{G}(n),\mathcal{H}(n));$$

the generalization to the general case is obvious. By induction hypothesis we have the terms $\mathbf{g}_\mathcal{G}$, $\mathbf{g}_\mathcal{H}$, and $\mathbf{g}_\mathcal{I}$. Then define

$$\mathbf{g}_\mathcal{F} := \lambda x.\boldsymbol{\tau}_\mathsf{N}(\lambda y_0.\boldsymbol{\tau}_\mathsf{N}(\lambda y_1.\mathbf{g}_\mathcal{I}(y_0,y_1),\mathbf{g}_\mathcal{H}x),\mathbf{g}_\mathcal{G}x)$$

and check that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves, for all $x \in \mathsf{N}$,

(5) $\mathbf{g}_\mathcal{F}x \in \mathsf{N} \leftrightarrow (\mathbf{g}_\mathcal{G}x \in \mathsf{N} \wedge \mathbf{g}_\mathcal{H}x \in \mathsf{N} \wedge \mathbf{g}_\mathcal{I}(\mathbf{g}_\mathcal{G}x,\mathbf{g}_\mathcal{H}x) \in \mathsf{N})$,

(6) $\mathbf{g}_\mathcal{F}x \in \mathsf{N} \to \mathbf{g}_\mathcal{F}x = \mathbf{g}_\mathcal{I}(\mathbf{g}_\mathcal{G}x,\mathbf{g}_\mathcal{H}x)$.

(iii) Minimization. For notational simplicity we restrict ourselves to the case that the unary $\mathcal{F}$ is defined from the binary $\mathcal{G}$ by minimization, i.e.,

$$\mathcal{F}(n) \text{ is the least } m \text{ with } \begin{cases} \mathcal{G}(n,m)=0 \text{ and} \\ \text{for all } k<m,\ \mathcal{G}(n,k) \text{ terminates,} \end{cases} \tag{$\sharp$}$$

if such $m$ exists; otherwise $\mathcal{F}(n)$ does not terminate. By the induction hypothesis we have a term $\mathbf{g}_\mathcal{G}$ representing $\mathcal{G}$ and define

$$\mathbf{g}_\mathcal{F} := \lambda x.\mathbf{min}(\lambda y.\mathbf{g}_\mathcal{G}(x,y)).$$

In view of Theorem 3.2 it is clear that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves, for all $x \in \mathsf{N}$,

(7) $\mathbf{g}_\mathcal{F}x \in \mathsf{N} \leftrightarrow (\exists y \in \mathsf{N})(\mathbf{g}_\mathcal{G}(x,y)=0 \wedge (\forall z<y)(\mathbf{g}_\mathcal{G}(x,z) \in \mathsf{N}))$,

(8) $\mathbf{g}_\mathcal{F}x \in \mathsf{N} \to (\mathbf{g}_\mathcal{G}(x,\mathbf{g}_\mathcal{F}x)=0 \wedge (\forall z<\mathbf{g}_\mathcal{F}x)(0<\mathbf{g}_\mathcal{G}(x,z)))$.

This finishes the proof of the definitional representation theorem for all partial recursive number-theoretic functions. ⊣

Then it is natural to ask for the numeralwise representation of the partial recursive functions. For this purpose, we need the following lemma.

Lemma 3.4 (evaluation of numerical terms). *For any closed term $t$, if $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $t \in \mathsf{N}$ then there exists a natural number $n$ such that $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $t = \overline{n}$.*

This lemma is proved in full detail in Rosebrock [17]. The underlying idea of its proof is also sketched in Section 5.1.

Theorem 3.5 (Part. rec. func.: numeralwise repr.). *For any (definition of a) partial recursive number-theoretic function $\mathcal{F}$, there is a closed term $\mathbf{g}_\mathcal{F}$ which numeralwise represents $\mathcal{F}$ in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.*

Proof. We can use the same closed term as in Theorem 3.3. For the case of the initial functions, the claim is trivial.

Let us consider the case of composition, namely $\mathcal{F}(n) \simeq \mathcal{I}(\mathcal{G}_0(n),\mathcal{G}_1(n))$. If $\mathcal{F}(n)=m$, then let $l_i := \mathcal{G}_i(n)$ for $i<2$. By the induction hypothesis, $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $\mathbf{g}_{\mathcal{G}_i}\overline{n} = \overline{l_i}$ for $i<2$ and $\mathbf{g}_\mathcal{I}(\overline{l_0},\overline{l_1}) = \overline{m}$. Therefore $\mathbf{g}_\mathcal{F}\overline{n} = \boldsymbol{\tau}_\mathsf{N}(\lambda y_0.\boldsymbol{\tau}_\mathsf{N}(\lambda y_1.\mathbf{g}_\mathcal{I}(y_0,y_1),\mathbf{g}_{\mathcal{G}_1}\overline{n}),\mathbf{g}_{\mathcal{G}_0}\overline{n}) = \overline{m}$ is provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

Conversely, if $\mathbf{g}_{\mathcal{F}}\overline{n} = \overline{m}$ is provable in $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$, then by the axioms $(\boldsymbol{\tau}_{\mathsf{N}}.2)$ and $(\boldsymbol{\tau}_{\mathsf{N}}.1)$, $\mathbf{g}_{\mathcal{G}_i}\overline{n} \in \mathsf{N}$ is provable for $i < 2$. By the last lemma, there exist $l_i$ for $i < 2$ such that $\mathbf{g}_{\mathcal{G}_i}\overline{n} = \overline{l_i}$ is provable. Then, by the induction hypothesis, $\mathcal{F}(n) = \mathcal{I}(l_0, l_1) = m$.

Next, we look at the case of minimization, namely ($\sharp$). If $\mathcal{F}(n) = m$, then $\mathcal{G}(n, m) = 0$ and $\mathcal{G}(n, k) > 0$ for $k < m$. By the induction hypothesis, $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ proves $\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m}) = 0$ and $0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{k})$ for any $k < m$. By Theorem 3.2(1), $\mathbf{min}(\lambda y.\mathbf{g}_{\mathcal{G}}(\overline{n}, y)) \in \mathsf{N}$ is provable. By the last lemma, there exists $l$ such that $\mathbf{min}(\lambda y.\mathbf{g}_{\mathcal{G}}(\overline{n}, y)) = \overline{l}$ is provable. By Theorem 3.2(2),

$$\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{l}) = 0 \wedge (\forall y < \overline{l})(0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, y))$$

is provable. Again by the induction hypothesis, $\mathcal{G}(n, l) = 0$. Therefore $m \leq l$. If $m < l$, then $0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m})$ is provable contradicting $\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m}) = 0$. Thus $m = l$ and $\mathbf{g}_{\mathcal{F}}\overline{n} = \overline{m}$ is provable.

Conversely, if $\mathbf{g}_{\mathcal{F}}\overline{n} = \overline{m}$ is provable, so is

$$\mathbf{g}_{\mathcal{G}}(\overline{n}, \overline{m}) = 0 \wedge (\forall y < \overline{m})(0 < \mathbf{g}_{\mathcal{G}}(\overline{n}, y))$$

in view of Theorem 3.2(2). By the induction hypothesis and the last lemma, $\mathcal{F}(n) = m$. ⊣

We close this section with the following easy lemma, which will be useful later. It asserts that subclasses of $\mathsf{N}$ that are represented as ranges of operations are *exactly* the projections of those represented as preimages of 0 under operations (cf. Definition 6.1).

LEMMA 3.6. *In* $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ *we can prove the following:*

(1) $\forall g\ \exists f\ (\forall x \in \mathsf{N})((\exists z \in \mathsf{N})(fz = x) \leftrightarrow (\exists y \in \mathsf{N})(g(\mathbf{pair}(x, y)) = 0))$,

(2) $\forall f\ \exists g\ (\forall x \in \mathsf{N})((\exists y \in \mathsf{N})(fy = x) \leftrightarrow (\exists y \in \mathsf{N})(g(\mathbf{pair}(x, y)) = 0))$.

PROOF. (1) Given $g$, take $f := \lambda z.\boldsymbol{\tau}_{\mathsf{N}}(\mathbf{d}_{\mathsf{N}}(\lambda v.\mathbf{proj}_0(z), \mathbf{nt}_{\mathsf{N}}, gz, 0), gz)$. It is easy to see $fz = x$ iff $gz = 0 \wedge \mathbf{proj}_0(z) = x$ for $x, z \in \mathsf{N}$.
(2) Given $f$, set $g := \lambda z.\boldsymbol{\tau}_{\mathsf{N}}(\mathbf{d}_{\mathsf{N}}(\lambda u.0, \mathbf{nt}_{\mathsf{N}}, f(\mathbf{proj}_1(z)), \mathbf{proj}_0(z)), f(\mathbf{proj}_1(z)))$. For $x, y \in \mathsf{N}$, it is easy to see $fy = x$ iff $g(\mathbf{pair}(x, y)) = 0$. ⊣

It is easy to check that all the arguments in this sections go through if we replace $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ by its intuitionistic counterpart.

**§4. Undefinability of $\boldsymbol{\tau}_{\mathsf{N}}$ in $\mathsf{BON}^+$.** We have seen how the truncation operator $\boldsymbol{\tau}_{\mathsf{N}}$ is used for a representation of the partial recursive functions within our operational framework. In this section, we prove that $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ is not a definable extension of $\mathsf{BON}^+$.

Our strategy is to show that there is no closed term $s$ such that $\mathsf{BON}^+$ proves

$$(\forall x \in \mathsf{N})(sx = 0) \wedge \forall x\ (sx \in \mathsf{N} \rightarrow x \in \mathsf{N}).$$

On the other hand such $s$ is easily definable from $\boldsymbol{\tau}_{\mathsf{N}}$ by $\lambda x.\boldsymbol{\tau}_{\mathsf{N}}(\lambda y.0, x)$.

To show this fact and for further unprovability results in Section 6 we make use of semantic considerations, and thus begin with introducing some basic notions.

DEFINITION 4.1. An *operational structure* is a 5-tuple

$$\mathfrak{M} = (M, App, Nat, \mathcal{S}, I)$$

with the following properties:

(1) $M$ is a nonempty set, the so-called *universe* of $\mathfrak{M}$, $App$ is a subset of $M^3$, unique in its last argument, $Nat$ is a subset of $M$, and $\mathcal{S}$ is a non-empty subset of the power set $\mathrm{Pow}(Nat)$ of $Nat$.

(2) $I$ is a mapping that assigns an element $I(r)$ of $M$ to any constant $r$ of the language $L$.

Furthermore, a *valuation* over this structure is a mapping $J$ that assigns an element $J(u)$ of $M$ to any individual variable $u$ and an element $J(U)$ of $\mathcal{S}$ to any relation variable $U$.

Given an operational structure $\mathfrak{M} = (M, App, Nat, \mathcal{S}, I)$ and the valuation $J$ over $\mathfrak{M}$, the value $\|r\|^J_{\mathfrak{M}}$ of a term $r$ is inductively defined as follows. If $r$ is an individual constant, then $\|r\|^J_{\mathfrak{M}} := I(r)$; if $r$ is an individual variable, then $\|r\|^J_{\mathfrak{M}} := J(r)$. If $r$ is the compound term $st$ we have to distinguish a few cases:

(1) If $\|s\|^J_{\mathfrak{M}}$ and $\|t\|^J_{\mathfrak{M}}$ are elements of $M$ and if there exists $m \in M$ such that $(\|s\|^J_{\mathfrak{M}}, \|t\|^J_{\mathfrak{M}}, m) \in App$, then this element $m$ is uniquely determined, and we set $\|r\|^J_{\mathfrak{M}} := m$;

(2) If $\|s\|^J_{\mathfrak{M}}$ and $\|t\|^J_{\mathfrak{M}}$ are elements of $M$ and if there exists no $m \in M$ such that $(\|s\|^J_{\mathfrak{M}}, \|t\|^J_{\mathfrak{M}}, m) \in App$, then $\|r\|^J_{\mathfrak{M}}$ is the value **undef**;

(3) If $\|s\|^J_{\mathfrak{M}}$ or $\|t\|^J_{\mathfrak{M}}$ is the value **undef**, then $\|r\|^J_{\mathfrak{M}}$ is the value **undef**.

Clearly, the value of a closed term does not depend on the valuation $J$ and, therefore, we simply write $\|r\|_{\mathfrak{M}}$ for the value of the closed term $r$ with respect to the operational structure $\mathfrak{M}$.

Similarly, the value $\|\varphi\|^J_{\mathfrak{M}}$ of an $L$ formula $\varphi$ with respect to the operational structure $\mathfrak{M} = (M, App, Nat, \mathcal{S}, I)$ and the valuation $J$ over $\mathfrak{M}$ is either $\mathbf{T}$ or $\mathbf{F}$. For atomic formulae we set the following:

(1) $\|t{\downarrow}\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|t\|^J_{\mathfrak{M}} \in M$, and $\|t{\downarrow}\|^J_{\mathfrak{M}} := \mathbf{F}$ if $\|t\|^J_{\mathfrak{M}}$ is the value **undef**;

(2) $\|s = t\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|s\|^J_{\mathfrak{M}} = \|t\|^J_{\mathfrak{M}} \in M$, and $\|s = t\|^J_{\mathfrak{M}} := \mathbf{F}$ if (at least) one of $\|s\|^J_{\mathfrak{M}}$ or $\|t\|^J_{\mathfrak{M}}$ is the value **undef** or if they are both in $M$ but different;

(3) $\|\mathsf{N}(t)\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|t\|^J_{\mathfrak{M}} \in Nat$, and $\|\mathsf{N}(t)\|^J_{\mathfrak{M}} := \mathbf{F}$ if $\|t\|^J_{\mathfrak{M}}$ is the value **undef** or an element of $M \setminus Nat$;

(4) $\|U(t)\|^J_{\mathfrak{M}} := \mathbf{T}$ if $\|t\|^J_{\mathfrak{M}} \in J(U)$, and $\|U(t)\|^J_{\mathfrak{M}} := \mathbf{F}$ if $\|t\|^J_{\mathfrak{M}}$ is the value **undef** or an element of $M \setminus J(U)$.

Starting off from this treatment of the atomic formulae, the values of the compound formulae are introduced as usual. We say that an $L$ formula $\varphi$ is *valid* in the operational structure $\mathfrak{M}$, in symbols $\mathfrak{M} \models \varphi$, iff $\|\varphi\|^J_{\mathfrak{M}} = \mathbf{T}$ for all valuations $J$ over this structure. Let $T$ be the theory $\mathsf{BON}^+$ or $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$. Then we call an operational structure $\mathfrak{M}$ a *model of* $T$ iff all axioms of $T$ are valid in $\mathfrak{M}$.

Recall that $\omega$ is the set of the standard natural numbers and in the following we write $\mathbb{N} = (\omega, \dots)$ for the standard model of Peano arithmetic $\mathsf{PA}$. We may assume without loss of generality that any model $\mathcal{M} = (M, \dots)$ of $\mathsf{PA}$ is an extension of $\mathbb{N}$ and that $\omega$ is an initial segment of $M$.

Models of Peano arithmetic $\mathsf{PA}$ can be easily extended to operational structures. Let $\{e\}$ be an indexing of the partial recursive functions, keeping in mind that there exists a $\Sigma_1$ formula *Kleene* of the language of $\mathsf{PA}$ that defines

$\{e\}(x) \simeq y$ in $\mathsf{PA}$ by $Kleene[e,x,y]$. If $N$ is either the set $\omega$ or the set $M$, the *$N$-extension of $\mathcal{M}$* is defined to be the operational structure

$$[\mathcal{M}, N] := (M, App_{\mathcal{M}}, N, \{\emptyset\}, I_\omega),$$

where $App_{\mathcal{M}}$ is defined to be the set

$$\{(e,x,y) \in M^3 : \mathcal{M} \models Kleene[e,x,y]\}$$

and $I_\omega$ is an arbitrary but fixed assignment of standard natural numbers to the constants of $L$ such that the axioms of $\mathsf{BON}^+$ are satisfied and any numeral $\overline{n}$ is interpreted as the natural number $n$; this is possible by formalizing ordinary recursion theory in $\mathsf{PA}$. Hence for any model $\mathcal{M} = (M, \dots)$ of $\mathsf{PA}$, the structures $[\mathcal{M}, \omega]$ and $[\mathcal{M}, M]$ are models of $\mathsf{BON}^+$. By the upward $\Sigma_1$ persistency, we have the following.

REMARK 4.2. If $t$ is a closed term that is defined in $[\mathbb{N}, \omega]$, i.e., $\|t\|_{[\mathbb{N},\omega]} \in \omega$, then for any model $\mathcal{M} = (M, \dots)$ of $\mathsf{PA}$,

$$\|t\|_{[\mathbb{N},\omega]} = \|t\|_{[\mathcal{M},\omega]} = \|t\|_{[\mathcal{M},M]}.$$

THEOREM 4.3. *There exists no closed term $s$ such that $\mathsf{BON}^+$ proves*

$$(\forall x \in \mathsf{N})(sx = 0) \wedge \forall x\ (sx \in \mathsf{N} \rightarrow x \in \mathsf{N}).$$

PROOF. For contradiction, let $s$ be a closed term such that $\mathsf{BON}^+$ proves

(i) $(\forall x \in \mathsf{N})(sx = 0)$,

(ii) $\forall x\ (sx \in \mathsf{N} \rightarrow x \in \mathsf{N})$.

Then we take any non-standard model $\mathcal{M} = (M, \dots)$ of Peano arithmetic and arbitrary $n \in M \setminus \omega$. In view of (i) and (ii) we thus have

(iii) $[\mathcal{M}, M] \models (\forall x \in \mathsf{N})(sx = 0)$,

(iv) $[\mathcal{M}, \omega] \models \forall x\ (sx \in \mathsf{N} \rightarrow x \in \mathsf{N})$.

From (iii) we conclude that $Kleene[m, n, 0]$ holds in $\mathcal{M}$ if $m$ is the value of $s$ in $[\mathcal{M}, M]$ which is also the value in $[\mathcal{M}, \omega]$. Together with (iv) we thus obtain $n \in \omega$, a contradiction. $\dashv$

COROLLARY 4.4. *The operator $\boldsymbol{\tau}_{\mathsf{N}}$ is not definable in $\mathsf{BON}^+$.*

This corollary can also be obtained by showing that another term cannot exist in $\mathsf{BON}^+$, see Theorem 4.7 below. This result, or better the strategy to show it, is interesting in its own and proceeds as follows.

Given a model $\mathcal{M}$ of $\mathsf{PA}$, we write $\mathbb{N} \prec_1 \mathcal{M}$ iff for every $\Sigma_1$ formula $\varphi[u]$ of the language of $\mathsf{PA}$ with at most $u$ free and all $n \in \omega$,

$$\mathbb{N} \models \varphi[\widehat{n}] \iff \mathcal{M} \models \varphi[\widehat{n}].$$

Here, $\widehat{n}$ is the numeral in the sense of the language of $\mathsf{PA}$ corresponding to $n \in \omega$. We use this different notation in order to avoid confusing the numerals in the sense of $\mathsf{BON}^+$ with those in the sense of $\mathsf{PA}$. The following observation is logical folklore and will play a central role in the proof of Theorem 4.7.

LEMMA 4.5. *There exists a model $\mathcal{M}$ of Peano arithmetic $\mathsf{PA}$ with $\mathbb{N} \not\prec_1 \mathcal{M}$.*

PROOF. Assume that $\mathbb{N} \prec_1 \mathcal{M}$ for all models $\mathcal{M}$ of $\mathsf{PA}$, and let $\varphi$ be a $\Sigma_1$ sentence logically equivalent to $\neg Con(\mathsf{PA})$. From $\mathbb{N} \not\models \varphi$ we thus obtain that $\mathcal{M} \models Con(\mathsf{PA})$ for all models $\mathcal{M}$ of $\mathsf{PA}$. By Gödel–Henkin's completeness this yields $\mathsf{PA} \vdash Con(\mathsf{PA})$, a contradiction. $\dashv$

There is a further well-known property of $\mathsf{PA}$, dealing with formalized recursion theory, that will be used in the proof of Theorem 4.7.

LEMMA 4.6. *Let $\varphi[u, v]$ be a $\Delta_0$ formula of the language of $\mathsf{PA}$ with at most $u$ and $v$ free. Then there exists a natural number $e_\varphi$ such that $\mathsf{PA}$ proves*

$$\forall x \left(\exists y\ \varphi[x, y] \leftrightarrow \{\widehat{e_\varphi}\}(x)\!\downarrow\right) \wedge \forall x \left(\{\widehat{e_\varphi}\}(x)\!\downarrow \rightarrow \varphi[x, \{\widehat{e_\varphi}\}(x)]\right).$$

The proof of this lemma is by a straightforward formalization of a "search from below" argument.

THEOREM 4.7. *There exists no closed term $t$ such that $\mathsf{BON}^+$ proves*

$$\forall f\ (tf\!\downarrow \wedge (\forall x \in \mathsf{N})(fx \in \mathsf{N} \leftrightarrow t(f, x) = 0)). \qquad (\flat)$$

PROOF. For contradiction assume that $\mathsf{BON}^+$ proves $(\flat)$ for a closed term $t$. By Lemma 4.5 take a model $\mathcal{M} = (M, \dots)$ of $\mathsf{PA}$ for which $\mathbb{N} \not\prec_1 \mathcal{M}$.

Now we pick an arbitrary $\Delta_0$ formula $\varphi[u, v]$ of the language of $\mathsf{PA}$ with at most $u$, $v$ free and choose $e_\varphi \in \omega$ according to the previous lemma such that

$$\mathsf{PA} \vdash \forall x \left(\exists y\ \varphi[x, y] \leftrightarrow \{\widehat{e_\varphi}\}(x)\!\downarrow\right) \wedge \forall x \left(\{\widehat{e_\varphi}\}(x)\!\downarrow \rightarrow \varphi[x, \{\widehat{e_\varphi}\}(x)]\right). \qquad (1)$$

In $\mathsf{BON}^+$, $(\flat)$ implies $t\,\overline{e_\varphi}\!\downarrow$, hence also $t\!\downarrow$ by strictness. This implies that the value of $t$ in $[\mathbb{N}, \omega]$ is a natural number and that

$$\|t\|_{[\mathbb{N},\omega]} = \|t\|_{[\mathcal{M},\omega]} = \|t\|_{[\mathcal{M},M]}$$

according to Remark 4.2. From $\mathsf{BON}^+ \vdash t\,\overline{e_\varphi}\!\downarrow$ we also obtain that there exists a natural number $m$ such that

$$[\mathbb{N}, \omega] \models t\,\overline{e_\varphi} = \overline{m} \qquad \text{and} \qquad [\mathcal{M}, M] \models t\,\overline{e_\varphi} = \overline{m}.$$

Since we assume the provability of $(\flat)$ in $\mathsf{BON}^+$, this implies

$$[\mathcal{M}, \omega] \models (\forall x \in \mathsf{N})(\overline{e_\varphi}\,x \in \mathsf{N} \leftrightarrow \overline{m}\,x = 0), \qquad (2)$$

$$[\mathcal{M}, M] \models (\forall x \in \mathsf{N})(\overline{e_\varphi}\,x \in \mathsf{N} \leftrightarrow \overline{m}\,x = 0). \qquad (3)$$

For any $n \in \omega$ we have the following equivalences. The first ones are consequences of (1) and the interpretation of $\mathsf{N}$ in $[\mathcal{M}, M]$,

$$\mathcal{M} \models \exists y\ \varphi[\widehat{n}, y] \iff \mathcal{M} \models \{\widehat{e_\varphi}\}(\widehat{n})\!\downarrow \iff [\mathcal{M}, M] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N}. \qquad (4)$$

Because of (3) we continue with

$$[\mathcal{M}, M] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N} \iff [\mathcal{M}, M] \models \overline{m}\,\overline{n} = 0. \qquad (5)$$

Then the interpretation of the application in $[\mathcal{M}, M]$ and $[\mathcal{M}, \omega]$ yields

$$[\mathcal{M}, M] \models \overline{m}\,\overline{n} = 0 \iff \mathcal{M} \models \{\widehat{m}\}(\widehat{n}) = 0 \iff [\mathcal{M}, \omega] \models \overline{m}\,\overline{n} = 0. \qquad (6)$$

Now we apply (2) and obtain

$$[\mathcal{M}, \omega] \models \overline{m}\,\overline{n} = 0 \iff [\mathcal{M}, \omega] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N}. \qquad (7)$$

By the interpretations of $\mathsf{N}$ and the application in $[\mathcal{M},\omega]$ we have

$$[\mathcal{M},\omega] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N} \iff \mathcal{M} \models \{\widehat{e_\varphi}\}(\widehat{n}) = \widehat{k} \text{ for some } k \in \omega. \tag{8}$$

By (1) and the absoluteness of $\varphi$ we further have

$$\mathcal{M} \models \{\widehat{e_\varphi}\}(\widehat{n}) = \widehat{k} \implies \mathcal{M} \models \varphi\Big[\widehat{n},\widehat{k}\Big] \iff \mathbb{N} \models \varphi\Big[\widehat{n},\widehat{k}\Big]$$

for any $k \in \omega$. Therefore, together with (8),

$$[\mathcal{M},\omega] \models \overline{e_\varphi}\,\overline{n} \in \mathsf{N} \implies \mathbb{N} \models \exists y\ \varphi[\widehat{n},y]. \tag{9}$$

Lines (4)–(9) plus the upward $\Sigma_1$ persistency thus give us

$$\mathcal{M} \models \exists y\ \varphi[\widehat{n},y] \iff \mathbb{N} \models \exists y\ \varphi[\widehat{n},y]$$

for all $n \in \omega$. Since $\varphi[u,v]$ has been an arbitrary $\Delta_0$ formula of the language of $\mathsf{PA}$, this is a contradiction to $\mathbb{N} \not\prec_1 \mathcal{M}$. ⊣

This also shows the undefinability of $\boldsymbol{\tau}_\mathsf{N}$ in $\mathsf{BON}^+$ since $t := \lambda fx.r_2(fx)$, where $r_2$ is from Lemma 6.5 below, cannot exist in $\mathsf{BON}^+$ according to this theorem.

**§5. $\omega$-models of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.** For the standard recursion-theoretic operational structure $[\mathbb{N},\omega]$ (also called *Kleene's first model*) with $\omega$ as universe and application $ab \simeq c$ interpreted as $\{a\}(b) \simeq c$ we can easily validate the $\boldsymbol{\tau}_\mathsf{N}$-axioms: Simply interpret $\boldsymbol{\tau}_\mathsf{N}$ as the identity operation. Hence $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ + (Tot-$\mathsf{N}$) is clearly consistent, and thus the $\boldsymbol{\tau}_\mathsf{N}$-axioms are justified with respect to this standard model of $\mathsf{BON}^+$, even under the additional assumption that all individuals are natural numbers.

In order to make a point that $\boldsymbol{\tau}_\mathsf{N}$ is a natural operator, we also look at further typical operational models: the canonical term model as well as two variants of Kleene's second model and of the graph model, respectively. In this article we confine ourselves to some basic definitions and results. A detailed analysis of these structures will be given in Rosebrock's forthcoming dissertation [17].

For the following, we call an operational structure $\mathfrak{M} = (M, App, Nat, \mathcal{S}, I)$ an *$\omega$-model* of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ iff it is a model of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ and, in addition,

$$Nat = \{\|\overline{n}\|_\mathfrak{M} : n \in \omega\}.$$

Before turning to some particular $\omega$-models, we summarize some general properties of $\omega$-models of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

Theorem 5.1. *Let $\mathfrak{M} = (M, \circ, Nat, \mathcal{S}, I)$ be an $\omega$-model of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ and $S \subseteq \omega$.*

(1) *If $S$ is $\Sigma^0_1$, there is $f \in M$ with $\mathfrak{M} \models f \in Char_2$ such that for all $m \in \omega$,*

$$m \in S \iff (f \circ \|\overline{m}\|_\mathfrak{M}) \circ \|\overline{n}\|_\mathfrak{M} = \big\|\overline{0}\big\|_\mathfrak{M} \text{ for some } n \in \omega.$$

(2) *The following are equivalent:*
- *There is $f \in M$ such that for all $m \in \omega$,*

$$m \in S \iff f \circ (\|\mathbf{pair}(\overline{m},\overline{n})\|_\mathfrak{M}) = \big\|\overline{0}\big\|_\mathfrak{M} \text{ for some } n \in \omega.$$

- *There is $g \in M$ such that for all $m \in \omega$,*

$$m \in S \iff g \circ \|\overline{n}\|_\mathfrak{M} = \|\overline{m}\|_\mathfrak{M} \text{ for some } n \in \omega.$$

| $r$ is of the form | $s$ is of the form |
|---|---|
| $\mathbf{k}(t_0, t_1)$ | $t_0$ |
| $\mathbf{s}(t_0, t_1, t_2)$ | $t_0(t_2, t_1 t_2)$ |
| $\mathbf{p}_0(\mathbf{p}(t_0, t_1))$ | $t_0$ |
| $\mathbf{p}_1(\mathbf{p}(t_0, t_1))$ | $t_1$ |
| $\mathbf{p}_\mathsf{N} 0$ | $0$ |
| $\mathbf{p}_\mathsf{N}(\mathbf{s}_\mathsf{N}\overline{n})$ | $\overline{n}$ |
| $\mathbf{d}_\mathsf{N}(t_0, t_1, \overline{m}, \overline{m})$ | $t_0$ |
| $\mathbf{d}_\mathsf{N}(t_0, t_1, \overline{m}, \overline{n})$ with $m \neq n$ | $t_1$ |
| $\boldsymbol{\tau}_\mathsf{N}(t_0, \overline{n})$ | $t_0\overline{n}$ |

TABLE 1. The relation $conv(r, s)$.

PROOF. For (1), let $S$ be $\Sigma^0_1$. There is a total recursive function $\alpha$ such that $S = \{m \in \omega : (\exists n \in \omega)(\alpha(m, n) = 0)\}$ and $\alpha(m, n) \in \{0, 1\}$ for any $m, n \in \omega$. Then $f := \mathbf{g}_\alpha$ from Theorem 3.5 is what is required, since $\mathfrak{M}$ is an $\omega$-model.

(2) follows immediately from Lemma 3.6. ⊣

**5.1. The canonical term model.** We begin with the canonical term model. On the closed terms a binary relation *conv* is introduced such that we have $conv(r, s)$ if and only if there exist closed terms $t_0$, $t_1$, $t_2$ as well as different natural numbers $m$ and $n$ for which one of the cases in Table 1 holds. If we have $conv(r, s)$ then $r$ is called a redex and $s$ the contractum of $r$.

Let $\approx$ be the smallest congruence relation, with respect to application, on the collection of all closed terms that contains *conv*. Given any closed term $r$, by $[r]$ we mean the equivalence class of $r$ modulo $\approx$.

Now we write $|\mathfrak{CT}|$ for the collection of all equivalence classes of the closed terms and define an application relation $\cdot^{\mathfrak{CT}}$ on $|\mathfrak{CT}|$ by setting, for all closed terms $r$ and $s$,

$$[r] \cdot^{\mathfrak{CT}} [s] := [rs].$$

DEFINITION 5.2. The *operational term structure* is the 5-tuple

$$\mathfrak{CT} = \Big(|\mathfrak{CT}|, \cdot^{\mathfrak{CT}}, Nat^{\mathfrak{CT}}, \mathrm{Pow}\Big(Nat^{\mathfrak{CT}}\Big), I^{\mathfrak{CT}}\Big),$$

where $Nat^{\mathfrak{CT}} = \{[\overline{n}] : n \in \omega\}$ and $I^{\mathfrak{CT}}(r) = [r]$ for every constant $r$ of $L$.

Essentially by exploiting the confluence property it is shown in Rosebrock [17] that $\mathfrak{CT}$ is a model of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$. We obtain also the second part of the following theorem, where the essence of its proof is

$$conv(r, s) \implies \mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) \vdash r\downarrow \to r = s,$$

but $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) \vdash s\downarrow \to r = s$ does not follow from $conv(r, s)$ in general.

THEOREM 5.3. *$\mathfrak{CT}$ is an $\omega$-model of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$. In addition, for all closed terms $r$ and $s$,*

$$r \approx s \implies \mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) \vdash (r\downarrow \wedge\, s\downarrow) \to r = s.$$

If $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ proves $t \in \mathsf{N}$ for some closed term $t$, then there exists $n \in \omega$ such that $t \approx \overline{n}$. Therefore Lemma 3.4 immediately follows.

**5.2. Two variants of Kleene's second model.** Kleene's second model provides a further interesting approach to constructing a model of a partial combinatory algebra; see, e.g., Beeson [2, VI.7.4] and Troelstra and van Dalen [19, 9.9.2]. First we have to introduce some notations.

In the following we will make use of the standard primitive recursive coding machinery: $\langle m_0, \dots, m_{n-1} \rangle$ stands for the primitive recursively formed $n$-tuple of the natural numbers $m_0, \dots, m_{n-1}$ and $*$ is the primitive recursive concatenation of the finite sequences, i.e.,

$$\langle m_0, \dots, m_{i-1} \rangle * \langle n_0, \dots, n_{j-1} \rangle = \langle m_0, \dots, m_{i-1}, n_0, \dots, n_{j-1} \rangle.$$

If $\alpha$ is a function from $\omega$ to $\omega$ and $n$ a natural number, then we write $\alpha \restriction n$ for the code of the initial segment of $\alpha$ up to $n-1$, i.e.,

$$\alpha \restriction n := \langle \alpha(0), \dots, \alpha(n-1) \rangle.$$

Finally, if $\alpha$ and $\beta$ are functions from $\omega$ to $\omega$ then $\alpha|\beta$ is the possibly partial function from $\omega$ to $\omega$ that is defined as follows:

$$(\alpha|\beta)(n) := \begin{cases} \alpha(\langle n \rangle * \beta \restriction m) - 1 & \text{if } m \text{ is minimal with } \alpha(\langle n \rangle * \beta \restriction m) > 0, \\ \text{undefined} & \text{if there is no such } m. \end{cases}$$

On functions from $\omega$ to $\omega$ we define a partial application relation by

$$\alpha \odot \beta := \begin{cases} \alpha|\beta & \text{if } \alpha|\beta \text{ is a total function from } \omega \text{ to } \omega, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Note that $\alpha|\beta$ is $\Sigma^0_1$ definable relative to $\alpha$ and $\beta$, and that the definedness of $\alpha \odot \beta$ is a $\Pi^0_2$ statement on $\alpha$ and $\beta$.

In the following we denote the collection of total recursive functions by *TRec*. Because of the $\Sigma^0_1$ definability, if $\alpha, \beta \in \mathit{TRec}$ and $\alpha \odot \beta$ exists then $\alpha \odot \beta \in \mathit{TRec}$.

The following lemma is easily proved by Kleene's normal form theorem. Together with the definition of $\odot$, it characterizes the functionals on Baire space $\omega^\omega$ described by this operation. It is not difficult to generalize it to the characterization of multi-argument functionals.

LEMMA 5.4. *We have the following results about the existence of specific functions:*

(1) *Any partial continuous functional on Baire space $\omega^\omega$ whose domain is a $G_\delta$ set can be expressed as $\beta \mapsto \alpha \odot \beta$ for some total function $\alpha$.*

(2) *For $\Sigma^0_1$ formulae $\varphi[n, \beta]$ and $\psi[n, m, \beta]$ without other parameters such that,*

$$\text{for any } \beta \in \omega^\omega \text{ if } (\forall n \in \omega)\varphi[n, \beta] \text{ then } (\forall n \in \omega)(\exists m \in \omega)\psi[n, m, \beta],$$

*there exists $\alpha \in \mathit{TRec}$ such that for any total function $\beta$ from $\omega$ to $\omega$,*

$$\begin{aligned} \alpha \odot \beta \text{ is defined } &\iff \varphi[n, \beta] \text{ for all } n \in \omega \\ &\implies \psi[n, (\alpha \odot \beta)(n), \beta] \text{ for all } n \in \omega. \end{aligned}$$

Note that (1) follows from the relativized version of (2). Thus (1) is a boldface version of (2). While the *lightface* version (2) is proved in Nemoto and Sato [15, 3.23(1)] (applied to $\varphi[n, \beta] \wedge \psi[n, m, \beta]$), the boldface one seems more popular in the literature; see, e.g., Rin and Walsh [16, 3.3] and Longley and Normann [11, 12.2.2]. This explains why the operations based on $\odot$ are sometimes called *partial continuous*, e.g., in Troelstra and van Dalen [19, 9.4.1].

$C_n$ is written for the constant function with value $n$ and

$$\mathit{Const} := \{C_n : n \in \omega\}.$$

In the structures below this is the interpretation of the predicate $\mathsf{N}$. It is known that the structures are models of $\mathsf{BON}^+$ (see, e.g., Beeson [2, VI.7.4.1, 7.5.1]) and we can easily extend them to those of $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ by using Lemma 5.4(2). The details will be shown in Rosebrock [17].

THEOREM 5.5 (Bold- and lightface Kleene's second model). *There exists an interpretation $I$ of the constants of $L$ in $TRec$ such that the operational structures*

$$\mathfrak{BK}_2 = (\omega^\omega, \odot, \mathit{Const}, \mathrm{Pow}(\mathit{Const}), I)$$
$$\text{and } \mathfrak{LK}_2 = (\mathit{TRec}, \odot, \mathit{Const}, \mathrm{Pow}(\mathit{Const}), I)$$

*are $\omega$-models of $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ and that $\|\overline{n}\|_{\mathfrak{LK}_2} = C_n$ for any $n \in \omega$.*

Despite the popularity of the boldface $\mathfrak{BK}_2$, later we will need the following result, which is specific to the lightface $\mathfrak{LK}_2$.

THEOREM 5.6. *For every subset $S$ of $\omega$ we have the following equivalences:*

(1) *$S$ is $\Sigma^0_1$ iff there is $\alpha \in TRec$ with $\mathfrak{LK}_2 \models \alpha \in Char_2$ such that for all $m \in \omega$,*

$$m \in S \iff (\alpha \odot C_m) \odot C_n = C_0 \text{ for some } n \in \omega.$$

(2) *$S$ is $\Pi^0_2$ iff there exists $\alpha \in TRec$ such that for all $m \in \omega$,*

$$m \in S \iff \alpha \odot C_m = C_0.$$

(3) *$S$ is $\Sigma^0_3$ iff there exists $\alpha \in TRec$ such that for all $m \in \omega$,*

$$m \in S \iff \alpha \odot C_n = C_m \text{ for some } n \in \omega.$$

PROOF. (1) The "only-if" part is by Theorem 5.1(1). For the "if" part, note that $(\alpha \odot C_m) \odot C_n = C_0$ iff $((\alpha|C_m)|C_n)(0) = 0$ by $\mathfrak{LK}_2 \models \alpha \in Char_2$.
(2) The "if" part is obvious. Let $S = \{n \in \omega : (\forall m \in \omega)\theta[m, n]\}$ with $\theta$ being $\Sigma^0_1$. Lemma 5.4(2) with $\varphi[n, \beta] \equiv \theta[n, \beta(0)]$ and $\psi[n, m, \beta] \equiv m = 0$ yields the required $\alpha \in TRec$.
(3) Theorem 5.1(2) asserts that $S$ satisfies the latter condition iff $S$ is a projection of a set satisfying the latter condition of (2). Hence (2) yields the statement. ⊣

**5.3. Two variants of the graph model.** The so-called graph model for the untyped lambda calculus was discovered independently by Engeler, Plotkin, and Scott.

The universes of our variants are included in $\mathrm{Pow}(\omega)$. To define the application relation, we let $(e_n : n \in \omega)$ be a standard enumeration of finite binary sequences where $e_n$ represents the finite set $\{i < |e_n| : e_n(i) = 1\}$. Here $|e_n|$ denotes the length of the sequence $e_n$ and for $i < |e_n|$, we let $e_n(i)$ is its $i$-th component. For arbitrary $P, Q \subseteq \omega$ we then set

$$P \cdot^{\mathfrak{G}} Q := \{m \in \omega : \langle n, m \rangle \in P \text{ and } e_n \subseteq Q \text{ for some } n \in \omega\},$$

where $e_n \subseteq Q$ means that the set represented by $e_n$ is a subset of $Q$. Clearly, this application is total on $\mathrm{Pow}(\omega)$ and the class $\Sigma^0_1$ is closed under it.

The next lemma is analogous to Lemma 5.4. Now $\mathrm{Pow}(\omega)$ is equipped with the so-called Scott topology, and must not be confused with Cantor space $2^\omega$.

LEMMA 5.7. *We have the following results about the existence of specific sets:*

(1) *Any continuous functional from the Scott domain* $\mathrm{Pow}(\omega)$ *to* $\mathrm{Pow}(\omega)$ *can be expressed as* $Q \mapsto P \cdot^{\mathfrak{G}} Q$ *for some* $P \subseteq \omega$.

(2) *For any* $\Sigma^0_1$ *formula* $\theta[n, Q]$ *without other parameters in which* $Q$ *occurs only positively, there exists a* $\Sigma^0_1$ *subset* $P$ *of* $\omega$ *such that, for any* $Q \subseteq \omega$,
$$P \cdot^{\mathfrak{G}} Q = \{n \in \omega : \theta[n, Q]\}.$$

Similarly to Lemma 5.4, (1) follows from the relativization of (2). As the Scott continuity is equivalent to the positive $\Sigma^0_1$ definability with set parameters, we can see the contrast between boldface and lightface again. While the boldface version (1) seems more popular (e.g., Barendregt [1, 18.1.8.(ii)], and Rin and Walsh [16, 3.6]), we can prove (2) easily by the $\Sigma^0_1$ normal form theorem in second order arithmetic with a modification for the positiveness of the set variables.

The natural numbers are represented by the singletons of elements of $\omega$, and we set
$$\mathit{Sing} := \{\{m\} : m \in \omega\}.$$

It is shown in, e.g., Beeson [2, VI.7.2.4, 7.5.2] that the following structures are models of $\mathsf{BON}^+$ and we can easily extend them to those of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ by using Lemma 5.7(2). The details of this result will also be shown in Rosebrock [17].

THEOREM 5.8 (Bold- and lightface graph model). *There exists an interpretation* $I$ *of the constants of* $L$ *in* $\mathrm{Pow}(\omega) \cap \Sigma^0_1$ *such that the operational structures*
$$\mathfrak{BG} = \big(\mathrm{Pow}(\omega), \cdot^{\mathfrak{G}}, \mathit{Sing}, \mathrm{Pow}(\mathit{Sing}), I\big)$$
$$\text{and } \mathfrak{LG} = \big(\mathrm{Pow}(\omega) \cap \Sigma^0_1, \cdot^{\mathfrak{G}}, \mathit{Sing}, \mathrm{Pow}(\mathit{Sing}), I\big)$$
*are* $\omega$*-models of* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\textsc{Tot-Ap})$ *and that* $\|\overline{n}\|_{\mathfrak{LG}} = \{n\}$ *for any* $n \in \omega$.

We can also have an analogue of Theorem 5.6 as follows. $\Sigma^0_1 \wedge \Pi^0_1$ denotes the class consisting of intersections of $\Sigma^0_1$ sets and $\Pi^0_1$ sets. This class must not be confused with $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$, the intersection of the classes $\Sigma^0_1$ and $\Pi^0_1$.

THEOREM 5.9. *For every subset* $S$ *of* $\omega$ *we have the following equivalences.*

(1) $S$ *is* $\Sigma^0_1$ *iff there is* $P \in \Sigma^0_1$ *with* $\mathfrak{LG} \models P \in \mathit{Char}_2$ *such that for all* $m \in \omega$,
$$m \in S \iff \big(P \cdot^{\mathfrak{G}} \{m\}\big) \cdot^{\mathfrak{G}} \{n\} = \{0\} \text{ for some } n \in \omega.$$

(2) $S$ *is* $\Sigma^0_1 \wedge \Pi^0_1$ *iff there exists* $P \in \Sigma^0_1$ *such that for all* $m \in \omega$,
$$m \in S \iff P \cdot^{\mathfrak{G}} \{m\} = \{0\}.$$

(3) $S$ *is* $\Sigma^0_2$ *iff there exists* $P \in \Sigma^0_1$ *such that for all* $m \in \omega$,
$$m \in S \iff P \cdot^{\mathfrak{G}} \{n\} = \{m\} \text{ for some } n \in \omega.$$

$$
\begin{array}{ccccccccc}
 & \Sigma^0_1 & & \Pi^0_1{\vee}\Sigma^0_1 & & \big(\Sigma^0_1{\wedge}\Pi^0_1\big){\vee}\Sigma^0_1 & & & \\
\nearrow & & \searrow\;\nearrow & & \searrow\;\nearrow & & \searrow & & \\
\Delta^0_1 & & \Delta\big(\Sigma^0_1{\wedge}\Pi^0_1\big) & & \Delta\big(\big(\Pi^0_1{\vee}\Sigma^0_1\big){\wedge}\Pi^0_1\big) & & \Delta\big(\big(\big(\Sigma^0_1{\wedge}\Pi^0_1\big){\vee}\Sigma^0_1\big){\wedge}\Pi^0_1\big) & \cdots\cdots & \Delta^0_2 \\
\searrow & & \nearrow\;\searrow & & \nearrow\;\searrow & & \nearrow & & \\
 & \Pi^0_1 & & \Sigma^0_1{\wedge}\Pi^0_1 & & \big(\Pi^0_1{\vee}\Sigma^0_1\big){\wedge}\Pi^0_1 & & &
\end{array}
$$

FIGURE 1. Semi-recursive difference hierarchy.

PROOF. (1) The "only-if" part is by Theorem 5.1(1). For the "if" part, note that $\big(P \cdot^{\mathfrak{G}} \{m\}\big) \cdot^{\mathfrak{G}} \{n\} = \{0\}$ iff $\big(P \cdot^{\mathfrak{G}} \{m\}\big) \cdot^{\mathfrak{G}} \{n\} \ni 0$ by $\mathfrak{LG} \models P \in \mathit{Char}_2$.
(2) For the 'if' part, the latter condition is equivalent to

$$\big(0 \in P \cdot^{\mathfrak{G}} \{m\}\big) \wedge (\forall k \in \omega)\big(k \in P \cdot^{\mathfrak{G}} \{m\} \rightarrow k = 0\big).$$

For the converse, let $S = \{m \in \omega : \varphi[m] \wedge \psi[m]\}$ with $\varphi$ and $\psi$ being $\Sigma^0_1$ and $\Pi^0_1$ respectively. Lemma 5.7(2) yields $P$ such that

$$P \cdot^{\mathfrak{G}} \{m\} = \{0 : \varphi[m]\} \cup \{1 : \neg\psi[m]\} \text{ for any } m \in \omega$$

by $\theta[n, Q] \equiv (\exists m \in Q)((n = 0 \wedge \varphi[m]) \vee (n = 1 \wedge \neg\psi[m]))$.
(3) Similar to Lemma 5.6(3), for projections of $\Sigma^0_1{\wedge}\Pi^0_1$ sets are exactly $\Sigma^0_2$ sets. ⊣

As the class $\Sigma^0_1{\wedge}\Pi^0_1$ is not so popular as the classes $\Sigma^0_n$, a short remark seems to be justified. Since the elements are of the form $R \setminus S$ with $R$ and $S$ being $\Sigma^0_1$, it is the second level of the lightface analogue of *Hausdorff–Kuratowski difference hierarchy*. The corresponding boldface class is denoted by $D_2(\boldsymbol{\Sigma}^0_1)$ in Louveau [12, 1.1], $(\boldsymbol{\Sigma}^0_1)_2$ in Nemoto [14], and would be by 2-$\boldsymbol{\Sigma}^0_1$ in the notation of Kanamori [10, Section 31] and $\boldsymbol{\Sigma}^0_{1,2}$ in that of Montalbán and Shore [13, 2.4]. Note however that they consider classes of subsets of Baire space $\omega^\omega$ or Cantor space $2^\omega$, whereas we consider classes of subsets of $\omega$. Even so, we can define a similar hierarchy by defining $\Pi^0_1{\vee}\Sigma^0_1$, $\big(\Sigma^0_1{\wedge}\Pi^0_1\big){\vee}\Sigma^0_1$ and so on in the obvious way. From a universal $\Sigma^0_1$ set, we can define universal sets for these classes. This yields the strictness of the hierarchy, similarly to the arithmetical hierarchy, as in Figure 1, where $\Delta\big(\Sigma^0_1{\wedge}\Pi^0_1\big)$ denotes $\big(\Sigma^0_1{\wedge}\Pi^0_1\big) \cap \big(\Pi^0_1{\vee}\Sigma^0_1\big)$ and so on. In particular, $\Sigma^0_1{\wedge}\Pi^0_1$ is properly between $\Sigma^0_1$ and $\Delta^0_2$.

**§6. Operational semi-decidability and the like.** Section 3 explains the role of $\boldsymbol{\tau}_{\mathsf{N}}$ for formalizing the basic parts of recursion theory within $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$. According to this, any partial recursive function is represented as a partial operator on $\mathsf{N}$ and moreover the basic closure properties of the structure of all partial recursive functions are also formalized as those of partial operators on $\mathsf{N}$. Therefore we could say that it also formalizes the structure of the partial recursive functions relative to some class of functions. In this sense, we could consider partial operations on $\mathsf{N}$ as "generalized" partial recursive functions.

Now let us go further with this paradigm, to the recursion-theoretic notions for sets of natural numbers. It is natural in our paradigm to call $U$ *operationally* *decidable* iff there exists an operation $f$ with

$$f \in \mathit{Char} \wedge (\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0).$$

Correspondingly, we call $U$ *operationally semi-decidable* iff there is $f$ with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0).$$

So the totality requirement is dropped in the case of semi-decidability.

In ordinary recursion theory a subset of $\omega$ is decidable iff the set itself and its complement are semi-decidable. As we will see below, this is not the case in our paradigm. Moreover, in ordinary recursion theory there are many (equivalent) ways how semi-decidability can be defined, but operationally the situation is more complex. The second part of the following definition lists some of the possible "standard" definitions of semi-decidability, tailored for our present context. Afterwards we will say more about their relationships.

Since ordinary recursion theory is typically developed in a classical context, we confine ourselves to classical arguments in the following. It would be interesting to see what would go through in a constructive context as well.

Definition 6.1. Given any $U$, we use the following abbreviations to express that $U$ is operationally decidable, semi-decidable, a projection of an operationally decidable set, a domain of an operation, a range of an operation or operationally enumerable:

$$\begin{aligned}
\mathit{OD}[U] &:= (\exists f \in \mathit{Char})(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0),\\
\mathit{OSD}[U] &:= \exists f\ (\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0),\\
\mathit{Pr}[U] &:= (\exists f \in \mathit{Char}_2)(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0)),\\
\mathit{Dom}[U] &:= \exists f\ (\forall x \in \mathsf{N})(x \in U \leftrightarrow fx \in \mathsf{N}),\\
\mathit{Rng}[U] &:= \exists f\ (\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = fy)),\\
\mathit{OE}[U] &:= U = \emptyset \vee (\exists f \in (\mathsf{N} \to \mathsf{N}))(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = fy)).
\end{aligned}$$

The notions $\mathit{OD}[\mathsf{N}\backslash U]$, $\mathit{OSD}[\mathsf{N}\backslash U]$, $\mathit{Pr}[\mathsf{N}\backslash U]$, $\ldots$ are defined accordingly.

We begin with the more or less obvious relationship between these notions.

Theorem 6.2. *In* $\mathsf{BON}^+$ *we can prove:*

(1) $\mathit{OD}[U] \to \mathit{OD}[\mathsf{N}\backslash U]$,
(2) $\mathit{OD}[U] \to \mathit{Pr}[U]$.

Proof. We assume $f \in \mathit{Char}$ and $(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0)$, and we set $r := \lambda u.\mathbf{d}_{\mathsf{N}}(\overline{1}, 0, fu, 0)$. Then $r \in \mathit{Char}$ and, for any $x \in \mathsf{N}$, $x \in \mathsf{N}\backslash U$ iff $rx = 0$. Hence we have (1). Furthermore, for $s := \lambda uv.fu$ we have $s \in \mathit{Char}_2$ and, for any $x \in \mathsf{N}$, $x \in U$ iff $(\exists y \in \mathsf{N})(s(x,y) = 0)$; thus we also have (2). ⊣

Theorem 6.3. *In* $\mathsf{BON}^+$ *we can prove*

$$\mathit{OD}[U] \leftrightarrow (\mathit{Pr}[U] \wedge \mathit{Pr}[\mathsf{N}\backslash U]).$$

Proof. By the previous theorem, the direction from left to right is obvious. For the converse, let $f, g \in \mathit{Char}_2$ be such that

$$x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0) \quad \text{and} \quad x \in \mathsf{N}\backslash U \leftrightarrow (\exists y \in \mathsf{N})(g(x,y) = 0)$$

for all $x \in \mathsf{N}$. Now we define

$$r := \lambda uv.\mathbf{d}_{\mathsf{N}}(0, g(u,v), f(u,v), 0) \qquad \text{and} \qquad s := \lambda u.f(u, \mathbf{min}_0(ru)).$$

Clearly, $rx \in \mathit{Char}$ and $(\exists y \in \mathsf{N})(r(x,y) = 0)$ for all $x \in \mathsf{N}$. Applying Theorem 2.5, we see $\mathbf{min}_0(rx) \in \mathsf{N}$ if $x \in \mathsf{N}$. This implies $s \in \mathit{Char}$. Assume now $x \in \mathsf{N}$ and $sx = 0$. Then $f(x, \mathbf{min}_0(rx)) = 0$. Thus, $x \in U$. Conversely, if $x \in \mathsf{N}$ with $sx = \overline{1}$, we conclude $0 = r(x, \mathbf{min}_0(rx)) = g(x, \mathbf{min}_0(rx))$ by Theorem 2.5 again. Hence, $x \in \mathsf{N} \backslash U$. We have shown $(\forall x \in \mathsf{N})(x \in U \leftrightarrow sx = 0)$. ⊣

THEOREM 6.4. *In* $\mathsf{BON}^+$ *we can prove*

$$Pr[U] \leftrightarrow OE[U].$$

PROOF. The equivalence is clearly provable for $U = \emptyset$. So let us assume $a \in U$. To show the direction from left to right we assume

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0)) \tag{$\natural$}$$

for some $f \in \mathit{Char}_2$ and set

$$r := \lambda u.\mathbf{d}_{\mathsf{N}}(\mathbf{proj}_0(u), a, f(\mathbf{proj}_0(u), \mathbf{proj}_1(u)), 0).$$

$r \in (\mathsf{N} \to \mathsf{N})$ is clear, and it remains to show that, for all $x \in \mathsf{N}$,

$$x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = ry).$$

Given $x \in U$, the equivalence ($\natural$) yields $f(x,y) = 0$ for some $y \in \mathsf{N}$. Hence $r(\mathbf{pair}(x,y)) = x$, and thus $(\exists z \in \mathsf{N})(x = rz)$. Conversely, if $x = rz$ for some $z \in \mathsf{N}$, then $x = a$ or $x = \mathbf{proj}_0(z) \wedge f(\mathbf{proj}_0(z), \mathbf{proj}_1(z)) = 0$. In both cases we have $x \in U$.

Turning to the direction from right to left of our theorem, assume $OE[U]$, say $g \in (\mathsf{N} \to \mathsf{N})$ with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(x = gy)).$$

Set $s := \lambda uv.\mathbf{d}_{\mathsf{N}}\big(0, \overline{1}, u, gv\big)$. Clearly, $s \in \mathit{Char}_2$ and

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(s(x,y) = 0)).$$

But this implies $Pr[U]$, as we had to show. ⊣

LEMMA 6.5. *There exist closed terms $r_1$, $r_2$, $r_3$ such that the following:*

(1) $\mathsf{BON}^+ \vdash r_1 0 = 0 \wedge (\forall x \in \mathsf{N})(x \neq 0 \to r_1 x \notin \mathsf{N})$,
(2) $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) \vdash \forall x\,(r_2 x = 0 \leftrightarrow x \in \mathsf{N})$,
(3) $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) \vdash \forall x\,(r_3 x \in \mathsf{N} \leftrightarrow x = 0)$.

PROOF. Let $\mathbf{nt}_{\mathsf{N}}$ be the closed term introduced in Corollary 2.3. Recall that $\mathsf{BON}^+$ proves $\mathbf{nt}_{\mathsf{N}}\downarrow$ and $\mathbf{nt}_{\mathsf{N}}(0) \notin \mathsf{N}$. For

$$r_1 := \lambda x.\mathbf{d}_{\mathsf{N}}(\lambda u.0, \mathbf{nt}_{\mathsf{N}}, x, 0)0 \qquad \text{and} \qquad r_2 := \lambda x.\boldsymbol{\tau}_{\mathsf{N}}(\lambda u.0, x),$$

(1) and (2) are immediately proved. For (3) consider

$$r_3 := \lambda x.\boldsymbol{\tau}_{\mathsf{N}}(r_1, x).$$

Then $x = 0$ implies $r_3 x = 0 \in \mathsf{N}$. Conversely $r_3 x \in \mathsf{N}$ yields $x \in \mathsf{N}$. Hence $r_1 x = r_3 x \in \mathsf{N}$ and thus $x = 0$. ⊣

The existence of the two closed terms $r_2$ and $r_3$ according to the previous lemma is also the core of the proof of (1) in the following theorem. As shown in Theorem 6.7, the converse directions of (2) and (3) do not hold in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

THEOREM 6.6. *In* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ *we can prove the following:*

(1) $\mathit{OSD}[U] \leftrightarrow \mathit{Dom}[U]$,
(2) $\mathit{Dom}[U] \rightarrow \mathit{Rng}[U]$,
(3) $\mathit{Pr}[U] \rightarrow \mathit{Dom}[U]$.

PROOF. (1) is easy by $r_2$ and $r_3$ from Lemma 6.5.

For (2), assume $\mathit{Dom}[U]$. Then (1) tells us $\mathit{OSD}[U]$, i.e., there exists $f$ with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow fx = 0).$$

Now we make use of Lemma 3.6. Set $t := \lambda u.f(\mathbf{proj}_0 u)$. Then obviously we have $(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(t(\mathbf{pair}(x,y)) = 0))$. By Lemma 3.6(1), there exists $g$ such that $(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists z \in \mathsf{N})(gz = x))$, implying $\mathit{Rng}[U]$.

We turn to (3). If $\mathit{Pr}[U]$ then there exists $f \in \mathit{Char}_2$ with

$$(\forall x \in \mathsf{N})(x \in U \leftrightarrow (\exists y \in \mathsf{N})(f(x,y) = 0)).$$

We first observe $fx \in (\mathsf{N} \rightarrow \mathsf{N})$ for all $x \in \mathsf{N}$ and set $t := \lambda u.\mathbf{min}(fu)$. Hence Theorem 3.2 gives us, for all $x \in \mathsf{N}$, that $tx \in \mathsf{N}$ iff $(\exists y \in \mathsf{N})(f(x,y) = 0)$. Consequently, $U$ is the domain of $t$. ⊣

Now we turn to some unprovability results that we directly obtain from the complexity results in connection with our lightface models, namely Theorems 5.6 and 5.9. In what follows, given a set $S$ of natural numbers and an operational structure $\mathfrak{M}$, we write $S^{\mathfrak{M}}$ for $\{\|\overline{n}\|_{\mathfrak{M}} : n \in S\}$.

THEOREM 6.7. *The following are <u>not</u> provable in* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$*:*

(1) $\mathit{Dom}[U] \rightarrow \mathit{Pr}[U]$,
(2) $\mathit{Rng}[U] \rightarrow \mathit{Dom}[U]$.

PROOF. To show the unprovability of (1), choose a universal $\Pi^0_1$ set $R$ of the natural numbers. $R$ is $\Sigma^0_1 \wedge \Pi^0_1$ but not $\Sigma^0_1$. In view of Theorems 5.9 and 6.6(1), the lightface graph model $\mathfrak{LG}$ satisfies $\mathit{Dom}\big[R^{\mathfrak{LG}}\big]$ but not $\mathit{Pr}\big[R^{\mathfrak{LG}}\big]$.

For establishing the unprovability of (2), we pick a universal $\Sigma^0_2$ set $S$ of the natural numbers. $S$ is $\Sigma^0_2$ but not $\Sigma^0_1 \wedge \Pi^0_1$. Then Theorem 5.9 tells us that $\mathfrak{LG}$ satisfies $\mathit{Rng}\big[S^{\mathfrak{LG}}\big]$ but not $\mathit{Dom}\big[S^{\mathfrak{LG}}\big]$. ⊣

Theorem 5.6 also gives us similar unprovability results but on $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ or on $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + \neg(\text{TOT-}\mathsf{N}) + \neg(\text{TOT-AP})$.

We summarize the results of Theorems 6.2, 6.4, 6.6, and 6.7 (together with an obvious observation) in Figure 2, where a black arrow means the provability of the corresponding implication in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ while a crossed arrow represents the unprovability in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$. It depicts the interdependencies of our (semi-)decidability notions, relative to the theory $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

Theorem 6.3 naturally leads us to be interested also in the "decidability notions", $\mathit{Pr}[U] \wedge \mathit{Pr}[\mathsf{N}\backslash U]$, $\mathit{Dom}[U] \wedge \mathit{Dom}[\mathsf{N}\backslash U]$ and $\mathit{Rng}[U] \wedge \mathit{Rng}[\mathsf{N}\backslash U]$. By Theorem 5.9 with help of Figure 1, we can similarly obtain the unprovability of the respective implications between them corresponding to Theorem 6.7, and

$$\begin{tikzcd}
OD[U] \arrow[r, bend right] \arrow[d, leftrightarrow] & Pr[U] \arrow[l, bend right] \arrow[r, bend right] \arrow[d, leftrightarrow] & Dom[U] \arrow[l, bend right] \arrow[r, bend right] \arrow[d, leftrightarrow] & Rng[U] \arrow[l, bend right] \\
OD[\mathsf{N}\backslash U] & OE[U] & OSD[U] &
\end{tikzcd}$$

FIGURE 2. Summary of our main results.

moreover the following unprovability of implications from the "decidabilities" to the "semi-decidabilities" (while the others of this type, e.g.,

$$Pr[U] \wedge Pr[\mathsf{N}\backslash U] \rightarrow Dom[U],$$

are obviously provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$, because of, e.g., Theorem 6.6(3)). In particular, in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$ or even in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$ we do not have that a relation on the natural numbers is operationally decidable iff the relation and its complement in the natural numbers are operationally semi-decidable.

THEOREM 6.8. *The following are <u>not</u> provable in* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$*:*

(1) $Dom[U] \wedge Dom[\mathsf{N}\backslash U] \rightarrow Pr[U]$,
(2) $Rng[U] \wedge Rng[\mathsf{N}\backslash U] \rightarrow Dom[U]$,
(3) $Rng[U] \wedge Rng[\mathsf{N}\backslash U] \rightarrow Pr[U]$.

How about the converses of the implications in Theorem 6.8? Trivially, they are all false in the ordinary recursion-theoretic operational structure, i.e., the variant $\mathfrak{K}_1$ of Kleene's first model where the relation variables vary over $\mathrm{Pow}(\omega)$, and hence not provable in $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-}\mathsf{N})$. However this does not show the unprovabilities on $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$. Theorems 5.6 and 5.9 do not show these unprovabilities, either. For this, we need the analogous results for the canonical term model $\mathfrak{CT}$, as follows.

LEMMA 6.9. *For any $S \subseteq \omega$, the following equivalences hold:*

$$S \text{ is } \Sigma^0_1 \iff \mathfrak{CT} \models Pr\big[S^{\mathfrak{CT}}\big] \iff \mathfrak{CT} \models Dom\big[S^{\mathfrak{CT}}\big] \iff \mathfrak{CT} \models Rng\big[S^{\mathfrak{CT}}\big].$$

PROOF. Theorem 5.1(1) yields the first $\Longrightarrow$ and (3) and (2) of Theorem 6.6 yield the second and the third. It remains to imply that $S$ is $\Sigma^0_1$ from $\mathfrak{CT} \models Rng\big[S^{\mathfrak{CT}}\big]$.

Code the closed terms by Gödel numbering. Then the relation *conv* is $\Delta^0_0$ and its congruent closure $\approx$ is $\Sigma^0_1$. Thus $\{m \in \omega : (\exists n \in \omega)(t\,\overline{n} \approx \overline{m})\}$ is $\Sigma^0_1$. $\dashv$

THEOREM 6.10. *The following are <u>not</u> provable in* $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N}) + (\text{TOT-AP})$*:*

(1) $Pr[U] \rightarrow Dom[U] \wedge Dom[\mathsf{N}\backslash U]$,
(2) $Dom[U] \rightarrow Rng[U] \wedge Rng[\mathsf{N}\backslash U]$,
(3) $Pr[U] \rightarrow Rng[U] \wedge Rng[\mathsf{N}\backslash U]$.

Table 2 summarizes our complexity results, namely Theorems 5.6 and 5.9 and Lemma 6.9. It characterizes the subsets $S$ of $\omega$ for which $S^{\mathfrak{M}}$ satisfies the semi-decidability notions in our four $\omega$-models $\mathfrak{M}$ of $\mathsf{BON}^+(\boldsymbol{\tau}_\mathsf{N})$.

| model / notion | Kleene's first $\mathfrak{K}_1$ | canonical term $\mathfrak{CT}$ | graph $\mathfrak{LG}$ | Kleene's second $\mathfrak{LK}_2$ |
|---|---|---|---|---|
| *Pr*, *OE* | $\Sigma^0_1$ | $\Sigma^0_1$ | $\Sigma^0_1$ | $\Sigma^0_1$ |
| *Dom*, *OSD* | $\Sigma^0_1$ | $\Sigma^0_1$ | $\Sigma^0_1 \wedge \Pi^0_1$ | $\Pi^0_2$ |
| *Rng* | $\Sigma^0_1$ | $\Sigma^0_1$ | $\Sigma^0_2$ | $\Sigma^0_3$ |

TABLE 2. Corresponding complexities to our semi-decidability notions.

The complexity results yield further interesting consequences. First, $\Sigma^0_1 \wedge \Pi^0_1$ is not closed under projections nor under binary unions (for otherwise $(\Sigma^0_1 \wedge \Pi^0_1) \vee \Sigma^0_1$ would be included in $\Sigma^0_1 \wedge \Pi^0_1$). Therefore, for example, $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) + (\text{TOT-AP})$ cannot prove the closure of *OSD* under binary unions, formalized as

$$(\forall x \in \mathsf{N})(x \in W \leftrightarrow x \in U \vee x \in V) \wedge \mathit{OSD}[U] \wedge \mathit{OSD}[V] \rightarrow \mathit{OSD}[W].$$

Second, since $\Pi^0_2$ is known to lack the reduction property (see, e.g., Hinman [7, III.1.10(ii)]), $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ cannot prove the operational form of the reduction property for *OSD*. (Also, since $\Sigma^0_1$ does not have the separation property, $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) + (\text{TOT-AP})$ cannot prove that of the separation property for *OSD*.) These results suggest that, despite its simple definition, the notion *OSD* as we defined is not a right operational formalization of semi-decidability, since it does not satisfy the basic properties which we expect from the word "semi-decidable".

We conclude this article with some open questions.

QUESTION 6.11. It might be interesting to ask

- if $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ proves the operational form of the reduction property of *Rng*;
- how to characterize a many-one degree $\Gamma$ for which there is an $\omega$-model $\mathfrak{M}$ of $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}})$ such that $S \in \Gamma$ iff $\mathfrak{M} \models \mathit{OSD}\left[S^{\mathfrak{M}}\right]$ for any $S \subseteq \omega$;
- if $\mathsf{BON}^+(\boldsymbol{\tau}_{\mathsf{N}}) + (\text{TOT-N})$ can prove the implications in Theorems 6.7 and 6.8.

**Acknowledgments.** This publication was made possible through the support of a grant from the John Templeton Foundation. The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the John Templeton Foundation. The work of the second author was also supported by the Swiss National Science Foundation.

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