DIMENSION INEQUALITY FOR A DEFINABLY COMPLETE UNIFORMLY LOCALLY O-MINIMAL STRUCTURE OF THE SECOND KIND

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ABSTRACT. Consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group. Let $f: X \to \mathbb{R}^n$ be a definable map, where X is a definable set and R is the universe of the structure. We demonstrate the inequality $\dim(f(X)) \leq \dim(X)$ in this paper. As a corollary, we get that the set of the points at which f is discontinuous is of dimension smaller than $\dim(X)$. We also show that the structure is defiably Baire in the course of the proof of the inequality.

1. INTRODUCTION

A uniformly locally o-minimal structure of the second kind was first defined and investigated in the author's previous work [5]. It enjoys several tame properties such as local monotonicity. In addition, it admits local definable cell decomposition when it is definably complete.

In [5], the author defined dimension of a set definable in a locally o-minimal structure admitting local definable cell decomposition. Many assertions on dimension known in o-minimal structures [2] also hold true for locally o-minimal structures admitting local definable cell decomposition which are not necessarily definably complete [5, Section 5.5]. An exception is the inequality $\dim(f(X)) \leq \dim(X)$, where $f: X \to \mathbb{R}^n$ is a definable map. The author gave an example which does not satisfy the above dimension inequality in [5, Remark 5.5]. The structure in the example is not definably complete. A question is whether the dimension inequality holds true when the structure is definably complete. This paper gives an affirmative answer to this question. Our main theorem is as follows:

Theorem 1.1. Let $\mathcal{R} = (R, <, +, 0, ...)$ be a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group. The inequality

 $\dim(f(X)) \le \dim(X)$

holds true for any definable map $f: X \to \mathbb{R}^n$.

We get the following corollary:

Corollary 1.2. Let $\mathcal{R} = (R, <, +, 0, ...)$ be the same structure as Theorem 1.1. Let $f: X \to R$ be a definable function. The set of the points at which f is discontinuous is of dimension smaller than dim(X).

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The author proved the dimension inequality in [6, Theorem 2.4] when the universe of the structure is the set of reals. This fact is not a direct corollary of the above theorem because the structure should be an expansion of an abelian group in the theorem.

The paper is organized as follows. In Section 2, we first review definitions used in the paper. We prove several basic facts in Section 3. Satisfaction of the dimension inequality is relevant to defiably Baire property introduced in [3]. Section 4 treats the definably Baire property. We show that a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group is definably Baire in the section. We finally demonstrate Theorem 1.1 in Section 5.

We introduce the terms and notations used in this paper. The term 'definable' means 'definable in the given structure with parameters' in this paper. A *CBD* set is a closed, bounded and definable set. For any set $X \subset \mathbb{R}^{m+n}$ definable in a structure $\mathcal{R} = (\mathbb{R}, \ldots)$ and for any $x \in \mathbb{R}^m$, the notation X_x denotes the fiber defined as $\{y \in \mathbb{R}^n \mid (x, y) \in X\}$. For a linearly ordered structure $\mathcal{R} = (\mathbb{R}, <, \ldots)$, an open interval is a definable set of the form $\{x \in \mathbb{R} \mid a < x < b\}$ for some $a, b \in \mathbb{R}$. It is denoted by (a, b) in this paper. We define a closed interval in the same manner and it is denoted by [a, b]. An open box in \mathbb{R}^n is the direct product of n open intervals. A closed box is defined similarly. Let A be a subset of a topological space. The notations $\operatorname{int}(A)$ and \overline{A} denote the interior and the closure of the set A, respectively. The notation |S| denotes the cardinality of a set S.

2. Definitions

We review the definitions given in the previous works. The definition of a definably complete structure is found in [8] and [1]. A locally o-minimal structure is defined and investigated in [9]. Readers can find the definitions of uniformly locally o-minimal structures of the second kind and locally o-minimal structures admitting local definable cell decomposition in [5]. We use D_{Σ} -sets introduced in [1].

Definition 2.1 (D_{Σ} -sets). Consider an expansion of a linearly ordered structure $\mathcal{R} = (R, <, 0, ...)$. A parameterized family of definable sets is the family of the fibers of a definable set. A parameterized family $\{X_{r,s}\}_{r>0,s>0}$ of CBD subsets of \mathbb{R}^n is called a D_{Σ} -family if $X_{r,s} \subset X_{r',s}$ and $X_{r,s'} \subset X_{r,s}$ whenever $r \leq r'$ and $s \leq s'$. A definable subset X of \mathbb{R}^n is a D_{Σ} -set if $X = \bigcup_{r>0,s>0} X_{r,s}$ for some

 D_{Σ} -family $\{X_{r,s}\}_{r>0,s>0}$.

A parameterized family of definable sets $\{X_s\}_{s>0}$ is a definable decreasing family of CBD sets if we have $X_s = X_{r,s}$ for some D_{Σ} -family $\{X_{r,s}\}_{r>0,s>0}$ with $X_{r_1,s} = X_{r_2,s}$ for all r_1, r_2 and s.

We next review definably Baire property introduced in [3].

Definition 2.2. Consider an expansion of a densely linearly ordered structure. A parameterized family of definable sets $\{X_r\}_{r>0}$ is called a *definable increasing* family if $X_r \subset X_{r'}$ whenever 0 < r < r'. A definably complete expansion of a densely linearly ordered structure is *definably Baire* if the union $\bigcup_{r>0} X_r$ of any definable increasing family $\{X_r\}_{r>0}$ with $\operatorname{int}(\overline{X_r}) = \emptyset$ has an empty interior.

The following proposition is a direct corollary of the local definable cell decomposition theorem [5, Theorem 4.2].

Proposition 2.3. Consider a definably complete uniformly locally o-minimal structure of the second kind. It is definably Baire if and only if the union $\bigcup_{r>0} X_r$ of any definable increasing family $\{X_r\}_{r>0}$ with $int(X_r) = \emptyset$ has an empty interior.

Proof. Because int $(\overline{X_r}) \neq \emptyset$ iff $\operatorname{int}(X_r) \neq \emptyset$ iff X_r contains an open cell in this case by [5, Theorem 4.2]. \square

The dimension of a set definable in a locally o-minimal structure admitting local definable cell decomposition is defined in [5, Section 5]. We get the following lemma on the dimension of the projection image. A lemma similar to it is found in [6], but we give a complete proof here.

Lemma 2.4. Consider a locally o-minimal structure $\mathcal{R} = (R, <, ...)$ admitting local definable cell decomposition. Let X be a definable subset of \mathbb{R}^{m+n} and $\pi: \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ R^m be a coordinate projection. Assume that the fibers X_x are of dimension ≤ 0 for all $x \in \mathbb{R}^m$. Then, we have dim $X \leq \dim \pi(X)$.

Proof. For any $(a,b) \in \mathbb{R}^m \times \mathbb{R}^n$, there exist open boxes $B_a \subset \mathbb{R}^m$ and $B_b \subset \mathbb{R}^n$ with $(a, b) \in B_a \times B_b$ and $\dim(X \cap (B_a \times B_b)) = \dim \pi(X \cap (B_a \times B_b))$ by [5, Lemma 5.4]. We have dim $\pi(X \cap (B_a \times B_b)) \leq \dim \pi(X)$ by [5, Lemma 5.1]. On the other $\dim(X \cap (B_a \times B_b))$ by [5, Corollary 5.3]. We hand, we have $\dim(X) =$ \sup $_{(a,b)\in \hat{R^m}\times R^n}$

have finished the proof.

3. Preliminaries

From now on, we consider a definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group $\mathcal{R} = (R, <)$ $, +, 0, \ldots$). We demonstrate several basic facts in this section.

Lemma 3.1. Let X be a bounded definable set. There exists a definable decreasing family of CBD sets $\{X_s\}_{s>0}$ with $X = \bigcup_{s>0} X_s$.

Proof. We demonstrate the lemma by the induction on $d = \dim(X)$. When d = 0, X is discrete and closed by [5, Corollary 5.3]. We have only to set $X_s = X$ for all s > 0 in this case.

We next consider the case in which d > 0. Let ∂X denote the frontier of X. We have dim $\partial \overline{X} < d$ by [5, Theorem 5.6]. We get dim $(X \cap \partial \overline{X}) < d$ by [5, Proposition 5.1]. There exists a definable decreasing family of CBD sets $\{Y_s\}_{s>0}$ with $X \cap \overline{\partial X} = \bigcup_{s>0} Y_s$ by the induction hypothesis. Set $Z_s = \{x \in \overline{X} \mid d(x, \overline{\partial X}) \geq 0\}$ s} for all s > 0, where the notation $d(x, \overline{\partial X})$ denotes the distance of the point x to the set $\overline{\partial X}$. They are CBD. It is obvious that $\bigcup_{s>0} Z_s = \overline{X} \setminus \overline{\partial X} = X \setminus \overline{\partial X}$. Set $X_s = Y_s \cup Z_s$. The family $\{X_s\}_{s>0}$ is a definable decreasing family we are looking for. \square

Lemma 3.2. Any definable set X is a D_{Σ} -set. That is, there exists a D_{Σ} -family ${X_{r,s}}_{r>0,s>0}$ with $X = \bigcup_{r>0,s>0} X_{r,s}$.

Proof. Let X be a definable subset of \mathbb{R}^n . Set $X_r = X \cap [-r, r]^n$. We can construct subsets $X_{r,s}$ of X_r satisfying the condition in the same manner as the proof of Lemma 3.2. We omit the details.

Lemma 3.3. Let X be a bounded definable set and $\{X_s\}_{s>0}$ be a definable decreasing family of CBD sets with $X = \bigcup_{s>0} X_s$. The CBD set X_s has a nonempty interior for some s > 0 if X has a nonempty interior.

Proof. We prove the lemma following the same strategy as the proof of [1, 3.1]. Let X be a definable subset of \mathbb{R}^n . We prove the lemma by the induction on n. We first consider the case in which n = 1. Assume that $\operatorname{int}(X_s) = \emptyset$ for all s > 0. Fix an arbitrary point $a \in \mathbb{R}$. There exist a positive integer N, an interval I with $a \in I$ and t > 0 such that, for any 0 < s < t, $I \cap X_s$ contains an open interval or consists of at most N points by [5, Theorem 4.2]. The sets $I \cap X_s$ consist of at most N points because $\operatorname{int}(X_s) = \emptyset$. We get $|X \cap I| = \left| \bigcup_{s>0} (I \cap X_s) \right| \leq N$. In particular, X has an empty interior.

We next consider the case in which n > 1. Assume that X has a nonempty interior. We show that the definable set X_s has a nonempty interior for some s > 0. A closed box $B = C \times I \subset \mathbb{R}^{n-1} \times \mathbb{R}$ is contained in X. We have $B = \bigcup_{s>0} (B \cap X_s)$. Hence, we may assume that X is a closed box B without loss of generality.

Shrinking B if necessary, we may assume that the fiber $(X_s)_x$ consists of at most M points and N closed intervals for some M > 0, N > 0 and any sufficiently small s > 0 and $x \in C$ by [5, Theorem 4.2]. Set $I = [c_1, c_2]$. Take 2N distinct points in the open interval (c_1, c_2) , say b_1, \ldots, b_{2N} . We may assume that $b_i < b_j$ whenever i < j. Set $b_0 = c_1$ and $b_{2N+1} = c_2$. Put $I_j = [b_{j-1}, b_j]$ for all $1 \le j \le 2N + 1$.

Consider the sets $Y_s^k = \{x \in C \mid I_k \subset (X_s)_x\}$ for all $x \ge j \ge 2N+1$. They are CBD. Therefore, $\{\bigcup_{k=1}^{2N+1} Y_s^k\}_{s>0}$ is a definable decreasing family of CBD sets. We demonstrate that $C = \bigcup_s \bigcup_{k=1}^{2N+1} Y_s^k$. Let $x \in C$ be fixed. We have only to show that $I_k \subset (X_s)_x$ for some k and s. For any k, there exists $s_k > 0$ such that $\operatorname{int}(I_k \cap (X_{s_k})_x) \neq \emptyset$ by the induction hypothesis because $\{I_k \cap (X_s)_x\}_{s>0}$ is a decreasing family of CBD sets with $I_k = \bigcup_s I_k \cap (X_s)_x$. Take $s = \min\{s_k \mid 1 \le k \le 2N+1\}$. We have $\operatorname{int}(I_k \cap (X_s)_x) \neq \emptyset$ for all $1 \le k \le 2N+1$. Assume that $I_k \not\subset (X_s)_x$ for all k. A maximal closed interval in $(X_s)_x$ should be contained in $I_k, I_k \cup I_{k+1}$ or $I_{k-1} \cup I_k$ for some k. Therefore, $\operatorname{int}(I_j \cap (X_s)_x)$ is empty for some $1 \le j \le 2N+1$. Contradiction. We have proven that $I_k \subset (X_s)_x$ for some k and s.

 $1 \leq j \leq 2N+1$. Contradiction. We have proven that $I_k \subset (X_s)_x$ for some k and s. Apply the induction hypothesis to $C = \bigcup_{s>0} \bigcup_{k=1}^{2N+1} Y_s^k$. The set $\bigcup_{k=1}^{2N+1} Y_s^k$ has a nonempty interior for some s > 0. The CBD set Y_s^k has a nonempty interior for some k by [5, Theorem 3.3]. The CBD set X_s has a nonempty interior because $I_k \times Y_s^k$ is contained in X_s .

Lemma 3.4. Assume that \mathcal{R} is definably Baire. Let X be a definable set and $\{X_{r,s}\}_{r>0,s>0}$ be a D_{Σ} -family with $X = \bigcup_{r>0,s>0} X_{r,s}$. The CBD set $X_{r,s}$ has a nonempty interior for some r > 0 and s > 0 if X has a nonempty interior.

Proof. Let X be a definable subset of R^n . Set $X'_{r,s} = X_{r,s} \cap [-r,r]^n$. We have $X = \bigcup_{r>0,s>0} X'_{r,s}$. We may assume that $X_r = \bigcup_{r>0,s>0} X_{r,s}$ is bounded considering $X'_{r,s}$ instead of $X_{r,s}$. The lemma is now immediate from Proposition 2.3 and Lemma 3.3.

4. On definably Baire property

We demonstrate that the structure \mathcal{R} is definably Baire.

Lemma 4.1. Let X be a bounded definable subset of \mathbb{R}^{n+1} . Set

 $S = \{ x \in \mathbb{R}^n \mid X_x \text{ contains an open interval} \}.$

The set S has an empty interior if X has an empty interior.

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Proof. Assume that S has a nonempty interior. There exists a definable decreasing family of CBD sets $\{X_s\}_{s>0}$ with $X = \bigcup_{s>0} X_s$ by Lemma 3.1. Set $S_s = \{x \in \mathbb{R}^n \mid \exists t \in \mathbb{R}, [t-s,t+s] \subset (X_s)_x\}$ for all s > 0. They are CBD by [8, Lemma 1.7] because they are the projection images of the CBD sets $S_s = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid [t-s,t+s] \subset (X_s)_x\}$. We have $S = \bigcup_{s>0} S_s$. In fact, it is obvious that $\bigcup_{s>0} S_s \subset S$ by the definition. Take a point $x \in S$. There exist $t \in \mathbb{R}$ and $s_1 > 0$ with $[t-s_1,t+s_1] \subset X_x$. In particular, we have $\operatorname{int}(X_x) \neq \emptyset$. We have $\operatorname{int}(X_{s_2})_x \neq \emptyset$ for some $s_2 > 0$ by Lemma 3.3. We may assume that $[t-s_1,t+s_1] \subset (X_{s_2})_x$ by taking new s_1 and t again. Set $s = \min\{s_1,s_2\}$, then we have $x \in S_s$. We have demonstrated that $S = \bigcup_{s>0} S_s$.

Again by Lemma 3.3, we have $int(S_s) \neq \emptyset$ for some s > 0. We obtain $int(X_s) \neq \emptyset$ by [1, 2.8(2)]. We get $int(X) \neq \emptyset$.

We reduce to the one-dimensional case.

Lemma 4.2. The structure \mathcal{R} is definably Baire if the union $\bigcup_{r>0} S_r$ of any definable increasing family $\{S_r\}_{r>0}$ of subsets of R has an empty interior whenever S_r have empty interiors for all r > 0.

Proof. Let $\{X_r\}_{r>0}$ be a definable increasing family of subsets of \mathbb{R}^n . Set $X = \bigcup_{r>0} X_r$. We have only to show that the definable set X_r has a nonempty interior for some r > 0 if X has a nonempty interior. The definable set X contains a bounded open box B. We may assume that X is a bounded open box B without loss of generality by considering B and $\{X_r \cap B\}_{r>0}$ in place of X and $\{X_r\}_{r>0}$, respectively.

We prove the lemma by the induction on n. The lemma is obvious when n = 0. We next consider the case in which n > 0. We lead to a contradiction assuming that X_r have empty interiors for all r > 0. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection forgetting the last coordinate. We have $B = B_1 \times I$ for some open box B_1 in \mathbb{R}^{n-1} and some open interval I. Consider the set $Y_r = \{x \in B_1 \mid \text{ the fiber } (X_r)_x \text{ contains an open interval} \}$ for all r > 0. They have empty interiors by Lemma 4.1. The union $\bigcup_{r>0} Y_r$ has an empty interior by the induction hypothesis. In particular, we have $B_1 \neq \bigcup_{r>0} Y_r$ and we can take a point $x \in B_1 \setminus (\bigcup_{r>0} Y_r)$. Since $x \notin \bigcup_{r>0} Y_r$, the fiber $(X_r)_x$ does not contain an open interval for any r > 0. Therefore, the union $\bigcup_{r>0} (X_r)_x$ has an empty interior by the assumption. On the other hand, we have $I = \bigcup_{r>0} (X_r)_x$ because $B = \bigcup_{r>0} X_r$. It is a contradiction.

We prove that \mathcal{R} is definably Baire now.

Theorem 4.3. A definably complete uniformly locally o-minimal expansion of the second kind of a densely linearly ordered abelian group is definably Baire.

Proof. Let $\mathcal{R} = (R, <, +, 0, ...)$ be the considered structure. Let $\{X_r\}_{r>0}$ be a definable increasing family of subsets of R. Set $X = \bigcup_{r>0} X_r$. We have only to show that the definable set X has an empty interior if X_r have empty interiors for all r > 0 by Lemma 4.2. Note that X_r are discrete and closed because the structure is locally o-minimal.

Assume that X has a nonempty interior. The definable set X contains an open interval. Take a point a contained in the open interval. Consider the definable function $f : \{r \in R \mid r > 0\} \rightarrow \{x \in R \mid x > a\}$ defined by $f(r) = \inf\{x > a \mid x \in X_r\}$. It is obvious that f is a decreasing function because $\{X_r\}_{r>0}$ is a definable increasing family. We demonstrate that $\lim_{r\to\infty} f(r) = a$. Let b be an arbitrary point sufficiently close to a with b > a. Since $X = \bigcup_{r>0} X_r$ contains a neighborhood of a, there exists a positive element $r \in R$ with $b \in X_r$. We have $a < f(r) \le b$ by the definition of f. We have shown that $\lim_{r\to\infty} f(r) = a$.

Consider the image $\operatorname{Im}(f)$ of the function f. Take a sufficiently small open interval I containing the point a with $I \subset X$. The intersection $I \cap \operatorname{Im}(f)$ is a finite union of points and open intervals because it is definable in the locally o-minimal structure \mathcal{R} . Take an arbitrary point $b \in \operatorname{Im}(f)$ and a point r > 0 with b = f(r). Since X_r is closed, we have $b \in X_r$. Any point $b' \in \operatorname{Im}(f)$ with b' > b is also contained in X_r . In fact, take a point r' > 0 with b' = f(r'). If r' > r, the set $X_{r'}$ contains the point b because $X_r \subset X_{r'}$. We have $b' = f(r') \leq b$ by the definition of the function f. It is a contradiction. If r' < r, we have $b' \in X_{r'} \subset X_r$.

Set $b_1 = \inf\{b' \in \operatorname{Im}(f) \mid b' > b\}$. We have $b_1 \in X_r$ and $b_1 > b$ because $\{b' \in \operatorname{Im}(f) \mid b' > b\} \subset X_r$ and X_r is closed and discrete. The open interval (b, b_1) has an empty intersection with $\operatorname{Im}(f)$. We have shown that $I \cap \operatorname{Im}(f)$ does not contain an open interval. The set $I \cap \operatorname{Im}(f)$ consists of finite points. It is a contradiction to the fact that $\lim_{r\to\infty} f(r) = a$.

Remark 4.4. It is already known that a definably complete expansion of an ordered field is definably Baire [7]. Our research target is a uniformly locally o-minimal structure of the second kind. A uniformly locally o-minimal expansion of the second kind of an ordered field is o-minimal by [5, Proposition 2.1]. In this case, it is trivially definably Baire by the definable cell decomposition theorem [2, Chapter 3, (2,11)]. We have more interest in the case in which the structure is not an expansion of an ordered field.

5. Proof of Theorem 1.1

We demonstrate Theorem 1.1 in this section. We first show that a definable map is continuous on an open subset of the domain of definition.

Lemma 5.1. A definable map $f: U \to R^n$ defined on an open set U is continuous on a nonempty definable open subset of U.

Proof. The structure \mathcal{R} is definably Baire by Theorem 4.3. We may use Lemma 3.4 in the proof.

Let U be a definable open subset of \mathbb{R}^m . Consider the projection $\pi : \mathbb{R}^{m+n} \to \mathbb{R}^m$ onto the first m coordinates. The notation $\Gamma(f)$ denotes the graph of f. There exists a D_{Σ} -family $\{X_{r,s}\}_{r,s}$ with $\Gamma(f) = \bigcup_{r,s} X_{r,s}$ by Lemma 3.2. Note that $\pi(X_{r,s})$ is CBD by [8, Lemma 1.7]. We have $U = \bigcup_{r,s} \pi(X_{r,s})$ and the fiber $\pi^{-1}(x) \cap \Gamma(f)$ is a singleton for any $x \in U$. Therefore, we obtain $X_{r,s} = \Gamma(f|_{\pi(X_{r,s})})$, where $f|_{\pi(X_{r,s})}$ is the restriction of f to $\pi(X_{r,s})$. Take a closed box B contained in U. The family $\{\pi(X_{r,s}) \cap B\}$ is a D_{Σ} -family and $B = \bigcup_{r,s} \pi(X_{r,s}) \cap B$. The CBD set $\pi(X_{r,s}) \cap B$ has a nonempty interior for some r and s by Lemma 3.4. Take a closed box B'contained in $\pi(X_{r,s}) \cap B$. The set $X_{r,s} \cap (B' \times \mathbb{R}^n) = \Gamma(f|_{B'})$ is closed. Therefore, f is continuous on $\operatorname{int}(B')$.

We finally prove Theorem 1.1.

Proof of Theorem 1.1. We prove the following assertion:

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(*): The inequality dim $(f(X)) \leq \dim(X)$ holds true for any definable map $f: X \to \mathbb{R}^n$.

Lemma 3.4 is available as in the proof of Lemma 5.1 for the same reason.

Set $d = \dim(f(X))$. We demonstrate that $\dim(X) \ge d$. We can reduce to the case in which the image f(X) is an open box B of dimension d. In fact, there exist an open box B in \mathbb{R}^d and a definable map $g: B \to f(X)$ such that the map g is a definable homeomorphism onto its image by the definition of dimension [5, Definition 5.1]. Set $Y = f^{-1}(g(B))$ and $h = g^{-1} \circ f|_Y : Y \to B$. When $\dim(Y) \ge d$, we get $\dim(X) \ge d$ by [5, Lemma 5.1] because Y is a subset of X. We may assume that f(X) = B by considering Y and h instead of X and f, respectively.

We next reduce to the case in which the map f is the restriction of a coordinate projection. Consider the graph $G = \Gamma(f) \subset R^{m+d}$ of the definable map f. Let $\pi: R^{m+d} \to R^d$ be the projection onto the last d coordinates. We have $\dim(X) \ge \dim(G) \ge d$ by Lemma 2.4 when $\dim(G) \ge d$. We may assume that $f: X \to B$ is the restriction of the projection $\pi: R^{m+d} \to R^d$ to X.

We have a D_{Σ} -family $\{X_{r,s}\}_{r>0,s>0}$ with $X = \bigcup_{r,s} X_{r,s}$ by Lemma 3.2. The family $\{f(X_{r,s})\}_{r>0,s>0}$ is also a D_{Σ} -family by [8, Lemma 1.7] because f is the restriction of a projection. We have $B = \bigcup_{r,s} f(X_{r,s})$. The CBD set $f(X_{r,s})$ has a nonempty interior for some r > 0 and s > 0 by Lemma 3.4. We fix such r > 0and s > 0. Take an open box U contained in $f(X_{r,s})$. Note that the inverse image $\{y \in X_{r,s} \mid f(y) = x\}$ of $x \in U$ is CBD because f is continuous. Consider a definable function $\varphi: U \to X_{r,s}$ given by $\varphi(x) = \operatorname{lexmin}\{y \in X_{r,s} \mid f(y) = x\}$, where the notation lexmin denotes the lexicographic minimum defined in [8]. We can get an open box V contained in U such that the restriction $\varphi|_V$ of φ to Vis continuous by Lemma 5.1. The definable set $X_{r,s}$ is of dimension $\geq d$ by the definition of dimension because it contains the graph of the definable continuous map $\varphi|_V$ defined on the open box V in \mathbb{R}^d . We have dim $X \geq \dim(X_{r,s}) \geq d$ by [5, Lemma 5.1]. We have proven Theorem 1.1.

The proof of Corollary 1.2 is the same as that of [6, Corollary 2.6]. We give a proof here because it is brief.

Proof of Corollary 1.2. Let \mathcal{D} be the set of points at which the definable function f is discontinuous. Assume that the domain of definition X is a definable subset of \mathbb{R}^m . Let G be the graph of f. We have $\dim(G) = \dim(X)$ by Lemma 2.4 and Theorem 1.1. Set $\mathcal{E} = \{(x, y) \in X \times \mathbb{R} \mid y = f(x) \text{ and } f \text{ is discontinuous at } x\}$. We get $\dim(\mathcal{E}) < \dim(G)$ by [5, Theorem 4.2, Corollary 5.3]. Let $\pi : \mathbb{R}^{m+1} \to \mathbb{R}^m$ be the projection forgetting the last coordinate. We have $\mathcal{D} = \pi(\mathcal{E})$ by the definitions of \mathcal{D} and \mathcal{E} . We finally obtain $\dim(\mathcal{D}) = \dim(\pi(\mathcal{E})) \leq \dim(\mathcal{E}) < \dim(G) = \dim(X)$ by Theorem 1.1.

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