

The reverse mathematics of theorems of Jordan and Lebesgue

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Abstract

The Jordan decomposition theorem states that every function $f: [0, 1] \rightarrow \mathbb{R}$ of bounded variation can be written as the difference of two non-decreasing functions. Combining this fact with a result of Lebesgue, every function of bounded variation is differentiable almost everywhere in the sense of Lebesgue measure. We analyse the strength of these theorems in the setting of reverse mathematics. Over RCA_0 , a stronger version of Jordan's result where all functions are continuous is equivalent to ACA_0 , while the version stated is equivalent to WKL_0 . The result that every function on $[0, 1]$ of bounded variation is almost everywhere differentiable is equivalent to WWKL_0 . To state this equivalence in a meaningful way, we develop a theory of Martin-Löf randomness over RCA_0 .

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1 Introduction

A main topic of reverse mathematics is to determine the axiomatic strength of theorems from classical analysis. For instance, the base system RCA_0 proves the intermediate value theorem. Over RCA_0 , the fact that every continuous real function on $[0, 1]$ is uniformly continuous is equivalent to the system WKL_0 , while the Bolzano-Weierstrass theorem (every bounded sequence of reals has a convergent subsequence) is equivalent to the stronger system ACA_0 (see [11, Thms. II.6.6, IV.2.3 and III.3.2], respectively).

Our purpose is to determine the strength of two important, interrelated theorems from analysis. Interpreting these theorems over RCA_0 necessitates to develop some theory of representations of functions and of Martin-Löf randomness over this weak base system.

1.1 The axiomatic strength of Jordan's decomposition theorem

Jordan's theorem, dating from 1879, states that every function $f: [0, 1] \rightarrow \mathbb{R}$ of bounded variation can be written as $g - h$ where g and h are nondecreasing functions (see e.g. [2] for background on real analysis). One calls the pair g, h a *Jordan decomposition* of f . In the setting of real analysis, the proof that a Jordan decomposition exists is simple: let $g(x)$ be the variation of f from 0 to x , and let $h = g - f$. However, even if f is computable in the usual sense of computable analysis, the function g is not necessarily computable: the variation of f , which equals $g(1)$, can be any non-negative left-c.e. real by Rettinger and Zheng [16, Thm. 3.3.(ii)]. They also give in Thm. 5.3 an example of a computable function of bounded variation without any computable Jordan decomposition. Since the computable sets form a model of RCA_0 , it follows that Jordan's theorem cannot be proved in RCA_0 .

Our first main topic is to determine the strength of Jordan's theorem. It turns out that its strength depends on which functions we admit in a decomposition. The version where all functions involved are continuous is equivalent to ACA_0 . The version where the non-decreasing functions g, h in the decomposition can be discontinuous is equivalent to WKL_0 . For the second version, we need to develop a theory of representing such functions g, h in models of RCA_0 . In Definition 4.1 we introduce rational presentations of functions, which broadly speaking provide information about all possible inequalities $g(p) < q$ and $g(p) > q$ for rationals p, q , while leaving open equalities.

Greenberg, Miller and Nies [5, Thm. 1.4 and Section 2.3], going back to unpublished work with Slaman, built a computable function of bounded variation such that any continuous Jordan decomposition computes the halting problem, and every Jordan decomposition allowing discontinuity computes a completion of Peano arithmetic. To prove some of our results above, we adapt their methods to the setting of reverse mathematics. This will require considerable additional effort.

1.2 The axiomatic strength of Lebesgue's theorem on a.e. differentiability

Lebesgue [7] proved that every nondecreasing function f is almost everywhere differentiable. By Jordan's theorem, it follows that the same conclusion holds for functions of bounded variation. See e.g. [2, Thm. 20.6 and Cor. 20.7].

Our second main topic is the strength of this theorem and of its corollary. We show that with reasonable interpretations of "almost everywhere" and "differentiable" that work over RCA_0 , both are equivalent to weak weak König's Lemma WWKL_0 introduced by Simpson and Yu [15],

which roughly speaking states that every tree of positive measure has a path. Showing this requires recasting a fair amount of the methods of Brattka, Miller and Nies [1] over RCA_0 . In one important place they used Σ_2^0 -bounding (in the form of the infinitary pigeon hole principle), which is not allowed in RCA_0 . So we have to circumvent this. To get around the fact that a computable function of bounded variation may not have a computable Jordan decomposition, they use a set computing a completion of Peano arithmetic, and relativize randomness to it. Since such sets are unavailable within RCA_0 , in Lemma 6.5 we will instead recast this idea using an argument of Simpson and Yokoyama [13]. They extend a model of WWKL_0 to a model of WKL_0 in a restrictive way, in that for each of the new sets A , some set in the given model is random relative to A . This is one of the few examples from earlier years where methods stemming from the algorithmic theory of randomness have been reviewed with the mindset of reverse mathematics.

It is interesting that of our two topics, proving Jordan decomposability requires the stronger systems, even though differentiation appears to be a more complex operation than taking a Jordan decomposition. In fact when we say that f is differentiable at z we cannot assert that the limit of slopes around z exists in the model of RCA_0 , as this would be equivalent to ACA_0 when considering suitable functions. To get around this we work with the concept of pseudo-differentiability going back to Demuth [3]: f is pseudo-differentiable at z if the slopes get closer and closer to each other as one zooms in on z (similar to the case of Cauchy sequences). If f is continuous at z and pseudo-differentiable at z , then f is differentiable at z (but the value of the derivative at z may still not exist in the model).

We mention that Shafer and the first author [9] have recently looked at further connections between reverse mathematics and randomness. They consider randomness notions for infinite bit sequences. For instance, they study the reverse mathematical content of a well-known result: the characterization of 2-randomness of a bit sequence Z via the plain Kolmogorov complexity of initial segments $Z \upharpoonright n$.

2 Preliminaries

Effectively uniformly continuous functions

We make the following definitions within RCA_0 , borrowing terminology from computable analysis. An *effectively* uniformly continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is presented by a *Cauchy name*: a sequence $(f_s)_{s \in \mathbb{N}}$ of rational polynomials (or, alternatively, polygonal functions with rational break-points) such that $\|f_s - f_r\|_\infty \leq 2^{-s}$ for all $r > s$. The sequence $(f_s)_{s \in \mathbb{N}}$ is intended to describe $f = \lim_{s \rightarrow \infty} f_s$. Within RCA_0 this definition is equivalent to the definition of continuous functions with a modulus of uniform continuity given in [11, Def. IV.2.1]. Note that a uniformly continuous function may not have a modulus of uniform continuity within RCA_0 . In contrast, within RCA_0 a continuous function with a Cauchy name always has a modulus of uniform continuity, and vice versa.

Functions of bounded variation

Suppose that $\Pi = \{t_0, \dots, t_n\}$ is a partition of an interval $[a, b]$, *i.e.*, $a = t_0 < t_1, \dots, t_n = b$ (abbreviated by $\Pi \triangleleft [a, b]$). We let

$$V(f, \Pi) = \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|.$$

We say that a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is of *bounded variation* if there is $k \in \mathbb{N}$ such that $V(f, \Pi) \leq k$ for every partition Π of $[0, 1]$. We define bounded variation in this way in order to avoid declaring that the supremum exists. We write $\mathbf{v}_f(t) = \sup_{\Pi \triangleleft [0, t]} V(f, \Pi)$, and $\mathbf{v}_f = \mathbf{v}_f(1)$ in case the sup exists. For a given rational number $q \in \mathbb{Q}$, we will use the assertion “ $\mathbf{v}_f(t) \leq q$ ” in the sense above. It can be expressed by a Π_1^0 formula independent of the sup exists.

3 Jordan decomposition for effectively uniformly continuous BV functions

Jordan’s theorem states that for every function f of bounded variation there is a pair of non-decreasing functions g, h , called a *Jordan decomposition*, such that $f = g - h$. For functions $f, g : [0, 1] \rightarrow \mathbb{R}$, write

$$f \leq_{\text{slope}} g \text{ iff } \forall x \forall y [x < y \rightarrow f(y) - f(x) \leq g(y) - g(x)];$$

i.e., the slopes of g are at least as big as the slopes of f . Finding a Jordan decomposition of f is equivalent to finding a non-decreasing function g such that $f \leq_{\text{slope}} g$: If $f = g - h$ for non-decreasing functions g, h , then $f \leq_{\text{slope}} g$. Conversely, if $f \leq_{\text{slope}} g$ for a non-decreasing function g , then $h = g - f$ is nondecreasing and $f = g - h$.

We consider a strong version of the Jordan decomposition theorem: the principle $\text{Jordan}_{\text{cont}}$, which states that for every continuous function f of bounded variation, there exist non-decreasing effectively uniformly continuous functions $g, h : [0, 1] \rightarrow \mathbb{R}$ such that $f = g - h$. Equivalently, there is a non-decreasing effectively uniformly continuous function $g : [0, 1] \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}} g$.

Theorem 3.1. *The following are equivalent over RCA_0 .*

1. ACA_0
2. $\text{Jordan}_{\text{cont}}$
3. *For every effectively uniformly continuous function f of bounded variation, there exist non-decreasing continuous functions $g, h : [0, 1] \rightarrow \mathbb{R}$ such that $f = g - h$.*

Proof. To show $1 \Rightarrow 2$, given a continuous function f of bounded variation, we construct a code for a continuous function \mathbf{v}_f . Note that within ACA_0 , $\mathbf{v}_f(t)$ always exists, and one can describe the function $t \mapsto \mathbf{v}_f(t)$ by an arithmetical formula. Thus, one can easily construct a code for \mathbf{v}_f by arithmetical comprehension. Then $g = \mathbf{v}_f$ is the desired function.

$2 \Rightarrow 3$ is trivial. To show $3 \Rightarrow 1$, let

$$q_n = 1 - 2^{-n-1}, \text{ and } q_{n,s} = q_n - 2^{-n-s-1}.$$

ACA_0 is equivalent to the following: if $h : \mathbb{N} \rightarrow \mathbb{N}$ is an injective function, then the range of h exists [11, Lemma III.1.3]. The plan is to encode the range of h into the variation of an effectively uniformly continuous function f .

For $v \in \mathbb{R}^+$ and $r \in \mathbb{N}$, we let $M_A(v, r)$ denote a “sawtooth” function on the interval A with r many teeth of height v . Given an injective function h , for each $s \in \mathbb{N}$, define a continuous function f_s as follows. On each interval of the form $I_k = [q_{h(k),k}, q_{h(k),k+1}]$ put

$$f_s = \begin{cases} M_{I_k}(2^{-k}, 2^{k-h(k)}) & \text{if } s \geq k > h(k), \\ 0 & \text{otherwise.} \end{cases}$$

Let $f_s = 0$ elsewhere. Note that on I_k , $f_s = f_t$ if $t > s \geq k$ or $k > t > s$, and $\|f_s - f_t\|_\infty \leq 2^{-s}$ if $t \geq k > s$. Thus, the sequence $(f_s)_{s \in \mathbb{N}}$ defines an effectively uniformly continuous function $f = \lim_{s \rightarrow \infty} f_s$. We show that f is of bounded variation with bound 1. Note that we only need to examine the variation of f on the disjoint intervals $[q_{h(k),k}, q_{h(k),k+1}]$ since $f = 0$ elsewhere.

Let $m \in \mathbb{N}$. For $k \in \{0, \dots, m\}$, let Π_k partition I_k . We estimate the variation of f on the interval $\bigcup_{k \leq m} I_k$. Without loss of generality we may assume that each partition contains the midpoints and endpoints of the sawteeth defined on I_k .¹ This allows us to easily compute the variation of f as the piece-wise combination of non-decreasing functions. For all $s \geq m$ one has

$$\sum_{k=0}^m V(f, \Pi_k) = \sum_{k=0}^m V(f_s, \Pi_k) \leq \sum_{k=0}^m 2^{-h(k)+1} < 1,$$

which establishes the desired bound.

By $\text{Jordan}_{\text{cont}}$, take $g : [0, 1] \rightarrow \mathbb{R}$ non-decreasing and continuous such that $f \leq_{\text{slope}} g$. Given that the range of h is encoded in the variation of f , we will use the (easily computable) variation of g on the interval $[q_{n,k}, q_{n,k+1}]$ to bound to possible pre-images of n under h .

Define a Δ_1^0 definable function $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ such that $g(q_n) - g(q_{n,\gamma(n)}) < 2^{-n}$ as follows. There is a Σ_0^0 formula $\theta(n, m, k)$ such that

$$\exists m \theta(n, m, k) \leftrightarrow g(q_n) - g(q_{n,k}) < 2^{-n}.$$

Since g is continuous and $\lim_{s \rightarrow \infty} q_{n,s} = q_n$ one has $\forall n \exists k \exists m \theta(n, m, k)$. As this sentence is Π_2^0 , given any instance of the variable n one can effectively obtain a witness $\langle m, k \rangle$ for the Σ_1^0 formula $\exists k \exists m \theta(n, m, k)$ (see, e.g. [11, Theorem II.3.5]). Thus we may put $\gamma(n) = k$, where $\langle m, k \rangle$ is least such that $\theta(n, m, k)$ holds.

Now if $h(k) = n < k$ then by the monotonicity of g ,

$$g(q_n) - g(q_{n,k}) \geq g(q_{n,k+1}) - g(q_{n,k}).$$

Let Π be a partition of $[q_{n,k}, q_{n,k+1}]$ containing the endpoints and midpoints of each sawtooth defined on that interval. Then since $g - f \leq g$ and the variation of an increasing function is the difference of its values at its endpoints one has

$$2^{-n+1} = V(f, \Pi) = V(g - (g - f), \Pi) \leq V(g, \Pi) + V(g - f, \Pi) \leq 2(g(q_{n,k+1}) - g(q_{n,k})).$$

Thus $g(q_n) - g(q_{n,k}) \geq 2^{-n}$, and then $k < \gamma(n)$. Hence

$$n \in \text{rng}(h) \leftrightarrow \exists k < \max\{\gamma(n), n + 1\} [h(k) = n],$$

so the range of h exists by Δ_1^0 comprehension. □

4 Jordan decomposition for BV functions of rational domain

In the foregoing section, we required that a Jordan decomposition consist of effectively uniformly continuous functions. Then the Jordan decomposition theorem has the same axiomatic strength as ACA_0 . To see this, we encoded the range of an injective function h into the variation of a function f

¹Indeed, this only refines the partition and provides an improved estimate.

of bounded variation. A Jordan decomposition of f into uniformly continuous functions allowed us to recover enough information to decide whether some number was the image of another under the injective function h .

We now relax the requirement on the Jordan decomposition by only stipulating that the decomposition is given by functions which are defined on the rationals. Such functions can be represented by finite strings that cumulatively describe the behaviour of the function at each rational. We will see that such simple objects do not allow the encoding of sets of high complexity.

Greenberg, Miller and Nies [5] proved that there is a computable function f on $[0, 1]$ of bounded variation such that every Jordan decomposition of f in this weak sense is PA-complete. One direction of our argument, $4 \Rightarrow 1$ of Theorem 4.10, is based on their proof; extra effort is required to make it work over RCA_0 as a base theory.

4.1 Rational presentations of functions

Let $[0, 1]_{\mathbb{Q}} := [0, 1] \cap \mathbb{Q}$. We present a function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ by a set $Z \subseteq [0, 1]_{\mathbb{Q}} \times \mathbb{Q}$ in the following way. We require that $(p, q) \in Z$ if $g(p) < q$, and $(p, q) \notin Z$ if $g(p) > q$. We leave open whether $(p, q) \in Z$ in case that $g(p) = q$. The formal definition follows.

Definition 4.1. A set $Z \subseteq [0, 1]_{\mathbb{Q}} \times \mathbb{Q}$ is called a *rational presentation* if

- (i) for any $p \in [0, 1]_{\mathbb{Q}}$, there exist $q, q' \in \mathbb{Q}$ such that $(p, q) \in Z$ and $(p, q') \notin Z$, and
- (ii) for any $p \in [0, 1]_{\mathbb{Q}}$ and for any $q, q' \in \mathbb{Q}$ with $q < q'$, $(p, q) \in Z$ implies $(p, q') \in Z$.

A rational presentation Z determines a function $g_Z : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ via

$$g_Z(p) = \inf\{q \in \mathbb{Q} : (p, q) \in Z\}.$$

We say that Z is a rational presentation of g_Z (and also of any function on $[0, 1]$ extending g_Z).

One can determine $g_Z(p)$ within RCA_0 since for any $n \in \mathbb{N}$, one can effectively find $q, q' \in \mathbb{Q}$ such that $(p, q) \in Z$, $(p, q') \notin Z$, and $|q - q'| \leq 2^{-n}$. Even though a rational presentation of a function is not unique if the function has some rational value, we sometimes identify Z with g_Z .

For given $x, y, z \in \mathbb{Q}$ and a rationally presented function $g_Z : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$, the assertion “ $g_Z(x) - g_Z(y) \leq z$ ” is expressed by a Π_1^0 formula with free variables Z, x, y, z :

$$\forall p_0, p_1, q_0, q_1 \in \mathbb{Q} [p_0 = x \wedge p_1 = y \wedge (p_0, q_0) \notin Z \wedge (p_1, q_1) \in Z \rightarrow q_0 - q_1 \leq z].$$

Similarly, “ $g_Z(x) - g_Z(y) \geq z$ ” can be expressed by a Π_1^0 formula, and “ $g_Z(x) - g_Z(y) < z$ ” and “ $g_Z(x) - g_Z(y) > z$ ” by Σ_1^0 formulas. Thus, the assertion “ $\mathbf{v}_{g_Z}(x) \leq z$ ” is also expressed by a Π_1^0 formula. (Here, we only consider partitions with rational end points.) We say that g_Z is of bounded variation if $\mathbf{v}_{g_Z}(1) \leq k$ for some $k \in \mathbb{N}$.

A function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ can be canonically encoded by a function $f : [0, 1]_{\mathbb{Q}} \times \mathbb{N} \rightarrow \mathbb{Q}$ such that $|f(p, n) - f(p, n + k)| \leq 2^{-n}$ for each $p \in [0, 1]_{\mathbb{Q}}$ and $n, k \in \mathbb{N}$. Rational presentations are essentially sufficient for presenting all real-valued functions on $[0, 1]_{\mathbb{Q}}$: as we show next, within RCA_0 any function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ has a rational presentation up to a vertical shift.

Lemma 4.2 (RCA_0). *For every function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$, there exists a real $a \in \mathbb{R}$ such that $f(p) + a \notin \mathbb{Q}$ for any $p \in [0, 1]_{\mathbb{Q}}$.*

Proof. Let $\{p_i\}_{i \in \mathbb{N}}$ be an enumeration of $[0, 1]_{\mathbb{Q}}$. We recursively define a sequence of rationals $\{a_i\}_{i \in \mathbb{N}}$ as follows. Let $a_0 = 0$.

For given $a_i \in \mathbb{Q}$ we let $a_{i+1} \in \mathbb{Q}$ such that $|a_i - a_{i+1}| < 4^{-i}$ and $|f(p_i) - a_{i+1}| > 4^{-i}/2$. One can always pick such a_{i+1} effectively since the required condition on a_{i+1} given a_i is Σ_1^0 . (Here we use the well-known fact that a dependent choice function for a Σ_1^0 binary predicate of numbers is available within RCA_0 . See, e.g., the argument in the proof of [11, Theorem II.5.8], or [14, Theorem 2.1].) Put $a = \lim_{n \rightarrow \infty} a_i$ and note that $|a - a_{i+1}| \leq 4^{-i}/3$. Therefore $|f(p_i) - a| > 4^{-i}/2 - 4^{-i}/3 > 0$. \square

Proposition 4.3.

- (i) WKL_0 proves that every function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ has a rational presentation.
- (ii) RCA_0 proves that every function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ has a rational presentation up to a vertical shift. That is, there exists a rational presentation Z and a real $r \in \mathbb{R}$ such that $f + r = g_Z$.

Proof. (i) Consider the Σ_1^0 -definable sets $\mathcal{A} = \{(p, q) \in [0, 1]_{\mathbb{Q}} \times \mathbb{Q} : f(p) < q\}$ and $\mathcal{B} = \{(p, q) \in [0, 1]_{\mathbb{Q}} \times \mathbb{Q} : f(p) > q\}$. To obtain a rational presentation for f , it suffices to find a set $Z \subseteq [0, 1]_{\mathbb{Q}} \times \mathbb{Q}$ such that $\mathcal{A} \subseteq Z \subseteq [0, 1]_{\mathbb{Q}} \times \mathbb{Q} \setminus \mathcal{B}$. Such a set Z is obtained by an instance of Σ_1^0 -separation, an axiom scheme which follows from WKL_0 over RCA_0 ([11, Lemma IV.4.4]).

(ii) We may assume that f avoids rational numbers after passing to a vertical shift of f via Lemma 4.2. Then both \mathcal{A} and \mathcal{B} are Δ_1^0 . \square

Corollary 4.4. (i) WKL_0 proves that the restriction to $[0, 1]_{\mathbb{Q}}$ of any continuous function $f : [0, 1] \rightarrow \mathbb{R}$ has a rational presentation.

(ii) RCA_0 proves that the restriction to $[0, 1]_{\mathbb{Q}}$ of any continuous function $f : [0, 1] \rightarrow \mathbb{R}$ has a rational presentation on $[0, 1]_{\mathbb{Q}}$ up to a vertical shift.

Proof. Within RCA_0 , one can effectively find the value of a continuous function. Thus, the restriction to $[0, 1]_{\mathbb{Q}}$ of a continuous function $f : [0, 1] \rightarrow \mathbb{R}$ has a canonical encoding. \square

Taking a vertical shift is essential in the above discussion: an effectively uniformly continuous function itself might not have a rational presentation within RCA_0 . To see this apply the next fact to a recursively inseparable pair.

Proposition 4.5. Given a disjoint pair A, B of c.e. sets, there is a computable nondecreasing function $f : [0, 1] \rightarrow \mathbb{R}$ such that every rational presentation computes a set X such that $A \subseteq X \subseteq \mathbb{N} \setminus B$.

Sketch of proof. We define a uniformly computable sequence of reals (r_e) such that r_e is very close to 2^{-e} ; say $|r_e - 2^{-e}| \leq 2^{-2e}$. The function f is then obtained by linear interpolation between the values $f(2^{-e}) = r_e$; in particular $f(0) = 0$ and f is computable in the usual sense of computable analysis.

We define r_e using a Cauchy name, as follows. Initially we let $r_e = 2^{-e}$. If stage s is least such that $s \geq 2e$ and $e \in A_s$ we subtract 2^{-s} to r_e and leave r_e at this value. If stage s is least such that $s \geq 2e$ and $e \in B_s$ we add 2^{-s} from r_e and leave r_e at this value.

If Z is a rational presentation of f , let $X = \{e : (2^{-e}, 2^{-e}) \in Z\}$. If $e \in A$ then $f(2^{-e}) < 2^{-e}$ and hence $e \in X$. If $e \in B$ then $f(2^{-e}) > 2^{-e}$ and hence $e \notin X$. \square

Proposition 4.6. The assertion “every effectively uniformly continuous function $f : [0, 1] \rightarrow \mathbb{R}$ has a rational presentation” implies WKL_0 over RCA_0 .

Proof. It is routine to transfer the computability theoretic proof above into an argument that the given assertion implies Σ_1^0 separation over RCA_0 . By [11, Lemma IV.4.4] the scheme of Σ_1^0 separation is equivalent to WKL_0 over RCA_0 . \square

The above discussions recast the problem of converting Cauchy representations and Dedekind cuts for reals studied by Hirst [6]. Proposition 4.6 can be viewed as a strengthening of [6, Theorem 7].

4.2 Jordan decomposition by rationally presented functions

We modify the \leq_{slope} notation for functions of rational domain. For $f, g : \subseteq [0, 1] \rightarrow \mathbb{R}$ we let

$$f \leq_{\text{slope}}^* g \text{ iff } \forall x, y \in [0, 1]_{\mathbb{Q}} [x < y \rightarrow (f(y) - f(x) \leq g(y) - g(x))].$$

We will use the following ‘‘folklore’’ fact for the next theorem.

Lemma 4.7 (WKL_0). *Every Π_1^0 definable infinite tree $T \subseteq 2^{<\mathbb{N}}$ has a path.*

Proof. Suppose that $\tau \in T \leftrightarrow \forall n \theta(n, \tau)$. By Δ_1^0 comprehension, there exists a tree

$$\bar{T} = \{\tau : \forall n \leq |\tau| \forall \sigma \preceq \tau \theta(n, \sigma)\}.$$

Then, $T \subseteq \bar{T}$, and any path of \bar{T} is a path of T by the definition of T . By WKL_0 , \bar{T} has a path, thus T has a path. \square

Theorem 4.8 (WKL_0). *For every rationally presented function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ of bounded variation, there is a rationally presented non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}}^* g$.*

Proof. Let $M \in \mathbb{N}$ such that $\mathbf{v}_f \leq M$. We fix an effective listing $(p_n, q_n)_{n \in \mathbb{N}}$ of all elements of $[0, 1]_{\mathbb{Q}} \times \mathbb{Q}$, and identify $Z : \mathbb{N} \rightarrow \{0, 1\}$ as $\{(p_n, q_n) : Z(n) = 1\} \subseteq [0, 1]_{\mathbb{Q}} \times \mathbb{Q}$. We construct a binary tree T such that any path Z through T encodes a non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ with $f \leq_{\text{slope}}^* g$. To do so, we ensure that the following conditions hold:

$$\mathcal{R}_0 : \text{ for any } r \in \mathbb{N}, 0 \leq g(p_r) \leq M;$$

$$\mathcal{R}_1 : \text{ for any } r, s \in \mathbb{N}, \text{ if } p_s \leq p_r, q_s \geq q_r, \text{ and } g(p_s) > q_s \text{ then } g(p_r) > q_r;$$

$$\mathcal{R}_2 : \text{ for any } r, s \in \mathbb{N}, \text{ if } p_s \leq p_r \text{ then } f(p_r) - f(p_s) \leq g(p_r) - g(p_s).$$

Here, \mathcal{R}_1 guarantees that any g encoded by a path Z_g through T is non-decreasing, and \mathcal{R}_2 guarantees the slope condition. Formally, we will consider a Π_1^0 definable tree T to be the set of all $\tau \in 2^{<\mathbb{N}}$ such that

$$(r0) \forall r < |\tau| \left[(q_r \leq 0 \rightarrow \tau(r) = 0) \wedge (q_r \geq M \rightarrow \tau(r) = 1) \right],$$

$$(r1) \forall r, s < |\tau| \left[(p_r \leq p_s \wedge q_r \geq q_s \wedge \tau(r) = 0) \rightarrow \tau(s) = 0 \right],$$

$$(r2) \forall r, s < |\tau| \left[(p_r \leq p_s \wedge \tau(r) = 0 \wedge \tau(s) = 1) \rightarrow |f(p_s) - f(p_r)| \leq q_s - q_r \right].$$

To see that T is infinite, notice that since f is of bounded variation, the string $Z_{v_f}|_k$ defined as $s \in Z_{v_f}|_k$ iff $\mathbf{v}_f(p_s) < q_s$ and $s < k$, which is available from bounded Σ_1^0 comprehension (see [11, Theorem II.3.10]), is an element of T for every $k \in \mathbb{N}$. Thus by Lemma 4.7, T has a path Z . By (r0) and (r1) (for the case $p_r = p_s$), Z encodes a rational presentation. Let g be the unique function $[0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ defined as $g = g_Z$.

Claim 4.8.1. *The function g is non-decreasing.*

Proof. Take $x, y \in [0, 1]_{\mathbb{Q}}$ with $x < y$. Let $q \in \mathbb{Q}$. It suffices to show that if $g(x) > q$ then $g(y) > q$. There are $r, s \in \mathbb{N}$ such that $p_s = y$, $q_s = q$, $p_r = x$, and $q_r = q$. If $g(p_r) > q_r$ then $Z(r) = 0$, and then by clause (r1), $Z(s) = 0$, which means $g(p_s) > q_s$. \diamond

Claim 4.8.2. $f \leq_{\text{slope}}^* g$.

Proof. Let $x, y \in [0, 1]_{\mathbb{Q}}$ such that $x < y$. It is enough to show that for any $q \in \mathbb{Q}$ such that $g(y) - g(x) < q$, $|f(y) - f(x)| < q$. By the definition of g , one can find $r, s \in \mathbb{N}$ such that $x = p_r$, $y = p_s$, $g(p_r) > q_r$, $g(p_s) < q_s$ and $q_s - q_r < q$. Then, by (r2) we have $|f(y) - f(x)| = |f(p_s) - f(p_r)| \leq q_s - q_r < q$. \diamond

Thus, this $g = g_Z$ is the desired function. \square

It is a well-known fact that every Π_1^0 -class with only finitely many members has a computable member. Greenberg, Nies and Slaman used this fact to build a computable function f on $[0, 1]$ of bounded variation such that any Jordan decomposition of f is PA-complete; see [5, Section 2.3 and in particular Prop. 2.9]. A natural formalization within RCA_0 of this fact is as follows: if an infinite tree T has only boundedly many incomparable nodes that are extendible, then T has a path that is computable relative to T . Simpson and Yokoyama [12] showed that this formalization already requires Σ_2^0 -induction. Instead, we will use the following lemma.

Lemma 4.9 (RCA_0). *Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree. If there is a bound on the cardinality of an arbitrary prefix-free subset of T , then T has a path.*

Proof. Take $K \in \mathbb{N}$ so that $|P| < K$ for any prefix-free set $P \subseteq T$. By Σ_1^0 induction, take

$$k = \max\{i \leq K : \text{there is a prefix-free set } P \subseteq T \text{ with } |P| = i\}. \quad (1)$$

Let $P_k \subseteq T$ witness (1). Let $\sigma = \max P_k$, where the max is taken with respect to the usual integer encoding of binary strings. Let $\ell = \max\{|\tau| : \tau \in P_k\}$. Any $\tau \in T$ with $|\tau| > \ell$ must extend an element of P_k , and can have at most one successor. By the pigeonhole principle (which is available from Σ_1^0 induction), there exists $\tau \in P_k$ with infinitely many extensions in T . Since each extension of τ of length exceeding ℓ has exactly one successor, we can effectively find a path through T extending τ . \square

We now establish the main theorem of this section.

Theorem 4.10. *The following are equivalent over RCA_0 .*

1. WKL_0 .
2. $\text{Jordan}_{\mathbb{Q}}$: for every rationally presented function f of bounded variation, there is a rationally presented non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}}^* g$.
3. For every continuous function f of bounded variation, there is a rationally presented non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}}^* g$.
4. For every effectively uniformly continuous function f of bounded variation which has a rational presentation, there is a rationally presented non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}}^* g$.

Proof. $1 \Rightarrow 2$ is Theorem 4.8, $2 \Rightarrow 3$ is a direct consequence of Corollary 4.4(ii), and $3 \Rightarrow 4$ is trivial. We show $4 \Rightarrow 1$. We reason within RCA_0 . Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite binary tree. Assume for a contradiction that T has no path. Let $\tilde{T} = \{\tau \in 2^{<\mathbb{N}} : \tau \notin T \wedge \tau|_{(|\tau|-1)} \in T\}$. Without loss of generality we may assume that \tilde{T} is infinite. Consider the Σ_1^0 definable set

$$\text{Nonext}(T) := \{\tau \in T : \tau \text{ has only finitely many extensions in } T\}.$$

Then, by [11, Lemma II.3.7], there exists a one-to-one function $h : \mathbb{N} \rightarrow \mathbb{N}$ such that $\text{rng}(h) = \text{Nonext}(T)$. (Here, we identify a binary string with its usual integer encoding.)

Let $(\tilde{\sigma}_k)_{k \in \mathbb{N}}$ be an enumeration of \tilde{T} such that $|\tilde{\sigma}_i| \leq |\tilde{\sigma}_{i+1}|$. Note that for any $k, \ell \in \mathbb{N}$,

$$|\tilde{\sigma}_k| \leq \ell \rightarrow k \leq 2^\ell. \quad (2)$$

For all $\sigma \in 2^{<\mathbb{N}}$ put $I_\sigma = [0.\sigma, 0.\sigma + 2^{-|\sigma|}]$. For each $s \in \mathbb{N}$ define a polygonal function $f_s : [0, 1] \rightarrow \mathbb{R}$ as follows. On the interval $I_{\tilde{\sigma}_k}$ set

$$f_s = \begin{cases} M_{I_{\tilde{\sigma}_k}}(2^{-k}, 2^{k-h(k)}) & \text{if } s \geq k > h(k), \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Let $f_s = 0$ elsewhere. Then $(f_s)_{s \in \mathbb{N}}$ defines an effectively uniformly continuous function $f = \lim_s f_s$. By Corollary 4.4(ii), one may replace f with a vertical shift and then assume that f has a rational presentation.

We show that f is of bounded variation. As in the proof of Theorem 3.1, we only need to consider the variation of f on the disjoint intervals $I_{\tilde{\sigma}_k}$. Let $m \in \mathbb{N}$, and for each $k \leq m$ let Π_k be a partition of $I_{\tilde{\sigma}_k}$ containing the midpoints and endpoints of each sawtooth defined on that interval. For all $s \geq m$ one has

$$\sum_{k=0}^m V(f, \Pi_k) = \sum_{k=0}^m V(f_s, \Pi_k) \leq \sum_{k=0}^m 2^{-h(k)+1} < 1,$$

as required.

By our hypothesis in (5.), there exists a rationally presented non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}}^* g$. Let $\Delta : \mathbb{N} \rightarrow \mathbb{R}$ be the function given by $\Delta(k) = \max\{g(0.\sigma + 2^{-|\sigma|}) - g(0.\sigma) : \sigma \in T \wedge |\sigma| = k\}$. Note that Δ is non-increasing.

There are two cases to consider. If $\lim_{n \rightarrow \infty} \Delta(n) = 0$, then g behaves like a continuous function. This provides a decomposition of f that allows us to use an argument similar to the one in Theorem 3.1 to prove the existence of $\text{rng}(h)$. One can then find a path through T by avoiding this set.

Otherwise, there is a jump-type discontinuity of g . The intervals around this point correspond to strings which form an infinite subtree \hat{T} of T . One can bound the size of any prefix-free subset of \hat{T} using the size of this jump, and thus effectively find a path through \hat{T} .

We now analyse the two cases in detail.

Case 1. $\lim_{n \rightarrow \infty} \Delta(n) = 0$. Take $\gamma : \mathbb{N} \rightarrow \mathbb{N}$ such that $\Delta(\gamma(n)) < 2^{-n}$. Such γ exists by Δ_1^0 comprehension since “ $\Delta(k) < 2^{-n}$ ” can be described by a Σ_1^0 formula. If $h(k) = n < k$ then by (3),

$$g(0.\tilde{\sigma}_k + 2^{-|\tilde{\sigma}_k|}) - g(0.\tilde{\sigma}_k) \geq V(M_{I_{\tilde{\sigma}_k}}(2^{-k}, 2^{k-h(k)}), \Pi_k) = 2^{-h(k)+1} \geq 2^{-n}.$$

Hence $|\tilde{\sigma}_k| \leq \gamma(n)$, and then by (2) $k \leq 2^{\gamma(n)}$. This gives

$$n \in \text{rng}(h) \leftrightarrow \exists k \leq \max\{2^{\gamma(n)}, n\} [h(k) = n],$$

so $\text{rng}(h) = \text{Nonext}(T)$ exists by Δ_1^0 comprehension. Thus $\text{Ext}(T) = T \setminus \text{Nonext}(T)$ exists. Now, one can construct a path of T by an easy primitive recursion: starting from the empty string, one can choose the left most immediate extension of a given string in $\text{Ext}(T)$.

Case 2. There exists $M \in \mathbb{N}$ such that $\forall m \exists n [n > m \wedge \Delta(n) > 2^{-M}]$. Then there are infinitely many strings $\sigma \in T$ such that

$$g(0.\sigma + 2^{-|\sigma|}) - g(0.\sigma) > 2^{-M}.$$

Without loss of generality, we may take $K \in \mathbb{N}$ such that $0 \leq g(0) < g(1) \leq K$. Let $L = \{i/2^{M+2} : 0 \leq i \leq K2^{M+2}\}$. Given a rational presentation Z of g , for $x, y, z \in \mathbb{Q}$, we write $g(y) - g(x) \geq_L z$ if there exist $r, s \in \mathbb{N}$ such that $x = p_r, y = p_s, q_r, q_s \in L, Z(r) = 1$ (which implies that $g(p_r) < q_r$), $Z(s) = 0$ (which implies that $g(p_s) > q_s$) and $q_s - q_r \geq z$. Since L is finite, $g(y) - g(x) \geq_L z$ is actually Δ_1^0 statement as we only need to check finitely many r and s . Note that if $g(y) - g(x) > 2^{-M}$, then $g(y) - g(x) \geq_L 2^{-M-2}$ since there are at least two points in $L \cap (g(x), g(y))$.

The tree $\widehat{T} \subseteq T$ defined by $\widehat{T} = \{\sigma \in T : g(0.\sigma + 2^{-|\sigma|}) - g(0.\sigma) \geq_L 2^{-M-2}\}$ is an infinite subtree of T . (The case assumption guarantees that \widehat{T} is infinite, and \widehat{T} is closed under prefixes because g is non-decreasing.) We verify the cardinality of any prefix-free subset of \widehat{T} is bounded. For any prefix-free $P \subseteq \widehat{T}$, we have

$$|P|2^{-M-2} \leq \sum_{\sigma \in P} g(0.\sigma + 2^{-|\sigma|}) - g(0.\sigma) \leq g(1) - g(0) \leq K.$$

Thus, $|P| \leq K2^{M+2}$. Hence by Lemma 4.9, \widehat{T} has a path, and thus T has a path. \square

We thank Paul Shafer for providing helpful comments on a previous version of this proof.

5 Martin-Löf randomness within RCA_0

To study Martin-Löf random reals within RCA_0 , we need to define a notion of uniform measure for open sets. A set of binary strings $S \subseteq 2^{<\mathbb{N}}$ is a *code* of an open set $U \subseteq 2^{\mathbb{N}}$ if $Z \in U \leftrightarrow \exists \sigma \in S [Z \succ \sigma]$. We write $[[S]]$ for the open set coded by S . (Note that a Σ_1^0 -definable set may be used to code an open set, but one can easily find an (existing) set which codes the same open set by Δ_1^0 -comprehension.) Given such a code $S \subseteq 2^{<\mathbb{N}}$, let $T_S := \{\sigma \in 2^{<\mathbb{N}} : \forall n < |\sigma| (\sigma \upharpoonright n \notin S)\}$. Note that T_S forms a tree, which we view as a code of the complement of U . We first define the measure for a code S of an open set, and also of its complementary code $T = T_S$:

$$\mu(S) := 1 - \mu(T) = 1 - \lim_{n \rightarrow \infty} \frac{|\{\sigma \in T : |\sigma| = n\}|}{2^n}.$$

Note that if S is prefix free, then $\mu(S) = \sum_{\sigma \in S} 2^{-|\sigma|}$. The existence of the limit is not guaranteed within RCA_0 , but one can still make assertions such as $\mu(S) \leq a$ or $\mu(T_S) \geq a$, which can be expressed by Π_1^0 -formulas.

$Z \in 2^{\mathbb{N}}$ is said to be Martin-Löf random relative to X if for any X -computable sequence of codes for open sets $\langle S_n : n \in \mathbb{N} \rangle$ such that $\mu(S_n) \leq 2^{-n}$, there exists $n \in \mathbb{N}$ such that $Z \notin [[S_n]]$. The assertion “for any X there exists a Martin-Löf random real relative to X ” is equivalent to WWKL by Simpson and Yu [15]. We always identify a real $z \in [0, 1]$ that is not a dyadic rational with its unique binary expansion viewed as an element of $2^{\mathbb{N}}$.

Besides the fact that the measure of an open set may not exist as a real in the model, there is another problem when developing measure theory within RCA_0 . There might exist two codes for open sets S_1 and S_2 such that $\forall x \in 2^{\mathbb{N}} (x \in [[S_1]] \leftrightarrow x \in [[S_2]])$ but $\mu(S_1) \neq \mu(S_2)$. Thus the value of μ depends on codes. We define the measure for an open set $U \subseteq 2^{\mathbb{N}}$ as

$$\bar{\mu}(U) := \sup\{\mu(S) \mid U = [[S]]\}.$$

This definition agrees with the internal measure of open sets defined in [15, p. 174]. Within WWKL_0 , $[[S_1]] = [[S_2]]$ implies $\mu(S_1) = \mu(S_2)$, thus μ and $\bar{\mu}$ coincide. Fortunately, the definition of Martin-Löf randomness will not be affected even if the two don't coincide. We take any of the two equivalent conditions below as a definition in the context of RCA_0 that Z is not Martin-Löf random relative to X .

Proposition 5.1 (RCA_0). *The following are equivalent for $Z, X \in 2^{\mathbb{N}}$.*

1. *There exists an X -computable sequence $\langle S_i \mid i \in \mathbb{N} \rangle$ of codes of open sets such that $\mu(S_i) \leq 2^{-i}$ and $Z \in \bigcap_{i \in \mathbb{N}} [[S_i]]$.*
2. *There exists an X -computable sequence $\langle S_i \mid i \in \mathbb{N} \rangle$ of codes of open sets such that $\bar{\mu}([[S_i]]) \leq 2^{-i}$ and $Z \in \bigcap_{i \in \mathbb{N}} [[S_i]]$.*

Proof. $2 \Rightarrow 1$ is trivial. To show $1 \Rightarrow 2$, let $\langle S_i \mid i \in \mathbb{N} \rangle$ be an X -computable sequence of codes for open sets such that $\mu(S_i) \leq 2^{-i}$ and $Z \in \bigcap_{i \in \mathbb{N}} [[S_i]]$. If Z is of the form $\sigma \frown 0^{\mathbb{N}}$, put $S'_i := \{\sigma \frown 0^i\}$. We have $\bar{\mu}([[S'_i]]) \leq 2^{-i}$ and $Z \in \bigcap_{i \in \mathbb{N}} [[S'_i]]$. Otherwise, put

$$S'_i := \{\sigma \frown 0^k \frown 1 : \sigma \in S_i, k \in \mathbb{N}\}.$$

Then $T_{S'_i} = \{\tau \in 2^{<\mathbb{N}} \mid \tau = \sigma \frown 0^k \text{ for some } \sigma \in T_i \text{ and } k \in \mathbb{N}\}$. Since $T_{S'_i} \supseteq T_{S_i}$ we have $\mu(T_{S'_i}) \leq \mu(T_{S_i})$. We still have $Z \in \bigcap_{i \in \mathbb{N}} [[S'_i]]$ by the case assumption on Z . On the other hand, for any $\hat{S} \subseteq 2^{<\mathbb{N}}$, if $\forall x \in 2^{\mathbb{N}} (x \in [[\hat{S}]] \rightarrow x \in [[S'_i]])$ then $T_{\hat{S}} \supseteq T_{S'_i}$ because $\sigma \in T_{S'_i} \rightarrow \sigma \frown 0^{\mathbb{N}} \in [T_{\hat{S}}]$. Thus, $\bar{\mu}([[S'_i]]) \leq \mu(S'_i) \leq 2^{-i}$. \square

6 Differentiability of functions of bounded variation in WWKL_0

Lebesgue's theorem states that functions on $[0, 1]$ of bounded variation are a.e. differentiable. The main result of this section, Theorem 6.8, shows that several versions of this result are equivalent to WWKL_0 over RCA_0 . For a function f and distinct reals a, b in the domain of f , we denote the slope by

$$S_f(a, b) = \frac{f(b) - f(a)}{b - a}.$$

Definition 6.1 (Section 2.3 of [1], going back to Demuth, within RCA_0). Let $f : \subseteq [0, 1] \rightarrow \mathbb{R}$ be a function with domain containing $[0, 1]_{\mathbb{Q}}$ (f may be a continuous function or a rationally presented function). For a given $h > 0$, the h -derivative of f at $x \in [0, 1]$ is the set of reals defined by

$$D_h f(x) = \{S_f(a, b) : a, b \in [0, 1]_{\mathbb{Q}} \wedge a \leq x \leq b \wedge 0 < b - a < h\}.$$

The function f is *pseudo-differentiable* at $z \in (0, 1)$ if $\lim_{h \rightarrow 0^+} \text{diam}(D_h f(z)) = 0$, or more formally, for any $\varepsilon > 0$, there exist $c, d \in \mathbb{Q}$ and $h > 0$ such that $d - c < \varepsilon$ and $D_h f(z) \subseteq [c, d]$.

The *upper* and *lower pseudo-derivatives* of f are defined by $\tilde{D}f(x) = \lim_{h \rightarrow 0^+} \sup D_h f(x)$ and $\underline{D}f(x) = \lim_{h \rightarrow 0^+} \inf D_h f(x)$. Then, the assertion $\lim_{h \rightarrow 0^+} \text{diam}(D_h f(z)) = 0$ formally means that $\underline{D}f(z) = \tilde{D}f(z)$ without referring to the limits themselves. The point is that we don't have to require that $f(z)$ be defined; for instance we could be interested in a function f only defined on rationals. In this way we can include in our equivalences with WWKL_0 in Theorem 6.8 a statement about functions with rational presentations. It follows from [1, Lemma 2.5] that if f is defined and continuous at z , then the pseudo-derivative at z exists iff the usual derivative exists, and they agree.

Note that the real $r = \tilde{D}f(x) = \underline{D}f(x)$ may fail to exist in a model of RCA_0 even if $\underline{D}f(z)$ and $\tilde{D}f(z)$ are equal. We will avoid mentioning the values $\tilde{D}f(x)$ or $\underline{D}f(x)$ and just consider inequality, as we already did in the case for bounded variation.

6.1 Pseudo-differentiability of non-decreasing functions within RCA_0

Lemma 6.2 (a version of part of [1, Theorem 4.3] that works within RCA_0). *Let $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ be a rationally presented non-decreasing function, and let $z \in [0, 1]$ be Martin-Löf random relative to f . Then f is pseudo-differentiable at z .*

Proof. Without loss of generality, we may suppose that $0 \leq f(0) \leq f(1) \leq 1$. Assume that f is not pseudo-differentiable at z . If z is rational, then z is not Martin-Löf random, so assume that z is irrational. We will consider the following two cases.

Case 1. $\underline{D}f(z) = \infty$. Thus, for any $m \in \mathbb{N}$, there exists $a_0 < z < b_0$ such that for any $a, b \in \mathbb{Q}$, $a_0 < a < z < b < b_0 \rightarrow S_f(a, b) > 2^m$. For a given $\sigma \in 2^{<\mathbb{N}}$, we let $l_\sigma = 0.\sigma$ and $r_\sigma = 0.\sigma + 2^{-|\sigma|}$. Let $\varphi(m, \sigma)$ be a Π_1^0 -formula saying that $S_f(l_\sigma, r_\sigma) \leq 2^m$. Write $\varphi(m, \sigma) \equiv \forall s \theta(s, m, \sigma)$ for a Σ_0^0 -formula θ , and put

$$T_m := \{\sigma \in 2^{<\mathbb{N}} : \forall \tau \preceq \sigma \forall s < |\sigma| \theta(s, m, \tau)\}.$$

Since $f(1) - f(0) \leq 1$, for each $m \in \mathbb{N}$ and $k \in \mathbb{N}$, there are at most 2^k strings of length $m + k$ which are not in T_m . Thus, $\mu(T_m) \geq 1 - 2^{-m}$, and hence $\langle 2^{\mathbb{N}} \setminus [T_m] : m \in \mathbb{N} \rangle$ forms a Martin-Löf test. By the assumption, $z \notin [T_m]$ for any $m \in \mathbb{N}$. Thus z is not Martin-Löf random relative to f .

Case 2. $\underline{D}f(z) < \infty$ and $\underline{D}f(z) < \tilde{D}f(z)$. We will follow the proof of (iii) \rightarrow (ii) of [1, Theorem 4.3] halfway within RCA_0 .

Since f is non-decreasing and $\underline{D}f(z) < \infty$, the limit $\lim_{y \rightarrow z} f(y)$ exists by nested interval completeness [11, Theorem II.4.8] which holds in RCA_0 . So we may assume that $f(z)$ exists and f is continuous at z ; this will be needed in the following argument when we apply a version of [1, Lemma 2.5] formalised within RCA_0 .

For given $p, q \in \mathbb{Q}$, an interval A is said to be a (p, q) -interval if it is of the form $A = (pi2^{-n} + q, p(i+1)2^{-n} + q)$ for some $n \in \mathbb{N}$ and $i \in \mathbb{Z}$. For a finite set $L \subseteq \mathbb{Q}^2$, an interval is said to be L -interval if it is a (p, q) -interval for some $(p, q) \in L$. One can formalise within RCA_0 the proofs of Lemma 2.5, Lemma 4.1 and most of the proof of Lemma 4.4 of [1]. To see this, note that these arguments only rely on elementary arithmetic, which can be formalised within RCA_0 . Hence we have the following.

Claim 6.2.1. *There exist rationals $\beta > \gamma > 0$ and a finite set $L \subseteq \mathbb{Q}^2$ such that*

$$\begin{aligned} \gamma &> \liminf_{h \rightarrow 0} \{S_f(A) : A \text{ is an } L\text{-interval} \wedge |A| \leq h \wedge z \in A\}, \\ \beta &< \limsup_{h \rightarrow 0} \{S_f(A) : A \text{ is an } L\text{-interval} \wedge |A| \leq h \wedge z \in A\}. \end{aligned}$$

In the final step of the proof of [1, Lemma 4.4] for both inequalities there, one picks (p, q) and (r, s) from L which by themselves witness the two inequalities above, respectively; that is, we only need to look at (p, q) intervals for the first, and at (r, s) -intervals for the second. However, this is impossible within RCA_0 since it requires an essential use of the infinite pigeonhole principle (also known as RT^1) which is equivalent to $\text{B}\Sigma_2^0$. Thus, we need to take a detour around this part of the proof.

We fix β, γ and $L \subseteq \mathbb{Q}^2$ as in Claim 6.2.1. An n -depth alternation L -sequence is a length $2n + 1$ decreasing sequence of L -intervals $[0, 1] \supseteq A_0 \supseteq A_1 \supseteq \dots \supseteq A_{2n}$ such that $S_f(A_{2i}) < \gamma$ for any $i \leq n$, $S_f(A_{2i+1}) > \beta$ for any $i < n$. For $(p, q), (r, s) \in L$, an n -depth alternation $(p, q); (r, s)$ -sequence is an n -depth alternation L -sequence such that A_{2i} is a (p, q) -interval for any $i \leq n$ and A_{2i+1} is an (r, s) -interval for any $i < n$. The last interval of an n -depth alternation $(p, q); (r, s)$ -sequence is called an n -depth $(p, q); (r, s)$ -interval. By Claim 6.2.1, for any $n \in \mathbb{N}$, there exists an n -depth alternation L -sequence such that $z \in A_{2n}$. Furthermore, by the finite pigeon hole principle, every $n|L|^2$ -depth alternation L -sequence contains an n -depth alternation $(p, q); (r, s)$ -subsequence for some $(p, q), (r, s) \in L$. Thus, we have the following claim.

Claim 6.2.2. *For any $n \in \mathbb{N}$ there exist $(p, q), (r, s) \in L$ and an n -depth $(p, q); (r, s)$ -interval A such that $z \in A$.*

Note that we cannot fix $(p, q), (r, s) \in L$ for all $n \in \mathbb{N}$ in this claim since it would require $\text{B}\Sigma_2^0$ again.

Next, we will calculate the size of n -depth $(p, q); (r, s)$ -intervals. Fix $(p, q), (r, s) \in L$ and let $\{A^s\}_{s < u}$ be a finite collection of n -depth $(p, q); (r, s)$ -intervals. Take an n -depth alternation $(p, q); (r, s)$ -sequence $A_0^s \supseteq \dots \supseteq A_{2n}^s = A^s$ for each $s < u$, and let $\bar{A}_i = \bigcup_{s < u} A_i^s$. For a finite union of intervals \bar{A} which is described by a finite disjoint union as $\bar{A} = \bigsqcup_{j < l} [a_j, b_j]$, put $|\bar{A}| := \sum_{j < l} (b_j - a_j)$ and $\Delta_f(\bar{A}) := \sum_{j < l} (f(b_j) - f(a_j))$. Since any two (p, q) -intervals (or (r, s) -intervals) are disjoint, or one includes the other, we have that $\Delta_f(\bar{A}_{2i})/|\bar{A}_{2i}| < \gamma$ for any $i \leq n$ and $\Delta_f(\bar{A}_{2i+1})/|\bar{A}_{2i+1}| > \beta$ for any $i < n$. Thus, for any $i < n$,

$$|\bar{A}_{2i+2}| \leq |\bar{A}_{2i+1}| < \frac{\Delta_f(\bar{A}_{2i+1})}{\beta} \leq \frac{\Delta_f(\bar{A}_{2i})}{\beta} < \frac{\gamma}{\beta} |\bar{A}_{2i}|.$$

Hence,

$$\bar{\mu} \left(\bigcup_{s < u} A^s \right) = |\bar{A}_{2n}| < \left(\frac{\gamma}{\beta} \right)^n |\bar{A}_0| \leq \left(\frac{\gamma}{\beta} \right)^n. \quad (4)$$

Now, put

$$U_n^{(p,q);(r,s)} := \bigcup \{A : A \text{ is an } n\text{-depth } (p, q); (r, s)\text{-interval}\},$$

$$U_n := \bigcup_{(p,q),(r,s) \in L} U_n^{(p,q);(r,s)}.$$

Note that one can enumerate all n -depth $(p, q); (r, s)$ -intervals f -computably. By (4), $\bar{\mu}(U_n^{(p,q);(r,s)}) \leq (\gamma/\beta)^n$. Thus, $\bar{\mu}(U_n) \leq (\gamma/\beta)^n |L|^2$. By Claim 6.2.2, $z \in \bigcap_{n \in \mathbb{N}} U_n$. Thus z is not Martin-Löf random relative to f . \square

Remark 6.3. By a careful formalization of the notion of test for computable randomness within RCA_0 , one can reformulate the above proof to obtain the following assertion within RCA_0 : for any rationally presented non-decreasing function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$, there exists a computable test relative

to f such that f is pseudo-differentiable at $z \in [0, 1]$ if z passes the test. On the other hand, one can easily see within RCA_0 that for any computable test relative to X , there exists a real computable from X which can pass it. Thus, within RCA_0 , every rationally presented non-decreasing function is pseudo-differentiable at some point. However, as we will see in Theorem 6.8, this does not imply that every rational presented non-decreasing function is pseudo-differentiable almost surely (as defined below).

6.2 A.e. pseudo-differentiability of functions of bounded variation

We introduce a notion of a.e. differentiability in a form that is appropriate within RCA_0 . In that setting, any open set $U \subseteq [0, 1]$ can be identified with an open set in $2^{\mathbb{N}}$ via the canonical surjection $\pi : 2^{\mathbb{N}} \rightarrow [0, 1]$ defined by $\pi(Z) = \sum_{n \in Z} 2^{-n}$. We define the measure for open sets in $[0, 1]$ by $\bar{\mu}(U) = \bar{\mu}(\pi^{-1}(U))$. This $\bar{\mu}$ coincides with the Lebesgue measure on $[0, 1]$ as in [15, p. 174].

Definition 6.4. A function $f : \subseteq [0, 1] \rightarrow \mathbb{R}$ with domain containing $[0, 1]_{\mathbb{Q}}$ is pseudo-differentiable *almost surely* if $\bar{\mu}(U) = 1$ for any open set $U \subseteq [0, 1]$ containing every point of pseudo-differentiability of f .

For the main results of this section we need two preliminaries. The first one is a model-theoretic generalization of the fact that any Martin-Löf random real is Martin-Löf random relative to some PA-degree by Downey, Hirschfeldt, Miller and Nies [4, Proposition 7.4] and Reimann and Slaman [10, Lemma 4.5].

Lemma 6.5 (Simpson/Yokoyama [13]). *For any countable model $(\mathcal{M}, \mathcal{S}) \models \text{WWKL}_0$ there is $\widehat{\mathcal{S}} \supseteq \mathcal{S}$ satisfying*

1. $(\mathcal{M}, \widehat{\mathcal{S}}) \models \text{WKL}_0$, and
2. for any $A \in \widehat{\mathcal{S}}$ there is $Z \in \mathcal{S}$ such that Z is Martin-Löf random relative to A .

The following is related to a well known result of Kučera; also see [8, Proposition 3.2.24]. We say that W is a tail of a set $Z \subseteq \mathbb{N}$ if there is n such that $W(i) = Z(n+i)$ for each i .

Lemma 6.6 (RCA_0). *Let $U \subseteq [0, 1]$ be an open set such that $\bar{\mu}(U) < 1$, and let $S \subseteq 2^{<\mathbb{N}}$ be a code for an open set such that $[[S]] = \pi^{-1}(U)$. Let $Z \in 2^{\mathbb{N}}$ be Martin-Löf random relative to S . There exists a tail W of Z such that $0.W = \pi(Z) \in [0, 1] \setminus U$.*

Proof. Choose $q \in \mathbb{Q}$ such that $\mu(S) \leq \bar{\mu}(U) \leq q < 1$. Then, $\mu(T_S) \geq 1 - q$. Let $\widetilde{T} = \{\tau \in 2^{<\mathbb{N}} : \tau \notin T_S \wedge \tau|_{(|\tau|-1)} \in T_S\}$. Put

$$T^n := \{\sigma_0 \widehat{\ } \dots \widehat{\ } \sigma_k : k < n \wedge [\forall i < k \sigma_i \in \widetilde{T}] \wedge \sigma_k \in T_S\}.$$

Then, we have $\mu(T^n) \geq 1 - q^n$. Thus, for large enough $l \in \mathbb{N}$, $\langle 2^{\mathbb{N}} \setminus [T^{ln}] : n \in \mathbb{N} \rangle$ forms a Martin-Löf test relative to S , and hence $Z \in [T^{ln}]$ for some $n \in \mathbb{N}$. By Σ_1^0 -induction, take

$$m = \max\{m' : \exists c \leq ln \exists \langle \sigma_i \in \widetilde{T} : i < c \rangle (Z|_{m'} = \sigma_0 \widehat{\ } \dots \widehat{\ } \sigma_{c-1})\}.$$

Then, the tail W of Z defined by $W(i) = Z(i+m)$ is in $[T_S]$, whence $0.W \in [0, 1] \setminus U$. \square

Theorem 6.7 (WWKL_0). *Every rationally presented function of bounded variation is pseudo-differentiable at some point, and is actually pseudo-differentiable almost surely.*

Proof. We show that the result holds in any countable model $(\mathcal{M}, \mathcal{S})$ of WWKL₀. Let $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ be a rationally presented function of bounded variation in $(\mathcal{M}, \mathcal{S})$, and let $U \subseteq [0, 1]$ be an open set such that $\bar{\mu}(U) < 1$. We will show that there exists a real $z \in [0, 1] \setminus U$ such that f is pseudo-differentiable at z . Let $(\mathcal{M}, \widehat{\mathcal{S}}) \models \text{WKL}_0$ be the model given by Lemma 6.5. By Theorem 4.10,

$$(\mathcal{M}, \widehat{\mathcal{S}}) \models \text{Jordan}_{\mathbb{Q}}.$$

Hence $\widehat{\mathcal{S}}$ contains a non-decreasing function $g : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ such that $f \leq_{\text{slope}}^* g$.

Within $(\mathcal{M}, \widehat{\mathcal{S}})$, define $h : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ by $h(x) = g(x) - f(x)$. By Lemma 6.5 again, there is a real $z \in [0, 1]$ such that $z \in \mathcal{S}$ and $z \in \text{MLR}^{g \oplus h \oplus U}$. By Lemma 6.6, we may assume that $z \in [0, 1] \setminus U$. The functions g and h are pseudo-differentiable at z in $(\mathcal{M}, \widehat{\mathcal{S}})$ by Lemma 6.2. Therefore f is pseudo-differentiable at z in $(\mathcal{M}, \widehat{\mathcal{S}})$, and hence in $(\mathcal{M}, \mathcal{S})$. \square

A continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be absolutely continuous if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $0 \leq a_0 \leq \dots \leq a_n \leq 1$ with $a_n - a_0 < \delta$, $\sum_{i < n} |f(a_{i+1}) - f(a_i)| < \varepsilon$. Note that every absolutely continuous function is of bounded variation within RCA₀.

Theorem 6.8. *The following are equivalent over RCA₀.*

1. WWKL₀
2. *Every rationally presented function of bounded variation is pseudo-differentiable almost surely.*
3. *Every rationally presented non-decreasing function is pseudo-differentiable almost surely.*
4. *Every continuous function of bounded variation is pseudo-differentiable almost surely.*
5. *Every effectively uniformly continuous and absolutely continuous function which has a rational presentation is pseudo-differentiable at some point.*

Proof. 1 \Rightarrow 2 is Theorem 6.7, 2 \Rightarrow 3 is trivial, 2 \Rightarrow 4 is straightforward from Corollary 4.4. For 4 \Rightarrow 5, within RCA₀, we have $\bar{\mu}(\emptyset) = 0$, since if $[[S]] = \emptyset$, then S is empty, so $\mu(S) = 0$. Thus, if an open set $U \subseteq [0, 1]$ has positive measure, then U is not empty. It remains to show $\neg 1 \Rightarrow \neg 3$ and $\neg 1 \Rightarrow \neg 5$.

We show $\neg 1 \Rightarrow \neg 3$. Assuming $\neg 1$ we first construct an open set $U \subseteq [0, 1]$ such that $\bar{\mu}(U) < 1$ and $[0, 1] \setminus U$ only contains rationals. The idea is similar to the one in the proof of Proposition 5.1: If WWKL fails, there is a tree T with no paths such that $\mu(T) \geq \varepsilon$ where $\varepsilon > 0$. Put

$$T' = \{\tau \in 2^{<\mathbb{N}} \mid \tau = \sigma \frown 0^k \text{ for some } \sigma \in T \text{ and } k < |\tau|\},$$

$$\widetilde{T} = \{\tau \in 2^{<\mathbb{N}} \mid \tau \notin T' \text{ and } \tau \upharpoonright_{|\tau|-1} \in T'\}, \text{ and}$$

$$U = \bigcup_{\tau \in \widetilde{T}} (0.\tau, 0.\tau + 2^{-|\tau|}) \subseteq [0, 1].$$

As in the proof of Proposition 5.1, we have $\bar{\mu}(2^{\mathbb{N}} \setminus [T']) \leq 1 - \mu(T) < 1$, and any path of T' is rational. Thus $\bar{\mu}(U) \leq \bar{\mu}(2^{\mathbb{N}} \setminus [T']) < 1$ and $[0, 1] \setminus U$ only contains rationals.

Next we construct a rationally presented non-decreasing function which is not pseudo-differentiable at any rational. Let $\{q_i\}_{i \in \mathbb{N}}$ be an enumeration of $[0, 1]_{\mathbb{Q}}$. Define a function $f : [0, 1]_{\mathbb{Q}} \rightarrow \mathbb{R}$ by $f(p) = \sum_{q_i < p} 2^{-i}$. Clearly, f is non-decreasing and not pseudo-differentiable at any rational. By Proposition 4.3, some vertical shift of f has a rational presentation. Thus we have $\neg 3$.

Finally, we show $\neg 1 \Rightarrow \neg 5$. This implication is related to [1, Theorem 6.7] (originally due to Demuth) in the setting of reverse mathematics. If WWKL fails, there is a tree T with no path such that $\mu(T) \geq \varepsilon$ where $\varepsilon > 0$. We construct a sequence of trees $\langle T_n : n \in \mathbb{N} \rangle$ such that no T_n has a path and $\mu(T_n) \geq 1 - 2^{-4n}$. Let the T^i be defined as in the proof of Lemma 6.6 where $q = 1 - \varepsilon$.

No T^i has a path, and $\mu(T^i) \geq 1 - (1 - \varepsilon)^i$. Thus, one can effectively choose $i_0 < i_1 < \dots$ so that $\mu(T^{i_n}) \geq 1 - 2^{-4n}$. Now let $T_n = T^{i_n}$.

Let $\tilde{T}_n = \{\tau \in 2^{<\mathbb{N}} : \tau \notin T_n \wedge \tau|_{|\tau|-1} \in T_n\}$. As before put $I_\sigma := [0, \sigma, 0, \sigma + 2^{-|\sigma|}]$. Since T_n has no path we have $[0, 1] = \bigcup_{\sigma \in \tilde{T}_n} I_\sigma$ for any n . Since $\mu(T_n) \geq 1 - 2^{-4n}$ we have $\sum_{\sigma \in \tilde{T}_n} |I_\sigma| \leq 2^{-4n}$. Note that if $\sigma \in \tilde{T}_n$ and $m < n$, there exists $\tau \in \tilde{T}_m$ such that $\tau \preceq \sigma$. Note also that $|\sigma| \geq n$ for any $\sigma \in \tilde{T}_n$.

For $v \in \mathbb{R}^+$ and $r \in \mathbb{N}$, recall $M_A(v, r)$ denotes a sawtooth function on the interval A with r many teeth of height v . For each $n, s \in \mathbb{N}$ define a polygonal function $f_{n,s} : [0, 1] \rightarrow \mathbb{R}$ as follows. For $\sigma \in \tilde{T}_n$, on the interval I_σ , set

$$f_{n,s} = \begin{cases} M_{I_\sigma}(2^{-2n-|\sigma|}, 2^{5n}) & \text{if } |\sigma| \leq s, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(f_{n,s})_{s \in \mathbb{N}}$ defines an effectively uniformly continuous function f_n . For these functions f_n one can check the following properties.

- (i) If $\sigma \in \tilde{T}_n$, $x \in I_\sigma$ and $m \geq n$, then $0 \leq f_m(x) \leq 2^{-2m-|\sigma|}$. In particular, $|f_n| \leq 2^{-2n}$.
- (ii) For any $0 \leq x < y \leq 1$ and for any $n \in \mathbb{N}$, $|f_n(x) - f_n(y)|/|x - y| \leq 2^{3n+1}$.
- (iii) $\mathbf{v}_{f_n} \leq 2^{-n+1}$.

(i) and (ii) follow from the definition. To see (iii),

$$\mathbf{v}_{f_n} = \sum_{\sigma \in \tilde{T}_n} \mathbf{v}_{M_{I_\sigma}(2^{-2n-|\sigma|}, 2^{5n})} = \sum_{\sigma \in \tilde{T}_n} 2^{3n-|\sigma|+1} \leq 2^{3n+1} 2^{-4n} = 2^{-n+1}.$$

Define an effectively uniformly continuous function f by $f = \sum_{n \in \mathbb{N}} f_n$. Then, f is of bounded variation since $\mathbf{v}_f = \sum_{n \in \mathbb{N}} \mathbf{v}_{f_n} \leq 2$. Actually, f is absolutely continuous. One can see this as follows. For any $x \in [0, 1]$ and $\varepsilon > 0$, take large enough $n \in \mathbb{N}$ so that $\sum_{j > n} \mathbf{v}_{f_j} < \varepsilon/2$. Since each f_i , $i \leq n$, is absolutely continuous by (ii), one can find $\delta > 0$ so that $\sum_{i \leq n} (f_i(x) - f_i(y)) < \varepsilon/2$ for any y such that $|x - y| < \delta$.

By Corollary 4.4(ii), after replacing f with a vertical shift we may assume that f has a rational presentation.

We will see that this f is not pseudo-differentiable at any point. Let $x \in [0, 1]$, $\delta > 0$ and $K \in \mathbb{N}$. We will find $a \leq x \leq b$ so that $b - a < \delta$ and $|S_f(a, b)| > K$. Take $n \in \mathbb{N}$ large enough so that $2^{3n-1} > K$ and $|I_\sigma| < \delta$ for any $\sigma \in \tilde{T}_n$. Since T_n has no path, there exists $\sigma \in \tilde{T}_n$ such that $x \in I_\sigma$. Let $a \leq x \leq b$ so that a, b are nearest to x yielding extreme values of the saw-tooth function $M_{I_\sigma}(2^{-2n-|\sigma|}, 2^{5n})$. Then, $|f_n(b) - f_n(a)| = 2^{-2n-|\sigma|}$ and $b - a = 2^{-5n-1-|\sigma|}$. By (i), $|f_j(b) - f_j(a)| \leq 2^{-2j-|\sigma|}$ for any $j > n$, and by (ii), $|f_i(b) - f_i(a)|/|b - a| \leq 2^{3i+1}$ for any $i < n$. Thus,

$$\begin{aligned} \frac{|f(b) - f(a)|}{|b - a|} &\geq \frac{|f_n(b) - f_n(a)|}{|b - a|} - \sum_{j > n} \frac{|f_j(b) - f_j(a)|}{|b - a|} - \sum_{i < n} \frac{|f_i(b) - f_i(a)|}{|b - a|} \\ &\geq \frac{2^{-2n-|\sigma|}}{2^{-5n-1-|\sigma|}} - \sum_{j > n} \frac{2^{-2j-|\sigma|}}{2^{-5n-1-|\sigma|}} - \sum_{i < n} 2^{3i+1} \\ &\geq 2^{3n+1} - 2^{3n} - 2^{3n-1} = 2^{3n-1}. \end{aligned}$$

Hence, $|S_f(a, b)| > K$. □

Remark 6.9. The equivalence of conditions 1 and 3 in the foregoing theorem seems rather strange compared to [1, Theorem 4.3] and our discussion in Remark 6.3 since there is no appearance of Martin-Löf random reals. Note that the existence of computable random reals doesn't imply WWKL over RCA_0 . This tricky situation may be understood that it is caused by a bad behavior of the Lebesgue measure within RCA_0 . For example, one cannot say that *every open set* $U \subseteq [0, 1]$ *is of measure 1 if* $[0, 1] \setminus U$ *is countable*. In fact, this is equivalent to WWKL by the argument in the previous proof.

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