Most(?) theories have Borel complete reducts

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Abstract

We prove that many seemingly simple theories have Borel complete reducts. Specifically, if a countable theory has uncountably many complete 1-types, then it has a Borel complete reduct. Similarly, if $Th(M)$ is not small, then M^{eq} has a Borel complete reduct, and if a theory T is not ω-stable, then the elementary diagram of some countable model of T has a Borel complete reduct.

1 Introduction

In their seminal paper [1], Friedman and Stanley define and develop a notion of *Borel reducibility* among classes of structures with universe ω in a fixed, countable language L that are Borel and invariant under permutations of ω. It is well known (see e.g., [3] or [2]) that such classes are of the form $\mathrm{Mod}(\Phi)$, the set of models of Φ whose universe is precisely ω for some sentence $\Phi \in L_{\omega_1,\omega}$, but here we concentrate on first-order, countable theories T. For countable theories T, S in possibly different language, a *Borel reduction* is a Borel function $f : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$ that satisfies $M \cong N$ if and only if $f(M) \cong f(N)$. One says that T is *Borel reducible* to S if there is a Borel reduction $f : \mathrm{Mod}(T) \to \mathrm{Mod}(S)$. As Borel reducibility is transitive, this induces a quasi-order on the class of all countable theories, where we say T and S are *Borel equivalent* if there are Borel reductions in both directions. In [1], Friedman and Stanley show that among Borel invariant classes (hence among countable first-order theories) there is a maximal class with respect to $\leq_B$.

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We say Φ is *Borel complete* if it is in this maximal class. Examples include the theories of graphs, linear orders, groups, and fields.

The intuition is that Borel complexity of a theory T is related to the complexity of invariants that describe the isomorphism types of countable models of T. Given an L-structure M, one naturally thinks of the reducts M_0 of M to be 'simpler objects' hence the invariants for a reduct 'should' be no more complicated than for the original M, but we will see that this intuition is incorrect. As a paradigm, let T be the theory of 'independent unary predicates' i.e., $T = Th(2^\omega, U_n)$, where each U_n is a unary predicate interpreted as $U_n = \{\eta \in 2^\omega : \eta(n) = 1\}$. The countable models of T are rather easy to describe. The isomorphism type of a model is specified by which countable, dense subset of 'branches' is realized, and how many elements realize each of those branches. However, with Theorem 3.2, we will see that T has a Borel complete reduct.

To be precise about reducts, we have the following definition.

Definition 1.1. Given an L-structure M, a *reduct M' of M* is an L'-structure with the same universe as M, and for which the interpretation every atomic L'-formula $\alpha(x_1, \ldots, x_k)$ is an L-definable subset of M^k (without parameters). An L'-theory T' is a *reduct of an L-theory* T if $T' = Th(M')$ for some reduct M' of some model M of T.

In the above definition, it would be equivalent to require that the interpretation in M' of every L'-formula $\theta(x_1, \ldots, x_k)$ is a 0-definable subset of M^k.

2 An engine for Borel completeness results

This section is devoted to proving Borel completeness for a specific family of theories. All of the theories T_h, are in the same language $L = \{E_n : n \in \omega\}$ and are indexed by strictly increasing functions $h : \omega \to \omega \setminus \{0\}$. For a specific choice of h, the theory T_h asserts that

- Each E_n is an equivalence relation with exactly $h(n)$ classes; and
- The E_n's cross-cut, i.e., for all nonempty, finite $F \subseteq \omega$, $E_F(x, y) := \bigwedge_{n \in F} E_n(x, y)$ is an equivalence relation with precisely $\Pi_{n \in F} h(n)$ classes.

It is well known that each of these theories T_h is complete and admits elimination of quantifiers. Thus, in any model of T_h, there is a unique 1-type. However, the strong type structure is complicated.[1] So much so, that the whole of this section is devoted to the proof of:

[1]Recall that in any structure M, two elements a, b have the same *strong type*, $\mathrm{stp}(a) = \mathrm{stp}(b)$, if $M \models E(a, b)$ for every 0-definable equivalence relation. Because of the quantifier elimination, in any model $M \models T_h$, $\mathrm{stp}(a) = \mathrm{stp}(b)$ if and only if $M \models E_n(a, b)$ for every $n \in \omega$.

Theorem 2.1. *For any strictly increasing $h : \omega \to \omega \setminus \{0\}$, T_h is Borel complete.*

Proof. Fix a strictly increasing function $h : \omega \to \omega \setminus \{0\}$. We begin by describing representatives $\mathcal{B}$ of the strong types and a group G that acts faithfully and transitively on $\mathcal{B}$. As notation, for each n, let $[h(n)]$ denote the $h(n)$-element set $\{1, \ldots, h(n)\}$ and let $Sym([h(n)])$ be the (finite) group of permutations of $[h(n)]$. Let

$$\mathcal{B} = \{f : \omega \to \omega : f(n) \in [h(n)] \text{ for all } n \in \omega\}$$

and let $G = \Pi_{n\in\omega} Sym([h(n)])$ be the direct product. As notation, for each $n \in \omega$, let $\pi_n : G \to Sym([h(n)])$ be the natural projection map. Note that G acts coordinate-wise on $\mathcal{B}$ by: For $g \in G$ and $f \in \mathcal{B}$, $g \cdot f$ is the element of $\mathcal{B}$ satisfying $g \cdot f(n) = \pi_n(g)(f(n))$.

Define an equivalence relation $\sim$ on $\mathcal{B}$ by:

$$f \sim f' \quad \text{if and only if} \quad \{n \in \omega : f(n) \neq f'(n)\} \text{ is finite.}$$

For $f \in \mathcal{B}$, let $[f]$ denote the $\sim$-class of f and, abusing notation somewhat, for $W \subseteq \mathcal{B}$

$$[W] := \bigcup \{[f] : f \in W\}.$$

Observe that for every $g \in G$, the permutation of $\mathcal{B}$ induced by the action of g maps $\sim$-classes onto $\sim$-classes, i.e., G also acts transitively on $\mathcal{B}/\sim$.

We first identify a countable family of $\sim$-classes that are 'sufficiently indiscernible'. Our first lemma is where we use the fact that the function h defining T_h is strictly increasing.

Lemma 2.2. *There is a countable set $Y = \{f_i : i \in \omega\} \subseteq \mathcal{B}$ such that whenever $i \neq j$, $\{n \in \omega : f_i(n) = f_j(n)\}$ is finite.*

Proof. We recursively construct Y in ω steps. Suppose $\{f_i : i < k\}$ have been chosen. Choose an integer N large enough so that $h(N) > k$ (hence $h(n) > k$ for all $n \geq N$). Now, construct $f_k \in \mathcal{B}$ to satisfy $f_k(n) \neq f_i(n)$ for all $n \geq N$ and all $i < k$. $\square$

Fix an enumeration $\langle f_i : i \in \omega\rangle$ of Y for the whole of the argument. The 'indiscernibility' of Y alluded to above is formalized by the following definition and lemma.

Definition 2.3. Given a permutation $\sigma \in Sym(\omega)$, a group element $g \in G$ *respects* σ if $g \cdot [f_i] = [f_{\sigma(i)}]$ for every $i \in \omega$.

Lemma 2.4. *For every permutation $\sigma \in Sym(\omega)$, there is some $g \in G$ respecting σ.*

Proof. Note that since h is increasing, $h(n) \geq n$ for every $n \in \omega$. Fix a permutation $\sigma \in Sym(\omega)$ and we will define some $g \in G$ respecting σ coordinate-wise. Using Lemma 2.2, choose a sequence

$$0 = N_0 \ll N_1 \ll N_2 \ll \dots$$

of integers such that for all $i \in \omega$, both $f_i(n) \neq f_j(n)$ and $f_{\sigma(i)}(n) \neq f_{\sigma(j)}(n)$ hold for all $n \geq N_i$ and all $j < i$.

Since $\{N_i\}$ are increasing, it follows that for each $i \in \omega$ and all $n \geq N_i$, the subsets $\{f_j(n) : j \leq i\}$ and $\{f_{\sigma(j)}(n) : j \leq i\}$ of $[h(n)]$ each have precisely $(i+1)$ elements. Thus, for each $i < \omega$ and for each $n \geq N_i$, there is a permutation $\delta_n \in Sym([h(n)])$ satisfying

$$\bigwedge_{j \leq i} \delta_n(f_j(n)) = f_{\sigma(j)}(n)$$

[Simply begin defining δ_n to meet these constraints, and then complete δ_n to a permutation of $[h(n)]$ arbitrarily.] Using this, define $g := \langle \delta_n : n \in \omega \rangle$, where each $\delta_n \in Sym([h(n)])$ is constructed as above. To see that g respects σ, note that for every $i \in \omega$, $(g \cdot f_i)(n) = f_{\sigma(i)}(n)$ for all $n \geq N_i$, so $(g \cdot f_i) \sim f_{\sigma(i)}$. $\square$

Definition 2.5. For distinct integers $i \neq j$, let $d_{i,j} \in \mathcal{B}$ be defined by:

$$d_{i,j}(n) := \begin{cases} f_i(n) & \text{if } n \text{ even;} \\ f_j(n) & \text{if } n \text{ odd.} \end{cases}$$

Let $Z := \{d_{i,j} : i \neq j\}$.

Note that $d_{i,j} \not\sim f_k$ for all distinct i, j and all $k \in \omega$, hence $\{[f_i] : i \in \omega\}$ and $\{[d_{i,j}] : i \neq j\}$ are disjoint.

Lemma 2.6. *For all $\sigma \in Sym(\omega)$, if $g \in G$ respects σ, then $g \cdot [d_{i,j}] = [d_{\sigma(i),\sigma(j)}]$ for all $i \neq j$.*

Proof. Choose $\sigma \in Sym(\omega)$, g respecting σ, and $i \neq j$. Choose N such that $(g \cdot [f_i])(n) = [f_{\sigma(i)}](n)$ and $(g \cdot [f_j])(n) = [f_{\sigma(j)}](n)$ for every $n \geq N$. Since $d_{i,j}(n) = f_i(n)$ for $n \geq N$ even,

$$(g \cdot d_{i,j})(n) = \pi_n(g)(d_{i,j}(n)) = \pi_n(g)(f_i(n)) = (g \cdot f_i)(n) = f_{\sigma(i)}(n)$$

Dually, $(g \cdot d_{i,j})(n) = f_{\sigma(j)}(n)$ when $n \geq N$ is odd, so $(g \cdot d_{i,j}) \sim d_{\sigma(i),\sigma(j)}$. $\square$

With the combinatorial preliminaries out of the way, we now prove that T_h is Borel complete. We form a highly homogeneous model $M^* \models T_h$ and thereafter, all models we consider will be countable, elementary substructures of M^*. Let $A = \{a_f : f \in \mathcal{B}\}$ and $B = \{b_f : f \in \mathcal{B}\}$ be disjoint sets and let M^* be the L-structure with universe $A \cup B$ and each E_n interpreted by the rules:

- For all $f \in \mathcal{B}$ and $n \in \omega$, $E_n(a_f, b_f)$; and
- For all $f, f' \in \mathcal{B}$ and $n \in \omega$, $E_n(a_f, a_{f'})$ iff $f(n) = f'(n)$.

with the other instances of E_n following by symmetry and transitivity. For any finite $F \subseteq \omega$, $\{f\restriction_F : f \in \mathcal{B}\}$ has exactly $\Pi_{n\in F} h(n)$ elements, hence $E_F(x, y) := \bigwedge_{n\in F} E_n(x, y)$ has $\Pi_{n\in F} h(n)$ classes in M^*. Thus, the $\{E_n : n \in \omega\}$ cross cut and $M^* \models T_h$.

Let $E_\infty(x, y)$ denote the (type definable) equivalence relation $\bigwedge_{n\in\omega} E_n(x, y)$. Then, in M^*, E_∞ partitions M^* into 2-element classes $\{a_f, b_f\}$, indexed by $f \in \mathcal{B}$. Note also that every $g \in G$ induces an L-automorphism $g^* \in Aut(M^*)$ by

$$g^*(x) := \begin{cases} a_{(g\cdot f)} & \text{if } x = a_f \text{ for some } f \in \mathcal{B} \\ b_{(g\cdot f)} & \text{if } x = b_f \text{ for some } f \in \mathcal{B} \end{cases}$$

Recall the set $Y = \{f_i : i \in \mathcal{B}\}$ from Lemma 2.2, so $[Y] = \{[f_i] : i \in \omega\}$. Let $M_0 \subseteq M^*$ be the substructure with universe $\{a_f : f \in [Y]\}$. As T_h admits elimination of quantifiers and as $[Y]$ is dense in $\mathcal{B}$, $M_0 \preceq M^*$. Moreover, every substructure M of M^* with universe containing M_0 will also be an elementary substructure of M^*, hence a model of T_h.

To show that $Mod(T_h)$ is Borel complete, we define a Borel mapping from {irreflexive graphs $\mathcal{G} = (\omega, R)$} to $Mod(T_h)$ as follows: Given $\mathcal{G}$, let $Z(R) := \{d_{i,j} \in Z : \mathcal{G} \models R(i, j)\}$, so $[Z(R)] = \bigcup\{[d_{i,j}] : d_{i,j} \in Z(R)\}$. Let $M_G \preceq M^*$ be the substructure with universe

$$M_0 \cup \{a_d, b_d : d \in [Z(R)]\}$$

That the map $\mathcal{G} \mapsto M_G$ is Borel is routine, given that Y and Z are fixed throughout.

Note that in M_G, every E_∞-class has has either one or two elements. Specifically, for each $d \in [Z(R)]$, the E_∞-class $[a_d]_\infty = \{a_d, b_d\}$, while the E_∞-class $[a_f]_\infty = \{a_f\}$ for every $f \in [Y]$.

We must show that for any two graphs $\mathcal{G} = (\omega, R)$ and $\mathcal{H} = (\omega, S)$, $\mathcal{G}$ and $\mathcal{H}$ are isomorphic if and only if the L-structures M_G and M_H are isomorphic.

To verify this, first choose a graph isomorphism $\sigma : (\omega, R) \to (\omega, S)$. Then $\sigma \in Sym(\omega)$ and, for distinct integers $i \neq j$, $d_{i,j} \in Z(R)$ if and only if $d_{\sigma(i),\sigma(j)} \in Z(S)$. Apply Lemma 2.4 to get $g \in G$ respecting σ and let $g^* \in Aut(M^*)$ be the L-automorphism induced by g. By Lemma 2.6 and Definition 2.3, it is easily checked that the restriction of g^* to M_G is an L-isomorphism between M_G and M_H.

Conversely, assume that $\Psi : M_G \to M_H$ is an L-isomorphism. Clearly, Ψ maps E_∞-classes in M_G to E_∞-classes in M_H. In particular, Ψ permutes the 1-element E_∞-classes $\{\{a_f\} : f \in [Y]\}$ of both M_G and M_H, and maps the 2-element E_∞-classes $\{\{a_d, b_d\} : d \in [Z(R)]\}$ of M_G onto the 2-element E_∞-classes $\{\{a_d, b_d\} : d \in [Z(S)]\}$ of M_H. That is, Ψ induces a bijection $F : [Y \sqcup Z(R)] \to [Y \sqcup Z(S)]$ that permutes $[Y]$.

As well, by the interpretations of the E_n's, for $f, f' \in [Y \sqcup Z(R)]$ and $n \in \omega$,

$$f(n) = f'(n) \quad \text{if and only if} \quad F(f)(n) = F(f')(n).$$

From this it follows that F maps $\sim$-classes onto $\sim$-classes. As F permutes $[Y]$ and as $[Y] = \bigcup\{[f_i] : i \in \omega\}$, F induces a permutation $\sigma \in Sym(\omega)$ given by $\sigma(i)$ is the unique $i^* \in \omega$ such that $F([f_i]) = [f_{i^*}]$.

We claim that this σ induces a graph isomorphism between $\mathcal{G} = (\omega, R)$ and $\mathcal{H} = (\omega, S)$. Indeed, choose any $(i, j) \in R$. Thus, $d_{i,j} \in Z(R)$. As F is $\sim$-preserving, choose N large enough so that $F(f_i)(n) = F(f_{\sigma(i)})(n)$ and $F(f_j)(n) = F(f_{\sigma(j)})(n)$ for every $n \geq N$. By definition of $d_{i,j}$, $d_{i,j}(n) = f_i(n)$ for $n \geq N$ even, so $F(d_{i,j})(n) = F(f_i)(n) = f_{\sigma(i)}(n)$ for such n. Dually, for $n \geq N$ odd, $F(d_{i,j})(n) = F(f_j)(n) = f_{\sigma(j)}(n)$. Hence, $F(d_{i,j}) \sim d_{\sigma(i),\sigma(j)} \in [Z(S)]$. Thus, $(\sigma(i), \sigma(j)) \in S$. The converse direction is symmetric (i.e., use Ψ^{-1} in place of Ψ and run the same argument). $\square$

Remark 2.7. If we relax the assumption that $h : \omega \to \omega \setminus \{0\}$ is strictly increasing, there are two cases. If h is unbounded, then the proof given above can easily be modified to show that the associated T_h is also Borel complete. Conversely, with Theorem 6.2 of [6] the authors prove that if $h : \omega \to \omega \setminus \{0\}$ is bounded, then T_h is not Borel complete. The salient distinction between the two cases is that when h is bounded, the associated group G has bounded exponent. However, even in the bounded case T_h has a Borel complete reduct by Lemma 3.1 below.

3 Applications to reducts

We begin with one easy lemma that, when considering reducts, obviates the need for the number of classes to be strictly increasing.

Lemma 3.1. *Let $L = \{E_n : n \in \omega\}$ and let $f : \omega \to \omega \setminus \{0, 1\}$ be any function. Then every model M of T_f, the complete theory asserting that each E_n is an equivalence relation with $f(n)$ classes, and that the $\{E_n\}$ cross-cut, has a Borel complete reduct.*

Proof. Given any function $f : \omega \to \omega \setminus \{0, 1\}$, choose a partition $\omega = \bigsqcup\{F_n : n \in \omega\}$ into non-empty finite sets for which $\Pi_{k\in F_n} f(k) < \Pi_{k\in F_m} f(k)$ whenever $n < m < \omega$. For each n, let $h(n) := \Pi_{k\in F_n} f(k)$ and let $E^*_n(x, y) := \bigwedge_{k\in F_n} E_k(x, y)$. Then, as h is strictly increasing and $\{E^*_n\}$ is a cross-cutting set of equivalence relations with each E^*_n having $h(n)$ classes.

Now let $M \models T_f$ be arbitrary and let $L' = \{E^*_n : n \in \omega\}$. As each E^*_n described above is 0-definable in M, there is an L'-reduct M' of M. It follows from Theorem 2.1 that $T' = Th(M')$ is Borel complete, so T_f has a Borel complete reduct. $\square$

Theorem 3.2. *Suppose T is a complete theory in a countable language with uncountably many 1-types. Then every model M of T has a Borel complete reduct.*

Proof. Let $M \models T$ be arbitrary. As usual, by the Cantor-Bendixon analysis of the compact, Hausdorff Stone space $S_1(T)$ of complete 1-types, choose a set $\{\varphi_\eta(x) : \eta \in 2^{<\omega}\}$ of 0-definable formulas, indexed by the tree $(2^{<\omega}, \trianglelefteq)$ ordered by initial segment, satisfying:

1. $M \models \exists x \varphi_\eta(x)$ for each $\eta \in 2^{<\omega}$;

2. For $\nu \trianglelefteq \eta$, $M \models \forall x(\varphi_\eta(x) \to \varphi_\nu(x))$;

3. For each $n \in \omega$, $\{\varphi_\eta(x) : \eta \in 2^n\}$ are pairwise contradictory.

By increasing these formulas slightly, we can additionally require

4. For each $n \in \omega$, $M \models \forall x(\bigvee_{\eta\in 2^n} \varphi_\eta(x))$.

Given such a tree of formulas, for each $n \in \omega$, define

$$\delta^0_n(x) := \bigwedge_{\eta\in 2^n} [\varphi_\eta(x) \to \varphi_{\eta\hat{\ }0}(x)] \quad \text{and} \quad \delta^1_n(x) := \bigwedge_{\eta\in 2^n} [\varphi_\eta(x) \to \varphi_{\eta\hat{\ }1}(x)]$$

Because of (4) above, $M \models \forall x(\delta^0_n(x) \vee \delta^1_n(x))$ for each n. Also, for each n, let

$$E_n(x, y) := [\delta^0_n(x) \leftrightarrow \delta^0_n(y)]$$

From the above, each E_n is a 0-definable equivalence relation with precisely two classes.

Claim. The equivalence relations $\{E_n : n \in \omega\}$ are cross-cutting.

Proof. It suffices to prove that for every $m > 0$, the equivalence relation $E^*_m(x, y) := \bigwedge_{n<m} E_n(x, y)$ has 2^m classes. So fix m and choose a subset $A_m = \{a_\eta : \eta \in 2^m\} \subseteq M$ forming a set of representatives for the formulas $\{\varphi_\eta(x) : \eta \in 2^m\}$. It suffices to show that $M \models \neg E^*_m(a_\eta, a_\nu)$ whenever $\eta \neq \nu$ are from 2^m. But this is clear. Fix distinct $\eta \neq \nu$ and choose any $k < m$ such that $\eta(k) \neq \nu(k)$. Then $M \models \neg E_k(a_\eta, a_\nu)$, hence $M \models \neg E^*_m(a_\eta, a_\nu)$. $\square$

Thus, taking the 0-definable relations $\{E_n\}$, M has a reduct that is a model of T_f (where f is the constant function 2). As reducts of reducts are reducts, it follows from Lemma 3.1 and Theorem 2.1 that M has a Borel complete reduct. $\square$

We highlight how unexpected Theorem 3.2 is with two examples. First, the theory of 'Independent unary predicates' mentioned in the Introduction has a Borel complete reduct.

Next, we explore the assumption that a countable, complete theory T is not small, i.e., for some k there are uncountably many k-types. We conjecture that some model of T has a Borel complete reduct. If $k = 1$, then by Theorem 3.2, every model of T has a Borel complete reduct. If $k > 1$ is least, then it is easily seen that there is some complete $(k-1)$ type $p(x_1, \ldots, x_{k-1})$ with uncountably many complete $q(x_1, \ldots, x_k)$ extending p. Thus, if M is any model of T realizing p, say by $\bar{a} = (a_1, \ldots, a_{k-1})$, the expansion $(M, a_1, \ldots, a_{k-1})$ has a Borel complete reduct, also by Theorem 3.2. Similarly, we have the following result.

Corollary 3.3. *Suppose T is a complete theory in a countable language that is not small. Then for any model M of T, M^{eq} has a Borel complete reduct.*

Proof. Let M be any model of T and choose k least such that T has uncountably many complete k-types consistent with it. In the language L^{eq}, there is a sort U_k and a definable bijection $f : M^k \to U_k$. Hence $Th(M^{eq})$ has uncountably many 1-types consistent with it, each extending U_k. Thus, M^{eq} has a Borel complete reduct by Theorem 3.2. $\square$

Finally, recall that a countable, complete theory is not ω-stable if, for some countable model M of T, the Stone space $S_1(M)$ is uncountable. From this, we immediately obtain our final corollary.

Corollary 3.4. *If a countable, complete T is not ω-stable, then for some countable model M of T, the elementary diagram of M in the language $L(M) = L \cup \{c_m : m \in M\}$ has a Borel complete reduct.*

Proof. Choose a countable M so that $S_1(M)$ is uncountable. Then, in the language $L(M)$, the theory of the expanded structure M_M in the language $L(M)$ has uncountably many 1-types, hence it has a Borel complete reduct by Theorem 3.2. $\square$

The results above are by no means characterizations. Indeed, there are many Borel complete ω-stable theories. In [5], the first author and Shelah prove that any ω-stable theory that has eni-DOP or is eni-deep is not only Borel complete, but also λ-Borel complete for all λ.[2] As well, there are ω-stable theories with only countably many countable models

[2]Definitions of eni-DOP and eni-deep are given in Definitions 2.3 and 6.2, respectively, of [5], and the definition of λ-Borel complete is recalled in Section 4 of this paper.

that have Borel complete reducts. To illustrate this, we introduce three interrelated theories. The first, T_0 in the language $L_0 = \{U, V, W, R\}$ is the paradigmatic DOP theory. T_0 asserts that:

- U, V, W partition the universe;
- $R \subseteq U \times V \times W$;
- $T_0 \models \forall x \forall y \exists^{\infty} z R(x,y,z)$; [more formally, for each n, $T_0 \models \forall x \forall y \exists^{\geq n} z R(x,y,z)$];
- $T_0 \models \forall x \forall x' \forall y \forall y' \forall z [R(x,y,z) \wedge R(x',y',z) \rightarrow (x = x' \wedge y = y')]$.

T_0 is both ω-stable and ω-categorical and its unique countable model is rather tame. The complexity of T_0 is only witnessed with uncountable models, where one can code arbitrary bipartite graphs in an uncountable model M by choosing the cardinalities of the sets $R(a,b,M)$ among $(a,b) \in U \times V$ to be either $\aleph_0$ or $|M|$.

To get bad behavior of countable models, we expand T_0 to an $L = L_0 \cup \{f_n : n \in \omega\}$-theory $T \supseteq T_0$ that additionally asserts:

- Each $f_n : U \times V \rightarrow W$;
- $\forall x \forall y R(x, y, f_n(x,y))$ for each n; and
- for distinct $n \neq m$, $\forall x \forall y (f_n(x,y) \neq f_m(x,y))$.

This T is ω-stable with eni-DOP and hence is Borel complete by Theorem 4.12 of [5].

However, T has an expansion T^* in a language $L^* := L \cup \{c, d, g, h\}$ whose models are much better behaved. Let T^* additionally assert:

- $U(c) \wedge V(d)$;
- $g : U \rightarrow V$ is a bijection with $g(c) = d$;
- Letting $W^* := \{z : R(c,d,z)\}$, $h : U \times V \times W^* \rightarrow W$ is an injective map that is the identity on W^* and, for each $(x,y) \in U \times V$, maps W^* onto $\{z \in W : R(x,y,z)\}$; and moreover
- h commutes with each f_n, i.e., $\forall x \forall y (h(x,y,f_n(c,d)) = f_n(x,y))$.

Then T^* is ω-stable and two-dimensional (the dimensions being $|U|$ and $|W^* \setminus \{f_n(c,d) : n \in \omega\}|$), hence T^* has only countably many countable models. However, T^* visibly has a Borel complete reduct, namely T.

4 Observations about the theories T_h

In addition to their utility in proving Borel complete reducts, the theories T_h in Section 2 illustrate some novel behaviors. First off, model theoretically, these theories are extremely simple. More precisely, each theory T_h is weakly minimal with the geometry of every strong type trivial (such theories are known as mutually algebraic in [4]).

Additionally, the theories T_h are the simplest known examples of theories that are Borel complete, but not λ-Borel complete for all cardinals λ. For λ any infinite cardinal, λ-Borel completeness was introduced in [5]. Instead of looking at L-structures with universe ω, we consider X_L^λ, the set of L-structures with universe λ. We topologize X_L^λ analogously; namely a basis consists of all sets

$$U_{\varphi(\alpha_1,\ldots,\alpha_n)} := \{M \in X_L^\lambda : M \models \varphi(\alpha_1, \ldots, \alpha_n)\}$$

for all L-formulas $\varphi(x_1, \ldots, x_n)$ and all $(\alpha_1, \ldots, \alpha_n) \in \lambda^n$. Define a subset of X_L^λ to be λ-*Borel* if it is is the smallest λ^+-algebra containing the basic open sets, and call a function $f : X_{L_1}^\lambda \to X_{L_2}^\lambda$ to be λ-*Borel* if the inverse image of every basic open set is λ-Borel. For T, S theories in languages L_1, L_2, respectively we say that $\mathrm{Mod}_\lambda(T)$ is λ-*Borel reducible* to $\mathrm{Mod}_\lambda(S)$ if there is a λ-Borel $f : \mathrm{Mod}_\lambda(T) \to \mathrm{Mod}_\lambda(S)$ preserving back-and-forth equivalence in both directions (i.e., $M \equiv_{\infty,\omega} N \Leftrightarrow f(M) \equiv_{\infty,\omega} f(N)$).

As back-and-forth equivalence is the same as isomorphism for countable structures, λ-Borel reducibility when $\lambda = \omega$ is identical to Borel reducibility. As before, for any infinite λ, there is a maximal class under λ-Borel reducibility, and we say a theory is λ-*Borel complete* if it is in this maximal class. All of the 'classical' Borel complete theories, e.g., graphs, linear orders, groups, and fields, are λ-Borel complete for all λ. However, the theories T_h are not.

Lemma 4.1. *If T is mutually algebraic in a countable language, then there are at most $\beth_2$ pairwise $\equiv_{\infty,\omega}$-inequivalent models (of any size).*

Proof. We show that every model M has an (∞, ω)-elementary substructure of size $2^{\aleph_0}$, which suffices. So, fix M and choose an arbitrary countable $M_0 \preceq M$. By Propositon 4.4 of [4], $M \setminus M_0$ can be decomposed into countable components, and any permutation of isomorphic components induces an automorphism of M fixing M_0 pointwise. As there are at most $2^{\aleph_0}$ non-isomorphic components over M_0, choose a substructure $N \subseteq M$ containing M_0 and, for each isomorphism type of a component, N contains either all of copies in M (if there are only finitely many) or else precisely $\aleph_0$ copies if M contains infinitely many copies. It is easily checked that $N \preceq_{\infty,\omega} M$. $\square$

Corollary 4.2. *No mutually algebraic theory T in a countable language is λ-Borel complete for $\lambda \geq \beth_2$. In particular, T_h is Borel complete, but not λ-Borel complete for large λ.*

Proof. Fix $\lambda \geq \beth_2$. It is readily checked that there is a family of 2^λ graphs that are pairwise not back and forth equivalent. As there are fewer than 2^λ $\equiv_{\infty,\omega}$-classes of models of T, there cannot be a λ-Borel reduction of graphs into $\mathrm{Mod}_\lambda(T)$. $\square$

In [8], another example of a Borel complete theory that is not λ-Borel complete for all λ is given (it is dubbed TK there) but the T_h examples are cleaner. In order to understand this behavior, in [8] we call a theory T *grounded* if every potential canonical Scott sentence σ of a model of T (i.e., in some forcing extension $\mathbb{V}[G]$ of $\mathbb{V}$, σ is a canonical Scott sentence of some model, then σ is a canonical Scott sentence of a model in $\mathbb{V}$. Proposition 5.1 of [8] proves that every theory of refining equivalence relations is grounded. By contrast, we have

Proposition 4.3. *If T is Borel complete with a cardinal bound on the number of $\equiv_{\infty,\omega}$-classes of models, then T is not grounded. In particular, T_h is not grounded.*

Proof. Let κ denote the number of $\equiv_{\infty,\omega}$-classes of models of T. If T were grounded, then κ would also bound the number of potential canonical Scott sentences. As the class of graphs has a proper class of potential canonical Scott sentences, it would follow from Theorem 3.10 of [8] that T could not be Borel complete. $\square$

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