

MODAL OPERATORS ON RINGS OF CONTINUOUS FUNCTIONS

G. BEZHANISHVILI, L. CARAI, P. J. MORANDI

ABSTRACT. It is a classic result in modal logic that the category of modal algebras is dually equivalent to the category of descriptive frames. The latter are Kripke frames equipped with a Stone topology such that the binary relation is continuous. This duality generalizes the celebrated Stone duality. Our goal is to further generalize descriptive frames so that the topology is an arbitrary compact Hausdorff topology. For this, instead of working with the boolean algebra of clopen subsets of a Stone space, we work with the ring of continuous real-valued functions on a compact Hausdorff space. The main novelty is to define a modal operator on such a ring utilizing a continuous relation on a compact Hausdorff space.

Our starting point is the well-known Gelfand duality between the category \mathbf{KHaus} of compact Hausdorff spaces and the category \mathbf{ubal} of uniformly complete bounded archimedean ℓ -algebras. We endow a bounded archimedean ℓ -algebra with a modal operator, which results in the category \mathbf{mbal} of modal bounded archimedean ℓ -algebras. Our main result establishes a dual adjunction between \mathbf{mbal} and the category \mathbf{KHF} of what we call compact Hausdorff frames; that is, Kripke frames equipped with a compact Hausdorff topology such that the binary relation is continuous. This dual adjunction restricts to a dual equivalence between \mathbf{KHF} and the reflective subcategory \mathbf{mubal} of \mathbf{mbal} consisting of uniformly complete objects of \mathbf{mbal} . This generalizes both Gelfand duality and the duality for modal algebras.

1. INTRODUCTION

In modal logic there is a well-established duality theory between categories of Kripke frames and the corresponding categories of boolean algebras with operators, which forms the backbone of modern studies of modal logic. One of the most fundamental such dualities establishes that the category of modal algebras is dually equivalent to the category of descriptive frames. This duality originates in the works of Jónsson and Tarski [19], Halmos [15], and Kripke [24]. In its current form it was developed by Esakia [11] and Goldblatt [14]. For a modern account we refer to [27] or the textbooks [9, 22, 8].

This duality generalizes the celebrated Stone duality between the categories of boolean algebras and Stone spaces (zero-dimensional compact Hausdorff spaces). Descriptive frames are Stone spaces equipped with a continuous relation. It is well known that a binary relation R on a Stone space X is continuous iff the corresponding map from X to the Vietoris space $\mathcal{V}X$, given by sending each $x \in X$ to its R -image, is a well-defined continuous map (see [11, Sec. 1] or [25, Sec. 3]). Since the Vietoris space $\mathcal{V}X$ of a compact Hausdorff space X is compact Hausdorff, the above consideration allows us to generalize the notion of a

2010 *Mathematics Subject Classification.* 03B45; 54C30; 06F25; 06E25; 06E15.

Key words and phrases. modal algebra, Kripke frame, real-valued function, ℓ -algebra, compact Hausdorff space, continuous relation.

descriptive frame to what we call a *compact Hausdorff frame*; that is, a compact Hausdorff space equipped with a continuous relation. The category **KHF** of compact Hausdorff frames was studied in [4] where Isbell [17] and de Vries [10] dualities for the category **KHaus** of compact Hausdorff spaces were generalized to **KHF**.

One of the best known (and oldest) dualities for **KHaus** is Gelfand duality, which establishes that **KHaus** is dually equivalent to the category *ubal* of uniformly complete bounded archimedean ℓ -algebras (see Section 2 for details). This duality is obtained by associating to each compact Hausdorff space X the ring $C(X)$ of continuous real-valued functions on X . For some time now there has been a desire to generalize Gelfand duality to a duality for **KHF**, but it remained elusive for at least two reasons. On the conceptual side, there was no agreement on what should be the definition of modal operators on the ring $C(X)$. On the technical side, it was unclear how to axiomatize attempted definitions of modal operators.

The goal of this paper is to resolve these issues. After recalling Gelfand duality, we define a modal operator on the ring $C(X)$ for each compact Hausdorff frame (X, R) , and study its basic properties. This motivates the definition of a modal operator on an arbitrary bounded archimedean ℓ -algebra, which is the main definition of the paper, giving rise to the category *mbal* of modal bounded archimedean ℓ -algebras. We show that there is a contravariant functor $(-)^*$ from **KHF** to *mbal*.

Next we define a contravariant functor $(-)_* : \mathit{mbal} \rightarrow \mathbf{KHF}$ in the opposite direction. Proving that $(-)_* : \mathit{mbal} \rightarrow \mathbf{KHF}$ is well defined is technically the most challenging part of the paper. Our main result establishes that the contravariant functors $(-)^*$ and $(-)_*$ yield a dual adjunction between *mbal* and **KHF**, which restricts to a dual equivalence between **KHF** and the reflective subcategory *mubal* of *mbal* consisting of uniformly complete objects of *mbal*.

Our result generalizes both Gelfand duality and the duality between modal algebras and descriptive frames. We also take first steps in developing correspondence theory for *mbal* by characterizing the classes of algebras in *mbal* such that the corresponding relations on the dual side are serial, reflexive, transitive, or symmetric. We conclude the paper outlining several possible future directions of this line of research.

2. GELFAND DUALITY

Gelfand duality has a long history. In [12], by working with continuous complex-valued functions, Gelfand and Naimark established that **KHaus** is dually equivalent to the category of commutative C^* -algebras. Independently, Stone [28] worked with continuous real-valued functions and established that **KHaus** is dually equivalent to the category of uniformly complete bounded archimedean ℓ -algebras. The two dualities are closely related as the two categories of algebras are equivalent, which can be seen directly without passing to **KHaus**. Indeed, the self-adjoint elements of a commutative C^* -algebra form an algebra that Stone worked with, and each such algebra A gives rise to a commutative C^* -algebra by taking the complexification $A \otimes_{\mathbb{R}} \mathbb{C}$ (see [5, Sec. 7] for details). Because of this, these two dualities are sometimes called by the unifying name of Gelfand-Naimark-Stone duality. We follow [18,

Sec. IV.4] in calling this Gelfand duality, although our approach is more closely related to Stone's.

We start by recalling several basic definitions (see [7, Ch. XIII and onwards] or [5]). All rings that we will consider in this paper are commutative and unital (have multiplicative identity 1).

Definition 2.1.

- (1) A ring A with a partial order \leq is an ℓ -ring (that is, a *lattice-ordered ring*) if (A, \leq) is a lattice, $a \leq b$ implies $a + c \leq b + c$ for each c , and $0 \leq a, b$ implies $0 \leq ab$.
- (2) An ℓ -ring A is *bounded* if for each $a \in A$ there is $n \in \mathbb{N}$ such that $a \leq n \cdot 1$ (that is, 1 is a *strong order unit*).
- (3) An ℓ -ring A is *archimedean* if for each $a, b \in A$, whenever $n \cdot a \leq b$ for each $n \in \mathbb{N}$, then $a \leq 0$.
- (4) An ℓ -ring A is an ℓ -algebra if it is an \mathbb{R} -algebra and for each $0 \leq a \in A$ and $0 \leq \lambda \in \mathbb{R}$ we have $0 \leq \lambda \cdot a$.
- (5) Let \mathbf{bal} be the category of bounded archimedean ℓ -algebras and unital ℓ -algebra homomorphisms.

Let $A \in \mathbf{bal}$. For $a \in A$, define the *absolute value* of a by

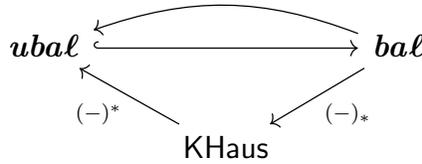
$$|a| = a \vee (-a)$$

and the *norm* of a by

$$\|a\| = \inf\{\lambda \in \mathbb{R} \mid |a| \leq \lambda\}.$$
¹

Then A is *uniformly complete* if the norm is complete. Let \mathbf{ubal} be the full subcategory of \mathbf{bal} consisting of uniformly complete ℓ -algebras.

Theorem 2.2 (Gelfand duality [12, 28]). *There is a dual adjunction between \mathbf{bal} and \mathbf{KHaus} which restricts to a dual equivalence between \mathbf{KHaus} and \mathbf{ubal} .*



The functors $(-)^* : \mathbf{KHaus} \rightarrow \mathbf{bal}$ and $(-)_* : \mathbf{bal} \rightarrow \mathbf{KHaus}$ establishing the dual adjunction are defined as follows. For a compact Hausdorff space X let X^* be the ring $C(X)$ of (necessarily bounded) continuous real-valued functions on X . For a continuous map $\varphi : X \rightarrow Y$ let $\varphi^* : C(Y) \rightarrow C(X)$ be defined by $\varphi^*(f) = f \circ \varphi$ for each $f \in C(Y)$. Then $(-)^* : \mathbf{KHaus} \rightarrow \mathbf{bal}$ is a well-defined contravariant functor.

For $A \in \mathbf{bal}$, we recall that an ideal I of A is an ℓ -ideal if $|a| \leq |b|$ and $b \in I$ imply $a \in I$, and that ℓ -ideals are exactly the kernels of ℓ -algebra homomorphisms. Let Y_A be the space of maximal ℓ -ideals of A , whose closed sets are exactly sets of the form

$$Z_\ell(I) = \{M \in Y_A \mid I \subseteq M\},$$

¹We view \mathbb{R} as an ℓ -subalgebra of A by identifying $\lambda \in \mathbb{R}$ with $\lambda \cdot 1 \in A$.

where I is an ℓ -ideal of A . The space Y_A is often referred to as the *Yosida space* of A , and it is well known that $Y_A \in \mathbf{KHaus}$. We then set $A_* = Y_A$. For a morphism α in \mathbf{bal} let $\alpha_* = \alpha^{-1}$. Then $(-)_* : \mathbf{bal} \rightarrow \mathbf{KHaus}$ is a well-defined contravariant functor, and the functors $(-)_*$ and $(-)^*$ yield a dual adjunction between \mathbf{bal} and \mathbf{KHaus} .

Moreover, for $X \in \mathbf{KHaus}$ we have that $\varepsilon_X : X \rightarrow (X^*)_*$ is a homeomorphism where

$$\varepsilon_X(x) = \{f \in C(X) \mid f(x) = 0\}.$$

Furthermore, for $A \in \mathbf{bal}$ define $\zeta_A : A \rightarrow (A_*)^*$ by $\zeta_A(a)(M) = \lambda$ where λ is the unique real number satisfying $a + M = \lambda + M$. Then ζ_A is a monomorphism in \mathbf{bal} separating points of Y_A . Therefore, by the Stone-Weierstrass theorem, we have:

Proposition 2.3.

- (1) *The uniform completion of $A \in \mathbf{bal}$ is $\zeta_A : A \rightarrow C(Y_A)$. Therefore, if A is uniformly complete, then ζ_A is an isomorphism.*
- (2) *\mathbf{ubal} is a reflective subcategory of \mathbf{bal} , and the reflector $\zeta : \mathbf{bal} \rightarrow \mathbf{ubal}$ assigns to each $A \in \mathbf{bal}$ its uniform completion $C(Y_A) \in \mathbf{ubal}$.*

Consequently, the dual adjunction restricts to a dual equivalence between \mathbf{ubal} and \mathbf{KHaus} , yielding Gelfand duality.

3. MODAL OPERATORS ON $C(X)$

In this section we define modal operators on rings of continuous real-valued functions on compact Hausdorff frames and study their basic properties. This motivates the definition of a modal operator on $A \in \mathbf{bal}$, giving rise to the category \mathbf{mbal} of modal bounded archimedean ℓ -algebras. We end the section by describing a contravariant functor from \mathbf{KHF} to \mathbf{mbal} .

We recall that a *Kripke frame* is a pair $\mathfrak{F} = (X, R)$ where X is a set and R is a binary relation on X . As usual, for $x \in X$ we write

$$R[x] = \{y \in X \mid xRy\} \quad \text{and} \quad R^{-1}[x] = \{y \in X \mid yRx\},$$

and for $U \subseteq X$ we write

$$R[U] = \bigcup \{R[u] \mid u \in U\} \quad \text{and} \quad R^{-1}[U] = \bigcup \{R^{-1}[u] \mid u \in U\}.$$

Definition 3.1. [4] A binary relation R on a compact Hausdorff space X is *continuous* if:

- (1) $R[x]$ is closed for each $x \in X$.
- (2) $F \subseteq X$ closed implies $R^{-1}[F]$ is closed.
- (3) $U \subseteq X$ open implies $R^{-1}[U]$ is open.

If R is a continuous relation on X , we call (X, R) a *compact Hausdorff frame*.

Notation 3.2. For a binary relation R on a set X let

$$\begin{aligned} D &= \{x \in X \mid R[x] \neq \emptyset\} = R^{-1}[X], \\ E &= X \setminus D = \{x \in X \mid R[x] = \emptyset\}. \end{aligned}$$

The next lemma is straightforward and we omit the proof.

Lemma 3.3. *If (X, R) is a compact Hausdorff frame, then D and E are clopen subsets of X .*

Definition 3.4. For a compact Hausdorff frame (X, R) , define \square_R on $C(X)$ by

$$(\square_R f)(x) = \begin{cases} \inf fR[x] & \text{if } x \in D \\ 1 & \text{otherwise.} \end{cases}$$

Remark 3.5. We define \diamond_R by

$$(\diamond_R f)(x) = \begin{cases} \sup fR[x] & \text{if } x \in D \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\diamond_R f = 1 - \square_R(1 - f) \quad \text{and} \quad \square_R f = 1 - \diamond_R(1 - f).$$

For, if $x \in D$, then

$$\begin{aligned} 1 - \square_R(1 - f)(x) &= 1 - \inf\{1 - f(y) \mid xRy\} = 1 - (1 - \sup\{f(y) \mid xRy\}) \\ &= \sup\{f(y) \mid xRy\} = \diamond_R f(x). \end{aligned}$$

If $x \in E$, then $(1 - \square_R(1 - f))(x) = 1 - 1 = 0 = (\diamond_R f)(x)$. Thus, $\diamond_R f = 1 - \square_R(1 - f)$, as desired. A similar argument yields $\square_R f = 1 - \diamond_R(1 - f)$. Therefore, each of \square_R and \diamond_R can be determined from the other.

Remark 3.6. Let (X, R) be a compact Hausdorff frame, $f \in C(X)$, and $x \in X$ with $R[x] \neq \emptyset$. Then $fR[x]$ is a nonempty compact subset of \mathbb{R} , and so it has least and greatest elements. Thus, we have

$$(\square_R f)(x) = \min fR[x] \quad \text{and} \quad (\diamond_R f)(x) = \max fR[x].$$

Lemma 3.7. *Let (X, R) be a compact Hausdorff frame. If $f \in C(X)$, then $\square_R f \in C(X)$.*

Proof. To see that $\square_R f$ is continuous, it is sufficient to show that for each $\lambda \in \mathbb{R}$, both $(\square_R f)^{-1}(\lambda, \infty)$ and $(\square_R f)^{-1}(-\infty, \lambda)$ are open in X . We first show that $(\square_R f)^{-1}(\lambda, \infty)$ is open. Let $x \in X$ and first suppose that $x \in D$. Then $fR[x]$ is a nonempty compact subset of \mathbb{R} , so it has a least element. Therefore,

$$\begin{aligned} x \in (\square_R f)^{-1}(\lambda, \infty) &\quad \text{iff} \quad (\square_R f)(x) > \lambda \\ &\quad \text{iff} \quad \min(fR[x]) > \lambda \\ &\quad \text{iff} \quad R[x] \subseteq f^{-1}(\lambda, \infty) \\ &\quad \text{iff} \quad x \in X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]. \end{aligned}$$

Next suppose that $x \in E$. Then $(\square_R f)(x) = 1$. Thus, $E \subseteq (\square_R f)^{-1}(\lambda, \infty)$ if $\lambda < 1$, and $E \cap (\square_R f)^{-1}(\lambda, \infty) = \emptyset$ otherwise. Consequently,

$$(\square_R f)^{-1}(\lambda, \infty) = [D \cap (X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)])] \cup E$$

if $\lambda < 1$, and

$$(\square_R f)^{-1}(\lambda, \infty) = D \cap (X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)])$$

if $1 \leq \lambda$. Since $f \in C(X)$ and R is continuous, $X \setminus R^{-1}[X \setminus f^{-1}(\lambda, \infty)]$ is open. Thus, $(\square_R f)^{-1}(\lambda, \infty)$ is open by Lemma 3.3.

We next show that $(\square_R f)^{-1}(-\infty, \lambda)$ is open. If $x \in D$, then

$$\begin{aligned} x \in (\square_R f)^{-1}(-\infty, \lambda) & \text{ iff } (\square_R f)(x) < \lambda \\ & \text{ iff } \min(fR[x]) < \lambda \\ & \text{ iff } R[x] \cap f^{-1}(-\infty, \lambda) \neq \emptyset \\ & \text{ iff } x \in R^{-1}[f^{-1}(-\infty, \lambda)]. \end{aligned}$$

If $\lambda \leq 1$, then $E \cap (\square_R f)^{-1}(-\infty, \lambda) = \emptyset$, and if $1 < \lambda$, then $E \subseteq (\square_R f)^{-1}(-\infty, \lambda)$. Therefore,

$$(\square_R f)^{-1}(-\infty, \lambda) = D \cap R^{-1}[f^{-1}(-\infty, \lambda)]$$

if $\lambda \leq 1$, and

$$(\square_R f)^{-1}(-\infty, \lambda) = [D \cap (R^{-1}[f^{-1}(-\infty, \lambda)])] \cup E$$

if $\lambda > 1$. Since $f \in C(X)$ and R is continuous, $R^{-1}[f^{-1}(-\infty, \lambda)]$ is open. Consequently, $(\square_R f)^{-1}(-\infty, \lambda)$ is open by Lemma 3.3. This completes the proof that if $f \in C(X)$, then $\square_R f \in C(X)$. \square

In the next lemma we describe the properties of \square_R . For this we recall (see, e.g., [5, Rem 2.2]) that if $A \in \mathbf{ba\ell}$ and $a \in A$, then the *positive* and *negative* parts of a are defined as

$$a^+ = a \vee 0 \quad \text{and} \quad a^- = -(a \wedge 0) = (-a) \vee 0.$$

Then $a^+, a^- \geq 0$, $a^+ \wedge a^- = 0$, $a = a^+ - a^-$, and $|a| = a^+ + a^-$.

Lemma 3.8. *Let (X, R) be a compact Hausdorff frame, $f, g \in C(X)$, and $\lambda \in \mathbb{R}$.*

- (1) $\square_R(f \wedge g) = \square_R f \wedge \square_R g$. In particular, \square_R is order preserving.
- (2) $\square_R \lambda = \lambda + (1 - \lambda)(\square_R 0)$. In particular, $\square_R 1 = 1$.
- (3) $\square_R(f^+) = (\square_R f)^+$.
- (4) $\square_R(f + \lambda) = \square_R f + \square_R \lambda - \square_R 0$.
- (5) If $0 \leq \lambda$, then $\square_R(\lambda f) = (\square_R \lambda)(\square_R f)$.

Proof. (1). For $x \in D$, we have

$$\begin{aligned} \square_R(f \wedge g)(x) &= \inf\{(f \wedge g)(y) \mid y \in R[x]\} = \inf\{\min\{f(y), g(y)\} \mid y \in R[x]\} \\ &= \min\{\inf\{f(y) \mid y \in R[x]\}, \inf\{g(y) \mid y \in R[x]\}\} \\ &= \min\{(\square_R f)(x), (\square_R g)(x)\} \\ &= (\square_R f \wedge \square_R g)(x). \end{aligned}$$

If $x \in E$, then $\square_R(f \wedge g)(x) = 1 = (\square_R f \wedge \square_R g)(x)$. Thus, $\square_R(f \wedge g) = \square_R f \wedge \square_R g$.

(2). For $x \in D$, if $\mu \in \mathbb{R}$, we have $(\square_R \mu)(x) = \inf\{\mu \mid y \in R[x]\} = \mu$. From this we see that $(\square_R \lambda)(x) = \lambda = (\lambda + (1 - \lambda)(\square_R 0))(x)$. If $x \in E$, then $(\square_R \lambda)(x) = 1 = (\lambda + (1 - \lambda)(\square_R 0))(x)$. Thus, $\square_R \lambda = \lambda = \lambda + (1 - \lambda)(\square_R 0)$. Setting $\lambda = 1$ yields $\square_R 1 = 1$.

(3). For $x \in D$, we have

$$\begin{aligned} (\Box_R(f^+))(x) &= \Box_R(f \vee 0)(x) = \inf\{\max\{f(y), 0\} \mid y \in R[x]\} \\ &= \max\{\inf\{f(y) \mid y \in R[x]\}, 0\} = \max\{\Box_R f(x), 0\} \\ &= (\Box_R f \vee 0)(x) = (\Box_R f)^+(x). \end{aligned}$$

If $x \in E$, then $(\Box_R(f^+))(x) = 1 = (\Box_R f)^+(x)$. Thus, $\Box_R(f^+) = (\Box_R f)^+$.

(4). For $x \in D$, we have

$$\begin{aligned} \Box_R(f + \lambda)(x) &= \inf\{f(y) + \lambda \mid y \in R[x]\} \\ &= \inf\{f(y) \mid y \in R[x]\} + \lambda \\ &= \Box_R f(x) + \lambda. \end{aligned}$$

On the other hand,

$$(\Box_R f + \Box_R \lambda - \Box_R 0)(x) = (\Box_R f)(x) + (\Box_R \lambda)(x) - (\Box_R 0)(x) = (\Box_R f)(x) + \lambda.$$

Therefore, $\Box_R(f + \lambda)(x) = (\Box_R f + \Box_R \lambda - \Box_R 0)(x)$. If $x \in E$, then $\Box_R(f + \lambda)(x) = 1 = (\Box_R f + \Box_R \lambda - \Box_R 0)(x)$. Thus, $\Box_R(f + \lambda) = \Box_R f + \Box_R \lambda - \Box_R 0$.

(5). Let $0 \leq \lambda$. For $x \in D$, we have

$$\begin{aligned} (\Box_R \lambda f)(x) &= \inf\{\lambda f(y) \mid y \in R[x]\} = \lambda \inf\{f(y) \mid y \in R[x]\} \\ &= \lambda(\Box_R f)(x) = (\Box_R \lambda)(x)(\Box_R f)(x) = (\Box_R \lambda \Box_R f)(x). \end{aligned}$$

If $x \in E$, then $(\Box_R \lambda f)(x) = 1 = (\Box_R \lambda)(\Box_R f)(x)$. Thus, $\Box_R(\lambda f) = (\Box_R \lambda)(\Box_R f)$. \square

Remark 3.9. Lemma 3.8 can be stated dually in terms of \Diamond_R as follows. Let (X, R) be a compact Hausdorff frame, $f, g \in C(X)$, and $\lambda \in \mathbb{R}$.

- (1) $\Diamond_R(f \vee g) = \Diamond_R f \vee \Diamond_R g$. In particular, \Diamond_R is order preserving.
- (2) $\Diamond_R \lambda = \lambda(\Diamond_R 1)$. In particular, $\Diamond_R 0 = 0$.
- (3) $\Diamond_R(f \wedge 1) = (\Diamond_R f) \wedge 1$.
- (4) $\Diamond_R(f + \lambda) = \Diamond_R f + \Diamond_R \lambda$.
- (5) If $0 \leq \lambda$, then $\Diamond_R(\lambda f) = \Diamond_R \lambda \Diamond_R f$.

The identities (1), (3), and (5) are direct translations of the corresponding identities for \Box_R . However, the identities (2) and (4) are simpler. We next show why \Diamond_R affords such simplifications.

For (2), since $\Diamond_R 1 = 1 - \Box_R 0$, by Lemma 3.8(2),

$$\Diamond_R \lambda = 1 - \Box_R(1 - \lambda) = 1 - (1 - \lambda + \lambda \Box_R 0) = \lambda(1 - \Box_R 0) = \lambda \Diamond_R 1.$$

For (4), by using (4) and (2) of Lemma 3.8, we have

$$\begin{aligned} \Diamond_R(f + \lambda) &= 1 - \Box_R(1 - (f + \lambda)) = 1 - \Box_R((1 - f) - \lambda) \\ &= 1 - (\Box_R(1 - f) + \Box_R(-\lambda) - \Box_R 0) = \Diamond_R f - \Box_R(-\lambda) + \Box_R 0 \\ &= \Diamond_R f - (-\lambda + (1 + \lambda)\Box_R 0) + \Box_R 0 = \Diamond_R f + \lambda(1 - \Box_R 0) = \Diamond_R f + \Diamond_R \lambda. \end{aligned}$$

In Remark 4.2 we explain why we prefer to work with \Box_R .

Lemmas 3.7 and 3.8 motivate the main definition of this paper.

Definition 3.10.

- (1) Let $A \in \mathbf{ba\ell}$. We say that a unary function $\square : A \rightarrow A$ is a *modal operator* on A provided \square satisfies the following axioms for each $a, b \in A$ and $\lambda \in \mathbb{R}$:
- (M1) $\square(a \wedge b) = \square a \wedge \square b$.
 - (M2) $\square \lambda = \lambda + (1 - \lambda)\square 0$.
 - (M3) $\square(a^+) = (\square a)^+$.
 - (M4) $\square(a + \lambda) = \square a + \square \lambda - \square 0$.
 - (M5) $\square(\lambda a) = (\square \lambda)(\square a)$ provided $\lambda \geq 0$.
- (2) If \square is a modal operator on $A \in \mathbf{ba\ell}$, then we call the pair $\mathfrak{A} = (A, \square)$ a *modal bounded archimedean ℓ -algebra*.
- (3) Let \mathbf{mbal} be the category of modal bounded archimedean ℓ -algebras and unital ℓ -algebra homomorphisms preserving \square .

Remark 3.11. We can define $\diamond : A \rightarrow A$ dual to \square by $\diamond a = 1 - \square(1 - a)$ for each $a \in A$. Then (A, \diamond) satisfies the axioms for \diamond dual to the ones for \square in Definition 3.10(1) (see Remark 3.9). While algebras in \mathbf{mbal} can be axiomatized either in the signature of \square or \diamond , we prefer to work with \square for the reasons given in Remark 4.2.

Remark 3.12. If $\square 0 = 0$, then (M2), (M4), and (M5) simplify to the following:

- (M2') $\square \lambda = \lambda$.
- (M4') $\square(a + \lambda) = \square a + \lambda$.
- (M5') $\square(\lambda a) = \lambda \square a$ provided $\lambda \geq 0$.

Moreover, (M2') follows from (M4') by setting $a = 0$. Furthermore, $\diamond a = -\square(-a)$. In Section 7 we will see that $\square 0 = 0$ holds iff the binary relation R_\square on the Yosida space of A is serial (see Definition 4.1).

Lemma 3.13. *Let $(A, \square) \in \mathbf{mbal}$, $a, b \in A$, and $\lambda \in \mathbb{R}$.*

- (1) $a \leq b$ implies $\square a \leq \square b$.
- (2) $\square 1 = 1$.
- (3) $a \geq 0$ implies $\square a \geq 0$.
- (4) $(\square 0)(\square a) = \square 0$. In particular, $\square 0$ is an idempotent.
- (5) $\square(a + \lambda) = \square a + \lambda(1 - \square 0)$.
- (6) $\diamond a = -\square(-a)(1 - \square 0)$.
- (7) $(\diamond a)(\square 0) = 0$.

Proof. (1). If $a \leq b$, then $a \wedge b = a$. Therefore, by (M1), $\square a = \square(a \wedge b) = \square a \wedge \square b$. Thus, $\square a \leq \square b$.

(2). This follows by substituting $\lambda = 1$ in (M2).

(3). From (M3) and $a \geq 0$ we have $\square a = \square(a^+) = (\square a)^+ \geq 0$.

(4). By (M5), $\square 0 = \square(0a) = (\square 0)(\square a)$. Setting $a = 0$ gives $(\square 0)^2 = \square 0$.

(5). By (M4), $\square(a + \lambda) = \square a + \square \lambda - \square 0$. By (M2), $\square \lambda = \lambda + (1 - \lambda)(\square 0) = \lambda(1 - \square 0) + \square 0$. Therefore, $\square \lambda - \square 0 = \lambda(1 - \square 0)$, and so (5) follows.

(6). By (M4), (2), and (4) we have

$$\begin{aligned} \diamond a &= 1 - \square(1 - a) = 1 - (\square(-a) + \square 1 - \square 0) \\ &= -\square(-a) + \square 0 = -\square(-a) + \square(-a)\square 0 \\ &= -\square(-a)(1 - \square 0). \end{aligned}$$

(7). Since $\square 0$ is an idempotent by (4), we have $(1 - \square 0)\square 0 = 0$. Multiplying both sides of (6) by $\square 0$ yields $\diamond a\square 0 = 0$. \square

As follows from Lemmas 3.7 and 3.8, if (X, R) is a compact Hausdorff frame, then $(C(X), \square_R) \in \mathbf{mbal}$. We now extend this correspondence to a contravariant functor. For this we recall the definition of a bounded morphism.

Definition 3.14.

- (1) A *bounded morphism* (or *p-morphism*) between Kripke frames $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ is a map $f : X \rightarrow Y$ satisfying $f(R[x]) = S[f(x)]$ for each $x \in X$ (equivalently, $f^{-1}(S^{-1}[y]) = R^{-1}[f^{-1}(y)]$ for each $y \in Y$).
- (2) Let **KHF** be the category of compact Hausdorff frames and continuous bounded morphisms.

Lemma 3.15. *If $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ are compact Hausdorff frames and $\varphi : X \rightarrow Y$ is a continuous bounded morphism, then φ^* is a morphism in \mathbf{mbal} .*

Proof. That φ^* is a **bal**-morphism follows from Gelfand duality. Therefore, it is sufficient to prove that φ^* preserves \square ; that is, $\varphi^*(\square_S f) = \square_R \varphi^*(f)$ for each $f \in C(Y)$. Since φ is a bounded morphism, $\varphi(R[x]) = S[\varphi(x)]$ for each $x \in X$. Let $x \in X$ and $f \in C(Y)$. If $R[x] \neq \emptyset$, then $S[\varphi(x)] \neq \emptyset$, so

$$\begin{aligned} \varphi^*(\square_S f)(x) &= (\square_S f \circ \varphi)(x) = (\square_S f)(\varphi(x)) = \inf(f(S[\varphi(x)])) \\ &= \inf(f(\varphi(R[x]))) = \inf((f \circ \varphi)(R[x])) = \square_R(f \circ \varphi)(x) \\ &= \square_R(\varphi^*(f))(x). \end{aligned}$$

If $R[x] = \emptyset$, then $S[\varphi(x)] = \emptyset$, so $\varphi^*(\square_S f)(x) = (\square_S f)(\varphi(x)) = 1 = (\square_R \varphi^*(f))(x)$. Thus, $\varphi^*(\square_S f) = \square_R \varphi^*(f)$. \square

Theorem 3.16. *There is a contravariant functor $(-)^* : \mathbf{KHF} \rightarrow \mathbf{mbal}$ which sends $\mathfrak{F} = (X, R)$ to $\mathfrak{F}^* = (C(X), \square_R)$ and a morphism φ in **KF** to φ^* .*

Proof. If $\mathfrak{F} \in \mathbf{KHF}$, then $\mathfrak{F}^* \in \mathbf{mbal}$ by Lemmas 3.7 and 3.8. If φ is a morphism in **KHF**, then φ^* is a morphism in \mathbf{mbal} by Lemma 3.15. It is elementary to see that $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$ and that $(-)^*$ preserves identity morphisms. Thus, $(-)^*$ is a contravariant functor. \square

4. CONTINUOUS RELATIONS ON THE YOSIDA SPACE

In this section we define a contravariant functor $(-)_* : \mathbf{mbal} \rightarrow \mathbf{KHF}$ in the other direction, which is technically the most involved part of the paper.

Let $A \in \mathbf{bal}$. For $S \subseteq A$ let

$$S^+ = \{a \in S \mid a \geq 0\}.$$

We point out that if I is an ℓ -ideal of A , then $I^+ = \{a^+ \mid a \in I\}$.

Definition 4.1. Let $(A, \square) \in \mathbf{mbal}$ and let Y_A be the Yosida space of A . Define R_\square on Y_A by

$$xR_\square y \text{ iff } \square y^+ \subseteq x, \text{ iff } y^+ \subseteq \square^{-1}x.$$

Remark 4.2. We have that $xR_\square y$ iff $(\forall a \geq 0)(a + y = 0 + y \Rightarrow \square a + x = 0 + x)$. If we work with \diamond instead of \square , since $\diamond a = 1 - \square(1 - a)$, the definition becomes $xR_\diamond y$ iff $(\forall b \leq 1)(b + y = 1 + y \Rightarrow \diamond b + x = 1 + x)$. Thus, $xR_\diamond y$ iff $\{1 - \diamond b \mid 1 - b \in y, b \leq 1\} \subseteq x$. This more complicated definition is one reason why we prefer to work with \square rather than \diamond . Another is that, as is standard in working with ordered algebras, using \square allows us to work with the positive cone rather than the set of elements below 1.

Let $A \in \mathbf{bal}$. We recall that the *zero set* of $a \in A$ is defined as

$$Z_\ell(a) = \{x \in Y_A \mid a \in x\}.$$

If $S \subseteq A$, then we set

$$Z_\ell(S) = \bigcap \{Z_\ell(a) \mid a \in S\} = \{x \in Y_A \mid S \subseteq x\}.$$

It is easy to see that if I is the ℓ -ideal generated by S , then $Z_\ell(S) = Z_\ell(I)$. We define the *cozero set* of S as

$$\text{coz}_\ell(S) = Y_A \setminus Z_\ell(S) = \{x \in Y_A \mid S \not\subseteq x\}.$$

Since the zero sets are exactly the closed sets, the cozero sets are exactly the open sets of Y_A . The family $\{\text{coz}_\ell(a) \mid a \in A\}$ then constitutes a basis for the topology on Y_A .

Remark 4.3. Let $A \in \mathbf{bal}$, Y_A be the Yosida space of A , $x \in Y_A$, and $a \in A$.

- (1) x is a prime ideal of A because $A/x \cong \mathbb{R}$ (see, e.g., [16, Cor. 2.7]).
- (2) Either $a^+ \in x$ or $a^- \in x$. This follows from (1) and $a^+a^- = 0$.
- (3) $a^+ \in x$ and $a^- \notin x$ iff $a + x < 0 + x$ (see [6, Rem. 2.11]).
- (4) $a^+ \in x$ iff $a + x \leq 0 + x$. For, if $a^+ \in x$, then $a + x = (a^+ - a^-) + x = -a^- + x \leq 0 + x$ since $a^- \geq 0$. Conversely, if $a + x \leq 0 + x$, then either $a + x < 0 + x$, in which case $a^+ \in x$ by (3), or $a + x = 0 + x$, in which case $a \in x$, so $a^+ \in x$.
- (5) $a^- \in x$ and $a^+ \notin x$ iff $a + x > 0 + x$ (see [6, Rem. 2.11]).
- (6) $a^- \in x$ iff $a + x \geq 0 + x$. The proof is similar to that of (4) but uses (5) instead of (3).

Recalling Notation 3.2, if (Y_A, R_\square) is the dual of $(A, \square) \in \mathbf{mbal}$, then we denote $R_\square^{-1}[Y_A]$ by D_A and $Y_A \setminus D_A$ by E_A .

Lemma 4.4. Let $(A, \square) \in \mathbf{mbal}$, $a \in A$, $\lambda \in \mathbb{R}$, and $x \in Y_A$.

- (1) If $x \in D_A$, then $\square 0 \in x$.
- (2) If $\square 0 \in x$, then $\square(a + \lambda) + x = (\square a + \lambda) + x$.
- (3) If $\square 0 \in x$, then $\square((a - \lambda)^+) \in x$ iff $(\square a - \lambda)^+ \in x$.
- (4) If $\square 0 \in x$, then $\diamond a + x = -\square(-a) + x$.
- (5) If $\square 0 \notin x$, then $1 - \square a \in x$.

(6) If $\diamond a \notin x$, then $\square 0 \in x$.

Proof. (1). If $x \in D_A$, then there is y with $xR_{\square}y$. Therefore, since $0 \in y^+$, we have $\square 0 \in x$.

(2). By (M4) and (M2), $\square(a+\lambda) = \square a + \lambda - \lambda \square 0$. Therefore, if $\square 0 \in x$, then $\square(a+\lambda) + x = (\square a + \lambda) + x$.

(3). This follows from (M3), Remark 4.3(4), and (2).

(4). Apply Lemma 3.13(6).

(5). By Lemma 3.13(4), $\square 0 = (\square 0)(\square a)$, so $(\square 0)(1 - \square a) = 0 \in x$. Since $\square 0 \notin x$ and x is a prime ideal, $1 - \square a \in x$.

(6). By Lemma 3.13(7), $(\diamond a)(\square 0) = 0 \in x$. Since x is a prime ideal and $\diamond a \notin x$, we have $\square 0 \in x$. \square

Proposition 4.5. $R_{\square}[x]$ is closed for every $x \in Y_A$.

Proof. We prove that $Y_A \setminus R_{\square}[x]$ is open for every $x \in Y_A$. Let $y \notin R_{\square}[x]$, so $y^+ \not\subseteq \square^{-1}x$. Therefore, there is $a \geq 0$ such that $a \in y$ and $\square a \notin x$. By Lemma 3.13(3), $\square a \geq 0$, so there is $0 \leq \lambda \in \mathbb{R}$ such that $(\square a - \lambda) + x > 0 + x$ but $(a - \lambda) + y < 0 + y$. By Remark 4.3(3), $(a - \lambda)^- \notin y$ and $(\square a - \lambda)^+ \notin x$. Thus, $y \in \text{coz}_{\ell}((a - \lambda)^-)$, and it remains to show that $\text{coz}_{\ell}((a - \lambda)^-) \cap R_{\square}[x] = \emptyset$. Suppose not. Then there is z such that $xR_{\square}z$ and $z \in \text{coz}_{\ell}((a - \lambda)^-)$. Since z is a prime ideal and $(a - \lambda)^- \notin z$, we have $(a - \lambda)^+ \in z$ (see Remark 4.3(2)). But $xR_{\square}z$ means $z^+ \subseteq \square^{-1}x$, so $\square 0, \square((a - \lambda)^+) \in x$. Thus, by (M3) and Lemma 4.4(3), $(\square a - \lambda)^+ \in x$, hence $(\square a - \lambda) + x \leq 0 + x$. The obtained contradiction proves that $\text{coz}_{\ell}((a - \lambda)^-) \cap R_{\square}[x] = \emptyset$, completing the proof. \square

For a topological space X and a continuous real-valued function f on X , we recall that the *zero set* of f is

$$Z(f) = \{x \in X \mid f(x) = 0\}$$

and the *cozero set* of f is

$$\text{coz}(f) = X \setminus Z(f) = \{x \in X \mid f(x) \neq 0\}.$$

The following lemma is a consequence of [13, Prob. 1D, p. 21].

Lemma 4.6. Let $A \in \mathbf{bal}$ and $a, s \in A$. If $Z_{\ell}(a) \subseteq \text{int } Z_{\ell}(s)$, then there is $f \in C(Y_A)$ such that $\zeta_A(s) = \zeta_A(a)f$ in $C(Y_A)$.

Proof. Observe that for each $t \in A$ we have $Z_{\ell}(t) = Z(\zeta_A(t))$. Therefore, $Z_{\ell}(a) \subseteq \text{int } Z_{\ell}(s)$ implies $Z(\zeta_A(a)) \subseteq \text{int } Z(\zeta_A(s))$. Now apply [13, Prob. 1D, p. 21]. \square

Lemma 4.7. Let $(A, \square) \in \mathbf{mbal}$, $x \in Y_A$, $S = (A \setminus \square^{-1}x)^+$, and $a \in (\square^{-1}x)^+$.

(1) $\bigcap \{\overline{\text{coz}_{\ell}(s)} \mid s \in S\} = \bigcap \{\overline{\text{coz}_{\ell}(s)} \mid s \in S\}$ for every $s \in S$.

(2) $\overline{\text{coz}_{\ell}(s)} \cap Z_{\ell}(a) \neq \emptyset$ for every $s \in S$.

(3) The family $\{\overline{\text{coz}_{\ell}(s)} \cap Z_{\ell}(a) \mid s \in S\}$ has the finite intersection property.

Proof. (1). The inclusion \subseteq is clear. To prove the reverse inclusion, it is sufficient to prove that for each $s \in S$ there is $t \in S$ such that $\overline{\text{coz}_{\ell}(t)} \subseteq \overline{\text{coz}_{\ell}(s)}$. Since $s \in S$, there is $\varepsilon \in \mathbb{R}$ with $\square s + x > \varepsilon + x > 0 + x$. Set $t = (s - \varepsilon)^+$. Then $t \geq 0$ and

$$\square t = \square(s - \varepsilon)^+ = (\square(s - \varepsilon))^+$$

by (M3). If $\Box t \in x$, then $\Box(s - \varepsilon) + x \leq 0 + x$, so $\Box s - \varepsilon(1 - \Box 0) + x \leq 0 + x$ by Lemma 3.13(5). We have $\Box 0 \in x$ by Lemma 4.4(5) as $\Box a \in x$, so $\Box s - \varepsilon \leq 0 + x$, and hence $\Box s + x \leq \varepsilon + x$. The obtained contradiction shows $\Box t \notin x$, so $t \in S$. Let $z \in Z_\ell(s)$. Then $z \in \zeta_A(s)^{-1}(-\varepsilon, \varepsilon)$, an open set. But $\overline{\zeta_A(s)^{-1}(-\varepsilon, \varepsilon)} \subseteq Z_\ell(t)$ by definition of t and Remark 4.3(3), so $\overline{Z_\ell(s)} \subseteq \text{int } Z_\ell(t)$. Thus, $\text{coz}_\ell(t) \subseteq \text{coz}_\ell(s)$.

(2). Note that $\text{coz}_\ell(s) \cap Z_\ell(a) \neq \emptyset$ means that $Z_\ell(a) \not\subseteq \text{int } Z_\ell(s)$. We argue by contradiction. Suppose $Z_\ell(a) \subseteq \text{int } Z_\ell(s)$. Then by Lemma 4.6, there is $f \in C(Y_A)$ such that $\zeta_A(s) = \zeta_A(a)f$ in $C(Y_A)$. Since $C(Y_A)$ is the uniform completion of A (see Proposition 2.3), there is a sequence $\{b_n\} \subseteq A$ such that $f = \lim \zeta_A(b_n)$. It is well known that multiplication is continuous with respect to the norm, so we have $\lim \zeta_A(ab_n) = \zeta_A(a)f = \zeta_A(s)$. Since $s \in S$, there is $\varepsilon > 0$ such that $\Box s + x > \varepsilon + x$, so $(\Box s - \varepsilon) + x > 0 + x$. There is N such that $\|s - ab_N\| < \varepsilon$. Therefore, $s < ab_N + \varepsilon$. Take $0 < \lambda \in \mathbb{R}$ such that $b_N \leq \lambda$. Then $s < \lambda a + \varepsilon$, so by Lemmas 3.13(1) and 4.4(2), and (M5),

$$\Box s + x \leq \Box(\lambda a + \varepsilon) + x = (\Box(\lambda a) + \varepsilon) + x = (\Box \lambda \Box a + \varepsilon) + x.$$

But $\Box a \in x$, so $\Box s + x \leq \varepsilon + x$, contradicting $\varepsilon + x < \Box s + x$.

(3). We first show that the intersection of any two members of the family contains another member of the family. Let $s, t \in S$. Then $\Box s, \Box t \notin x$. Since x is a maximal ℓ -ideal, $A/x \cong \mathbb{R}$ is totally ordered, so

$$(\Box s \wedge \Box t) + x = \min\{\Box s + x, \Box t + x\} \neq 0 + x,$$

and hence $\Box s \wedge \Box t \notin x$. By (M1), this shows $\Box(s \wedge t) \notin x$, which gives $s \wedge t \in S$. Since $\text{coz}_\ell(s \wedge t) = \text{coz}_\ell(s) \cap \text{coz}_\ell(t)$, we have:

$$\begin{aligned} \overline{(\text{coz}_\ell(s) \cap Z_\ell(a))} \cap \overline{(\text{coz}_\ell(t) \cap Z_\ell(a))} &= \overline{\text{coz}_\ell(s)} \cap \overline{\text{coz}_\ell(t)} \cap Z_\ell(a) \\ &\supseteq \overline{\text{coz}_\ell(s) \cap \text{coz}_\ell(t)} \cap Z_\ell(a) \\ &= \overline{\text{coz}_\ell(s \wedge t)} \cap Z_\ell(a). \end{aligned}$$

Because $s \wedge t \in S$, we have that $\overline{\text{coz}_\ell(s \wedge t)} \cap Z_\ell(a)$ is in the family. An easy induction argument then completes the proof because every element of the family is nonempty by (2). \square

Proposition 4.8. *Let $(A, \Box) \in \mathbf{mbal}$ and $x \in Y_A$. Then $(\Box^{-1}x)^+ = \bigcup\{y^+ \mid y \in R_\Box[x]\}$.*

Proof. The right-to-left inclusion follows from the definition of R_\Box . For the left-to-right inclusion, let $a \in (\Box^{-1}x)^+$. By Lemma 4.7(1),

$$\bigcap\{\text{coz}_\ell(s) \cap Z_\ell(a) \mid s \in S\} = \bigcap\{\overline{\text{coz}_\ell(s)} \cap Z_\ell(a) \mid s \in S\}.$$

By Lemma 4.7(3) and compactness of Y_A , this intersection is nonempty. Therefore, there is $y \in \bigcap\{\text{coz}_\ell(s) \cap Z_\ell(a) \mid s \in S\}$. This means that $a \in y$ and $y \cap S = \emptyset$, so $y^+ \subseteq \Box^{-1}x$. Thus, a is contained in some $y \in R_\Box[x]$, completing the proof. \square

Lemma 4.9. *Let $(A, \Box) \in \mathbf{mbal}$.*

- (1) $R_\Box^{-1}[Z_\ell(a)] = Z_\ell(\Box a)$ for every $0 \leq a \in A$.
- (2) $D_A = Z_\ell(\Box 0)$.

Proof. (1). Let $x \in R_{\square}^{-1}[Z_{\ell}(a)]$. Then there is $y \in Y_A$ such that $xR_{\square}y$ and $a \in y$. Therefore, $a \in y^+ \subseteq \square^{-1}x$. Thus, $\square a \in x$, and so $x \in Z_{\ell}(\square a)$.

For the other inclusion, let $x \in Z_{\ell}(\square a)$. Since $\square a \in x$ and $\square a \geq 0$, we have $a \in (\square^{-1}x)^+$. By Proposition 4.8, there is $y \in Y_A$ such that $xR_{\square}y$ and $a \in y$. Thus, $x \in R_{\square}^{-1}[Z_{\ell}(a)]$.

(2). This follows from (1) by setting $a = 0$ and using $Y_A = Z_{\ell}(0)$. \square

We will use Lemma 4.9 to prove that $R_{\square}^{-1}[F]$ is closed for each closed subset F of Y_A . For this we require Esakia's lemma, which is an important tool in modal logic (see, e.g., [9, Sec. 10.3]). The original statement is for descriptive frames, but it has a straightforward generalization to the setting of compact Hausdorff frames (see [4, Lem. 2.17]). We call a relation R on a compact Hausdorff space X *point-closed* if $R[x]$ is closed for each $x \in X$.

Lemma 4.10 (Esakia's lemma). *If R is a point-closed relation on a compact Hausdorff space X , then for each (nonempty) down-directed family $\{F_i \mid i \in I\}$ of closed subsets of X we have*

$$R^{-1} \left[\bigcap \{F_i \mid i \in I\} \right] = \bigcap \{R^{-1}[F_i] \mid i \in I\}.$$

Remark 4.11. Let $(A, \square) \in \mathbf{mbal}$ and S be a set of nonnegative elements of A closed under addition. Since $Z_{\ell}(a+b) \subseteq Z_{\ell}(a) \cap Z_{\ell}(b)$ for each $a, b \in S$, we have that $\{Z_{\ell}(a) \mid a \in S\}$ is a down-directed family of closed subsets of Y_A . Then, by Esakia's lemma and Lemma 4.9, we have:

$$\begin{aligned} R_{\square}^{-1}[Z_{\ell}(S)] &= R_{\square}^{-1} \left[\bigcap \{Z_{\ell}(a) \mid a \in S\} \right] = \bigcap \{R_{\square}^{-1}[Z_{\ell}(a)] \mid a \in S\} \\ &= \bigcap \{Z_{\ell}(\square a) \mid a \in S\} = Z_{\ell}(\square S). \end{aligned}$$

In particular, for an ℓ -ideal I , since $Z_{\ell}(I) = Z_{\ell}(I^+)$, we have

$$R_{\square}^{-1}Z_{\ell}(I) = R_{\square}^{-1}Z_{\ell}(I^+) = \bigcap \{Z_{\ell}(\square a) \mid a \in I^+\}.$$

Proposition 4.12. $R_{\square}^{-1}[F]$ is closed for every closed subset F of Y_A .

Proof. Since F is a closed subset of Y_A , there is an ℓ -ideal I such that $F = Z_{\ell}(I)$. By Remark 4.11,

$$R_{\square}^{-1}Z_{\ell}(I) = \bigcap \{Z_{\ell}(\square a) \mid a \in I^+\},$$

which is closed because it is an intersection of closed subsets of Y_A . \square

Lemma 4.13. *If $\diamond a \in x$ and $xR_{\square}y$, then $a^+ \in y$.*

Proof. Suppose that $xR_{\square}y$ and $a^+ \notin y$. Then $a + y > 0 + y$, so there is $0 < \lambda \in \mathbb{R}$ such that $a + y = \lambda + y$. Therefore, $\lambda - a \in y$, so $(\lambda - a)^+ \in y$. Since $y^+ \subseteq \square^{-1}x$, we have $(\square(\lambda - a))^+ \in x$ by (M3), so $(\lambda + \square(-a))^+ \in x$ by Lemma 4.4(3). Thus, $(\lambda + \square(-a)) + x \leq 0 + x$, so $\lambda + x \leq -\square(-a) + x$, and hence $\lambda + x \leq \diamond a + x$ by Lemma 4.4(4). Since $\lambda + x > 0 + x$, this shows $\diamond a \notin x$. \square

Lemma 4.14. $R_{\square}^{-1}[\text{coz}_{\ell}(a)] = \text{coz}_{\ell}(\diamond a)$ for every $0 \leq a \in A$.

Proof. For the left-to-right inclusion, suppose $x \notin \text{coz}_\ell(\diamond a)$. Then $\diamond a \in x$. Consider $y \in R_\square[x]$. By Lemma 4.13, $a = a^+ \in y$, so $y \notin \text{coz}_\ell(a)$. Therefore, $x \notin R_\square^{-1}[\text{coz}_\ell(a)]$.

For the right-to-left inclusion, let $x \in \text{coz}_\ell(\diamond a)$. Then $\diamond a \notin x$, so $\square 0 \in x$ by Lemma 4.4(6). Therefore, by Lemma 4.4(4), $0 + x \neq \diamond a + x = -\square(-a) + x$, and hence $\square(-a) \notin x$. Since $-a \leq 0$, we have $\square(-a) + x \leq \square 0 + x = 0 + x$. Thus, there is $\lambda \in \mathbb{R}$ with $\lambda < 0$ and $\square(-a) + x = \lambda + x$, so $\square(-a) - \lambda \in x$. By Lemma 4.4(3), we have

$$\square((-a - \lambda)^+) \in x \text{ iff } (\square(-a) - \lambda)^+ \in x.$$

Consequently, by Proposition 4.8,

$$(-a - \lambda)^+ \in (\square^{-1}x)^+ = \bigcup \{y^+ \mid y \in R_\square[x]\}.$$

Hence, there is $y \in R_\square[x]$ such that $(-a - \lambda)^+ \in y$. This means that $(-a - \lambda) + y \leq 0 + y$, so $a + y \geq -\lambda + y > 0 + y$. Therefore, $a \notin y$, and so $y \in \text{coz}_\ell(a)$. Thus, $x \in R_\square^{-1}[\text{coz}_\ell(a)]$. \square

Proposition 4.15. $R_\square^{-1}[U]$ is open for every open subset U of Y_A .

Proof. Open subsets of Y_A are of the form $\text{coz}_\ell(I) = \bigcup \{\text{coz}_\ell(a) \mid a \in I\}$ for some ℓ -ideal I . Since $\text{coz}_\ell(I) = \bigcup \{\text{coz}_\ell(a) \mid a \in I, a \geq 0\}$ and R_\square^{-1} commutes with arbitrary unions, by Lemma 4.14, we have

$$\begin{aligned} R_\square^{-1} \text{coz}_\ell(I) &= R_\square^{-1} \bigcup \{\text{coz}_\ell(a) \mid a \in I, a \geq 0\} \\ &= \bigcup \{R_\square^{-1} \text{coz}_\ell(a) \mid a \in I, a \geq 0\} \\ &= \bigcup \{\text{coz}_\ell(\diamond a) \mid a \in I, a \geq 0\}, \end{aligned}$$

which is open because it is a union of open subsets of Y_A . \square

Putting Propositions 4.5, 4.12, and 4.15 together yields:

Theorem 4.16. If $(A, \square) \in \mathbf{mbal}$, then $(Y_A, R_\square) \in \mathbf{KHF}$.

We finish the section by showing how to extend the object correspondence of Theorem 4.16 to a contravariant functor $(-)_* : \mathbf{mbal} \rightarrow \mathbf{KHF}$.

Lemma 4.17. Let $(A, \square), (B, \square) \in \mathbf{mbal}$ and $\alpha : A \rightarrow B$ be a morphism in \mathbf{mbal} . Then $\alpha_* : Y_B \rightarrow Y_A$ is a bounded morphism.

Proof. For each $y \in Y_A$, we have that y^+ and $\alpha(y^+)$ are sets of nonnegative elements closed under addition, so Remark 4.11 applies. Therefore, since $Z(y^+) = \{y\}$,

$$(\alpha_*)^{-1}(R_\square^{-1}[y]) = (\alpha_*)^{-1}(R_\square^{-1}[Z_\ell(y^+)]) = (\alpha_*)^{-1}(Z_\ell(\square y^+))$$

and

$$Z_\ell(\square \alpha(y^+)) = R_\square^{-1}[Z_\ell(\alpha(y^+))].$$

The definition of α_* shows that $(\alpha_*)^{-1}(Z_\ell(\square y^+)) = Z_\ell(\alpha(\square y^+))$ and $(\alpha_*)^{-1}(Z_\ell(y^+)) = Z_\ell(\alpha(y^+))$. This yields

$$(\alpha_*)^{-1}(R_\square^{-1}[y]) = (\alpha_*)^{-1}(Z_\ell(\square y^+)) = Z_\ell(\alpha(\square y^+))$$

and

$$R_{\square}^{-1}[(\alpha_*)^{-1}(y)] = R_{\square}^{-1}[(\alpha_*)^{-1}(Z_{\ell}(y^+))] = R_{\square}^{-1}[Z_{\ell}(\alpha(y^+))] = Z_{\ell}(\square\alpha(y^+)).$$

Consequently, since α commutes with \square , we have $(\alpha_*)^{-1}(R_{\square}^{-1}[y]) = R_{\square}^{-1}[(\alpha_*)^{-1}(y)]$, which proves that α_* is a bounded morphism. \square

Putting Theorem 4.16 and Lemma 4.17 together and remembering that $(-)_* : \mathbf{bal} \rightarrow \mathbf{KHaus}$ is a contravariant functor yields:

Theorem 4.18. $(-)_* : \mathbf{mbal} \rightarrow \mathbf{KHF}$ is a contravariant functor.

5. DUALITY

In this section we prove our main results. We show that $(-)_*$ and $(-)^*$ yield a dual adjunction between \mathbf{mbal} and \mathbf{KHF} which restricts to a dual equivalence between the category of uniformly complete members of \mathbf{mbal} and \mathbf{KHF} .

Definition 5.1. Let \mathbf{mubal} be the full subcategory of \mathbf{mbal} consisting of uniformly complete objects of \mathbf{mbal} .

Proposition 5.2. \mathbf{mubal} is a reflective subcategory of \mathbf{mbal} .

Proof. By Proposition 2.3(2), \mathbf{ubal} is a reflective subcategory of \mathbf{bal} , where $\zeta : \mathbf{bal} \rightarrow \mathbf{ubal}$ is the reflector. We first show that ζ_A is an \mathbf{mbal} -morphism for each $(A, \square) \in \mathbf{mbal}$. Let $x \in Y_A$. Recall that

$$(\square_{R_{\square}}\zeta_A(a))(x) = \begin{cases} \inf\{\zeta_A(a)(y) \mid xR_{\square}y\} & \text{if } x \in D_A \\ 1 & \text{otherwise.} \end{cases}$$

If $x \in E_A$, then $\square 0 \notin x$ by Lemma 4.9(2). Therefore, $\square a - 1 \in x$ by Lemma 4.4(5), and hence $\zeta_A(\square a)(x) = 1 = (\square_{R_{\square}}\zeta_A(a))(x)$. Now let $x \in D_A$. Then $(\square_{R_{\square}}\zeta_A(a))(x) = \inf\{\zeta_A(a)(y) \mid xR_{\square}y\}$. We first show that $\zeta_A(\square a)(x) \leq \inf\{\zeta_A(a)(y) \mid xR_{\square}y\}$. Suppose that $xR_{\square}y$, so $y^+ \subseteq \square^{-1}x$. Let $\lambda = \zeta_A(a)(y)$. Then $a - \lambda \in y$, so $(a - \lambda)^+ \in y^+ \subseteq \square^{-1}x$, and hence $(\square a - \lambda)^+ \in x$ iff $\square((a - \lambda)^+) \in x$ by Lemma 4.4(3). Therefore,

$$0 = \zeta_A((\square a - \lambda)^+)(x) = \max\{\zeta_A(\square a)(x) - \lambda, 0\},$$

so $\zeta_A(\square a)(x) - \lambda \leq 0$, and hence $\zeta_A(\square a)(x) \leq \lambda = \zeta_A(a)(y)$. Thus, $\zeta_A(\square a)(x) \leq \inf\{\zeta_A(a)(y) \mid xR_{\square}y\}$.

We next show that $\zeta_A(\square a)(x) \geq \inf\{\zeta_A(a)(y) \mid xR_{\square}y\}$. Let $\mu = \zeta_A(\square a)(x)$. We have $\square((a - \mu)^+) \in x$ iff $(\square a - \mu)^+ \in x$. Therefore, by Proposition 4.8,

$$(a - \mu)^+ \in (\square^{-1}x)^+ = \bigcup\{y^+ \mid xR_{\square}y\}.$$

So there is $y \in R_{\square}[x]$ such that $(a - \mu)^+ \in y$. Thus, $\max\{\zeta_A(a)(y) - \mu, 0\} = 0$. This yields $\zeta_A(a)(y) - \mu \leq 0$, and so $\zeta_A(a)(y) \leq \mu = \zeta_A(\square a)(x)$. Consequently, $\inf\{\zeta_A(a)(y) \mid y \in R_{\square}[x]\} \leq \zeta_A(\square a)(x)$.

Next, let $\alpha : A \rightarrow B$ be an **mbal**-morphism with $B \in \mathbf{mubal}$. Since α is a **bal**-morphism, there is a unique **bal**-morphism $\gamma : C(Y_A) \rightarrow C(Y_B)$, given by $\gamma = \zeta_B^{-1} \circ C(\alpha_*)$, such that $\gamma \circ \zeta_A = \alpha$.

$$\begin{array}{ccc} A & \xrightarrow{\zeta_A} & C(Y_A) \\ \alpha \downarrow & \nearrow \gamma & \downarrow C(\alpha_*) \\ B & \xleftarrow{\zeta_B^{-1}} & C(Y_B) \end{array}$$

As we saw in the paragraph above, ζ_B is an **mbal**-morphism. Also, $C(\alpha_*) : C(Y_A) \rightarrow C(Y_B)$ is an **mbal**-morphism by Lemmas 4.17 and 3.15. Therefore, γ is an **mbal**-morphism, concluding the proof. \square

Theorem 5.3. *The functors $(-)_* : \mathbf{mbal} \rightarrow \mathbf{KHF}$ and $(-)^* : \mathbf{KHF} \rightarrow \mathbf{mbal}$ yield a dual adjunction of the categories, which restricts to a dual equivalence between \mathbf{mubal} and \mathbf{KHF} .*

$$\begin{array}{ccc} & \curvearrowright & \\ \mathbf{mubal} & \xleftrightarrow{\quad} & \mathbf{mbal} \\ & \curvearrowleft & \\ & \text{KHF} & \end{array}$$

(−)* (−)*

Proof. By Gelfand duality, the functors $(-)_* : \mathbf{bal} \rightarrow \mathbf{KHaus}$ and $(-)^* : \mathbf{KHaus} \rightarrow \mathbf{bal}$ yield a dual adjunction between **bal** and **KHaus** that restricts to a dual equivalence between **ubal** and **KHaus**. The natural transformations are given by $\zeta : 1_{\mathbf{bal}} \rightarrow (-)^* \circ (-)_*$ and $\varepsilon : 1_{\mathbf{KHaus}} \rightarrow (-)_* \circ (-)^*$ where we recall from Section 2 that $\varepsilon_X : X \rightarrow X_{C(X)}$ is defined by

$$\varepsilon_X(x) = M_x = \{f \in C(X) \mid f(x) = 0\}.$$

Therefore, it is sufficient to show that ζ_A is a morphism in **mbal** for each $(A, \square) \in \mathbf{mbal}$ and that ε_X is a bounded morphism for each $(X, R) \in \mathbf{KHF}$. We showed in the proof of Proposition 5.2 that $\zeta_A(\square a) = \square_{R\square} \zeta_A(a)$ for each $(A, \square) \in \mathbf{mbal}$ and $a \in A$. Thus, ζ_A is a morphism in **mbal**, and hence it remains to show that xRy iff $\varepsilon_X(x)R_{\square_R}\varepsilon_X(y)$ for each $(X, R) \in \mathbf{KHF}$.

To see this recall that $\varepsilon_X(x)R_{\square_R}\varepsilon_X(y)$ means that $M_y^+ \subseteq \square_R^{-1}M_x$. First suppose that xRy and $f \in M_y^+$. Then $f(y) = 0$ and $f \geq 0$. We have $(\square_R f)(x) = \inf\{f(z) \mid xRz\} = 0$. Therefore, $\square_R f \in M_x$, and so $f \in \square_R^{-1}M_x$. This gives $M_y^+ \subseteq \square_R^{-1}M_x$. Next suppose that $x \not R y$, so $y \notin R[x]$. If $R[x] = \emptyset$, then $(\square_R 0)(x) = 1$, so $0 \in M_y^+$ but $\square_R 0 \notin M_x$, yielding $M_y^+ \not\subseteq \square_R^{-1}M_x$. On the other hand, if $R[x] \neq \emptyset$, since $R[x]$ is closed, by Urysohn's Lemma there is $f \geq 0$ such that $f(y) = 0$ and $f(R[x]) = \{1\}$. Thus, $f \in M_y^+$ and $\square_R f \notin M_x$. Consequently, $M_y^+ \not\subseteq \square_R^{-1}M_x$. \square

6. CONNECTIONS WITH MODAL ALGEBRAS AND DESCRIPTIVE FRAMES

In this section we connect Theorem 5.3 to the duality between **MA** and **DF**. Recall that a *modal algebra* is a pair $\mathfrak{A} = (A, \square)$ where A is a boolean algebra and \square is a unary function on A preserving finite meets (including 1). As usual, the dual function \diamond is defined by $\diamond a = \neg \square \neg a$, and is axiomatized as a unary function preserving finite joins (including 0). Let

\mathbf{MA} be the category of modal algebras and modal homomorphisms (boolean homomorphisms preserving \square).

We recall from the Introduction that a *descriptive frame* is a pair $\mathfrak{F} = (X, R)$ where X is a Stone space and R is a continuous relation on X , and that \mathbf{DF} is the category of descriptive frames and continuous bounded morphisms. As we already pointed out, Stone duality generalizes to the following duality:

Theorem 6.1 ([11, 14]). *\mathbf{MA} is dually equivalent to \mathbf{DF} .*

The functors $(-)^* : \mathbf{DF} \rightarrow \mathbf{MA}$ and $(-)_* : \mathbf{MA} \rightarrow \mathbf{DF}$ are defined as follows. For a descriptive Kripke frame $\mathfrak{F} = (X, R)$ let $\mathfrak{F}^* = (\mathbf{Clop}(X), \square_R)$ where $\mathbf{Clop}(X)$ is the boolean algebra of clopen subsets of X and $\square_R U = X \setminus R^{-1}[X \setminus U]$ (alternatively, $\diamond_R U = R^{-1}[U]$). For a bounded morphism f let $f^* = f^{-1}$. Then $(-)^* : \mathbf{DF} \rightarrow \mathbf{MA}$ is a well-defined contravariant functor.

For a modal algebra $\mathfrak{A} = (A, \square)$ let $\mathfrak{A}_* = (Y_A, R_\square)$ where Y_A is the set of ultrafilters of A and

$$xR_\square y \quad \text{iff} \quad (\forall a \in A)(\square a \in x \Rightarrow a \in y) \quad \text{iff} \quad \square^{-1}x \subseteq y$$

(alternatively, $xR_\square y$ iff $(\forall a \in A)(a \in y \Rightarrow \diamond a \in x)$ iff $y \subseteq \diamond^{-1}x$). For a modal algebra homomorphism h let $h_* = h^{-1}$. Then $(-)_* : \mathbf{MA} \rightarrow \mathbf{DF}$ is a well-defined contravariant functor, and the functors $(-)_*$ and $(-)^*$ yield a dual equivalence of \mathbf{MA} and \mathbf{DF} .

To define a functor from \mathbf{mbal} to \mathbf{MA} we recall that for each commutative ring A with 1, the idempotents of A form a boolean algebra $\text{Id}(A)$, where the boolean operations on $\text{Id}(A)$ are defined as follows:

$$e \wedge f = ef, \quad e \vee f = e + f - ef, \quad \neg e = 1 - e.$$

We point out that if $A \in \mathbf{bal}$, then the lattice operations on A restrict to those on $\text{Id}(A)$.

Remark 6.2. We will use the following two identities of f -rings (see [7, Sec. XIII.3] and [7, Cor. XVII.5.1]):

$$(a \wedge b) + c = (a + c) \wedge (b + c) \quad \text{and} \quad (a \wedge b)d = (ad) \wedge (bd) \quad \text{for } d \geq 0.$$

Lemma 6.3. *If $(A, \square) \in \mathbf{mbal}$, then \square sends idempotents to idempotents.*

Proof. First observe that $e \in A$ is an idempotent iff $1 \wedge 2e = e$. To see this, if e is an idempotent, by Remark 6.2,

$$(1 \wedge 2e) - e = (1 - e) \wedge e = \neg e \wedge e = 0.$$

Therefore, $1 \wedge 2e = e$. Conversely, suppose that $1 \wedge 2e = e$. Then $(1 - e) \wedge e = 0$ by the same calculation. Since each $A \in \mathbf{bal}$ is an f -ring (see, e.g., [7, Lem. XVII.5.2]), from $(1 - e) \wedge e = 0$ it follows that $(1 - e)e = 0$ (see, e.g., [7, Lem. XVII.5.1]). Thus, $e^2 = e$, and hence e is an idempotent.

For each $a \in A$, by (M5), (M2), and Lemma 3.13(4) we have

$$\square(2a) = \square 2\square a = (2 - \square 0)\square a = (2 - 2\square 0 + \square 0)\square a = 2\square a(1 - \square 0) + \square 0.$$

By Lemma 3.13(3), $\square 0 \geq 0$, so Lemma 3.13(4) and Remark 6.2 imply

$$(1 \wedge 2\square a)\square 0 = \square 0 \wedge 2\square a\square 0 = \square 0 \wedge 2\square 0 = \square 0.$$

Now suppose e is an idempotent, so $e = 1 \wedge 2e$. Since $\square 0 \leq \square 1 = 1$, we have $1 - \square 0 \geq 0$. Thus, by Remark 6.2 and the two identities just proved,

$$\begin{aligned} \square e &= \square(1 \wedge 2e) = 1 \wedge \square(2e) \\ &= ((1 - \square 0) + \square 0) \wedge \square(2e) \\ &= ((1 - \square 0) + \square 0) \wedge (2\square e(1 - \square 0) + \square 0) \\ &= ((1 - \square 0) \wedge 2\square e(1 - \square 0)) + \square 0 \\ &= (1 \wedge 2\square e)(1 - \square 0) + \square 0 \\ &= (1 \wedge 2\square e)(1 - \square 0) + (1 \wedge 2\square e)\square 0 \\ &= 1 \wedge 2\square e. \end{aligned}$$

Therefore, $\square e$ is idempotent. □

Lemma 6.4. *If $(A, \square) \in \mathbf{mbal}$, then $(\text{Id}(A), \square) \in \mathbf{MA}$.*

Proof. Since $A \in \mathbf{bal}$, we have that $\text{Id}(A)$ is a boolean algebra. By Lemma 6.3, \square is well defined on $\text{Id}(A)$. That \square preserves finite meets in $\text{Id}(A)$ follows from (M1) and Lemma 3.13(2). Thus, $(\text{Id}(A), \square) \in \mathbf{MA}$. □

Define $\text{Id} : \mathbf{mbal} \rightarrow \mathbf{MA}$ by sending $(A, \square) \in \mathbf{mbal}$ to $(\text{Id}(A), \square) \in \mathbf{MA}$ and a morphism $A \rightarrow B$ in \mathbf{mbal} to its restriction $\text{Id}(A) \rightarrow \text{Id}(B)$. The next lemma is an easy consequence of Lemma 6.4.

Lemma 6.5. *$\text{Id} : \mathbf{mbal} \rightarrow \mathbf{MA}$ is a well-defined covariant functor.*

We recall (see [26] and the references therein) that a commutative ring A is *clean* if each element is the sum of an idempotent and a unit.

Definition 6.6. Let \mathbf{cubal} be the full subcategory of \mathbf{ubal} consisting of those $A \in \mathbf{ubal}$ where A is clean.

Remark 6.7. By Stone duality for boolean algebras and [5, Prop. 5.20], the following diagram commutes (up to natural isomorphism), and the functor Id yields an equivalence of \mathbf{cubal} and \mathbf{BA} .

$$\begin{array}{ccc} \mathbf{cubal} & \xrightarrow{\text{Id}} & \mathbf{BA} \\ & \swarrow^{(-)*} \quad \searrow^{(-)*} & \\ & \text{Stone} & \end{array}$$

Definition 6.8. Let \mathbf{mcubal} be the full subcategory of \mathbf{mubal} consisting of those $(A, \square) \in \mathbf{mubal}$ where A is clean.

As a corollary of Theorems 5.3, 6.1 and Remark 6.7, we obtain:

Theorem 6.9. *The diagram below commutes (up to natural isomorphism) and the functor Id yields an equivalence of \mathbf{mcubal} and \mathbf{MA} .*

$$\begin{array}{ccc}
 \mathbf{mcubal} & \xrightarrow{\text{Id}} & \mathbf{MA} \\
 \swarrow^{(-)^*} & & \swarrow^{(-)^*} \\
 & & \mathbf{DF} \\
 \nwarrow_{(-)^*} & & \nwarrow_{(-)^*}
 \end{array}$$

7. SOME CORRESPONDENCE RESULTS

In this section we take the first steps towards the correspondence theory for \mathbf{mbal} by characterizing algebraically what it takes for the relation R_\square to satisfy additional first-order properties, such as seriality, reflexivity, transitivity, and symmetry.

Convention 7.1. To simplify notation, we denote the dual (Y_A, R_\square) of $(A, \square) \in \mathbf{mbal}$ simply by (Y, R) .

We recall that a relation R on X is *serial* if $R[x] \neq \emptyset$ for each $x \in X$.

Proposition 7.2. *Let $(A, \square) \in \mathbf{mbal}$. Then R is serial iff $\square 0 = 0$ in A .*

Proof. Suppose that R is serial. Then $R[x] \neq \emptyset$, so $(\square_R 0)(x) = 0$ for each $x \in Y$. Thus, $\square_R 0 = 0$. Since (A, \square) embeds into $(C(Y), \square_R)$, we conclude that $\square 0 = 0$ in A . Conversely, suppose that $\square 0 = 0$ in A . Since $Y = Z_\ell(0)$, by Lemma 4.9(2), we have $D_A = Z_\ell(\square 0) = Z_\ell(0) = Y$. Thus, R is serial. \square

Proposition 7.3. *Let $(A, \square) \in \mathbf{mbal}$. Then R is reflexive iff $\square a \leq a$ for each $a \in A$.*

Proof. Suppose that R is reflexive and $f \in C(Y)$. For each $x \in Y$, we have $x \in R[x]$. Thus, $(\square_R f)(x) = \inf f R[x] \leq f(x)$. Since (A, \square) embeds into $(C(Y), \square_R)$, we conclude that $\square a \leq a$ for each $a \in A$. Conversely, suppose $\square a \leq a$ for each $a \in A$. Let $x \in Y$ and $a \in x^+$. Then $0 \leq \square a \leq a \in x$. Thus, $x^+ \subseteq \square^{-1}x$, and so xRx . \square

Proposition 7.4. *Let $(A, \square) \in \mathbf{mbal}$. Then R is transitive iff $\square a \leq \square(\square a(1 - \square 0) + a\square 0)$ for each $a \in A$.*

Proof. First suppose that R is transitive. Let $f \in C(Y)$ and $x \in Y$. If $R[x] = \emptyset$, then by definition of \square_R

$$(\square_R f)(x) = 1 = \square_R(\square_R f(1 - \square_R 0) + f\square_R 0)(x).$$

Suppose that $R[x] \neq \emptyset$. Then $(\square_R f)(x) = \inf f R[x]$ and

$$\square_R(\square_R f(1 - \square_R 0) + f\square_R 0)(x) = \inf\{(\square_R f)(y)(1 - \square_R 0)(y) + f(y)(\square_R 0)(y) \mid xRy\}.$$

We have

$$(\square_R f)(y)(1 - \square_R 0)(y) + f(y)(\square_R 0)(y) = \begin{cases} f(y) & \text{if } R[y] = \emptyset \\ (\square_R f)(y) & \text{if } R[y] \neq \emptyset. \end{cases}$$

It is therefore sufficient to prove that, for each $y \in R[x]$, if $R[y] = \emptyset$ then $(\square_R f)(x) \leq f(y)$ and if $R[y] \neq \emptyset$ then $(\square_R f)(x) \leq (\square_R f)(y)$. Suppose $R[y] = \emptyset$. Since $R[x] \neq \emptyset$, we have

$$(\square_R f)(x) = \inf\{f(z) \mid z \in R[x]\} \leq f(y).$$

If $R[y] \neq \emptyset$, then by transitivity of R we have $R[y] \subseteq R[x]$, so

$$(\square_R f)(x) = \inf\{f(z) \mid z \in R[x]\} \leq \inf\{f(w) \mid w \in R[y]\} = (\square_R f)(y).$$

Thus, $\square_R f \leq \square_R(\square_R f(1 - \square_R 0) + f \square_R 0)$. Since (A, \square) embeds into $(C(Y), \square_R)$, we conclude that $\square a \leq \square(\square a(1 - \square 0) + a \square 0)$ for each $a \in A$.

Conversely, suppose $\square a \leq \square(\square a(1 - \square 0) + a \square 0)$ for each $a \in A$. Let $x, y, z \in Y$ with xRy and yRz . Then $y^+ \subseteq \square^{-1}x$ and $z^+ \subseteq \square^{-1}y$. Suppose that $a \in z^+$. Then $\square a \in y^+$. Since $0 \in z^+$, we have $\square 0 \in y^+$. Thus, since y is an ideal, $\square a(1 - \square 0) + a \square 0 \in y$. Because $\square a(1 - \square 0) + a \square 0 \geq 0$, we have $\square(\square a(1 - \square 0) + a \square 0) \in x$. By hypothesis, $0 \leq \square a \leq \square(\square a(1 - \square 0) + a \square 0) \in x$. Thus, $\square a \in x$. This shows that $z^+ \subseteq \square^{-1}x$, and hence xRz . \square

Proposition 7.5. *Let $(A, \square) \in \mathbf{mbal}$. Then R is symmetric iff $\diamond \square a(1 - \square 0) \leq a(1 - \square 0)$ for each $a \in A$.*

Proof. First suppose that R is symmetric. Let $f \in C(Y)$ and $x \in Y$. If $R[x] = \emptyset$, then $(1 - \square_R 0)(x) = 0$ so

$$(\diamond_R \square_R f)(x)(1 - \square_R 0)(x) = 0 = f(x)(1 - \square_R 0)(x).$$

If $R[x] \neq \emptyset$, then $(1 - \square_R 0)(x) = 1$, so it is sufficient to prove that $(\diamond_R \square_R f)(x) \leq f(x)$. For any $y \in R[x]$ we have $x \in R[y]$ by symmetry. Therefore,

$$(\square_R f)(y) = \inf\{f(z) \mid z \in R[y]\} \leq f(x).$$

Thus, recalling Remark 3.5, we have

$$(\diamond_R \square_R f)(x) = \sup\{(\square_R f)(y) \mid y \in R[x]\} \leq f(x).$$

Since (A, \square) embeds into $(C(Y), \square_R)$, we conclude that $\diamond \square a(1 - \square 0) \leq a(1 - \square 0)$ for each $a \in A$.

Conversely, suppose $\diamond \square a(1 - \square 0) \leq a(1 - \square 0)$ for each $a \in A$. Let $x, y \in Y$ with xRy . Then $y^+ \subseteq \square^{-1}x$, so $0 \in y^+$ implies $\square 0 \in x$. Thus,

$$\diamond \square a + x = \diamond \square a(1 - \square 0) + x \leq a(1 - \square 0) + x = a + x.$$

To see that yRx , let $a \in x^+$. If $\square a \notin y$, then $0 + y < \square a + y$ because $\square a \geq 0$. So there is $0 < \lambda \in \mathbb{R}$ such that $\lambda - \square a \in y$. Thus, $(\lambda - \square a)^+ \in y^+$. Since xRy , by (2) and (4) of Lemma 4.4, we have

$$(\lambda - \diamond \square a)^+ + x = (\lambda + \square(-\square a))^+ + x = (\square(\lambda - \square a))^+ + x = \square(\lambda - \square a)^+ + x = 0 + x.$$

Because $\diamond \square a + x \leq a + x$ we have $(\lambda - a) + x \leq (\lambda - \diamond \square a) + x$. Therefore,

$$0 \leq (\lambda - a)^+ + x \leq (\lambda - \diamond \square a)^+ + x = 0 + x.$$

This implies $(\lambda - a)^+ \in x$. Thus, by Remark 4.3(4), $0 + x < \lambda + x \leq a + x$, which contradicts $a \in x^+$. Therefore, $\square a \in y$, which yields $x^+ \subseteq \square^{-1}y$. Thus, yRx . \square

Remark 7.6. If we work with \diamond instead of \square , then Propositions 7.2—7.5 can be stated as follows.

- (1) R is serial iff $\diamond 1 = 1$.
- (2) R is reflexive iff $a \leq \diamond a$ for each $a \in A$.
- (3) R is transitive iff $\diamond(\diamond a + a(1 - \diamond 1)) \leq \diamond a$ for each $a \in A$.
- (4) R is symmetric iff $\diamond \square a \leq a \diamond 1$ for each $a \in A$.

Remark 7.7. Let $(A, \square) \in \mathbf{mbal}$. If $\square 0 = 0$, then the transitivity and symmetry axioms simplify to $\square a \leq \square \square a$ and $\diamond \square a \leq a$, which are standard transitivity and symmetry axioms in modal logic.

Definition 7.8.

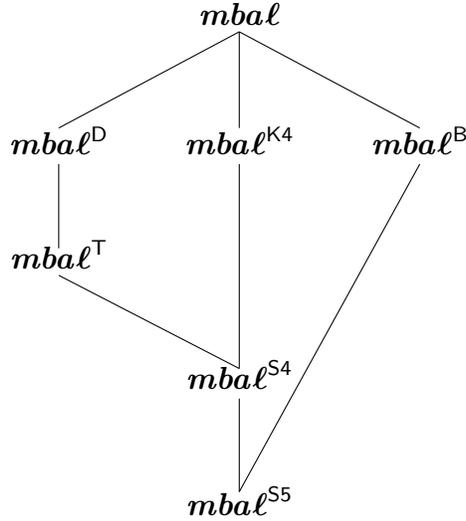
- (1) Let \mathbf{mbal}^D be the full subcategory of \mathbf{mbal} consisting of objects $(A, \square) \in \mathbf{mbal}$ satisfying $\square 0 = 0$.
- (2) Let \mathbf{mbal}^T be the full subcategory of \mathbf{mbal} consisting of objects $(A, \square) \in \mathbf{mbal}$ satisfying $\square a \leq a$.
- (3) Let \mathbf{mbal}^{K4} be the full subcategory of \mathbf{mbal} consisting of objects $(A, \square) \in \mathbf{mbal}$ satisfying $\square a \leq \square(\square a(1 - \square 0) + a \square 0)$.
- (4) Let \mathbf{mbal}^B be the full subcategory of \mathbf{mbal} consisting of objects $(A, \square) \in \mathbf{mbal}$ satisfying $\diamond \square a(1 - \square 0) \leq a(1 - \square 0)$.
- (5) Let $\mathbf{mbal}^{S4} = \mathbf{mbal}^T \cap \mathbf{mbal}^{K4}$.
- (6) Let $\mathbf{mbal}^{S5} = \mathbf{mbal}^{S4} \cap \mathbf{mbal}^B$.

Remark 7.9. Since the reflexivity axiom implies the seriality axiom, we obtain that $(A, \square) \in \mathbf{mbal}^{S4}$ iff $(A, \square) \in \mathbf{mbal}^T$ and $\square a \leq \square \square a$ for each $a \in A$. Similarly, $(A, \square) \in \mathbf{mbal}^{S5}$ iff $(A, \square) \in \mathbf{mbal}^{S4}$ and $\diamond \square a \leq a$ for each $a \in A$.

Remark 7.10. The notation of Definition 7.8 is motivated by the standard notation in modal logic:

- (1) D denotes the least normal modal logic containing the axiom $\diamond \top$.
- (2) T denotes the least normal modal logic containing the axiom $\square p \rightarrow p$.
- (3) K4 denotes the least normal modal logic containing the axiom $\square p \rightarrow \square \square p$.
- (4) B denotes the least normal modal logic containing the axiom $\diamond \square p \rightarrow p$.
- (5) S4 denotes the join of T and K4.
- (6) S5 denotes the join of S4 and B.

As with the corresponding classes of modal algebras, we have the following inclusions between the subclasses of algebras in \mathbf{mbal} given in Definition 7.8:



Similarly to Definition 7.8, for $X \in \{D, T, K4, B, S4, S5\}$ we define the following categories:

- The categories $mubal^X$ are defined similarly to $mbal^X$ but with $mbal$ replaced by $mubal$.
- The categories $mcubal^X$ are defined similarly to $mbal^X$ but with $mbal$ replaced by $mcubal$.
- The categories MA^X are defined similarly to $mbal^X$ but with $mbal$ replaced by MA .
- The categories KHF^X are defined by adding the corresponding properties on the relation R to the definition of KHF .
- The categories DF^X are defined as KHF^X by restricting KHF to DF .

Theorems 5.3 and 6.9, Propositions 7.2—7.5, and the corresponding versions of Theorem 6.1 yield the following result.

Theorem 7.11. *Suppose that $X \in \{D, T, K4, B, S4, S5\}$.*

- (1) *The category $mubal^X$ is dually equivalent to KHF^X .*
- (2) *The categories $mcubal^X$ and MA^X are dually equivalent to DF^X , and hence are equivalent.*

8. CONCLUDING REMARKS

We finish the paper with several remarks, which indicate a number of possible directions for future research.

Remark 8.1.

- (1) As we pointed out in the Introduction, there are other dualities for $KHaus$. For example, in pointfree topology we have Isbell duality [17] (see also [1] or [18, Sec. III.1]) and de Vries duality [10] (see also [2]). The two are closely related, see [3]. Isbell and de Vries dualities were generalized to the setting of KHF in [4]. We plan to compare the results of [4] to the ones obtained in this paper.
- (2) Another relevant duality was established by Kakutani [20, 21], the Krein brothers [23], and Yosida [29], who also worked with continuous real-valued functions, but

their signature was that of a vector lattice instead of an ℓ -algebra. Gelfand duality has a natural counterpart in this setting. Let **bav** be the category of bounded archimedean vector lattices and let **ubav** be its reflective subcategory consisting of uniformly complete objects. Then there is a dual adjunction between **bav** and **KHaus**, which restricts to a dual equivalence between **ubav** and **KHaus**. This duality is known as Yosida duality (or Kakutani-Krein-Yosida duality). In our axiomatization of **mbal** (see Definition 3.10), the only axiom involving multiplication is (M5). In the serial case (M5) simplifies to (M5') of Remark 3.12, which only involves scalar multiplication. In the non-serial case, (M5) can be replaced by the following two axioms

- $\Box(\lambda a) = \lambda \Box a + (1 - \lambda) \Box 0$ provided $\lambda \geq 0$,
- $\Box 0 \wedge (1 - \Box a)^+ = 0$,

which again only involve vector lattice operations. This yields the category **mbav** of modal bounded archimedean vector lattices and its reflective subcategory **mubav** consisting of uniformly complete objects. The results of Section 5 then generalize to the setting of **mbav** and **mubav**, and provide a generalization of Yosida duality.

- (3) Our definition of a modal operator on a bounded archimedean ℓ -algebra can be further adjusted to the settings of ℓ -rings, ℓ -groups, and MV-algebras. In this regard, it would be interesting to develop logical systems corresponding to these algebras.
- (4) It would be natural to develop the correspondence theory for **mbal** by generalizing the results of Section 7, with the final goal towards a Sahlqvist type correspondence (see, e.g., [8, Ch. 3]).

REFERENCES

- [1] B. Banaschewski and C. J. Mulvey, *Stone-Čech compactification of locales. I*, Houston J. Math. **6** (1980), no. 3, 301–312.
- [2] G. Bezhanishvili, *Stone duality and Gleason covers through de Vries duality*, Topology Appl. **157** (2010), no. 6, 1064–1080.
- [3] ———, *De Vries algebras and compact regular frames*, Appl. Categ. Structures **20** (2012), no. 6, 569–582.
- [4] G. Bezhanishvili, N. Bezhanishvili, and J. Harding, *Modal compact Hausdorff spaces*, J. Logic Comput. **25** (2015), no. 1, 1–35.
- [5] G. Bezhanishvili, P. J. Morandi, and B. Olberding, *Bounded Archimedean ℓ -algebras and Gelfand-Neumark-Stone duality*, Theory Appl. Categ. **28** (2013), Paper No. 16, 435–475.
- [6] ———, *A functional approach to Dedekind completions and the representation of vector lattices and ℓ -algebras by normal functions*, Theory Appl. Categ. **31** (2016), Paper No. 37, 1095–1133.
- [7] G. Birkhoff, *Lattice theory*, third ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I., 1979.
- [8] P. Blackburn, M. de Rijke, and Y. Venema, *Modal logic*, Cambridge Tracts in Theoretical Computer Science, vol. 53, Cambridge University Press, Cambridge, 2001.
- [9] A. Chagrov and M. Zakharyashev, *Modal logic*, Oxford Logic Guides, vol. 35, The Clarendon Press, Oxford University Press, New York, 1997.
- [10] H. de Vries, *Compact spaces and compactifications. An algebraic approach*, Ph.D. thesis, University of Amsterdam, 1962.
- [11] L. L. Esakia, *Topological Kripke models*, Dokl. Akad. Nauk SSSR **214** (1974), 298–301.

- [12] I. Gelfand and M. Neumark, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Rec. Math. [Mat. Sbornik] N.S. **12(54)** (1943), 197–213.
- [13] L. Gillman and M. Jerison, *Rings of continuous functions*, The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto-London-New York, 1960.
- [14] R. I. Goldblatt, *Metamathematics of modal logic*, Rep. Math. Logic (1976), no. 6, 41–77.
- [15] P. R. Halmos, *Algebraic logic. I. Monadic Boolean algebras*, Compositio Math. **12** (1956), 217–249.
- [16] M. Henriksen and D. G. Johnson, *On the structure of a class of Archimedean lattice-ordered algebras*, Fund. Math. **50** (1961/1962), 73–94.
- [17] J. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5–32.
- [18] P. T. Johnstone, *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1982.
- [19] B. Jónsson and A. Tarski, *Boolean algebras with operators. I*, Amer. J. Math. **73** (1951), 891–939.
- [20] S. Kakutani, *Weak topology, bicomact set and the principle of duality*, Proc. Imp. Acad. Tokyo **16** (1940), 63–67.
- [21] ———, *Concrete representation of abstract (M)-spaces. (A characterization of the space of continuous functions.)*, Ann. of Math. (2) **42** (1941), 994–1024.
- [22] M. Kracht, *Tools and techniques in modal logic*, Studies in Logic and the Foundations of Mathematics, vol. 142, North-Holland Publishing Co., Amsterdam, 1999.
- [23] M. Krein and S. Krein, *On an inner characteristic of the set of all continuous functions defined on a bicomact Hausdorff space*, C. R. (Doklady) Acad. Sci. URSS (N.S.) **27** (1940), 427–430.
- [24] S. A. Kripke, *Semantical considerations on modal logic*, Acta Philos. Fenn. **Fasc.** (1963), 83–94.
- [25] C. Kupke, A. Kurz, and Y. Venema, *Stone coalgebras*, Theoret. Comput. Sci. **327** (2004), no. 1-2, 109–134.
- [26] W. W. McGovern, *Neat rings*, J. Pure Appl. Algebra **205** (2006), no. 2, 243–265.
- [27] G. Sambin and V. Vaccaro, *Topology and duality in modal logic*, Ann. Pure Appl. Logic **37** (1988), no. 3, 249–296.
- [28] M. H. Stone, *A general theory of spectra. I*, Proc. Nat. Acad. Sci. U.S.A. **26** (1940), 280–283.
- [29] K. Yosida, *On vector lattice with a unit*, Proc. Imp. Acad. Tokyo **17** (1941), 121–124.

Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, guram@nmsu.edu, lcarai@nmsu.edu, pmorandi@nmsu.edu