# MUTUAL INTERPRETABILITY OF WEAK ESSENTIALLY UNDECIDABLE 

## THEORIES

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#### Abstract

[Abstract: Kristiansen and Murwanashyaka recently proved that Robinson arithmetic, Q , is interpretable in an elementary theory of full binary trees,T. We prove that, conversely, $T$ is interpretable in Q by producing a formal interpretation of T in an elementary concatenation theory $\mathrm{QT}^{+}$, thereby also establishing mutual interpretability of T with several well-known weak essentially undecidable theories of numbers, strings and sets. We also introduce a "hybrid" elementary theory of strings and trees, WQT*, and establish its mutual interpretability with Robinson's weak arithmetic R, the weak theory of trees WT of Kristiansen and Murwanashyaka and the weak concatenation theory WTC ${ }^{-\varepsilon}$ of Higuchi and Horihata.]


Key words: interpretability, full binary trees, Robinson arithmetic, concatenation theory, strings, essential undecidability
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The classic monograph work of Tarski Mostowski and Robinson [8] isolated two weak formal theories of arithmetic, R and Q , as minimal "basis theories" for metamathematical arguments of foundational significance involving formalizing computation, incompleteness, undecidability, etc. The two theories were singled out as essentially undecidable, in that neither can consistently be extended to a decidable theory. The work introduced a powerful method for establishing incompleteness and undecidability of a wide range of mathematical theories built around the notion of relative interpretability of one theory in another. Roughly, a formula with a single free variable is chosen in the language of the second theory - the interpreting theory -- to define the "universe of the interpretation", and suitable definitions for the non-logical vocabulary of the first theory - the interpreted theory -are given in the language of the interpreting theory. Formulae of the interpreted theory are then translated into formulae of the interpreting theory based on those definitions, in such a way that the logical operations are preserved under the translation and, crucially, all occurrences of quantifiers become relativized to the universe of the interpretation. Consequently, deductive relations between formulae are preserved: in particular, theorems of the interpreted theory are translated into theorems of the interpreting theory. In this specific sense reasoning in one theory is formally simulated in
another theory, establishing relative consistency of the former in the latter. Once it is shown that R or Q is interpretable in some given theory, it follows from Tarski's methods that the latter is also essentially undecidable.

It was only within the last two decades that some light has been shed on what makes R and Q special, a result of work of many researchers, including (earlier work by) Collins and Halpern, Wilkie, Grzegorczyk, Zdanowski, Švejdar, Ganea, and, especially, Visser. One approach was to characterize them as mutually interpretable with concatenation theories (theories of strings) or weak subsystems of set theory, each naturally motivated and of independent interest in their own right (see [1] for further references). Another is to produce a "coordinate-free" characterization independent of a particular axiomatic presentation in some formal language, as, e.g., in the remarkable theorem of Visser [10]: a recursively axiomatizable theory is interpretable in $R$ if and only if it is locally finitely satisfiable, that is, each finite subset of its non-logical axioms has a finite model.

An important new angle on these issues was recently introduced in the work of Kristiansen and Murwanashyaka [6]. They consider two elementary
axiomatizations, WT and T, whose intended models are simple inductively generated structures like trees or terms, and rigorously develop a direct and novel approach to formalization of computation by ultra-elementary means. T is formulated in the language $\mathcal{L}_{\mathrm{T}}=\{0,(), \subseteq\}$ with a single individual constant 0 , a binary operation symbol (, ) and a 2-place relational symbol 드 with the following axioms:
(T1) $\quad \forall x, y \neg(x, y)=0$,
(T2) $\forall x, y, z, w[(x, y)=(z, w) \rightarrow x=z \& y=w]$
(T3) $\forall x[x \sqsubseteq 0 \leftrightarrow x=0]$

$$
\begin{equation*}
\forall x, y, z[x \sqsubseteq(y, z) \leftrightarrow x=(y, z) v x \sqsubseteq y v x \sqsubseteq z] \tag{T4}
\end{equation*}
$$

On the other hand, the theory WT is formulated in the same vocabulary, but has infinitely many axioms given by the two schemas
(WT1) $\neg(\mathrm{s}=\mathrm{t}) \quad$ for any distinct variable-free terms $\mathrm{s}, \mathrm{t}$ of $\mathcal{L}_{\mathrm{T}}$,
(WT2) $\quad \forall \mathrm{x}\left(\mathrm{x} \subseteq \mathrm{t} \leftrightarrow \mathrm{V}_{\mathrm{s} \in \mathcal{S}(\mathrm{t})} \mathrm{x}=\mathrm{s}\right) \quad$ for each variable-free term t of $\mathcal{L}_{\mathrm{T}}$, where $S(\mathrm{t})$ is the set of all subterms of t .

The theory WT, which turns out to be contained in T, is proved to be mutually interpretable with R. The stronger theory T, which can be thought of as the basic theory of full binary trees, even though lacking induction is shown to be sufficiently strong to allow for a formal interpretation of basic arithmetical operations validating the axioms of Q. Kristiansen and Murwanashyaka further conjectured that, conversely, T is also formally interpretable in Q .

In this paper we prove that $T$ is indeed interpretable in $Q$, by formally interpreting T in a theory of concatenation, $\mathrm{QT}^{+}$, previously investigated in [1] and established to be mutually interpretable with Q along with a host of other theories whose intended interpretations are natural numbers, strings or sets. Hence T and Q are mutually interpretable. Further we formulate a weak theory of concatenation, $\mathrm{WQT}^{*}$, and a "pseudo-concatenation" theory WQT , and establish their mutual interpretability with Robinson's R. (While R is deductively contained, hence also interpretable, in $Q$, the latter, being finitely axiomatized but having no finite model, by Visser's Theorem is not interpretable in R.)

Several distinct formulations of concatenation theory which have been put forward as standard axiomatizations and as such extensively studied are not deductively co-extensive. Some, like Grzegorczyk's theory TC, are centered around what came to be known as Tarski's Law (or Editor Axiom), and some of the variants include the empty string as a unit element. Others, such as the theory $\mathrm{QT}^{+}$used in [1] and here, and a closely related theory F originally introduced by Tarski in [8], are on their face more explicitly theories of semigroups with two generators. Nonetheless, all these theories turn out to be mutually interpretable on account of their mutual interpretability with Q. Our choice of QT+ is motivated by the "ground-up" approach exemplified in the formula-selection method expounded below in §3.

In $\S \S 1-2$ we give a preview of our interpretation of T in concatenation theory. In §3 we introduce the concatenation theory $\mathrm{QT}^{+}$, explain the main methodological tool used throughout the paper, the formula selection method applied to tractable strings and string forms, and develop elements of formal concatenation theory $\mathrm{QT}^{+}$related to tallies, adding of tallies and parts of strings. §4 we describe the essentials of the coding methods subsequently used in formalization of definitions by string recursion in §5. The resulting
formal schema of definition is applied to obtain definitions of counting functions $\alpha$ and $\beta$ which we rely on to construct the formal interpretation introduced in §§1-2. In §6 the interpretation is formally defined, and translations of the axioms of T formally verified. There we state the main result of the paper, the First Mutual Interpretability Theorem of Weak Essentially Undecidable Theories, relating T and $\mathrm{QT}^{+}$to a number of wellknown theories of numbers, strings and sets. Finally, in §7 we introduce concatenation variants WQT and WQT* of Robinson's theory R and establish the corresponding Second Mutual Interpretability Theorem with the weak theory of trees WT.

Many of our arguments involve construction of specific formulas and tedious verifications of their specific properties. Most of these details can be found in the Appendix. The entire formal construction ultimately rests on coding of sets of strings by strings within QT $^{+}$, which is given in complete detail in [2]. We provide specific references as needed.

## 1. Trees as Strings

The intended domain of interpretation of the theory T is the set of variablefree $\mathcal{L}_{\mathrm{T}}$-terms

$$
{ }^{*} \text { ) } \quad 0,(00),(0(00)),((00) 0),((00)(00)), \ldots
$$

Alternatively, we may think of the domain as consisting of finite full binary trees - also called 2-trees -- trees in which every node other than the endnodes has two immediate descendants. In order to interpret T in concatenation theory, we need some way of representing these objects terms or trees - by binary strings. We would like to do this directly, without having to rely on a coding of sets or sequences.

For this purpose we will use a variant of Polish notation to read binary strings as codes for inductively generated objects having the structure characteristic of terms or trees. Thus, e.g., the terms in $\left(^{*}\right)$ will be coded, respectively, by
(**) a, baa, babaa, bbaaa, bbaabaa, ...

To obtain the string code from a given variable-free $\mathcal{L}_{\mathrm{T}}$-term we proceed from left to right by replacing the left parentheses by b's and 0's by a's, ignoring the right parentheses.

Looking at the strings that are examples of term codes in (**), we note that they all share the following features:
(c1) the total number of a's in the string exceeds the total number of b's exactly by 1 ,
(c2) each proper initial segment of the string has at least as many b's as a's.

In other words, each of these strings is its own smallest initial segment in which the number of $a$ 's strictly exceeds the number of $b$ 's. We will take this to be the defining property of binary term/tree codes. We offer the following as informal justification. Each $b$ indicates a branching vertex, incurring a "debt" of two "open places", which need to filled by completing the branchings. That can be done either immediately by simply writing $a$, an end node, or by opening another branching, temporarily increasing the "debt of open places". Each successive a reduces the "debt" of places to be filled by one, until all open branchings are completed and the last two remaining
"places" filled with $a$ 's, resulting in a full binary tree. Ultimately, $b$ 's in the binary code track the number of branchings, i.e. non-terminal nodes, and a's the number of terminal nodes in the tree.

To define the domain of the formal interpretation of T in concatenation theory we will need to be able to single out by means of a formula of concatenation theory those among arbitrary strings that are term codes. Key role in this connection will be played by functions $\alpha$ and $\beta$ that count the number of occurrences of the letters $a, b$, resp., in a given binary string. They are defined as follows:

$$
\begin{array}{ll}
\alpha(\mathrm{a})=1 & \beta(\mathrm{a})=0 \\
\alpha(\mathrm{~b})=0 & \beta(\mathrm{~b})=1 \\
\alpha\left(\mathrm{x}^{*} \mathrm{a}\right)=\alpha(\mathrm{x})+1 & \beta\left(\mathrm{x}^{*} \mathrm{a}\right)=\beta(\mathrm{x}) \\
\alpha\left(\mathrm{x}^{*} \mathrm{~b}\right)=\alpha(\mathrm{x}) & \beta\left(\mathrm{x}^{*} \mathrm{~b}\right)=\beta(\mathrm{x})+1
\end{array}
$$

Call a string x is almost even, writing $Æ(x)$, if (c1) $\alpha(x)=\beta(x)+1$, and
(c2) for each proper initial segment $u$ of $\mathrm{x}, \alpha(\mathrm{u}) \leq \beta(\mathrm{u})$.

Within concatenation theory the values of $\alpha, \beta$ will be expressed by b-tallies, i.e., strings of consecutive $b$ 's. The functions $\alpha$ and $\beta$ are additive in that

$$
\alpha\left(x^{*} y\right)=\alpha(x)+\alpha(y) \quad \text { and } \quad \beta\left(x^{*} y\right)=\beta(x)+\beta(y) .
$$

To express and verify these properties in concatenation theory we will need to introduce a suitable operation Addtally having the requisite properties of addition on non-negative integers. But the main problem to be solved is showing that $\alpha$ and $\beta$, which are defined by recursion on strings, can actually be defined in concatenation theory.

## 2. Outline of the Interpretation

The language $\mathcal{L}_{\mathrm{C}}=\{\mathrm{a}, \mathrm{b}, *\}$ of concatenation theory has two individual constants $a, b$, and a single binary operation symbol *. Its intended interpretation $\Sigma^{*}$ has as its domain the set of all non-empty finite strings of $a$ 's and $b$ 's, the constants ' a ', ' b ', resp., stand for the digits $\mathrm{a}, \mathrm{b}$ (or 0,1 , resp.), and, for given strings $\mathrm{x}, \mathrm{y}$ from the domain of $\Sigma^{*}$, we take $\mathrm{x}^{*} \mathrm{y}$ to be the string obtained by concatenation (i.e., juxtaposition) of the successive digits of $y$ to the right of the end digit of $x$. Simply put, for variable-free terms $s, t$ of $\mathcal{L}_{\mathcal{L}}$, an
atomic formula ' $\mathrm{s}=\mathrm{t}^{\prime}$ ' is true in $\Sigma^{*}$ just in case s and t denote the very same binary string. For the purpose of informal exposition of the basic idea behind the interpretation we will avail ourselves, "as a first approximation", of formulations couched in the first-order theory $\operatorname{Th}\left(\Sigma^{*}\right)$ consisting of all true sentences of $\mathcal{L}_{\mathrm{C}}$ in $\Sigma^{*}$. Specifically, at this point we will simply assume that the graphs of the functions $\alpha, \beta$, are expressible by some formulas $A^{\sharp}(x, y)$, $\mathrm{B}^{\#}(\mathrm{x}, \mathrm{y})$, resp., of $\mathcal{L}_{\mathrm{C}}$ along with the graph of Addtally, and carry on reasoning informally within $\operatorname{Th}\left(\Sigma^{*}\right)$. In subsequent sections we turn to the detailed technical work of actually proving these assumptions by formalizing string recursion in concatenation theory and verifying the corresponding translations into $\mathcal{L}_{\mathrm{C}}$ of the axioms of T , all of which has to be formally carried out within an extremely weak subtheory $\mathrm{QT}^{+}$of $\mathrm{Th}\left(\Sigma^{*}\right)$.

First, some abbreviations. Let $\mathrm{xBy} \equiv \exists \mathrm{zx} \mathrm{x}^{*} \mathrm{z}=\mathrm{y}$ and $\mathrm{xEy} \equiv \exists \mathrm{zz} \mathrm{x}=\mathrm{y}$. Then let

$$
x \subseteq_{p} y \equiv x=y \vee x B y \vee x E y \vee \exists y_{1} \exists y_{2} y=y_{1}{ }^{*}\left(x^{*} y_{2}\right)
$$

(Often, we shall write xy for $\mathrm{x}^{*} \mathrm{y}$.)
2.1(a) $\Sigma^{*} \vDash Æ(x) \rightarrow \mathrm{x}=\mathrm{a} v(\mathrm{bBx} \& \mathrm{aaEx})$.
(b) $\Sigma^{*}$ F $\notin(\mathrm{x}) \& \mathrm{x}_{2} \mathrm{Ex} \rightarrow \alpha\left(\mathrm{x}_{2}\right) \geq \beta\left(\mathrm{x}_{2}\right)+1$.
(c) $\Sigma^{*} \vDash \notin(x) \& Æ(\mathrm{u}) \& \mathrm{xy}=\mathrm{uv} \rightarrow \mathrm{x}=\mathrm{u} \& \mathrm{u}=\mathrm{v}$.

Proof: (a) Clearly, $\Sigma^{*} \neq Æ(a)$. Assume $\Sigma^{*} \vDash \notin(x) \& x \neq a$. Then $\Sigma^{*} \vDash \neg a B x$, by (c2). So $\Sigma^{*}$ F bBx. Note that $\left.\Sigma^{*} \vDash \neg \mathbb{E}(\mathrm{aa}) \& \neg \mathbb{( a b}\right) \& \neg \notin(\mathrm{ba}) \& \neg \mathbb{E}(\mathrm{bb})$.

Hence any x such that $\Sigma^{*} \mathrm{~F} \mathbb{E}(\mathrm{x})$ must have a (proper) endsegment of length
2. Suppose $\sum^{*} \vDash x=x_{1} a b v x=x_{1} b a v x=x_{1} b b$, that is, $a b E x v b a E x v b b E x$.

By (c1) and (c2), $\Sigma^{*}$ F $\alpha(\mathrm{x})=\beta(\mathrm{x})+1$, and $\Sigma^{*} \mathrm{~F} \alpha\left(\mathrm{x}_{1}\right) \leq \beta\left(\mathrm{x}_{1}\right)$. If $\Sigma^{*} \mathrm{~F}$ abEx or $\Sigma^{*}$ f baEx, then $\Sigma^{*} \vDash \alpha(\mathrm{x})=\alpha\left(\mathrm{x}_{1}\right)+1$ and $\Sigma^{*} \vDash \beta(\mathrm{x})=\beta\left(\mathrm{x}_{1}\right)+1$. But then $\Sigma^{*} \mathrm{~F} \alpha(\mathrm{x})=\beta(\mathrm{x})+1=\left(\beta\left(\mathrm{x}_{1}\right)+1\right)+1=\beta\left(\mathrm{x}_{1}\right)+2 \geq \alpha\left(\mathrm{x}_{1}\right)+2>\alpha\left(\mathrm{x}_{1}\right)+1=\alpha(\mathrm{x})$,
a contradiction. On the other hand, if $\Sigma^{*} \vDash \mathrm{bbEx}$, then $\Sigma^{*} \vDash \alpha(\mathrm{x})=\alpha\left(\mathrm{x}_{1}\right)$ and $\Sigma^{*} \vDash \beta(x)=\beta\left(x_{1}\right)+2$. But then

$$
\Sigma^{*} \mathrm{~F} \alpha(\mathrm{x})=\beta(\mathrm{x})+1=\left(\beta\left(\mathrm{x}_{1}\right)+2\right)+1=\beta\left(\mathrm{x}_{1}\right)+3 \geq \alpha\left(\mathrm{x}_{1}\right)+3=\alpha(\mathrm{x})+3>\alpha(\mathrm{x}),
$$

a contradiction again. Hence $\Sigma^{*}$ F $\neg$ abEx \& $\neg$ baEx $\& \neg$ bbEx. But then we must have $\Sigma^{*}$ f aaEx.
(b) Assume $\Sigma^{*} \vDash \notin(x) \& x_{2} E x$. Then $\Sigma^{*} \vDash \exists \mathrm{x}_{1} \mathrm{x}=\mathrm{x}_{1} \mathrm{x}_{2}$, hence
$\Sigma^{*} \vDash \alpha\left(\mathrm{x}_{1}\right) \leq \beta\left(\mathrm{x}_{1}\right)$. But $\Sigma^{*}$ f $\alpha(\mathrm{x})=\beta(\mathrm{x})+1$ and

$$
\Sigma^{*} \vDash \alpha(\mathrm{x})=\alpha\left(\mathrm{x}_{1} \mathrm{x}_{2}\right)=\alpha\left(\mathrm{x}_{1}\right)+\alpha\left(\mathrm{x}_{2}\right)
$$

whereas $\Sigma^{*} f \beta(x)=\beta\left(x_{1} x_{2}\right)=\beta\left(x_{1}\right)+\beta\left(x_{2}\right)$. Then

$$
\Sigma^{*} \vDash \alpha\left(\mathrm{x}_{1}\right)+\alpha\left(\mathrm{x}_{2}\right)=\beta\left(\mathrm{x}_{1}\right)+\beta\left(\mathrm{x}_{2}\right)+1,
$$

whence from $\Sigma^{*}$ F $\alpha\left(\mathrm{x}_{1}\right) \leq \beta\left(\mathrm{x}_{1}\right)$ we have $\alpha\left(\mathrm{x}_{2}\right) \geq \beta\left(\mathrm{x}_{2}\right)+1$, as claimed.
(c) Assume $\Sigma^{*} \vDash \notin(x) \& \mathbb{E}(\mathrm{u}) \& x y=u v$. We have that

$$
\Sigma^{*} f(x=u \& y=v) v x B u v u B x .
$$

Suppose $\Sigma^{*}$ F xBu . From $\Sigma^{*}$ F $Æ(\mathrm{u}), \Sigma^{*}$ F $\alpha(\mathrm{x}) \leq \beta(\mathrm{x})$, and from $\Sigma^{*} \vDash \notin(x)$, $\Sigma^{*} k \alpha(x)=\beta(x)+1$. But then $\Sigma^{*} \vDash \beta(x)+1 \leq \beta(x)$, a contradiction.

Likewise if $\Sigma^{*}$ ह uBx. Hence $\Sigma^{*}$ ह $\mathrm{x}=\mathrm{u} \& \mathrm{y}=\mathrm{v}$.
2.2 $\quad \Sigma^{*} \vDash Æ(\mathrm{x}) \leftrightarrow \mathrm{x}=\mathrm{a}$ v $\exists!\mathrm{y}, \mathrm{z}(\mathrm{x}=\mathrm{b}(\mathrm{yz}) \& Æ(\mathrm{y}) \& Æ(\mathrm{z}))$.

Proof: $(\Leftarrow)$ Assume $\Sigma^{*} \vDash \notin(y) \& Æ(z) \& x=b y z$. Then

$$
\Sigma^{*} \vDash \alpha(\mathrm{y})=\beta(\mathrm{y})+1 \& \alpha(\mathrm{z})=\beta(\mathrm{z})+1 .
$$

Now, $\quad \Sigma^{*} F \alpha(\mathrm{x})=\alpha(\mathrm{byz})=\alpha(\mathrm{yz})=\alpha(\mathrm{y})+\alpha(\mathrm{z})$
and $\quad \Sigma^{*} \vDash \beta(x)=\beta(b y z)=\beta(b)+\beta(y z)=\beta(y)+\beta(z)+1$. Then
$\Sigma^{*} \vDash \alpha(\mathrm{x})=\alpha(\mathrm{y})+\alpha(\mathrm{z})=(\beta(\mathrm{y})+1)+(\beta(\mathrm{z})+1)=(\beta(\mathrm{y})+\beta(\mathrm{z})+1)+1=\beta(\mathrm{x})+1$
which verifies (c1). For (c2), assume $\Sigma^{*}$ fuBx, i.e., $\Sigma^{*}$ f uBbyz.

Then $\quad \Sigma^{*} \mathrm{fu}=\mathrm{b} \mathrm{v}$ uBby $\mathrm{v} u=$ by $\mathrm{v} \exists \mathrm{z}_{1}\left(\mathrm{z}_{1} \mathrm{Bz} \& \mathrm{u}=\mathrm{byz}_{1}\right)$.

To illustrate the proof, we consider the case $\Sigma^{*} \vDash \exists \mathrm{z}_{1}\left(\mathrm{z}_{1}\right.$ By \& $\left.\mathrm{u}=\mathrm{byz}_{1}\right)$.

Then from $\Sigma^{*} \vDash Æ(\mathrm{z}), \Sigma^{*} \vDash \alpha\left(\mathrm{z}_{1}\right) \leq \beta\left(\mathrm{z}_{1}\right)$, and from $\Sigma^{*} \vDash \notin(\mathrm{y})$,
$\Sigma^{*} \vDash \alpha(\mathrm{y})=\beta(\mathrm{y})+1$. Then $\Sigma^{*} \vDash \alpha(\mathrm{u})=\alpha\left(\mathrm{byz}_{1}\right)=\alpha\left(\mathrm{yz}_{1}\right)=\alpha(\mathrm{y})+\alpha\left(\mathrm{z}_{1}\right)$ and
$\Sigma^{*} \vDash \beta(u)=\beta\left(b y z_{1}\right)=\beta(b)+\beta\left(y z_{1}\right)=\beta(y)+\beta\left(z_{1}\right)+1$. Hence
$\Sigma^{*} \vDash \alpha(\mathrm{u})=\alpha(\mathrm{y})+\alpha\left(\mathrm{z}_{1}\right)=(\beta(\mathrm{y})+1)+\alpha\left(\mathrm{z}_{1}\right) \leq$

$$
\leq(\beta(\mathrm{y})+1)+\beta\left(\mathrm{z}_{1}\right)=\beta(\mathrm{y})+\beta\left(\mathrm{z}_{1}\right)+1=\beta(\mathrm{u}) .
$$

Thus $\Sigma^{*} \vDash \alpha(\mathrm{u}) \leq \beta(\mathrm{u})$. This completes the proof of (c2). So $\Sigma^{*} \vDash Æ(\mathrm{x})$.
$(\Rightarrow)$ Assume $\Sigma^{*} \vDash \notin(x) \& x \neq a$. Then, by 2.1(a), $\Sigma^{*} \vDash b B x \& a a E x$, that is, $\Sigma^{*} \vDash \exists \mathrm{x}_{1} \mathrm{x}=\mathrm{bx} \mathrm{x}_{1} \& \exists \mathrm{x}_{2} \mathrm{x}=\mathrm{x}_{2}$ aa.

So $\Sigma^{*} \mathrm{Fbx}=\mathrm{x}_{2}$ aa. We may assume that $\Sigma^{*} \mathrm{FbBx} 2$, for if $\Sigma^{*} \mathrm{~F} \mathrm{x}_{2}=\mathrm{b}$, then $\Sigma^{*} \vDash \mathrm{x}=\mathrm{b}(\mathrm{aa})$ and we may take $\mathrm{y}=\mathrm{a}$ and $\mathrm{z}=\mathrm{a}$. So $\sum^{*} \vDash \exists \mathrm{x}_{3} \mathrm{x}_{2}=\mathrm{bx} \mathrm{x}_{3}$, and $\Sigma^{*} \mathrm{Fx}=\mathrm{bx}_{1}=\mathrm{x}_{2}(\mathrm{aa})=\mathrm{bx}_{3}(\mathrm{aa})$, whence $\quad \Sigma^{*} \mathrm{~F} \mathrm{x}_{1}=\mathrm{x}_{3}(\mathrm{aa})$. Let $\mathrm{y}_{\mathrm{j}}$ be a proper initial segment of $x_{1}$, and $z_{j}$ the corresponding endsegment of $x_{1}$ such that $\Sigma^{*} \mathrm{~F}_{\mathrm{y}}^{\mathrm{j}} \mathrm{z}_{\mathrm{j}}=\mathrm{x}_{1}$. At least one $\mathrm{y}_{\mathrm{j}}$ has the property

$$
\begin{equation*}
\Sigma^{*} \vDash \alpha\left(y_{\mathrm{j}}\right)=\beta\left(\mathrm{y}_{\mathrm{i}}\right)+1 . \tag{*}
\end{equation*}
$$

Consider, e.g., $\mathrm{x}_{3} \mathrm{a}$. From hypothesis $\Sigma^{*} \vDash \mathbb{E}(\mathrm{x})$ we have $\Sigma^{*} \vDash \alpha(\mathrm{x})=\beta(\mathrm{x})+1$.

But $\sum^{*}$ : $\alpha(\mathrm{x})=\alpha\left(\mathrm{b}\left(\left(\mathrm{x}_{3} \mathrm{a}\right) \mathrm{a}\right)\right)=\alpha(\mathrm{b})+\alpha\left(\mathrm{x}_{3} \mathrm{a}\right)+\alpha(\mathrm{a})=\alpha\left(\mathrm{x}_{3} \mathrm{a}\right)+1$ and
$\Sigma^{*} \vDash \beta(x)=\beta\left(b\left(\left(x_{3} a\right) a\right)\right)=\beta(b)+\beta\left(x_{3} a\right)+\beta(a)=1+\beta\left(x_{3} a\right)$.

Then $\Sigma^{*} \vDash \alpha\left(x_{3} a\right)=\alpha(x)-1=\beta(x)=\beta\left(x_{3} a\right)+1$.

Let $y_{i}$ be the shortest initial segment of $x_{1}$ with the property (*). Then

$$
\Sigma^{*} \vDash \mathrm{x}_{1}=\mathrm{y}_{\mathrm{i}} \mathrm{z}_{\mathrm{i}} \& \alpha\left(\mathrm{y}_{\mathrm{i}}\right)=\beta\left(\mathrm{y}_{\mathrm{i}}\right)+1 .
$$

We claim that (i) $\Sigma^{*} \vDash \alpha\left(\mathrm{z}_{\mathrm{i}}\right)=\beta\left(\mathrm{z}_{\mathrm{i}}\right)+1$, (ii) $\Sigma^{*} \mathrm{~F} \forall \mathrm{u}\left(\mathrm{uBy}_{\mathrm{i}} \rightarrow \alpha(\mathrm{u}) \leq \beta(\mathrm{u})\right)$, and (iii) $\Sigma^{*} \vDash \forall \mathrm{v}\left(\mathrm{vBz}_{\mathrm{i}} \rightarrow \alpha(\mathrm{v}) \leq \beta(\mathrm{v})\right)$.

For (i) we have $\Sigma^{*} f \alpha(\mathrm{x})=\alpha\left(\mathrm{bx}_{1}\right)=\alpha\left(\mathrm{x}_{1}\right)=\alpha\left(\mathrm{y}_{\mathrm{i}} \mathrm{z}_{\mathrm{i}}\right)=\alpha\left(\mathrm{y}_{\mathrm{i}}\right)+\alpha\left(\mathrm{z}_{\mathrm{i}}\right)$
and $\quad \Sigma^{*} F \beta(\mathrm{x})=\beta\left(\mathrm{bx}_{1}\right)=1+\beta\left(\mathrm{x}_{1}\right)=1+\beta\left(\mathrm{y}_{\mathrm{i}} \mathrm{z}_{\mathrm{i}}\right)=1+\beta\left(\mathrm{y}_{\mathrm{i}}\right)+\beta\left(\mathrm{z}_{\mathrm{i}}\right)$.

Then $\Sigma^{*}$ \& $\alpha\left(y_{i}\right)+\alpha\left(z_{i}\right)=\left(1+\beta\left(y_{i}\right)+\beta\left(z_{i}\right)\right)+1$, and from $\Sigma^{*}$ ह $\alpha\left(y_{i}\right)=\beta\left(y_{i}\right)+1$ we obtain $\Sigma^{*}$ k $\alpha\left(\mathrm{z}_{\mathrm{i}}\right)=\beta\left(\mathrm{z}_{\mathrm{i}}\right)+1$.

For (ii), suppose $\Sigma^{*}$ FuByi. Since $\Sigma^{*} k x_{1}=y_{i} z_{i}$, we then have $\Sigma^{*} k u B x_{1}$. But then, by the choice of $y_{i}, \Sigma^{*} \vDash \alpha(\mathrm{u}) \leq \beta(\mathrm{u})$. For (iii), suppose $\Sigma^{*}$ F $\mathrm{vBz} \mathrm{i}_{\mathrm{i}}$. Then $\Sigma^{*} \vDash \exists \mathrm{w} \mathrm{z}_{\mathrm{i}}=\mathrm{vw}$, whence $\Sigma^{*}$ f wEx. From $\Sigma^{*} \vDash$ Æ(x), by 2.1(b),
$\Sigma^{*} \vDash \alpha(w) \geq \beta(w)+1$. But $\Sigma^{*} \vDash \alpha\left(z_{i}\right)=\alpha(v)+\alpha(w)$ and
$\Sigma^{*} \vDash \beta\left(\mathrm{z}_{\mathrm{i}}\right)=\beta(\mathrm{v})+\beta(\mathrm{w}) . \mathrm{By}(\mathrm{i}), \quad \Sigma^{*}$ F $\alpha(\mathrm{v})+\alpha(\mathrm{w})=\beta(\mathrm{v})+\beta(\mathrm{w})+1$.

Then from $\Sigma^{*} \vDash \alpha(\mathrm{w}) \geq \beta(\mathrm{w})+1$, we have $\Sigma^{*} \vDash \alpha(\mathrm{v}) \leq \beta(\mathrm{v})$.

From (i)-(iii) we have that $\Sigma^{*} \vDash Æ\left(y_{i}\right) \& Æ\left(z_{i}\right)$. The uniqueness of $y, z$ follows from 2.1(c).

The proof of 2.2 yields an algorithm for extracting the description of a tree from a given $Æ$ string $x$. (i) Drop the initial $b$. (ii) If the next digit is $a$, that is the left node Æ string; the rest of the string is the right node Æ string. (iii) If the next digit is $b$, take the shortest initial segment $y$ of the remainder of the original string such that $\alpha(y)=\beta(y)+2$; then the string by is the left node $\notin$ string, and the endsegment of the remainder corresponding to by is the right node $\not Æ$ string. Repeat steps (i)-(iii) until no $b$ 's are left.
$2.3 \Sigma^{*} \vDash Æ(\mathrm{x}) \& Æ(\mathrm{y}) \& Æ(\mathrm{z}) \rightarrow\left(\mathrm{x} \subseteq_{\mathrm{p}} \mathrm{byz} \rightarrow \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{v} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p} z} \mathrm{z}\right)$.

Proof: Assume $\Sigma^{*} \vDash \mathrm{x} \subseteq_{\mathrm{p}}$ byz where $\Sigma^{*} \mathrm{~F}$ Æ(x) \& $(\mathrm{y})$ \& Æ(z). Now, we have that

$$
\Sigma^{*} \vDash \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x}=\mathrm{b} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{yz} \mathrm{v} \exists \mathrm{u}(\mathrm{uByz} \& \mathrm{x}=\mathrm{bu}) .
$$

Suppose that $\Sigma^{*} \vDash \exists \mathrm{u}(\mathrm{uByz} \& \mathrm{x}=\mathrm{bu})$. From $\Sigma^{*} \vDash \AA(\mathrm{y}) \& \AA(\mathrm{z})$, by 2.2, $\Sigma^{*} \vDash \notin(b y z)$. From $\Sigma^{*}$ fuByz, we have $\Sigma^{*} \vDash \exists v u v=y z$, whence $\Sigma^{*}$ f buBb(yz). Thus $\Sigma^{*}$ F $\mathrm{xBb}(\mathrm{yz})$. But from $\Sigma^{*}$ F Æ(byz), $\Sigma^{*} \vDash \alpha(\mathrm{x}) \leq \beta(\mathrm{x})$, which contradicts $\Sigma^{*} \vDash Æ(\mathrm{x})$. So $\Sigma^{*} \vDash \exists \mathrm{u}(\mathrm{uByz} \& \mathrm{x}=\mathrm{bu})$ is ruled out.

By 2.1(a), so is $\Sigma^{*} \vDash \mathrm{x}=\mathrm{b}$. So we are left with $\Sigma^{*} \vDash \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{byz} \rightarrow \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{yz}$. Supposing $\Sigma^{*} \mathrm{Fx} \subseteq_{\mathrm{p}} \mathrm{yz}$, we have that
$\Sigma^{*} \vDash x=y z \vee x \subseteq_{p y} v x \subseteq_{p z} v \exists y_{1}\left(y_{1} E y \& x=y_{1} z\right) v$

$$
v \exists z_{1}\left(z_{1} B z \& x=y z_{1}\right) v \exists y_{1}, z_{1}\left(y_{1} E y \& z_{1} B z \& x=y_{1} z_{1}\right) .
$$

Assume $\Sigma^{*}$ F $\mathrm{x}=\mathrm{yz}$. Then from $\Sigma^{*}$ ह Æ(y) \& Æ(z), we have
$\Sigma^{*} \vDash \alpha(\mathrm{y})=\beta(\mathrm{y})+1$ and $\alpha(\mathrm{z})=\beta(\mathrm{z})+1$. But $\Sigma^{*}$ F $\alpha(\mathrm{yz})=\alpha(\mathrm{y})+\alpha(\mathrm{z})$, so

$$
\Sigma^{*} \vDash \alpha(y z)=(\beta(y)+1)+(\beta(\mathrm{z})+1)=\beta(\mathrm{y})+\beta(\mathrm{z})+2 .
$$

On the other hand, $\Sigma^{*} \vDash \beta(y z)=\beta(y)+\beta(\mathrm{z})$. Thus $\Sigma^{*}$ f $\alpha(\mathrm{yz})=\beta(\mathrm{yz})+2$, whence from $\Sigma^{*} \vDash \mathrm{x}=\mathrm{yz}$, we derive $\Sigma^{*} \vDash \alpha(\mathrm{x})=\beta(\mathrm{x})+2$, contradicting $\Sigma^{*} \vDash Æ(\mathrm{x})$. So $\Sigma^{*} \vDash \mathrm{x}=\mathrm{yz}$ is ruled out.

Suppose now that $\Sigma^{*} \vDash \exists y_{1}\left(y_{1}\right.$ Ey \& $\left.\mathrm{x}=\mathrm{y}_{1} \mathrm{z}\right)$, so $\Sigma^{*} \vDash \mathrm{y}_{1} \mathrm{Bx}$. From $\Sigma^{*} \vDash \notin(\mathrm{x})$, $\Sigma^{*} \vDash \alpha\left(\mathrm{y}_{1}\right) \leq \beta\left(\mathrm{y}_{1}\right)$. But from $\Sigma^{*} \vDash \notin(\mathrm{y}) \& \mathrm{y}_{1} \mathrm{Ey}$, we obtain, by 2.1(b), $\sum^{*} \vDash \alpha\left(y_{1}\right) \geq \beta\left(y_{1}\right)+1$, a contradiction.

Suppose that $\Sigma^{*} \vDash \exists \mathrm{z}_{1}\left(\mathrm{z}_{1} \mathrm{Bz} \& \mathrm{x}=\mathrm{yz}_{1}\right)$, so $\Sigma^{*} \mathrm{~F} \mathrm{yBx}$. But then from $\Sigma^{*} \vDash Æ(\mathrm{x})$, we have $\Sigma^{*} \vDash \alpha(y) \leq \beta(y)$, and from $\Sigma^{*} \vDash \notin(y), \quad \Sigma^{*} \vDash \alpha(y)=\beta(y)+1$, again a contradiction. If $\Sigma^{*} \vDash \exists y_{1}, z_{1}\left(y_{1} E y \& z_{1} B z \& x=y_{1} z_{1}\right)$, we derive a contradiction by reasoning as in either of the two preceding cases.

The other cases having been ruled out, we conclude under the principal hypothesis that $\Sigma^{*} \mathrm{Fx} \subseteq_{\mathrm{pyz}} \rightarrow \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{pz}}$, and further that

$$
\Sigma^{*} \vDash \quad \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{byz} \rightarrow \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \text { v x} \subseteq_{\mathrm{pz}},
$$

as required.

If we take the domain to consists of $Æ$ strings, 2.1(c), 2.2 and 2.3 suffice to give the "first approximation" of our interpretation of T in concatenation
theory: translations of (T1)-(T4) will be validated in $\Sigma^{*}$ if we model the term/tree-building operation $\mathrm{x}, \mathrm{y} \mapsto(\mathrm{xy})$ by $b x y$, the subterm/subtree relation $\subseteq$ by the substring relation $\subseteq_{p}$ between $Æ$ strings, and the digit $a$ is taken to stand for the simple term 0 . The entire project, however, hinges on definability of the counting functions $\alpha$ and $\beta$ in concatenation theory. Showing that the latter contains resources needed to formally justify definitions by elementary recursion on strings requires, first, that we precisely formulate concatenation theory as a formal theory, and second, that we introduce codings for ordered pairs of strings, sequences of such, etc., and verify their properties relevant to the argument in that formal theory. We now turn to that task. In the process we shall make crucial use of the method of formula selection explained in [1].

## 3. Formal Concatenation Theory

We shall work within a first-order theory formulated in $\mathcal{L}_{C}=\left\{\mathrm{a}, \mathrm{b},{ }^{*}\right\}$, with the universal closures of the following conditions as axioms:
(QT1) $\quad x^{*}\left(y^{*} z\right)=\left(x^{*} y\right)^{*} z$
(QT2) $\neg\left(x^{*} y=a\right) \& \neg\left(x^{*} y=b\right)$
(QT3) $\left(x^{*} a=y^{*} a \rightarrow x=y\right) \&\left(x^{*} b=y^{*} b \rightarrow x=y\right) \&$ \& ( $\left.a^{*} x=a^{*} y \rightarrow x=y\right) \&\left(b^{*} x=b^{*} y \rightarrow x=y\right)$
(QT4) $\neg\left(\mathrm{a}^{*} \mathrm{x}=\mathrm{b}^{*} \mathrm{y}\right) \& \neg\left(\mathrm{x}^{*} \mathrm{a}=\mathrm{y}^{*} \mathrm{~b}\right)$
(QT5) $\mathrm{x}=\mathrm{avx}=\mathrm{bv}\left(\exists \mathrm{y}\left(\mathrm{a}^{*} \mathrm{y}=\mathrm{x} v \mathrm{~b}^{*} \mathrm{y}=\mathrm{x}\right) \& \exists \mathrm{z}\left(\mathrm{z}^{*} \mathrm{a}=\mathrm{x} v \mathrm{z}^{*} \mathrm{~b}=\mathrm{x}\right)\right)$
On account of (QT1), we sometimes omit parentheses and * when writing ( $\left.\mathrm{x}^{*} \mathrm{y}\right)$.

It is convenient to have a function symbol for a successor operation on strings:

$$
\begin{equation*}
S x=y \leftrightarrow\left((x=a \& y=b) v\left(\neg x=a \& x^{*} b=y\right)\right) \tag{QT6}
\end{equation*}
$$

Since (QT6) is basically a definition, adding it to the rest results in an inessential (i.e. conservative) extension. We call this theory QT $^{+}$.

Let $\quad x R y \equiv(x=a \& \neg y=a) v x B y$.
Provably in QT $^{+}, \mathrm{xRy} \mathrm{vx}=\mathrm{y}$ is a discrete preordering of strings (see [1]).

We shall call a formula $\mathrm{I}(\mathrm{x})$ in the language of $\mathrm{QT}^{+}$a string form if
$\mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{a}), \quad \mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{b}), \quad \mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{x}) \rightarrow \mathrm{I}\left(\mathrm{x}^{*} \mathrm{a}\right) \quad$ and $\quad \mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{x}) \rightarrow \mathrm{I}\left(\mathrm{x}^{*} \mathrm{~b}\right)$.
(Note: in [1] and [2] such formulae were called string concepts.) String forms will allow us to restrict our attention, systematically step-by-step, to strings
that satisfy conditions expressible by specifically selected formulas provided the latter can be proved in QT+ to apply to "sufficiently many" strings. We say that a string form J is stronger than I if $\mathrm{QT}^{+} \vdash \forall \mathrm{x}(\mathrm{J}(\mathrm{x}) \rightarrow \mathrm{I}(\mathrm{x}))$ and write $\mathrm{J} \subseteq \mathrm{I}$.

Let $\mathrm{I}_{0}(\mathrm{x}) \equiv \forall \mathrm{y}(\mathrm{yRx} v \mathrm{y}=\mathrm{x} \rightarrow \neg \mathrm{yRy})$. We call $\mathrm{I}_{0}$ strings tractable.
3.1(a) $\mathrm{I}_{0}(\mathrm{x})$ is a string form.
(b) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \forall \mathrm{y}\left(\mathrm{~J}(\mathrm{x}) \& \mathrm{~J}(\mathrm{y}) \rightarrow \mathrm{J}\left(\mathrm{x}^{*} \mathrm{y}\right)\right) .
$$

(c) For any string form $I \subseteq I_{0}$ there is a string form $\mathrm{J} \leq \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x}(\mathrm{~J} \leq(\mathrm{x}) \& \mathrm{y} \leq \mathrm{x} \rightarrow \mathrm{~J} \leq(\mathrm{y})) .
$$

(d) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{y}\left(\mathrm{y} \subseteq_{\mathrm{p} X} \rightarrow \mathrm{~J}(\mathrm{y})\right) .
$$

(e) For any string form $I \subseteq I_{0}$ there is a string form $J \equiv I_{L C} \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{~J} \forall \mathrm{x}, \mathrm{y}\left(\mathrm{z}^{*} \mathrm{x}=\mathrm{z}^{*} \mathrm{y} \rightarrow \mathrm{x}=\mathrm{y}\right) .
$$

(f) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{~J} \forall \mathrm{x}, \mathrm{y}\left(\mathrm{x}^{*} \mathrm{z}=\mathrm{y}^{*} \mathrm{z} \rightarrow \mathrm{x}=\mathrm{y}\right) .
$$

For proofs, see [1], and [2], (3.2), (3.3), (3.13), (3.7) and (3.6).

Parts (b)-(c) tell us that when establishing that a given string form I may be strengthened to a string form J with another property, we can always strengthen the string form J to one that is also closed with respect to * or downward closed with respect to $\leq$ or $\subseteq_{\mathrm{p}}$.

We define $\quad \operatorname{Tally}_{\mathrm{a}}(\mathrm{x}) \equiv \forall \mathrm{y} \subseteq_{\mathrm{p}} \mathrm{x}(\operatorname{Digit}(\mathrm{y}) \rightarrow \mathrm{y}=\mathrm{a})$
and $\operatorname{Tallyb}_{\mathrm{b}}(\mathrm{x}) \equiv \forall \mathrm{y} \subseteq_{\mathrm{p}} \mathrm{x}(\operatorname{Digit}(\mathrm{y}) \rightarrow \mathrm{y}=\mathrm{b})$ where $\operatorname{Digit}(\mathrm{x}) \equiv \mathrm{x}=\mathrm{av} \mathrm{x}=\mathrm{b}$.

Write $\mathrm{x}<\mathrm{y}$ for $\mathrm{I}_{0}(\mathrm{x}) \& \mathrm{I}_{0}(\mathrm{y}) \& \mathrm{xRy}$. As usual, $\mathrm{x} \leq \mathrm{y}$ stands for $\mathrm{x}<\mathrm{y} \mathrm{v} \mathrm{x}=\mathrm{y}$.

The following properties of tallies are easily established:
3.2 (a) $\mathrm{QT}^{+} \vdash \mathrm{Tallyb}(\mathrm{y}) \rightarrow$ Tallyb(Sy).
(b) $\quad \mathrm{QT}^{+} \vdash \mathrm{Tally}_{\mathrm{b}}(\mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{bv} \mathrm{Jy}_{1}\left(\mathrm{Tally}_{\mathrm{b}}\left(\mathrm{y}_{1}\right) \& \mathrm{y}=\mathrm{Sy}_{1}\right)$.
(c) $\quad \mathrm{QT}^{+} \vdash \forall \mathrm{v}, \mathrm{u}\left(\mathrm{Tallyb}_{\mathrm{b}}(\mathrm{v}) \& \mathrm{u}<\mathrm{v} \rightarrow \mathrm{Su} \leq \mathrm{v}\right)$.
(d) $\quad \mathrm{QT}^{+}+\mathrm{Tallyb}_{\mathrm{b}}(\mathrm{y}) \rightarrow(\mathrm{x}<\mathrm{y} \leftrightarrow \mathrm{Sx}<$ Sy $)$.

For some further properties we have to resort to string forms:
3.3(a) For any string form $I \subseteq I_{0}$ there is a string form $\mathrm{J} \equiv \mathrm{I}_{\mathrm{CT}} \subseteq \mathrm{I}$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{~J} \forall \mathrm{y}\left(\mathrm{Tallyb}_{\mathrm{b}}(\mathrm{y}) \& \operatorname{Tally\mathrm {b}}(\mathrm{z}) \rightarrow \operatorname{Tally} \mathrm{b}\left(\mathrm{y}^{*} \mathrm{z}\right)\right) .
$$

(b) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that
$\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{J} \forall \mathrm{x}\left(\operatorname{Tallyb}_{\mathrm{b}}(\mathrm{x}) \& \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{z}) \rightarrow \mathrm{x} \leq \mathrm{z} \mathrm{v} \mathrm{z} \leq \mathrm{x}\right)$.
(c) For any string form $\mathrm{I} \subseteq \mathrm{I}_{0}$ there is a string form $\mathrm{J} \equiv \mathrm{I}_{3.3(\mathrm{c})} \subseteq \mathrm{I}$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{u} \in \mathrm{~J}\left(\operatorname{Tallyb}_{\mathrm{b}}(\mathrm{u}) \rightarrow \mathrm{u}^{*} \mathrm{~b}=\mathrm{b}^{*} \mathrm{u}\right) .
$$

(d) For any string form $\mathrm{I} \subseteq \mathrm{I}_{0}$ there is a string form $\mathrm{J} \subseteq$ I such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{~J} \forall \mathrm{x}\left(\operatorname{Tally\mathrm {b}}(\mathrm{x}) \& \operatorname{Tally\mathrm {b}}(\mathrm{y}) \rightarrow \mathrm{Sx} \mathrm{x}^{*} \mathrm{y}=\mathrm{x}^{*} \mathrm{Sy}=\mathrm{S}\left(\mathrm{x}^{*} \mathrm{y}\right)\right) .
$$

(e) For any string form $I \subseteq I_{3.3(c)}$ there is a string form $\mathrm{J} \equiv \mathrm{I}_{\text {сомм }} \subseteq \mathrm{I}_{\text {such }}$ that $\mathrm{QT}^{+} \vdash \forall \mathrm{u}, \mathrm{v} \in \mathrm{J}\left(\mathrm{Tallyb}_{\mathrm{b}}(\mathrm{u}) \& \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{v}) \rightarrow \mathrm{u}^{*} \mathrm{v}=\mathrm{v}^{*} \mathrm{u}\right)$.

For proofs, see [2], (4.5), (4.6), (4.8) and (4.10).

Let Addtally ( $\mathrm{x}, \mathrm{y}, \mathrm{z}$ ) abbreviate the formula
(Tallyb(x) \& Tallyb(y) \& ( $(x=b \& z=y) v(y=b \& z=x) v$

$$
\begin{aligned}
v \exists x_{1}, y_{1}\left(\operatorname{Tally}_{\mathrm{b}}\left(\mathrm{x}_{1}\right) \& \mathrm{x}=\mathrm{Sx}_{1} \&\right. & \text { Tally } \left.\left.\mathrm{y}_{\mathrm{b}}\left(\mathrm{y}_{1}\right) \& \mathrm{y}=\text { Sy }_{1} \& \mathrm{z}=\mathrm{x}^{*} \mathrm{y}_{1}\right)\right) \mathrm{v} \\
\mathrm{v} & \left(\left(\neg \operatorname{Tally}_{\mathrm{b}}(\mathrm{x}) \mathrm{v} \neg \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{y})\right) \& \mathrm{z}=\mathrm{b}\right)
\end{aligned}
$$

We want to show that, provably in $\mathrm{QT}^{+}$, Addtally $(\mathrm{x}, \mathrm{y}, \mathrm{z})$ behaves like the graph of addition function on natural numbers. The following are immediate consequences of definitions:
3.4(a) $\mathrm{QT}^{+} \vdash \operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{v})$ \& Addtally $(\mathrm{x}, \mathrm{y}, \mathrm{w}) \rightarrow \mathrm{v}=\mathrm{w}$.
(b) $\mathrm{QT}^{+} \vdash \mathrm{Tally}_{\mathrm{b}}(\mathrm{x}) \rightarrow$ Addtally $(\mathrm{x}, \mathrm{b}, \mathrm{x})$.
("x+0 = x")
(c) $\mathrm{QT}^{+} \vdash \mathrm{Tallyb}_{\mathrm{b}}(\mathrm{y}) \rightarrow$ Addtally $(\mathrm{b}, \mathrm{y}, \mathrm{y})$.
(" $0+y=y$ ")
(d) $\mathrm{QT}^{+} \vdash \operatorname{Tallyb}(\mathrm{x}) \rightarrow \operatorname{Addtally}(\mathrm{x}, \mathrm{bb}, \mathrm{Sx}) . \quad(\mathrm{X} \mathrm{x}+1=\mathrm{Sx}$ ")
(e) $\mathrm{QT}^{+} \vdash \mathrm{Tally}_{\mathrm{b}}(\mathrm{x}) \& \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{y}) \rightarrow(\operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{z}) \rightarrow$ Addtally $(\mathrm{x}, \mathrm{yb}, \mathrm{zb}))$.

$$
(" x+S y=S(x+y) ")
$$

We also have:
3.5(a) For any string form $I \subseteq I_{0}$ there is a string form $J \equiv I_{\text {Add }} \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{x}, \mathrm{y} \in \mathrm{J} \exists \mathrm{z} \mathrm{z} \in \mathrm{J}(\operatorname{Tallyb} \mathrm{z}) \& \operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{z})$ ).
(b) $\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{I}_{0}\left(\mathrm{Tallyb}_{\mathrm{b}}(\mathrm{u}) \& \mathrm{Tallyb}_{\mathrm{b}}(\mathrm{v}) \&\right.$

$$
\begin{aligned}
& \& \operatorname{Addtally}(\mathrm{x}, \mathrm{u}, \mathrm{y}) \& \operatorname{Addtally}(\mathrm{x}, \mathrm{v}, \mathrm{z}) \& \mathrm{u} \leq \mathrm{v} \rightarrow \mathrm{y} \leq \mathrm{z}) . \\
& \qquad\left({ }^{(" \mathrm{u}} \leq \mathrm{v} \rightarrow \mathrm{x}+\mathrm{u} \leq \mathrm{x}+\mathrm{v}\right. \text { ")}
\end{aligned}
$$

(c) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\text { QT }^{+} \vdash \forall y \in J\left(T a l l y b(y) \rightarrow \text { Addtally }(\mathrm{bb}, \mathrm{y}, \mathrm{Sy}) . \quad\left(" 1+\mathrm{y}=\mathrm{Sy}^{\prime}\right)\right.
$$

(d) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that QT+ $\vdash \forall \mathrm{y} \in \mathrm{J} \forall \mathrm{x}, \mathrm{z}\left(\mathrm{Tally}_{\mathrm{b}}(\mathrm{x}) \& \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{y}) \& \operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{z}) \rightarrow\right.$ Addtally $\left.(\mathrm{xb}, \mathrm{y}, \mathrm{zb})\right)$
("Sx+y=S(x+y")
(e) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{J} \forall \mathrm{y}, \mathrm{z}, \mathrm{v}\left(\operatorname{Tally}_{\mathrm{b}}(\mathrm{x}) \& \operatorname{Tally}_{\mathrm{b}}(\mathrm{y}) \& \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{z}) \rightarrow\right.$ $\rightarrow(\operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{v}) \& \operatorname{Addtally}(\mathrm{x}, \mathrm{z}, \mathrm{v}) \rightarrow \mathrm{y}=\mathrm{z}))$.

$$
(" x+y=x+z \rightarrow y=z \text { ") }
$$

(f) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that
$\mathrm{QT}^{+}+\forall \mathrm{y} \in \mathrm{J} \forall \mathrm{x}\left(\operatorname{Tallyb}_{\mathrm{b}}(\mathrm{x}) \& \operatorname{Tally}_{\mathrm{b}}(\mathrm{y}) \rightarrow\right.$

$$
\left.\rightarrow\left(\mathrm{x} \leq \mathrm{y} \leftrightarrow \exists \mathrm{z}\left(\operatorname{Tall}_{\mathrm{b}}(\mathrm{z}) \& \operatorname{Addtally}(\mathrm{z}, \mathrm{x}, \mathrm{y})\right)\right)\right)
$$

$$
\text { ("x } x \leq y \leftrightarrow \exists z z+x=y \text { " })
$$

(g) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x}, \mathrm{y} \in \mathrm{~J}(\operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{z}) \rightarrow \text { Addtally }(\mathrm{y}, \mathrm{x}, \mathrm{z})) . \quad(\text { " } \mathrm{x}+\mathrm{y}=\mathrm{y}+\mathrm{x"})
$$

(h) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{x}, \mathrm{y}, \mathrm{z} \in \mathrm{J}\left(\operatorname{Addtally}(\mathrm{x}, \mathrm{y}, \mathrm{u})\right.$ \& Addtally $\left(\mathrm{u}, \mathrm{z}, \mathrm{v}_{1}\right)$ \& Addtally $(\mathrm{y}, \mathrm{z}, \mathrm{w})$ \&

$$
\begin{aligned}
& \left.\& \text { Addtally }\left(\mathrm{x}, \mathrm{w}, \mathrm{~V}_{2}\right) \rightarrow \mathrm{v}_{1}=\mathrm{v}_{2}\right) \\
& \quad\left({ }^{\prime \prime}(\mathrm{x}+\mathrm{y})+\mathrm{z}=\mathrm{x}+(\mathrm{y}+\mathrm{z})\right. \text { ") }
\end{aligned}
$$

(i) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{x}_{2}, \mathrm{y}_{1}, \mathrm{y}_{2} \in \mathrm{~J} \forall \mathrm{x}_{1}, \mathrm{z}_{1}, \mathrm{z}_{2}\left(\operatorname{Tally}_{\mathrm{b}}\left(\mathrm{x}_{2}\right) \& \operatorname{Tally}_{\mathrm{b}}\left(\mathrm{y}_{1}\right) \& \operatorname{Tally}_{\mathrm{b}}\left(\mathrm{y}_{2}\right) \&\right.$ \& Addtally $\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{z}_{1}\right)$ \& Addtally $\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{z}_{2}\right) \& \mathrm{x}_{1} \leq \mathrm{y}_{1} \& \mathrm{z}_{1}=\mathrm{Sz}_{2} \rightarrow$

$$
\left.\rightarrow \mathrm{Sy}_{2} \leq \mathrm{x}_{2}\right) .
$$

$$
\left(" x_{1}+x_{2}=\left(y_{1}+y_{2}\right)+1 \& x_{1} \leq y_{1} \rightarrow y_{2}+1 \leq x_{2} \text { " }\right)
$$

Proof: For (a), let J $\equiv \mathrm{I}_{\text {ctc }}$ from 3.3(a). For (c) and (d), let J be as in 3.3(c). For (e), let $\mathrm{J} \equiv \mathrm{I}_{\mathrm{Lc}}$ from 3.1(d). For (f) and (g), let J $\equiv \mathrm{I}_{\text {сомм }}$ from 3.3(e). For (h), let $\mathrm{J} \equiv \mathrm{J}_{1} \& \mathrm{~J}_{2}$ where $\mathrm{J}_{1}$ is $\mathrm{I}_{\mathrm{ctc}}$ and $\mathrm{J}_{2}$ as in 3.3(c). Finally, for (i), let $\mathrm{J} \equiv \mathrm{I}_{\text {Lc }} \& \mathrm{I}_{\text {стс }} \& \mathrm{I}_{3.3(\mathrm{c})}$ \& $\mathrm{I}_{\text {сомм }}$ and see Appendix.

We now turn to the part-of relation $\subseteq_{\mathrm{p}}$ between strings. To prevent unpleasant surprises, we want to make sure that this relation has natural properties we would normally expect it to have.
3.6(a)

$$
\mathrm{QT}^{+} \vdash \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \& \mathrm{y} \subseteq_{\mathrm{p}} \mathrm{z} \rightarrow \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{z} .
$$

(b) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \neg \mathrm{xEx} .
$$

(c) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \neg \exists \mathrm{x}_{1}, \mathrm{x}_{2}\left(\mathrm{x}_{1} \mathrm{xx}_{2}=\mathrm{x}\right) .
$$

(d) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{y}\left(\mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \& \mathrm{y} \subseteq_{\mathrm{p}} \mathrm{x} \rightarrow \mathrm{x}=\mathrm{y}\right) .
$$

(e) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{y}\left(\neg \mathrm{xy} \subseteq_{\mathrm{px}} \& \neg \mathrm{yx} \subseteq_{\mathrm{px}}\right) .
$$

Proof: For (b) and (c), see [2], (3.4) and (3.5). For (d) and (e), see [2], (3.11) and (3.12).

We now specifically consider proper initial segments and endsegments. The initial segments of arbitrary strings can be totally ordered by the initial-segment-of relation $B$, rendering the partial ordering <in which $a$ is the least element tree-like:
3.7(a) For any string form $\mathrm{I} \subseteq \mathrm{I}_{0}$ there is a string form $\mathrm{J}_{\text {LoIS }} \subseteq \mathrm{I}$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{u}, \mathrm{v}(\mathrm{uBx} \& \mathrm{vBx} \rightarrow \mathrm{u}=\mathrm{v} \mathrm{v} u B v \mathrm{v} \mathrm{vBu}) .
$$

(b) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{y}, \mathrm{z} \in \mathrm{~J} \forall \mathrm{x}(\mathrm{xByz} \leftrightarrow \mathrm{xBy} \mathrm{v} \mathrm{x}=\mathrm{y} v \exists \mathrm{w}(\mathrm{wBz} \& \mathrm{yw}=\mathrm{x})) .
$$

(c) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{x}, \mathrm{y} \in \mathrm{J} \forall \mathrm{u}\left(\mathrm{uBb}(\mathrm{xy}) \rightarrow \mathrm{u}=\mathrm{b} v \mathrm{uBbx} \mathrm{v} \mathrm{u}=\mathrm{bx} v \exists \mathrm{y}_{1}\left(\mathrm{y}_{1} \mathrm{By} \& \mathrm{u}=\mathrm{bx} \mathrm{y}_{1}\right)\right)$.
(d) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{u}, \mathrm{v}(\mathrm{uEx} \& \mathrm{vEx} \rightarrow \mathrm{u}=\mathrm{v} \mathrm{v} \text { uEv v vEu}) .
$$

(e) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{y}, \mathrm{z} \in \mathrm{J} \forall \mathrm{x}(\mathrm{xEyz} \leftrightarrow \mathrm{xEz} \mathrm{vx}=\mathrm{zv} \exists \mathrm{w}(\mathrm{wEy} \& \mathrm{wz}=\mathrm{x})$ ).
(f) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+}+\forall \mathrm{y}, \mathrm{z} \in \mathrm{J} \forall \mathrm{x}, \mathrm{x}_{1}, \mathrm{x}_{2}\left(\mathrm{x}_{1} \mathrm{xx}_{2}=\mathrm{yz} \rightarrow\right.$

$$
\left.\rightarrow \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \text { v } \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{z} \vee \exists \mathrm{y}_{1}, \mathrm{z}_{1}\left(\mathrm{y}_{1} \operatorname{Ey} \& \mathrm{z}_{1} \mathrm{Bz} \& \mathrm{x}=\mathrm{y}_{1} \mathrm{z}_{1}\right)\right) .
$$

(g) For any string concept $\mathrm{I} \subseteq \mathrm{I}_{0}$ there is a string concept $\mathrm{J} \subseteq \mathrm{I}$ such that

$$
\begin{aligned}
Q T+\vdash \forall y, z & \in J \forall x\left(x \subseteq_{p y z} \rightarrow x=y z v x \subseteq_{p y} v x \subseteq_{p z} v \exists y_{1}\left(y_{1} E y \& x=y_{1} z\right) v\right. \\
& \left.v \exists z_{1}\left(z_{1} B z \& x=y z_{1}\right) v \exists y_{1}, z_{1}\left(y_{1} E y \& z_{1} B z \& x=y_{1} z_{1}\right)\right) .
\end{aligned}
$$

(h) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{y}, \mathrm{z} \in \mathrm{~J} \forall \mathrm{x}\left(\mathrm{x} \subseteq_{\mathrm{p}} \mathrm{~b}(\mathrm{yz}) \rightarrow\right.
$$

$$
\left.\rightarrow \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x}=\mathrm{b} \vee \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{yz} \vee \exists \mathrm{u}_{2}\left(\mathrm{u}_{2} \operatorname{Byz} \& \mathrm{x}=\mathrm{bu} u_{2}\right)\right) .
$$

Proof: For (a), see [2], (3.8). For (b) and (c), let J 三 $\mathrm{I}_{\text {LC }}$ \& $\mathrm{I}_{\text {LoIs. }}$. For (d), see
[2], (3.10), and then (e) is proved analogously to (b). For (f) take J s in (b), and (g) follows from (b)-(f). Then (h) is obtained as a special case of (g).

## 4. Coding sequences and pairs of strings by strings

Formalizing recursion requires coding of sequences, and since the kind of recursion used to define the counting functions $\alpha$ and $\beta$ proceeds on strings, to carry out the formalization of such definitions in concatenation theory we will need to be able to code sequences of strings by strings. The general idea behind the coding goes back to Quine [7], and more recently to Visser [9], but the key for our purposes is to show that the relevant properties of the coding are provable in $\mathrm{QT}^{+}$. We make use of the coding scheme described in [2], pp.86-88 and summarized in [1], §§7-8. (Predicates 'Pref(x, t)',
 machinery needed to demonstrate that, modulo the methodology of formula selection, all of the necessary reasoning can indeed be carried out in $\mathrm{QT}^{+}$is presented in detail in [2], pp.89-263.) In particular, we can establish:
4.1(a) SINGLETON LEMMA. For any string form $I \subseteq I_{0}$ there is a string form ISNGL $\subseteq$ I such that
$\mathrm{QT}^{+}+\forall \mathrm{x} \in \mathrm{I}_{\text {SNGL }} \forall \mathrm{u}, \mathrm{t}_{1}, \mathrm{t}_{2}\left(\operatorname{Set}(\mathrm{x}) \& \operatorname{Firstf}\left(\mathrm{x}, \mathrm{t}_{1}\right.\right.$, aua, $\left.\mathrm{t}_{2}\right) \& \mathrm{x}=\mathrm{t}_{1}$ aual $_{2} \rightarrow$ $\rightarrow \forall \mathrm{w}(\mathrm{w} \varepsilon \mathrm{x} \leftrightarrow \mathrm{w}=\mathrm{u}))$.
(b) APPENDING LEMMA. For any string form $I \subseteq I_{0}$ there is a string form $\mathrm{I}_{\mathrm{APP}} \subseteq \mathrm{I}$ such that
$\mathrm{QT}^{+}+\forall \mathrm{x}, \mathrm{y} \in \mathrm{I}_{\mathrm{APP}} \forall \mathrm{t}, \mathrm{t}_{2}, \mathrm{t}_{3}\left(\operatorname{Env}\left(\mathrm{t}_{2}, \mathrm{x}\right) \& \operatorname{Env}(\mathrm{t}, \mathrm{y}) \&\left(\mathrm{t}_{3} \mathrm{a}\right) \operatorname{By}\right.$ \& Tallyb$\left(\mathrm{t}_{3}\right) \& \mathrm{t}_{2}<\mathrm{t}_{3} \&$

$$
\& \neg \exists \mathrm{u}(\mathrm{u} \varepsilon \mathrm{x} \& \mathrm{u} \varepsilon \mathrm{y}) \rightarrow \exists \mathrm{z} \in \mathrm{I}_{\mathrm{APP}}(\operatorname{Env}(\mathrm{t}, \mathrm{z}) \& \forall \mathrm{u}(\mathrm{u} \varepsilon \mathrm{z} \leftrightarrow \mathrm{u} \varepsilon \mathrm{x} v \mathrm{u} \varepsilon \mathrm{y})) .
$$

(c) DOUBLETON LEMMA. For any string form $I \subseteq I_{0}$ there is a string form IDBL $\subseteq I$ such that

$$
\begin{array}{r}
\mathrm{QT}^{+}+\forall \mathrm{x} \in \mathrm{I}_{\mathrm{DBL}} \forall \mathrm{t}_{1}, \mathrm{t}_{2}, \mathrm{t}_{3}, \mathrm{u}, \mathrm{v}\left(\operatorname{Pref}\left(\text { aua }, \mathrm{t}_{1}\right) \& \operatorname{Pref}\left(\mathrm{ava}, \mathrm{t}_{2}\right) \& \mathrm{t}_{1}<\mathrm{t}_{2} \& \mathrm{t}_{2}=\mathrm{t}_{3} \& \mathrm{u} \neq \mathrm{v} \&\right. \\
\& \mathrm{x}=\mathrm{t}_{1} \text { auat }_{2} \mathrm{avat}_{3} \rightarrow \operatorname{Set}(\mathrm{x}) \& \forall \mathrm{w}(\mathrm{w} \varepsilon \mathrm{x} \leftrightarrow(\mathrm{w}=\mathrm{u} v \mathrm{w}=\mathrm{v})) .
\end{array}
$$

Proof: See [2], (5.21), (5.46) and (5.58).

To use the coding of sets to code sequences of strings, we need to populate the coded sets with ordered pairs of arbitrary strings.

Let $\quad \operatorname{Pair}[\mathrm{x}, \mathrm{y}, \mathrm{z}] \equiv \exists \mathrm{t} \subseteq_{\mathrm{p}} \mathrm{z}\left(\mathrm{z}=\right.$ taxatayat \& $\left.\operatorname{MinMax}^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{xay})\right)$.
(The predicate 'MinMax ${ }^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{u})$ ' expressing ' t is a shortest non-occurrent b-tally in string u' is defined in [1], §10.) We then have:
4.2 PAIRING LEMMA. (a) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x}, \mathrm{y} \in \mathrm{~J} \exists \mathrm{z} \in \mathrm{~J}\left(\operatorname{Pair}[\mathrm{x}, \mathrm{y}, \mathrm{z}] \& \forall \mathrm{z}^{\prime}\left(\operatorname{Pair}\left[\mathrm{x}, \mathrm{y}, \mathrm{z}^{\prime}\right] \rightarrow \mathrm{z}^{\prime}=\mathrm{z}\right)\right) .
$$

(b) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{~J} \forall \mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{y}_{1}, \mathrm{y}_{2}\left(\operatorname{Pair}\left[\mathrm{x}_{1}, \mathrm{y}_{1}, \mathrm{z}\right] \& \operatorname{Pair}\left[\mathrm{x}_{2}, \mathrm{y}_{2}, \mathrm{z}\right] \rightarrow \mathrm{x}_{1}=\mathrm{x}_{2} \& \mathrm{y}_{1}=\mathrm{y}_{2}\right) .
$$

In (a), choose J as in [2], (6.8). For (b), referring to [2], let $\mathrm{J} \equiv \mathrm{I}_{3.6}$ \& $\mathrm{I}_{4.20}$ \& $\mathrm{I}_{4.23 b}$.

## 5. String recursion

Let $p, q$ be strings, and $f_{1}, f_{2}$ be functions on strings. Informally, we say that $h$ is defined by string recursion from $f_{1}, f_{2}$ if

$$
\begin{array}{cl}
h(a)=p & h(b)=q \\
h\left(y^{*} a\right)=f_{1}(y, h(y)) & h\left(y^{*} b\right)=f_{2}(y, h(y))
\end{array}
$$

We want to justify definitions of this sort in $\mathrm{QT}^{+}$.

Let $I^{0}$ be the string form that is the conjunction of the string forms used to obtain the SINGLETON LEMMA, the APPENDING LEMMA, the DOUBLETON LEMMA and the PAIRING LEMMA. The theorem below asserts that, given strings $p, q$ and operations $F_{1}, F_{2}$ given by formulae satisfying the principal hypothesis, any string form I stronger than $I^{\diamond}$ can in turn be strengthened to a string form J containing arbitrarily long length indices for computations of uniquely determined values for successive arguments from J obtained by string recursion from $\mathrm{p}, \mathrm{q}, \mathrm{F}_{1}, \mathrm{~F}_{2}$.

STRING RECURSION THEOREM. Let $\mathrm{F}_{1}(\mathrm{y}, \mathrm{z}, \mathrm{u})$ and $\mathrm{F}_{2}(\mathrm{y}, \mathrm{z}, \mathrm{u})$ be $\mathcal{L}_{\mathrm{C}}$ formulae, and let $\mathrm{I} \subseteq \mathrm{I}^{\ominus}$ closed under * and downward closed under $\subseteq_{\mathrm{p}}$. Suppose that

$$
\begin{aligned}
& \mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{p}) \& \mathrm{I}(\mathrm{q}), \\
& \mathrm{QT}^{+} \vdash \forall \mathrm{y}, \mathrm{z} \in \mathrm{I} \exists!\mathrm{u} \in \mathrm{I} \mathrm{~F}_{1}(\mathrm{y}, \mathrm{z}, \mathrm{u}), \text { and } \quad \mathrm{QT}^{+} \vdash \forall \mathrm{y}, \mathrm{z} \in \mathrm{I} \exists!\mathrm{u} \in \mathrm{I} \mathrm{~F}_{2}(\mathrm{y}, \mathrm{z}, \mathrm{u}) .
\end{aligned}
$$

Then there is an $\mathcal{L}_{\mathcal{C}}$ formula $\mathrm{H}(\mathrm{y}, \mathrm{z})$ and a string form $\mathrm{J} \subseteq$ I such that
(i) $Q^{+}+\vdash \forall y \in J \exists!z \in I H(y, z)$,
(iia) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{a}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{p})$,
(iib) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{b}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{q})$,
(iiia) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{J} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{H}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{H}\left(\mathrm{y}^{*} \mathrm{a}, \mathrm{z}\right) \leftrightarrow \mathrm{F}_{1}(\mathrm{y}, \mathrm{u}, \mathrm{z})\right)\right)$,
and (iiib) $Q^{+}+\vdash \forall y \in J \forall u, z \in I\left(H(y, u) \rightarrow\left(H(y * b, z) \leftrightarrow F_{2}(y, u, z)\right)\right)$.
(We read " $\exists!x \in J(\ldots)$ " as " $\exists x(J(x) \&(\ldots) \& \forall y(J(y) \&(\ldots) \rightarrow y=x))$ ").

Proof: Let Comp( $u, \mathrm{~m}$ ) abbreviate
$\operatorname{Set}(u) \&\left(a \leq m \rightarrow \exists v \subseteq_{p} u(\operatorname{Pair}[a, p, v] \& v \varepsilon u)\right) \&$
$\&\left(b \leq m \rightarrow \exists v \subseteq_{p} u(\operatorname{Pair}[b, q, v] \& v \varepsilon u)\right) \&$
$\& \forall \mathrm{z}<\mathrm{m} \forall \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{v}_{1}\left(\operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{~F}_{1}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right) \rightarrow\right.$

$$
\left.\rightarrow \exists \mathrm{v}_{2} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{a}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{2} \varepsilon \mathrm{u}\right)\right) \&
$$

$\& \forall \mathrm{z}<\mathrm{m} \forall \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{v}_{1}\left(\operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{~F}_{2}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right) \rightarrow\right.$

$$
\left.\rightarrow \exists \mathrm{v}_{2} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{~b}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{2} \varepsilon \mathrm{u}\right)\right) \&
$$

$\& \forall z, \mathbf{u}_{1}, \mathbf{u}_{2}, \mathrm{v}_{1}, \mathrm{v}_{2}\left(\operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{~V}_{1}\right] \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{v}_{2} \varepsilon \mathrm{u} \rightarrow\right.$

$$
\left.\rightarrow \mathrm{u}_{1}=\mathrm{u}_{2} \& \mathrm{v}_{1}=\mathrm{v}_{2}\right) .
$$

Comp( $u, m$ ) means, roughly, that $u$ is a set code for a computation determined
by $\mathrm{p}, \mathrm{q}, \mathrm{F}_{1}, \mathrm{~F}_{2}$, in at least m steps where the length indices m are strings ordered by the tree-like ordering $\leq$.

Let MinComp( $u, m$ ) abbreviate
$\operatorname{Comp}(u, m) \& \forall u^{\prime}\left(\operatorname{Comp}\left(u^{\prime}, m\right) \rightarrow \forall y\left(y \varepsilon u \rightarrow y \varepsilon u^{\prime}\right)\right) \&$
$\& \forall \mathrm{z}, \mathrm{v}, \mathrm{w}(\operatorname{Pair}[\mathrm{z}, \mathrm{v}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u} \rightarrow(\mathrm{m}=\mathrm{a} \& \mathrm{z}=\mathrm{a}) \mathrm{v}(\mathrm{m}=\mathrm{b} \& \mathrm{z}=\mathrm{b}) \mathrm{v}$ $\mathrm{v} \exists \mathrm{n}<\mathrm{m}(\mathrm{z} \leq$ na $\mathrm{v} \mathrm{z} \leq \mathrm{nb})$ ).
Let $\mathrm{J}(\mathrm{m})$ abbreviate
$I(m) \& \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, m) \& \operatorname{Pair}[m, y, w] \& w \varepsilon u)$.

Finally, let $\mathrm{H}(\mathrm{m}, \mathrm{y})$ abbreviate

$$
\exists \mathrm{u}, \mathrm{w}(\operatorname{MinComp}(\mathrm{u}, \mathrm{~m}) \& \operatorname{Pair}[\mathrm{~m}, \mathrm{y}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u}) .
$$

For detailed verification that J and $H$ have the desired properties see Appendix.

We are now in the position to define the counting functions $\alpha$ and $\beta$.
Let $p=b b, q=b, \quad F_{1}(y, z, u) \equiv y=y \& z=S u$ and $F_{2}(y, u, z) \equiv y=y \& z=u$.
Then the principal hypothesis of the String Recursion Theorem holds trivially. Applying the Theorem we obtain a formula $\mathrm{A}^{\#}(\mathrm{y}, \mathrm{z})$ and a string form $\mathrm{I}_{\alpha} \subseteq \mathrm{I}$ such that
(ia) $\quad \mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}_{\alpha} \exists!\mathrm{z} \in \mathrm{I} \mathrm{A}^{\#}(\mathrm{y}, \mathrm{z})$,
(iia $\left.{ }^{\alpha}\right) \quad \mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{I}\left(\mathrm{A}^{\#}(\mathrm{a}, \mathrm{z}) \leftrightarrow \mathrm{z}=\mathrm{bb}\right)$,
(iib $\left.{ }^{\alpha}\right) \quad \mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{I}\left(\mathrm{A}^{\#}(\mathrm{~b}, \mathrm{z}) \leftrightarrow \mathrm{z}=\mathrm{b}\right)$,
(iiia $\left.{ }^{\alpha}\right) \quad \mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}_{\alpha} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{A}^{\#}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{A}^{\#}\left(\mathrm{y}^{*} \mathrm{a}, \mathrm{z}\right) \rightarrow \mathrm{z}=\mathrm{u}^{*} \mathrm{~b}\right)\right)$,
(iiib ${ }^{\alpha}$ )
$\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}_{\alpha} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{A}^{\#}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{A}^{\#}\left(\mathrm{y}^{*} \mathrm{~b}, \mathrm{z}\right) \rightarrow \mathrm{z}=\mathrm{u}\right)\right)$.

Informally speaking, $A^{\#}(y, z)$ defines the graph of the function $\alpha$.
Exactly analogously, by letting p and q , and $\mathrm{F}_{1}, \mathrm{~F}_{2}$, respectively, exchange places, we apply the Theorem to obtain a formula $B^{\#}(y, z)$ defining the graph of the function $\beta$ and a string form $\mathrm{I}_{\beta} \subseteq \mathrm{I}$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}_{\beta} \exists!\mathrm{z} \in \mathrm{I} \mathrm{~B}^{\#}(\mathrm{y}, \mathrm{z}),
$$

(iia ${ }^{\beta}$ )
$\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{I}\left(\mathrm{B}^{\#}(\mathrm{a}, \mathrm{z}) \leftrightarrow \mathrm{z}=\mathrm{b}\right)$,
$\mathrm{QT}^{+} \vdash \forall \mathrm{z} \in \mathrm{I}\left(\mathrm{B}^{\#}(\mathrm{~b}, \mathrm{z}) \leftrightarrow \mathrm{z}=\mathrm{bb}\right)$,
$\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}_{\beta} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{B}^{\#}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{B}^{\#}\left(\mathrm{y}^{*} \mathrm{a}, \mathrm{z}\right) \rightarrow \mathrm{z}=\mathrm{u}\right)\right)$,
(iiib ${ }^{\beta}$ )
$\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}_{\beta} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{B}^{\#}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{B}^{\#}\left(\mathrm{y}^{*} \mathrm{~b}, \mathrm{z}\right) \rightarrow \mathrm{z}=\mathrm{u}^{*} \mathrm{~b}\right)\right)$.

We can then prove that $\alpha$ and $\beta$ correctly count b's in b-tallies:
5.1(a) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\begin{aligned}
& \mathrm{QT}^{+} \vdash \mathrm{A}^{\#}(\mathrm{a}, \mathrm{Sb}) \& \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{y} \in \mathrm{I}\left(\mathrm{Tally}_{\mathrm{b}}(\mathrm{x}) \& \mathrm{~A}^{\#}(\mathrm{x}, \mathrm{y}) \rightarrow \mathrm{y}=\mathrm{b}\right) \text { and } \\
& \mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{y} \in \mathrm{I}\left(\mathrm{Tally}_{\mathrm{a}}(\mathrm{x}) \& \mathrm{~B}^{\#}(\mathrm{x}, \mathrm{y}) \rightarrow \mathrm{y}=\mathrm{b}\right) .
\end{aligned}
$$

(I.e., ' $\alpha(\mathrm{a})=1$ ' and ' $\mathrm{Tallyb}_{\mathrm{b}}(\mathrm{x}) \rightarrow \alpha(\mathrm{x})=0$ ', and 'Tally $(\mathrm{x}) \rightarrow \beta(\mathrm{x})=0$ '.)
(b) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{y} \in \mathrm{I}\left(\operatorname{Tally\mathrm {b}}(\mathrm{x}) \& \mathrm{~B}^{\#}(\mathrm{x}, \mathrm{y}) \rightarrow \mathrm{y}=\mathrm{x} * \mathrm{~b}\right) .
$$

Informally, $\operatorname{Tallyb}_{\mathrm{b}}(\mathrm{x}) \rightarrow \beta(\mathrm{x})=$ length $(\mathrm{x})$.

We now verify that the functions $\alpha$ and $\beta$ are indeed additive. Let $\mathrm{I}_{\text {Add }}$ be as in 3.5(a).
5.2(a) For any string form $I \subseteq I_{\alpha}$ and $I \subseteq I_{\text {Add }}$ there is a string form $\mathrm{J} \equiv \mathrm{I}_{\text {Add } \alpha} \subseteq \mathrm{I}$ such that
$\mathrm{QT}^{+} \vdash \forall \mathrm{x}, \mathrm{y} \in \mathrm{J} \forall \mathrm{u}, \mathrm{v}, \mathrm{w}\left(\mathrm{A}^{\sharp}(\mathrm{x}, \mathrm{u}) \& \mathrm{~A}^{\sharp}(\mathrm{y}, \mathrm{v}) \& \operatorname{AddTally}(\mathrm{u}, \mathrm{v}, \mathrm{w}) \rightarrow \mathrm{A}^{\#}\left(\mathrm{x}^{*} \mathrm{y}, \mathrm{w}\right)\right)$.

$$
\left(" \alpha\left(x^{*} y\right)=\alpha(x)+\alpha(y) "\right)
$$

(b) For any string form $I \subseteq I_{\beta}$ and $I \subseteq I_{\text {Add }}$ there is a string form $J \equiv I_{\text {Add } \beta} \subseteq I$ such that

$$
\begin{aligned}
Q T^{+} \vdash \forall x, y \in J \forall u, v, w\left(B^{\#}(x, u) \& B^{\#}(y, v) \& \operatorname{AddTally}(u, v, w)\right. & \left.\rightarrow B^{\#}\left(x^{*} y, w\right)\right) . \\
\left(" \beta\left(x^{*} y\right)\right. & =\beta(x)+\beta(y) ")
\end{aligned}
$$

Proof: See Appendix.

## 6. Formal Construction of the Interpretation

Let $\mathrm{I}_{\text {Add } \alpha}$ be the string form obtained from $\mathrm{I}_{0}$ by the series of modifications described in §§3-5 up to and including 5.2(a). Analogously for and $\mathrm{I}_{\text {Add }}$ and 5.2(b).

Let $\quad J^{*} \equiv \mathrm{I}_{\text {Add } \alpha} \& \mathrm{I}_{\text {Add } \beta}$.
Then $\mathrm{J}^{*} \subseteq \mathrm{I}_{\text {Add } \alpha}$ and $\mathrm{J}^{*} \subseteq \mathrm{I}_{\text {Add }}$ and $\mathrm{J}^{*} \subseteq \mathrm{I}_{\text {Add }}$ as well as $\mathrm{J}^{*} \subseteq \mathrm{I}^{\diamond}$. We may also assume that J* is closed under *, and downward closed under $\leq$ and $\subseteq_{\mathrm{p}}$. Hence it may be assumed that the string form J* is also closed under Addtally and the functions $\alpha$ and $\beta$.

We then formally define $Æ(x)$ as
$\exists y, z\left(A^{\#}(x, y) \& B^{\#}(x, z) \& y=S z\right) \&$

$$
\& \forall u, v, w\left(u B x \& A^{\#}(u, v) \& B^{\#}(u, w) \rightarrow v \leq w\right) .
$$

(These are conditions (c1)-(c2) from §1.)
We set

$$
I^{*}(\mathrm{x}) \equiv Æ(\mathrm{x}) \& \mathrm{~J}^{*}(\mathrm{x})
$$

The formula $\mathrm{I}^{*}(\mathrm{x})$ will formally define in $\mathrm{QT}^{+}$the domain of interpretation of theory T. We now proceed to formally verify the translations of the axioms of T by derivations in $\mathrm{QT}^{+}$.
6.1(a) $\mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{x}_{2} \mathrm{Ex} \rightarrow \forall \mathrm{u}, \mathrm{v}\left(\mathrm{A}^{\#}(\mathrm{x} 2, \mathrm{u}) \& \mathrm{~B}^{\#}(\mathrm{x} 2, \mathrm{v}) \rightarrow \mathrm{Sv} \leq \mathrm{u}\right)$.
(b) $\quad \mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{y}) \& \mathrm{z}=\mathrm{bxy} \rightarrow \mathrm{I}^{*}(\mathrm{z})$.
(c) $\quad \mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{u}) \&$ bxy=buv $\rightarrow \mathrm{x}=\mathrm{u} \& \mathrm{y}=\mathrm{v}$.
(d) $\mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \rightarrow\left(\mathrm{x} \subseteq_{\mathrm{p}} \mathrm{a} \leftrightarrow \mathrm{x}=\mathrm{a}\right)$.
(e) $\quad \mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{y}) \& \mathrm{I}^{*}(\mathrm{z}) \rightarrow\left(\mathrm{x} \complement_{\mathrm{p}} \mathrm{byz} \leftrightarrow \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \mathrm{vx} \subseteq_{\mathrm{p}} \mathrm{z}\right)$.

Proof: See Appendix. We give the details of the proof of (e) to illustrate the flavor of the type of formal argument used.

Assume $M \vDash x \subseteq_{p}$ byz where $M \vDash I^{*}(x) \& I^{*}(y) \& I^{*}(z)$.
Then $M \vDash J^{*}(x) \& J^{*}(y) \& J^{*}(z)$ and also $M \vDash Æ(x) \& Æ(y) \& Æ(z)$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), M \vDash \exists!x_{1} \in J^{*} A^{\#}\left(x, x_{1}\right) \& \exists!x_{2} \in J^{*} B^{\#}\left(x, x_{2}\right)$.
From $\mathrm{M} \vDash \mathrm{x} \subseteq_{\mathrm{p}}$ byz by $3.7(\mathrm{~h})$ we have that
$\mathrm{M} \vDash \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x}=\mathrm{b} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{yz} \mathrm{v} \exists \mathrm{J}(\mathrm{uByz} \& \mathrm{x}=\mathrm{bu})$.
We distinguish the cases:
(1) $M \vDash \exists u(u B y z \& x=b u)$.

Then by (QT2), $M \neq x \neq a$. From $M \vDash I^{*}(y) \& I^{*}(z)$, by 6.1(b), $M \neq I^{*}(b y z)$.
From $M \vDash u B y z, ~ M \vDash \exists v u v=y z$, hence $M \vDash b(u v)=b(y z)$, also
$M \vDash(b u) v=b(y z)$. Thus $M \neq b u B b(y z)$, hence $M \vDash x B b(y z)$.
From $M \neq I^{*}(b y z), M \neq \mathbb{E}(b y z)$, whence $M \vDash x_{1} \leq x_{2}$. But from $M \vDash \notin(x)$,
$M \vDash x_{1}=S x_{2}$, and we have $M \vDash x_{1} \leq x_{2}<S x_{2}=x_{1}$, contradicting $M \vDash I_{0}\left(x_{1}\right)$.
Hence (1) is ruled out.
(2) $M \vDash x=b$.

Then by (QT2), $M \neq x \neq a$, and from $M \vDash \notin(x)$, we have $M \vDash b B x$. But then $\mathrm{M} \vDash \mathrm{bBb}$, contradicting (QT2). Hence (2) is also ruled out.
(3) $M \vDash x \subseteq_{p y z}$.

By 3.7 (g), $\quad \mathrm{M} \vDash \mathrm{x}=\mathrm{yz} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y}$ v $\mathrm{x} \subseteq_{\mathrm{pz}} \mathrm{v} \quad \exists \mathrm{y}_{1}\left(\mathrm{y}_{1} \mathrm{Ey} \& \mathrm{x}=\mathrm{y}_{1} \mathrm{z}\right) \mathrm{v}$

$$
\mathrm{v} \exists \mathrm{z}_{1}\left(\mathrm{z}_{1} \mathrm{Bz} \& \mathrm{x}=\mathrm{yz}_{1}\right) \text { v } \exists \mathrm{y}_{1}, \mathrm{z}_{1}\left(\mathrm{y}_{1} \mathrm{Ey} \& \mathrm{z}_{1} \mathrm{Bz} \& \mathrm{x}=\mathrm{y}_{1} \mathrm{z}_{1}\right) .
$$

(3i) $M \vDash x=y z$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), \quad M \vDash \exists!y_{1} \in J^{*} A^{\#}\left(y, y_{1}\right) \& \exists!y_{2} \in J^{*} B^{\#}\left(y, y_{2}\right)$,
and further $M \vDash \exists!z_{1} \in J^{*} A^{\#}\left(z, z_{1}\right) \& \exists!z_{2} \in J^{*} B^{\#}\left(z, z_{2}\right)$.
From $M \neq \mathbb{E}(y), \quad M \neq y_{1}=S_{2}$, and from $M \vDash \notin(z), \quad M \vDash z_{1}=S z_{2}$.
By 3.5(a), $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{p}_{1}\right)\right.$ \& Addtally $\left.\left(\mathrm{y}_{1}, \mathrm{z}_{1}, \mathrm{p}_{1}\right)\right)$
and $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{2} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}\left(\mathrm{p}_{2}\right)\right.$ \& Addtally $\left.\left(\mathrm{y}_{2}, \mathrm{z}_{2}, \mathrm{p}_{2}\right)\right)$.
Then from $M \neq A^{\#}\left(y, y_{1}\right) \& A^{\#}\left(z, z_{1}\right)$, by $5.2(a), M \neq A^{\#}\left(y^{*} z, p_{1}\right)$, and
from $M \neq B^{\#}\left(y_{1}, y_{2}\right) \& B^{\#}\left(z, z_{2}\right)$, by $5.2(b), M \neq B^{\#}\left(y^{*} z_{2}, p_{2}\right)$,
$\Rightarrow$ from $\mathrm{M} \vDash \mathrm{y}_{1}=\mathrm{Sy}_{2} \& \mathrm{z}_{1}=\mathrm{Sz}_{2}, \mathrm{M} \vDash \operatorname{Addtally}\left(\mathrm{Sy}_{2}, \mathrm{Sz}_{2}, \mathrm{p}_{1}\right)$.
On the other hand, from $M=\operatorname{Addtally}\left(\mathrm{y}_{2}, \mathrm{z}_{2}, \mathrm{p}_{2}\right)$, by $3.4(\mathrm{e})$, $\mathrm{M} \vDash$ Addtally $\left(\mathrm{y}_{2}, \mathrm{Sz}_{2}, \mathrm{Sp}_{2}\right)$, whence by $3.5(\mathrm{~d}), \mathrm{M} \neq \operatorname{Addtally}\left(\mathrm{Sy}_{2}, \mathrm{Sz}_{2}, \mathrm{SSp}_{2}\right)$.

By single-valuedness of Addtally, we then have $\quad \mathrm{M} \vDash \mathrm{p}_{1}=\mathrm{SSp}_{2}$.
From hypothesis $M \neq x=y z \& A^{\#}\left(y^{*} z, p_{1}\right) \& B^{\#}\left(y^{*} z, p_{2}\right)$,

$$
\mathrm{M} \vDash \mathrm{~A}^{\#}\left(\mathrm{x}, \mathrm{p}_{1}\right) \& \mathrm{~B}^{\#}\left(\mathrm{x}, \mathrm{p}_{2}\right)
$$

Hence from $M \neq A^{\#}\left(x, x_{1}\right) \& B^{\#}\left(x, x_{2}\right)$, by single-valuedness of $A^{\#}$ and $B^{\#}$,

$$
\mathrm{M} \vDash \mathrm{p}_{1}=\mathrm{x}_{1} \& \mathrm{p}_{2}=\mathrm{x}_{2}
$$

Thus from $M \neq p_{1}=S S p_{2}$, we have $M \vDash x_{1}=S S x_{2}$. But from $M \vDash \notin(x)$ we have
$M \vDash x_{1}=S x_{2}$, whence $M \vDash x_{1}=S x_{1}$. But then from $M \vDash x_{1}<S x_{1}$, we obtain
$\mathrm{M} \vDash \mathrm{x}_{1}<\mathrm{x}_{1}$, contradicting $\mathrm{M} \vDash \mathrm{I}_{0}\left(\mathrm{x}_{1}\right)$. Hence (3i) is ruled out.
(3ii) $M \neq \exists y_{1}\left(y_{1} E y \& x=y_{1} z\right)$.

Then $\mathrm{M}=\mathrm{y}_{1} \mathrm{Bx}$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), M \vDash \exists!u_{1} \in J^{*} A^{\#}\left(y_{1}, u_{1}\right) \& \exists!u_{2} \in J^{*} B^{\#}\left(y_{2}, u_{2}\right)$.
From $M \neq \mathbb{E}(x) \& y_{1} B x, \quad M \vDash u_{1} \leq u_{2}$, whereas from $M \vDash I^{*}(y) \& y_{1} E y$, by 6.1(a), $M \vDash S u_{2} \leq u_{1}$. But then $M \vDash u_{2}<S u_{2} \leq u_{2}$, contradicting $M \vDash I_{0}\left(u_{2}\right)$. This rules out (3ii).
(3iii) $\mathrm{M} \vDash \exists \mathrm{z}_{1}\left(\mathrm{z}_{1} \mathrm{Bz} \& \mathrm{x}=\mathrm{yz} \mathrm{z}_{1}\right)$.
Then $M \neq y B x$. By $\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), \quad M \neq \exists!y_{1} \in J^{*} A^{\#}\left(y_{,} y_{1}\right) \& \exists!y_{2} \in J^{*} B^{\#}\left(y, y_{2}\right)$.
From $M \neq \mathbb{E}(x) \& y B x, M \neq y_{1} \leq y_{2}$. But from $M \neq \mathbb{E}(y), M \neq y_{1}=S y_{2}$, and we obtain $\mathrm{M} \vDash \mathrm{y}_{1} \leq \mathrm{y}_{2}<\mathrm{Sy}_{2}=\mathrm{y}_{1}$, contradicting $\mathrm{M} \vDash \mathrm{I}_{0}\left(\mathrm{y}_{1}\right)$. Hence (3iii) is ruled out.
(3iv) $M \vDash \exists y_{1}, z_{1}\left(y_{1}\right.$ Ey \& $\left.z_{1} B z \& x=y_{1} z_{1}\right)$.
This is ruled out by reasoning as in either (3ii) or (3iii).
We then conclude under the principal hypothesis that

$$
\mathrm{M} \vDash \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{yz} \rightarrow \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \text { v } \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{z}
$$

and further that $\quad \mathrm{M} \vDash \mathrm{x} \subseteq_{\mathrm{p}}$ byz $\rightarrow \mathrm{x}=\mathrm{byz} \mathrm{vx} \subseteq_{\mathrm{p}} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{z}$.
The converse is immediate from the definition of $\subseteq_{\mathrm{p} z}$.

Taking the formula $\mathbb{E}(\mathrm{x})$ from $\S 6$ to define the domain, and interpreting the non-logical vocabulary $\mathcal{L}_{\mathrm{T}}=\{0,(), \sqsubseteq\}$ of T by $a, b x y$ and $\subseteq_{\mathrm{p}}$, resp., as explained in §2, we have that 6.1(b)-(e), along with the fact that $\mathrm{QT}^{+} \vdash \mathrm{bxy} \neq \mathrm{a}$, suffice to establish formal interpretability of T in $\mathrm{QT}^{+}$. On the
other hand, from [1], building on previous work of Halpern and Collins, Wilkie, Visser, Grzegorczyk and Ganea, we have that

$$
\mathrm{TC} \equiv_{1} \mathrm{QT}+\equiv_{1} \mathrm{AST} \equiv_{1} \mathrm{AST}+\mathrm{EXT} \equiv_{1} \mathrm{Q} \equiv_{1}
$$

Since, by [6], $\mathrm{Q} \leq_{\mathrm{I}} \mathrm{T}$, this suffices to establish

WEAK ESSENTIALLY UNDECIDABLE THEORIES: FIRST MUTUAL
INTERPRETABILITY THEOREM.

$$
\mathrm{T} \equiv_{\mathrm{I}} \mathrm{QT}^{+} \equiv_{\mathrm{I}} \mathrm{QT} \mathrm{~T}_{0} \equiv_{\mathrm{I}} \mathrm{TC} \equiv_{\mathrm{I}} \mathrm{Q} \equiv_{\mathrm{I}} \mathrm{AST} .
$$

In addition, each of the theories above is mutually interpretable with
AST+EXT, and Buss's theory $\mathrm{S}_{2}^{1} \quad$ (see Ferreira and Ferreira [4]).
§7. $R$ and its variants

We now consider the expanded vocabulary $\mathcal{L}_{\mathrm{C}, \varsigma^{*}}=\left\{\mathrm{a}, \mathrm{b},{ }^{*}, \complement^{*}\right\}$ with two individual constants - the digits $a, b$ - a single binary operation symbol * and a 2-place relational symbol $\sqsubseteq^{*}$. Each variable-free term of $\mathcal{L} \mathrm{C}, \varsigma^{*}$ represents a finite string of $a$ 's and/or $b$ 's, and each such string may have multiple variable-free terms as its representations, differing in the arrangement of
parentheses indicating the order of applications of the term operation *.
Recalling the theory WT described in the introduction formulated in $\mathcal{L}_{\mathrm{T}}=\{0,(), \sqsubseteq\}$, we are going to single out $\mathcal{L}_{\mathrm{C}, \complement^{*}}$ terms that represent tree-like strings obtained from variable-free terms of $\mathcal{L}_{\mathrm{T}}$ as described in $\S 1$. With each variable-free term v of $\mathcal{L}_{\mathrm{T}}$ we associate a unique $\mathcal{L}_{\mathrm{C}, \Sigma^{*}}$ term $\mathrm{V}^{\tau}$ as follows:

$$
0^{\tau} \equiv \mathrm{a} \quad(u, v)^{\tau} \equiv b^{*}\left(u^{\tau *} v^{\tau}\right)
$$

The $\mathcal{L}_{\mathrm{QT}, \varsigma^{*}}$ term $\mathrm{V}^{\tau}$ is an $\nVdash$ string that codes v .

If $S(\mathrm{v})$ is the set of all (variable-free) subterms of v , let

$$
\Sigma(\mathrm{t})=\left\{\mathrm{u}^{\mathrm{\tau}} \mid \text { for some } \mathcal{L}_{\mathrm{T}} \text {-term } \mathrm{v}, \mathrm{u} \in S(\mathrm{v}) \text { and } \mathrm{t}=\mathrm{v}^{\tau}\right\} .
$$

We then let $\Sigma^{\tau}=\mathrm{U}_{\mathrm{v} \in S} \Sigma\left(\mathrm{v}^{\tau}\right)$, where $S$ is the set all variable-free terms of $\mathcal{L}_{\mathrm{T}}$. A straightforward induction on the complexity of $\mathcal{L}_{\mathrm{T}}$ terms establishes that the mapping $\tau$ is $1-1$.

Let WQT be the first-order theory formulated in $\mathcal{L}_{\mathrm{C}, \mathrm{E}^{*}}$ with the following axioms:

$$
\begin{aligned}
& \text { (WQT1) } \neg(\mathrm{s}=\mathrm{t}) \quad \text { for any distinct terms } \mathrm{s}, \mathrm{t} \in \Sigma^{\mathrm{\tau}} \text {, } \\
& \text { (WQT2) } \forall \mathrm{z}\left(\mathrm{z} \sqsubseteq^{*} \mathrm{~b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \leftrightarrow \mathrm{z}=\mathrm{b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \mathrm{v} \mathrm{z} \sqsubseteq^{*} \mathrm{~s} \vee \mathrm{z} \sqsubseteq^{*} \mathrm{t}\right)
\end{aligned}
$$

$$
\text { (WQT3) } \quad \forall \mathrm{z}\left(\mathrm{z} \underline{C}^{*} \mathrm{a} \leftrightarrow \mathrm{z}=\mathrm{a}\right) .
$$

Here, (WQT1) and (WQT2) are axiom schemas with infinitely many instances.

We now define a formal interpretation $(\tau)$ of WT in WQT. Let the formula

$$
\mathrm{T}^{*}(\mathrm{x}) \equiv \mathrm{x}=\mathrm{a} v \exists \mathrm{y}, \mathrm{zx}=\mathrm{b}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right)
$$

define the domain. Interpret 0 by a, the binary term building operation (,) of $\mathcal{L}_{\mathrm{T}}$ by $b^{*}\left(\mathrm{x}^{*} \mathrm{y}\right)$, and $\subseteq$ by $\complement^{*}$. We then have immediately:

$$
\begin{aligned}
& \mathrm{WQT} \vdash \mathrm{~T}^{*}\left(0^{\tau}\right), \\
& \mathrm{WQT} \vdash \mathrm{~T}^{*}(\mathrm{y}) \& \mathrm{~T}^{*}(\mathrm{z}) \rightarrow \mathrm{T}^{*}\left(\mathrm{~b}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right)\right) .
\end{aligned}
$$

A trivial induction on the complexity of $\mathcal{L}_{\mathrm{T}}$ terms verifies that each $\mathrm{v} \in S$ is interpreted by $V^{\tau} \in \Sigma^{\tau}$ in WQT. Since the map $\tau$ is $1-1$, we have that

$$
\mathrm{WQT} \vdash(\neg(\mathrm{u}=\mathrm{v}))^{(\tau)},
$$

$\neg\left(u^{\tau}=v^{\tau}\right)$ being the translations $(\neg(u=v))^{(\tau)}$ of the instances of axiom schema (WT1) of WT, for distinct $\mathrm{u}, \mathrm{v} \in S$.

Consider now an instance of the schema (WT2), for some $\mathrm{v} \in S$ :

$$
\forall \mathrm{x}\left(\mathrm{x} \sqsubseteq \mathrm{v} \leftrightarrow \mathrm{~V}_{\mathrm{u} \in S}(\mathrm{v}) \mathrm{x}=\mathrm{u}\right) .
$$

If v is the atomic term 0 , we have that $S(\mathrm{v})=\{0\}$. Hence the formula in question is $\quad \forall x(x \sqsubseteq 0 \leftrightarrow x=0)$.

But, by (WQT3), $\quad W Q T \vdash \forall x\left(x{ }^{*} a \leftrightarrow x=a\right)$.

Hence, a fortiori, $\mathrm{WQT} \vdash \forall \mathrm{x}\left(\mathrm{T}^{*}(\mathrm{x}) \rightarrow\left(\mathrm{x} \varrho^{*} \mathrm{a} \leftrightarrow \mathrm{x}=\mathrm{a}\right)\right)$, which is the
( $\tau$ )-translation of the above instance of (WT2).

Consider now $\mathrm{t} \in S$ of the form ( $\mathrm{u}, \mathrm{v}$ ). Note that

$$
S(\mathrm{t})=S(\mathrm{u}) \cup S(\mathrm{v}) \cup\{\mathrm{t}\}
$$

Hence $\quad \Sigma\left(\mathrm{t}^{\tau}\right)=\Sigma\left(\mathrm{u}^{\tau}\right) \cup \Sigma\left(\mathrm{v}^{\tau}\right) \cup\left\{\mathrm{t}^{\tau}\right\}$.

Assume now that
$\mathrm{WQT} \vdash\left[\forall \mathrm{z}\left(\mathrm{z} \subseteq \mathrm{u} \leftrightarrow \mathrm{V}_{\mathrm{s} \in S(\mathrm{u})} \mathrm{z}=\mathrm{s}\right)\right]^{(\mathrm{T})}$ and $\left.\mathrm{WQT} \vdash\left[\forall \mathrm{z}\left(\mathrm{z} \subseteq \mathrm{v} \leftrightarrow \mathrm{V}_{\mathrm{s} \in S(\mathrm{v})} \mathrm{z}=\mathrm{s}\right)\right]\right]^{(\mathrm{\tau})}$.

Then

$$
\mathrm{WQT} \vdash \forall \mathrm{z}\left(\mathrm{~T}^{*}(\mathrm{z}) \rightarrow\left(\mathrm{z} \sqsubseteq^{*} \mathrm{u}^{\tau} \leftrightarrow \mathrm{V}_{s^{\tau} \in \Sigma\left(\mathrm{u}^{\tau}\right)} \mathrm{z}=\mathrm{s}^{\tau}\right)\right)
$$

$$
\mathrm{WQT} \vdash \forall \mathrm{z}\left(\mathrm{~T}^{*}(\mathrm{z}) \rightarrow\left(\mathrm{z} \sqsubseteq^{*} \mathrm{~V}^{\tau} \leftrightarrow \mathrm{V}_{s^{\tau} \in \Sigma\left(\mathrm{v}^{\tau}\right)} \mathrm{z}=\mathrm{s}^{\tau}\right)\right) .
$$

Let $M$ be a model of WQT. Assume $M \vDash T^{*}(x)$ and consider $M \vDash x \sqsubseteq^{*} t$.

We have that $t^{\tau}$ is in fact $b^{*}\left(u^{\tau} v^{\tau}\right)$. Hence
$\Leftrightarrow M \vDash x \sqsubseteq^{*} b^{*}\left(u^{\tau} * v^{\tau}\right) \Leftrightarrow$ by (WQT2), MF $x=b^{*}\left(u^{\tau *} v^{\tau}\right) v x \sqsubseteq^{*} u^{\tau} v x \sqsubseteq^{*} v^{\tau} \Leftrightarrow$
$\Leftrightarrow M \vDash x=b^{*}\left(u^{\tau} V^{\tau}\right) v V_{s^{\tau} \in \Sigma\left(u^{\tau}\right)} x=s^{\tau} v V_{s}{ }^{\tau} \in \Sigma\left(v^{\tau}\right) X=s^{\tau} \Leftrightarrow$
$\Leftrightarrow M \vDash V_{s}{ }^{\tau} \in \Sigma\left(t^{\tau}\right) x=s^{\tau}$
using ( $\dagger$ ). Therefore,
that is, $\quad W Q T \vdash\left[\forall \mathrm{x}\left(\mathrm{x} \subseteq \mathrm{t} \leftrightarrow \mathrm{V}_{\mathrm{s} \in S(\mathrm{t})} \mathrm{x}=\mathrm{s}\right)\right]^{(\tau)}$.

Hence the ( $\tau$ )-translation of each instance of (WT2) is also provable in WQT.

We conclude that

## 7.1. $\quad W T \leq{ }_{I} W Q T$.

The theory WQT is not recognizably a concatenation theory: the axioms make no substantive assumptions about the binary operation *, not even associativity. On that account, it might be considered at best as a "pseudoconcatenation" notational variant of WT. We now consider another firstorder theory, $\mathrm{WQT}^{*}$, formulated in the same vocabulary $\mathcal{L}_{\mathrm{C}, \complement^{*}}=\left\{\mathrm{a}, \mathrm{b}\right.$, , $\left.^{*}, \sqsubseteq^{*}\right\}$ as WQT, with the following axioms: for each variable-free term $t$ of $\mathcal{L}_{\mathrm{C}, \Sigma^{*},}$
(WQT* $\left.{ }^{*}\right) \quad \forall \mathrm{x}, \mathrm{y}, \mathrm{z}\left(\mathrm{x}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right) \sqsubseteq_{\mathrm{p}} \mathrm{t} v\left(\mathrm{x}^{*} \mathrm{y}\right)^{*} \mathrm{z} \coprod_{\mathrm{p}} \mathrm{t} \rightarrow \mathrm{x}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right)=\left(\mathrm{x}^{*} \mathrm{y}\right)^{*} \mathrm{z}\right)$

$$
\begin{aligned}
\left(\text { WQT }^{*}\right) \quad & \forall x, y\left(x^{*} y \sqsubseteq_{p} t \rightarrow \neg\left(x^{*} y=a\right) \& \neg\left(x^{*} y=b\right)\right) \\
\left(W Q T^{*} 3\right) \quad & \forall x, y\left(\left(a^{*} x \sqsubseteq_{p t} \& a^{*} y \sqsubseteq_{p} t \rightarrow\left(a^{*} x=a^{*} y \rightarrow x=y\right)\right) \&\right. \\
& \&\left(b^{*} x \sqsubseteq_{p} t \& b^{*} y \sqsubseteq_{p} t \rightarrow\left(b^{*} x=b^{*} y \rightarrow x=y\right)\right) \& \\
& \&\left(x^{*} a \sqsubseteq_{p t} \& y^{*} a \sqsubseteq_{p t} \rightarrow\left(x^{*} a=y^{*} a \rightarrow x=y\right)\right) \\
& \left.\&\left(x^{*} b \sqsubseteq_{p} t \& x^{*} y \sqsubseteq_{p t} t\left(x^{*} b=y^{*} b \rightarrow x=y\right)\right)\right)
\end{aligned}
$$

(WQT*4) $\quad \forall \mathrm{x}, \mathrm{y}\left(\left(\mathrm{a}^{*} \mathrm{x} \sqsubseteq_{\mathrm{p}} \mathrm{t} \& \mathrm{~b}^{*} \mathrm{y} \sqsubseteq_{\mathrm{p}} \mathrm{t} \rightarrow \neg\left(\mathrm{a}^{*} \mathrm{x}=\mathrm{b}^{*} \mathrm{y}\right)\right) \&\right.$

$$
\left.\&\left(\mathrm{x}^{*} \mathrm{a} \sqsubseteq_{\mathrm{p}} \mathrm{t} \& \mathrm{y}^{*} \mathrm{~b} \sqsubseteq_{\mathrm{p}} \mathrm{t} \rightarrow \neg\left(\mathrm{x}^{*} \mathrm{a}=\mathrm{y}^{*} \mathrm{~b}\right)\right)\right)
$$

$\left(W_{Q T}{ }^{*} 5\right) \quad \forall x \sqsubseteq_{\mathrm{p}} \mathrm{t}(\mathrm{x}=\mathrm{a} v \mathrm{x}=\mathrm{b} v((\mathrm{aBx} v \mathrm{bBx}) \&(\mathrm{aEx} v \mathrm{bEx})))$
$\left(\right.$ WQT $\left.^{*} 6\right) \quad \forall y, z\left(\mathrm{~b}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right) \sqsubseteq^{*} \mathrm{t} \rightarrow\right.$

$$
\left.\rightarrow \forall \mathrm{x}\left(\mathrm{x} \sqsubseteq^{*} \mathrm{~b}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right) \leftrightarrow \mathrm{x}=\mathrm{b}^{*}\left(\mathrm{y}^{*} \mathrm{z}\right) \mathrm{vx} \sqsubseteq^{*} \mathrm{y} v \mathrm{x} \sqsubseteq^{*} \mathrm{z}\right)\right)
$$

(WQT*7) $\quad \forall \mathrm{z}\left(\mathrm{z}\right.$ $\left.{ }^{*} \mathrm{a} \leftrightarrow \mathrm{z}=\mathrm{a}\right)$
(WQT*8) $\quad \forall x, y\left(x \sqsubseteq^{*} y \& y \sqsubseteq^{*} x \rightarrow x=y\right)$
$\left(W_{Q T}{ }^{*} 9\right) \quad \forall \mathrm{x}, \mathrm{y}\left(\mathrm{x} \sqsubseteq^{*} \mathrm{y} \& \mathrm{y} \coprod^{*} \mathrm{z} \rightarrow \mathrm{y} \subseteq^{*} \mathrm{z}\right)$

Here we use the following abbreviations:

$$
x B y \equiv \exists z y=x^{*} z \quad x E y \equiv \exists z y=z^{*} x,
$$

and $\mathrm{x} \sqsubseteq_{\mathrm{p}} \mathrm{y} \equiv \mathrm{x}=\mathrm{y} v \mathrm{xBy} v \mathrm{xEy} \vee \exists \mathrm{z}_{1}, \mathrm{z}_{2} \mathrm{y}=\mathrm{z}_{1}{ }^{*}\left(\mathrm{x}^{*} \mathrm{z}_{2}\right)$ $) \exists \mathrm{z}_{1}, \mathrm{z}_{2} \mathrm{y}=\left(\mathrm{z}_{1}{ }^{*} \mathrm{x}\right) \mathrm{z}_{2}$.

Then $\forall x \sqsubseteq_{\mathrm{p}} u \varphi \equiv \forall \mathrm{x}\left(\mathrm{x} \sqsubseteq_{\mathrm{p}} u \rightarrow \varphi\right)$, where x does not occur in the term u . Also, $\forall \mathrm{x} \sqsubseteq^{*} \mathrm{u} \varphi \equiv \forall \mathrm{x}\left(\mathrm{x} \unrhd^{*} \mathrm{u} \rightarrow \varphi\right)$.
(WQT*1)-(WQT**) are axiom schemas with infinitely many instances, one for each variable-free term $t$. The schemas (WQT*1)-(WQT*5) are "bounded" versions of the axioms (QT1)-(QT5) of QT+. Schema (WQT*6) is a "bounded" generalization of schema (WQT2) of WQT. In light of that, WQT * may be naturally interpreted as a hybrid basic theory of finite strings and trees: the intended domain are the finite strings of $a$ 's and/or $b$ 's, * is interpreted as the concatenation operation, and $\sqsubseteq^{*}$ as the substring relation between $Æ$ strings.
$\mathrm{WQT}^{*}$ is an extension of WQT. First, we note the following:
7.2. For any distinct terms $s, t \in \Sigma^{\tau}, \quad W Q T^{*} \vdash \neg(s=t)$.

Proof: We argue by (meta-theoretic) induction on the number of digits in $\mathrm{s}, \mathrm{t}$. If either one of $s$ or $t$ is the single digit $a$, this is immediate by ( $\mathrm{WQT} \mathrm{T}^{*} 2$ ). If neither s nor t are single digits, let $\mathrm{s}_{1} \ldots \mathrm{~s}_{\mathrm{m}}$ and $\mathrm{t}_{1} \ldots \mathrm{t}_{\mathrm{n}}$ be their successive
digits (ignoring parentheses), and let $\mathrm{s}_{\mathrm{i}} \neq \mathrm{t}_{\mathrm{i}}$ be the leftmost digit where s and t differ. Then $\mathrm{s}=\mathrm{s}_{1} \ldots \mathrm{si}_{\mathrm{i}-1} \mathrm{~s}_{\mathrm{i}} \mathrm{s}^{-}$and $\mathrm{t}=\mathrm{t}_{1} \ldots \mathrm{t}_{\mathrm{i}-1} \mathrm{t}_{\mathrm{i}} \mathrm{t}^{-}$where $\mathrm{s}^{-}=\mathrm{s}_{\mathrm{i}+1} \ldots \mathrm{~s}_{\mathrm{m}}$ and $\mathrm{t}^{-}=\mathrm{t}_{\mathrm{i}+1 \ldots} \ldots \mathrm{t}_{\mathrm{n}}$. By $\left(\mathrm{WQT}^{*} 4\right), \quad \mathrm{WQT}^{*} \vdash \neg\left(\mathrm{~s}_{\mathrm{i}} \mathrm{s}^{-}=\mathrm{t}_{\mathrm{i}} \mathrm{t}^{-}\right)$. By repeatedly applying (WQT* ${ }^{*}$ ) and (WQT* 3 ) we obtain $\mathrm{WQT}^{*} \vdash \neg\left(\mathrm{~s}_{1} \ldots \mathrm{~s}_{\mathrm{i}-1} \mathrm{~s}_{\mathrm{i}} \mathrm{s}^{-}=\mathrm{t}_{1} \ldots \mathrm{t}_{\mathrm{i}-1} \mathrm{t}_{\mathrm{i}} \mathrm{t}^{-}\right)$, that is, $\mathrm{WQT}^{*} \vdash \neg(\mathrm{~s}=\mathrm{t})$, as required.

Hence in particular all instances of schema (WQT1) are provable in WQT*. Consider an instance of (WQT2) for terms $s, t \in \Sigma^{\tau}$,

$$
\forall \mathrm{z}\left(\mathrm{z} \varrho^{*} \mathrm{~b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \leftrightarrow \mathrm{z}=\mathrm{b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \mathrm{v} \mathrm{z} \varrho^{*} \mathrm{~S} \mathrm{v} \mathrm{z} \varrho^{*} \mathrm{t}\right)
$$

Now, $\mathrm{WQT}^{*} \vdash \mathrm{~b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right)=\mathrm{b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right)$, so $\mathrm{WQT} \mathrm{T}^{*} \vdash \mathrm{~b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \sqsubseteq_{\mathrm{p}} \mathrm{b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right)$. From (WQT* 6$)$,

$$
\mathrm{WQT}^{*} \vdash \forall \mathrm{x}\left(\mathrm{x} \varrho^{*} \mathrm{~b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \leftrightarrow \mathrm{x}=\mathrm{b}^{*}\left(\mathrm{~s}^{*} \mathrm{t}\right) \mathrm{v} \mathrm{x} \varrho^{*} \mathrm{~s} \mathrm{v} \mathrm{x} \varrho^{*} \mathrm{t}\right) .
$$

Hence each instance of (WQT2) is provable in WQT*. Given that (WQT3) is (WQT*7), this is enough to establish that $\mathrm{WQT}^{*}$ is an extension of WQT .

On the other hand, we also have:
7.3. $\mathrm{WQT}^{*}$ is locally finitely satisfiable.

That is, each finite subset of its non-logical axioms has a finite model.

Proof: See Appendix

By Visser's Theorem, it follows that $\mathrm{WQT}^{*}$ is interpretable in R .
Since by [6], R $\leq{ }_{I}$ WT, we then have

WEAK ESSENTIALLY UNDECIDABLE THEORIES: SECOND MUTUAL
INTERPRETABILITY THEOREM. $\quad \mathrm{R} \equiv_{I} \mathrm{WTC}^{-\varepsilon} \equiv_{I} \mathrm{WT} \equiv_{I} \mathrm{WQT} \equiv_{I} \mathrm{WQT}^{*}$.

For definition of the theory WTC ${ }^{-\varepsilon}$ see [5].

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## APPENDIX

$2.2 \quad \Sigma^{*} \vDash \notin(\mathrm{x}) \leftrightarrow \mathrm{x}=\mathrm{a}$ v $\exists!\mathrm{y}, \mathrm{z}(\mathrm{x}=\mathrm{b}(\mathrm{yz}) \& Æ(\mathrm{y}) \& Æ(\mathrm{z}))$.

Proof: $(\Leftarrow)$ Assume $\Sigma^{*} \neq Æ(y) \& Æ(z) \& x=b y z$. Then

$$
\Sigma^{*} \vDash \alpha(y)=\beta(y)+1 \& \alpha(z)=\beta(z)+1 .
$$

Now, $\quad \Sigma^{*} \mathrm{~F} \alpha(\mathrm{x})=\alpha(\mathrm{byz})=\alpha(\mathrm{yz})=\alpha(\mathrm{y})+\alpha(\mathrm{z})$
and $\quad \Sigma^{*}$ f $\beta(\mathrm{x})=\beta(\mathrm{byz})=\beta(\mathrm{b})+\beta(\mathrm{yz})=\beta(\mathrm{y})+\beta(\mathrm{z})+1$. Then
$\Sigma^{*} \mathrm{~F} \alpha(\mathrm{x})=\alpha(\mathrm{y})+\alpha(\mathrm{z})=(\beta(\mathrm{y})+1)+(\beta(\mathrm{z})+1)=(\beta(\mathrm{y})+\beta(\mathrm{z})+1)+1=\beta(\mathrm{x})+1$
which verifies (c1). For (c2), assume $\Sigma^{*}$ fuBx, i.e., $\Sigma^{*}$ f uBbyz.

Then $\quad \Sigma^{*} \mathrm{Fu}=\mathrm{b} v$ uBby $\mathrm{v} u=$ by $\mathrm{v} \exists \mathrm{z}_{1}\left(\mathrm{z}_{1} \mathrm{Bz} \& \mathrm{u}=\mathrm{byz}_{1}\right)$.

If (a) $\Sigma^{*} \mathrm{Fu}=\mathrm{b}$, then $\Sigma^{*} \mathrm{~F} \alpha(\mathrm{u})=\alpha(\mathrm{b})=0<1=\beta(\mathrm{b})=\beta(\mathrm{u})$.

If (b) $\Sigma^{*}$ f uBby, then $\Sigma^{*} \vDash \exists y_{1}\left(\mathrm{y}_{1} B y \quad \& \mathrm{u}=\mathrm{by}_{1}\right)$. Then from $\Sigma^{*} \vDash \mathbb{E}(\mathrm{y})$,
$\Sigma^{*} \vDash \alpha\left(\mathrm{y}_{1}\right) \leq \beta\left(\mathrm{y}_{1}\right)$, whence $\Sigma^{*}$ ह $\alpha(\mathrm{u})=\alpha\left(\mathrm{by}_{1}\right)=\alpha\left(\mathrm{y}_{1}\right)$ and
$\Sigma^{*} \vDash \beta(u)=\beta\left(b y_{1}\right)=\beta(b)+\beta\left(y_{1}\right)=\beta\left(y_{1}\right)+1$.

Hence $\Sigma^{*}$ f $\alpha(\mathrm{u})=\alpha\left(\mathrm{y}_{1}\right) \leq \beta\left(\mathrm{y}_{1}\right)<\beta\left(\mathrm{y}_{1}\right)+1=\beta(\mathrm{u})$.

Suppose (c) $\Sigma^{*} \vDash \mathrm{u}=\mathrm{by}$. Then from $\Sigma^{*} \vDash Æ(\mathrm{y}), \Sigma^{*} \vDash \alpha(\mathrm{y})=\beta(\mathrm{y})+1$, and we have $\Sigma^{*}$ F $\alpha(\mathrm{u})=\alpha(\mathrm{by})=\alpha(\mathrm{y})$ and
$\Sigma^{*}{ }^{k} \beta(\mathrm{u})=\beta(\mathrm{by})=\beta(\mathrm{b})+\beta(\mathrm{y})=\beta\left(\mathrm{y}_{1}\right)+1$.

Hence $\Sigma^{*}$ F $\alpha(\mathrm{u})=\alpha(\mathrm{y})=\beta(\mathrm{y})+1=\beta(\mathrm{u})$, so $\Sigma^{*}$ F $\alpha(\mathrm{u}) \leq \beta(\mathrm{u})$.

Finally, suppose (d) $\Sigma^{*} \vDash \exists \mathrm{z}_{1}\left(\mathrm{z}_{1}\right.$ By $\left.\& \mathrm{u}=\mathrm{byz} \mathrm{z}_{1}\right)$. Then from $\Sigma^{*} \vDash Æ(\mathrm{z})$,
$\Sigma^{*} \vDash \alpha\left(\mathrm{z}_{1}\right) \leq \beta\left(\mathrm{z}_{1}\right)$, and from $\Sigma^{*} \vDash Æ(\mathrm{y}), \Sigma^{*} \vDash \alpha(\mathrm{y})=\beta(\mathrm{y})+1$. Then
$\Sigma^{*}$ ह $\alpha(\mathrm{u})=\alpha\left(\mathrm{byz}_{1}\right)=\alpha\left(\mathrm{yz}_{1}\right)=\alpha(\mathrm{y})+\alpha\left(\mathrm{z}_{1}\right)$ and
$\Sigma^{*} \vDash \beta(\mathrm{u})=\beta\left(\mathrm{byz} z_{1}\right)=\beta(\mathrm{b})+\beta\left(\mathrm{yz} \mathrm{z}_{1}\right)=\beta(\mathrm{y})+\beta\left(\mathrm{z}_{1}\right)+1$. Hence
$\Sigma^{*} \vDash \alpha(\mathrm{u})=\alpha(\mathrm{y})+\alpha\left(\mathrm{z}_{1}\right)=(\beta(\mathrm{y})+1)+\alpha\left(\mathrm{z}_{1}\right) \leq$

$$
\leq(\beta(\mathrm{y})+1)+\beta\left(\mathrm{z}_{1}\right)=\beta(\mathrm{y})+\beta\left(\mathrm{z}_{1}\right)+1=\beta(\mathrm{u}) .
$$

Thus $\Sigma^{*}$ F $\alpha(\mathrm{u}) \leq \beta(\mathrm{u})$. This completes the proof of (c2). So $\Sigma^{*} \vDash \mathbb{E}(\mathrm{x})$.
3.5(i) For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that $\mathrm{QT}^{+} \vdash \forall \mathrm{x}_{2}, \mathrm{y}_{1}, \mathrm{y}_{2} \in \mathrm{~J} \forall \mathrm{x}_{1}, \mathrm{z}_{1}, \mathrm{z}_{2}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{x}_{2}\right) \& \operatorname{Tally\mathrm {y}}\left(\mathrm{y}_{1}\right) \& \operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{y}_{2}\right) \&\right.$ \& Addtally $\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{z}_{1}\right) \& \operatorname{Addtally}\left(\mathrm{y} 1, \mathrm{y}_{2}, \mathrm{z}_{2}\right) \& \mathrm{x}_{1} \leq \mathrm{y}_{1} \& \mathrm{z}_{1}=\mathrm{Sz}_{2} \rightarrow$ $\left.\rightarrow \mathrm{Sy}_{2} \leq \mathrm{x}_{2}\right)$.

Let $\mathrm{J}(\mathrm{y}) \equiv \mathrm{I}_{\text {LC }} \& \mathrm{I}_{\text {ctc }} \& \mathrm{I}_{6.2(\mathrm{a})}$ \& $\mathrm{I}_{\text {сомм }}$.
Assume $\mathrm{M}=\operatorname{Addtally}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{z}_{1}\right)$ \& Addtally $\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{z}_{2}\right)$
where $\mathrm{M} F \mathrm{x}_{1} \leq \mathrm{y}_{1} \& \mathrm{z}_{1}=\mathrm{Sz}_{2}$ and $\mathrm{M} \vDash \operatorname{Tally}_{\mathrm{b}}\left(\mathrm{x}_{2}\right) \& \operatorname{Tally}_{\mathrm{b}}\left(\mathrm{y}_{1}\right)$ \& Tallyb$\left(\mathrm{y}_{2}\right)$ and $\mathrm{M} \vDash \mathrm{J}\left(\mathrm{y}_{1}\right)$.

From $M \vDash \operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{y}_{1}\right) \& \mathrm{x}_{1} \leq \mathrm{y}_{1}, \quad \mathrm{M} \vDash \operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{x}_{1}\right)$.
By (f), $\mathrm{M} \vDash \exists \mathrm{u}_{1}\left(\operatorname{Tally} \mathrm{f}_{\mathrm{b}}\left(\mathrm{u}_{1}\right)\right.$ \& Addtally $\left.\left(\mathrm{u}_{1}, \mathrm{x}_{1}, \mathrm{y}_{1}\right)\right)$, whereas by 3.5(a),
$\mathrm{M} \vDash \exists!\mathrm{p}_{1} \in \mathrm{~J}$ Addtally $\left(\mathrm{u}_{1}, \mathrm{x}_{1}, \mathrm{p}_{1}\right)$.
Then, by single-valuedness of Addtally, $\mathrm{M} \vDash \mathrm{y}_{1}=\mathrm{p}_{1}$, whence from hypothesis
$\mathrm{M}=\operatorname{Addtally}\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{z}_{2}\right), \quad \mathrm{M}=\operatorname{Addtally}\left(\mathrm{p}_{1}, \mathrm{y}_{2}, \mathrm{z}_{2}\right)$.
On the other hand, $\mathrm{M} \vDash \exists!\mathrm{p}_{2} \in \mathrm{~J}$ Addtally $\left(\mathrm{x}_{1}, \mathrm{u}_{1}, \mathrm{p}_{2}\right)$, whence by ( g ),
$\mathrm{M}=\operatorname{Addtally}\left(\mathrm{u}_{1}, \mathrm{x}_{1}, \mathrm{p}_{2}\right)$.
But then from $\mathrm{M} \vDash \operatorname{Addtally}\left(\mathrm{u}_{1}, \mathrm{X}_{1}, \mathrm{p}_{1}\right)$, by single-valuedness of Addtally,

$$
\mathrm{M}=\mathrm{p}_{1}=\mathrm{p}_{2} .
$$

Hence $\mathrm{M}=\operatorname{Addtally}\left(\mathrm{p}_{2}, \mathrm{y}_{2}, \mathrm{z}_{2}\right)$, and from 3.4(e) we obtain

$$
\mathrm{M}=\operatorname{Addtally}\left(\mathrm{p}_{2}, \mathrm{y}_{2} * \mathrm{~b}, \mathrm{z}_{2} * \mathrm{~b}\right) .
$$

From M \& Addtally $\left(\mathrm{y}_{1}, \mathrm{y}_{2}, \mathrm{z}_{2}\right)$, by 3.5(a), $\mathrm{M} \vDash \operatorname{Tally}_{\mathrm{b}}\left(\mathrm{z}_{2}\right)$.
So from $\mathrm{M} \vDash \operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{y}_{2}\right) \& \operatorname{Tallyb}\left(\mathrm{z}_{2}\right), \quad \mathrm{M} \vDash \operatorname{Addtally}\left(\mathrm{p}_{2}, \mathrm{Sy}_{2}, \mathrm{Sz}_{2}\right)$.
Again by 3.5(a),
$\mathrm{M} \vDash \exists!\mathrm{v}_{1} \in \mathrm{~J}$ Addtally $\left(\mathrm{u}_{1}, \mathrm{Sy}_{2}, \mathrm{v}_{1}\right)$ and $\mathrm{M} \vDash \exists!\mathrm{w}_{1} \in \mathrm{~J}$ Addtally $\left(\mathrm{x}_{1}, \mathrm{v}_{1}, \mathrm{w}_{1}\right)$.
From $M \neq \operatorname{Addtally}\left(\mathrm{x}_{1}, \mathrm{u}_{1}, \mathrm{p}_{2}\right)$ by (h), $\mathrm{M} \vDash \mathrm{Sz}_{2}=\mathrm{w}_{1}$.
From hypothesis $\mathrm{M} \vDash \mathrm{z}_{1}=S \mathrm{z}_{2}, \mathrm{M} \vDash \mathrm{z}_{1}=\mathrm{w}_{1}$, so $\mathrm{M} \vDash \operatorname{Addtally}\left(\mathrm{x}_{1}, \mathrm{v}_{1}, \mathrm{z}_{1}\right)$.
On the other hand, from hypothesis $\mathrm{M}=\operatorname{Addtally}\left(\mathrm{x}_{1}, \mathrm{x}_{2}, \mathrm{Z}_{1}\right)$, by (e),

$$
\mathrm{M} \neq \mathrm{X}_{2}=\mathrm{v}_{1},
$$

Hence from $\mathrm{M} \vDash \operatorname{Addtally}\left(\mathrm{u}_{1}, \mathrm{Sy}_{2}, \mathrm{v}_{1}\right), \quad \mathrm{M} \vDash \operatorname{Addtally}\left(\mathrm{u}_{1}, \mathrm{Sy}_{2}, \mathrm{X}_{2}\right)$.
But then from $M \vDash \operatorname{Tally}_{b}\left(u_{1}\right)$, by (h), $M \vDash S y_{2} \leq x_{2}$, as required.

Let $\quad \operatorname{MaxT}_{\mathrm{b}}(\mathrm{t}, \mathrm{w}) \equiv \operatorname{Tally}_{\mathrm{b}}(\mathrm{t}) \& \forall \mathrm{t}^{\prime}\left(\operatorname{Tally}_{\mathrm{b}}\left(\mathrm{t}^{\prime}\right) \& \mathrm{t}^{\prime} \subseteq_{\mathrm{p}} \mathrm{W} \rightarrow \mathrm{t}^{\prime} \subseteq_{\mathrm{p}} \mathrm{t}\right)$.

Let us say, further, when a b-tally t is longer than any b-tally in x :

$$
\operatorname{Max}^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{x}) \equiv \operatorname{Max}_{\mathrm{b}}(\mathrm{t}, \mathrm{x}) \& \neg \mathrm{t} \subseteq_{\mathrm{px}} .
$$

We then define when a string $u$ is a preframe indexed by $t$ :

$$
\operatorname{Pref}(\mathrm{u}, \mathrm{t}) \equiv \exists \mathrm{y} \subseteq_{\mathrm{p}} \mathrm{u}\left(\mathrm{aya}=\mathrm{u} \& \operatorname{Max}+\mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{u})\right) ;
$$

when $t_{1} u t_{2}$ is (the) first frame in the string $x, \operatorname{Firstf}\left(x, t_{1}, u, t_{2}\right):$
$\operatorname{Pref}\left(\mathrm{u}, \mathrm{t}_{1}\right) \& \operatorname{Tally}_{\mathrm{b}}\left(\mathrm{t}_{2}\right) \&\left(\left(\mathrm{t}_{1}=\mathrm{t}_{2} \& \mathrm{t}_{1} \mathrm{ut}_{2}=\mathrm{x}\right) \mathrm{v}\left(\mathrm{t}_{1}<\mathrm{t}_{2} \&\left(\mathrm{t}_{1} \mathrm{ut}_{2} \mathrm{a}\right) \mathrm{Bx}\right)\right) ;$
when $\mathrm{t}_{1} \mathrm{ut}_{2}$ is (the) last frame in $\mathrm{x}, \operatorname{Lastf}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right)$ :
$\operatorname{Pref}\left(\mathrm{u}, \mathrm{t}_{1}\right) \& \mathrm{t}_{1}=\mathrm{t}_{2} \&\left(\mathrm{t}_{1} \mathrm{ut}_{2}=\mathrm{x}\right.$ v $\left.\exists \mathrm{w}\left(\mathrm{wat}_{1} \mathrm{ut}_{2}=\mathrm{x} \& \operatorname{Max}^{+} \mathrm{T}_{\mathrm{b}}\left(\mathrm{t}_{1}, \mathrm{w}\right)\right)\right) ;$
and when $t_{1} u t_{2}$ is an intermediate frame in $x$ immediately following an initial segment wof $x, \operatorname{Intf}\left(\mathrm{x}, \mathrm{w}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right)$ :

$$
\operatorname{Pref}\left(u, t_{1}\right) \& \operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{t}_{2}\right) \& \mathrm{t}_{1}<\mathrm{t}_{2} \& \exists \mathrm{w}_{1}\left(\mathrm{wat}_{1} \mathrm{ut}_{2} \mathrm{aw}_{1}=\mathrm{x}\right) \& \operatorname{Max}^{+} \mathrm{T}_{\mathrm{b}}\left(\mathrm{t}_{1}, \mathrm{w}\right) .
$$

Then we define when a string u is $\mathrm{t}_{1} \mathrm{t}_{2}$-framed in x :

$$
\operatorname{Fr}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right) \equiv \operatorname{Firstf}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right) \mathrm{v} \exists \mathrm{w} \operatorname{Intf}\left(\mathrm{x}, \mathrm{w}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right) \mathrm{v} \operatorname{Lastf}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right),
$$ We say that $\mathrm{t}_{1}$ is the initial, and $\mathrm{t}_{2}$ terminal tally marker in the frame.

Next we define "t envelops $x$ ", $\operatorname{Env}(\mathrm{t}, \mathrm{x})$, to be the conjunction of the following five conditions:
(a) $\operatorname{MaxT}_{\mathrm{b}}(\mathrm{t}, \mathrm{x})$
(b) $\exists \mathrm{u} \subseteq_{\mathrm{px}} \exists \mathrm{t}_{1}, \mathrm{t}_{2} \operatorname{Firstf}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right)$
(c) $\exists \mathrm{u} \subseteq_{\mathrm{p}} \mathrm{x} \operatorname{Lastf}(\mathrm{x}, \mathrm{t}, \mathrm{u}, \mathrm{t})$

" t is a longest b -tally in x ",
"x has a first frame",
" x has a last frame with t as its initial and terminal marker"
(d) $\forall \mathrm{u} \subseteq_{\mathrm{p} x} \forall \mathrm{t}_{1}, \mathrm{t}_{2}, \mathrm{t}_{3}, \mathrm{t}_{4}\left(\operatorname{Fr}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right) \& \operatorname{Fr}\left(\mathrm{x}, \mathrm{t}_{3}, \mathrm{u}, \mathrm{t}_{4}\right) \rightarrow \mathrm{t}_{1}=\mathrm{t}_{3}\right)$
"different initial tally markers frame distinct strings",
(e) $\forall \mathrm{u}_{1}, \mathrm{u}_{2} \subseteq_{\mathrm{p}} \forall \mathrm{t}^{\prime}, \mathrm{t}_{1}, \mathrm{t}_{2}\left(\operatorname{Fr}\left(\mathrm{x}, \mathrm{t}^{\prime}, \mathrm{u}_{1}, \mathrm{t}_{1}\right) \& \operatorname{Fr}\left(\mathrm{x}, \mathrm{t}^{\prime}, \mathrm{u}_{2}, \mathrm{t}_{2}\right) \rightarrow \mathrm{u}_{1}=\mathrm{u}_{2}\right)$
"distinct strings are framed by different initial tally markers"

Now we say x is a set code if x is aa or else x is enveloped by some b -tally:

$$
\operatorname{Set}(\mathrm{x}) \equiv \mathrm{x}=\mathrm{aa} \mathrm{v} \exists \mathrm{t} \subseteq_{\mathrm{p} x} \operatorname{Env}(\mathrm{t}, \mathrm{x})
$$

Finally, we say that a string $y$ is a member of the set coded by string $x$ if $x$ is enveloped by some $b$-tally $t$ and the juxtaposition of the string $y$ with single tokens of digit a is framed in x :

$$
\mathrm{y} \varepsilon \mathrm{x} \equiv \exists \mathrm{t} \subseteq_{\mathrm{px}} \mathrm{x}\left(\operatorname{Env}(\mathrm{t}, \mathrm{x}) \& \exists \mathrm{u} \subseteq_{\mathrm{p}} \mathrm{x} \exists \mathrm{t}_{1}, \mathrm{t}_{2}\left(\operatorname{Fr}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{u}, \mathrm{t}_{2}\right) \& \mathrm{u}=\mathrm{aya}\right)\right) .
$$

We can then establish:

SINGLETON LEMMA. For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\begin{aligned}
& \text { QT }{ }^{+}+\forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{u}, \mathrm{t}_{1}, \mathrm{t}_{2}\left(\operatorname{Set}(\mathrm{x}) \& \operatorname{Firstf}\left(\mathrm{x}, \mathrm{t}_{1}, \mathrm{aua}, \mathrm{t}_{2}\right) \& \mathrm{x}=\mathrm{t}_{1} \text { aual }_{2} \rightarrow\right. \\
& \rightarrow \forall \mathrm{w}(\mathrm{w} \varepsilon \mathrm{x} \leftrightarrow \mathrm{w}=\mathrm{u})) .
\end{aligned}
$$

(See [2], (5.21).)
APPENDING LEMMA. For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\begin{aligned}
& \mathrm{QT}^{+}+\forall \mathrm{x}, \mathrm{y} \in \mathrm{~J} \forall \mathrm{t}, \mathrm{t}_{2}, \mathrm{t}_{3}\left(\operatorname{Env}\left(\mathrm{t}_{2}, \mathrm{x}\right) \& \operatorname{Env}(\mathrm{t}, \mathrm{y}) \&\left(\mathrm{t}_{3} \mathrm{a}\right) \operatorname{By} \& \operatorname{Tallyb}\left(\mathrm{t}_{3}\right) \& \mathrm{t}_{2}<\mathrm{t}_{3} \&\right. \\
& \& \neg \exists \mathrm{u}(\mathrm{u} \varepsilon \mathrm{x} \& \mathrm{u} \varepsilon \mathrm{y}) \rightarrow \exists \mathrm{z} \in \mathrm{~J}(\operatorname{Env}(\mathrm{t}, \mathrm{z}) \& \forall \mathrm{u}(\mathrm{u} \varepsilon \mathrm{z} \leftrightarrow \mathrm{u} \varepsilon \mathrm{x} v \mathrm{u} \varepsilon \mathrm{y})) .
\end{aligned}
$$

(See [2], (5.46).)
We then derive:

DOUBLETON LEMMA. For any string form $I \subseteq I_{0}$ there is a string form $J \subseteq I$ such that

$$
\begin{aligned}
\mathrm{QT}^{+}+\forall \mathrm{x} \in \mathrm{~J} \forall \mathrm{t}_{1}, \mathrm{t}_{2}, \mathrm{t}_{3}, \mathrm{u}, \mathrm{v}\left(\operatorname{Pref}\left(\text { aua, } \mathrm{t}_{1}\right) \& \operatorname{Pref}\left(\mathrm{ava}, \mathrm{t}_{2}\right) \& \mathrm{t}_{1}<\mathrm{t}_{2} \& \mathrm{t}_{2}=\mathrm{t}_{3} \& \mathrm{u} \neq \mathrm{v} \&\right. \\
\& \mathrm{x}=\mathrm{t}_{1} \text { auat }_{2} \text { avat }_{3} \rightarrow \operatorname{Set}(\mathrm{x}) \& \forall \mathrm{w}(\mathrm{w} \varepsilon \mathrm{x} \leftrightarrow(\mathrm{w}=\mathrm{u} v \mathrm{w}=\mathrm{v})) .
\end{aligned}
$$

(See [2], (5.58).)

Let

$$
\operatorname{MinMax}^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{u}) \equiv \operatorname{Max}^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{u}) \& \forall \mathrm{t}^{\prime}\left(\operatorname{Max}^{+} \mathrm{T}_{\mathrm{b}}\left(\mathrm{t}^{\prime}, \mathrm{u}\right) \rightarrow \mathrm{t} \leq \mathrm{t}^{\prime}\right)
$$

In that case we say that $t$ is a shortest non-occurrent b-tally in string $u$.

We then have:

SHORTEST NON-OCCURRENT TALLY LEMMA. For any string form I $\subseteq \mathrm{I}_{0}$ there is a string form $J \subseteq I$ such that

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{x} \in \mathrm{~J} \exists!\mathrm{t} \in \mathrm{~J} \operatorname{MinMax}+\mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{x}) .
$$

STRING RECURSION THEOREM. Let $\mathrm{F}_{1}(\mathrm{y}, \mathrm{z}, \mathrm{u})$ and $\mathrm{F}_{2}(\mathrm{y}, \mathrm{z}, \mathrm{u})$ be formulae, and let $\mathrm{I} \subseteq \mathrm{I}^{\triangleright}$ closed under * and downward closed under $\subseteq_{\mathrm{p}}$. Suppose that

$$
\begin{aligned}
& \mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{p}) \& \mathrm{I}(\mathrm{q}), \\
& \mathrm{QT}^{+} \vdash \forall \mathrm{y}, \mathrm{z} \in \mathrm{I} \exists!\mathrm{u} \in \mathrm{I} \mathrm{~F}_{1}(\mathrm{y}, \mathrm{z}, \mathrm{u}),
\end{aligned}
$$

and $Q T^{+} \vdash \forall y, z \in I \exists!u \in I F_{2}(y, z, u)$.

Then there is a formula $H(y, z)$ and a string form $J \subseteq I$ such that
(i) $Q T^{+} \vdash \forall y \in J ~ \exists!z \in I H(y, z)$,
(iia) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{a}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{p})$,
(iib) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{b}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{q})$,
(iiia) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{J} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{H}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{H}\left(\mathrm{y}^{*} \mathrm{a}, \mathrm{z}\right) \leftrightarrow \mathrm{F}_{1}(\mathrm{y}, \mathrm{u}, \mathrm{z})\right)\right)$,
and (iiib) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{J} \forall \mathrm{u}, \mathrm{z} \in \mathrm{I}\left(\mathrm{H}(\mathrm{y}, \mathrm{u}) \rightarrow\left(\mathrm{H}\left(\mathrm{y}^{*} \mathrm{~b}, \mathrm{z}\right) \leftrightarrow \mathrm{F}_{2}(\mathrm{y}, \mathrm{u}, \mathrm{z})\right)\right)$.

Proof: Let Comp( $u, \mathrm{~m}$ ) abbreviate
$\operatorname{Set}(u) \&\left(a \leq m \rightarrow \exists v \subseteq_{p} u(\operatorname{Pair}[a, p, v] \& v \varepsilon u)\right) \&$

$$
\&\left(b \leq m \rightarrow \exists v \subseteq_{p} u(\operatorname{Pair}[b, q, v] \& v \varepsilon u)\right) \&
$$

$\& \forall \mathrm{z}<\mathrm{m} \forall \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{v}_{1}\left(\operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{~F}_{1}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right) \rightarrow\right.$

$$
\left.\rightarrow \exists \mathrm{v}_{2} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{a}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{2} \varepsilon \mathrm{u}\right)\right) \&
$$

$\& \forall \mathrm{z}<\mathrm{m} \forall \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{v}_{1}\left(\operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{~F}_{2}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right) \rightarrow\right.$

$$
\left.\rightarrow \exists \mathrm{v}_{2} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{~b}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{2} \varepsilon \mathrm{u}\right)\right) \&
$$

$\& \forall \mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{v}_{1}, \mathrm{v}_{2}\left(\operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{v}_{2} \varepsilon \mathrm{u} \rightarrow\right.$

$$
\left.\rightarrow \mathrm{u}_{1}=\mathrm{u}_{2} \& \mathrm{v}_{1}=\mathrm{v}_{2}\right) .
$$

Let (C1)-(C6) be the successive conjuncts that make up Comp(u,m,x). Then (C4) and (C5) express the usual conditions that a sequence code u should satisfy to represent the course of a recursion. The last clause, (C6), is a uniqueness condition. Then $\operatorname{Comp}(u, m)$ means, roughly, that $u$ is a set code
for a computation determined by $p, q, F_{1}, F_{2}$, in at least $m$ steps where the length indices $m$ are strings ordered by the tree-like ordering $\leq$.

Let $\operatorname{MinComp}(u, m)$ abbreviate

$$
\begin{array}{r}
\operatorname{Comp}(\mathrm{u}, \mathrm{~m}) \& \forall \mathrm{u}^{\prime}\left(\operatorname{Comp}\left(\mathrm{u}^{\prime}, \mathrm{m}\right) \rightarrow \forall \mathrm{y}\left(\mathrm{y} \varepsilon \mathrm{u} \rightarrow \mathrm{y} \varepsilon \mathrm{u}^{\prime}\right)\right) \& \\
\& \forall \mathrm{z}, \mathrm{v}, \mathrm{w}(\operatorname{Pair}[\mathrm{z}, \mathrm{v}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u} \rightarrow(\mathrm{~m}=\mathrm{a} \& \mathrm{z}=\mathrm{a}) \mathrm{v}(\mathrm{~m}=\mathrm{b} \& \mathrm{z}=\mathrm{b}) \mathrm{v} \\
\\
\mathrm{v} \exists \mathrm{n}<\mathrm{m}(\mathrm{z} \leq \mathrm{na} \mathrm{v} \mathrm{z} \leq \mathrm{nb})) .
\end{array}
$$

Let J(m) abbreviate
$I(m) \& \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, m) \& \operatorname{Pair}[m, y, w] \& w \varepsilon u)$.

Finally, let H(m,y) abbreviate
$\exists \mathrm{u}, \mathrm{w}(\operatorname{MinComp}(\mathrm{u}, \mathrm{m}) \& \operatorname{Pair}[\mathrm{~m}, \mathrm{y}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u})$.

Let (C1)-(C6) be the successive conjuncts that make up Comp (u,m). Then (C4) and (C5) express the usual conditions that a sequence code u should satisfy to represent the course of a recursion. The last clause, (C6), is a uniqueness condition.

The proof consists of ten claims.

Claim 1: $\mathrm{QT}^{+} \vdash \mathrm{J}(\mathrm{a})$.

By the principal hypothesis, $\quad \mathrm{QT}^{+} \vdash \mathrm{I}(\mathrm{p})$.
By the Pairing Lemma, $\quad \mathrm{QT}^{+} \vdash \exists!\mathrm{w} \in \mathrm{I} \operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{w}]$.
By the Shortest Non-Occurrent Tally Lemma, M $\vDash \exists!t \in I$ MinMax ${ }^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{awa})$.
$\Rightarrow \mathrm{M} \vDash \mathrm{Max}^{+} \mathrm{T}_{\mathrm{b}}(\mathrm{t}, \mathrm{awa})$.
Let $\mathrm{u}=$ tawat.
Then $M \neq I(u)$.
$\Rightarrow M \neq \operatorname{Firstf}(\mathrm{u}, \mathrm{t}, \mathrm{awa}, \mathrm{t}) \& \operatorname{Lastf}(\mathrm{u}, \mathrm{t}, \mathrm{awa}, \mathrm{t})$,
$\Rightarrow$ by the Singleton Lemma, $M \vDash \operatorname{Set}(\mathrm{u}) \& \forall \mathrm{z}(\mathrm{z} \varepsilon \mathrm{u} \leftrightarrow \mathrm{z}=\mathrm{w})$,
$\Rightarrow \mathrm{M} \vDash \operatorname{Set}(\mathrm{u}) \& \operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u}$,
which suffices to establish parts (C1) and (C2) of $M \neq \operatorname{Comp}(u, a)$.
Since $\mathrm{QT}^{+} \vdash \neg(\mathrm{b} \leq \mathrm{a})$ and $\mathrm{QT}^{+} \vdash \forall \mathrm{z} \neg(\mathrm{z}<\mathrm{a})$, parts (C3)-(C5) hold trivially.

For (C6), assume that
$\mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{v}_{2} \varepsilon \mathrm{u}$.
$\Rightarrow$ by choice of $u, M \neq v_{1}=v_{2}$,
$\Rightarrow$ since $M \neq \mathrm{v}_{1} \subseteq_{\mathrm{p}} \mathrm{u} \& \mathrm{I}(\mathrm{u}), \mathrm{M} \vDash \mathrm{I}\left(\mathrm{v}_{1}\right)$,
$\Rightarrow M \neq \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{1}\right] \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{2}, \mathrm{v}_{1}\right]$,
$\Rightarrow$ by the Pairing Lemma, $M \vDash \mathrm{u}_{1}=\mathrm{u}_{2}$,
$\Rightarrow \mathrm{M} \vDash \mathrm{u}_{1}=\mathrm{u}_{2} \& \mathrm{v}_{1}=\mathrm{v}_{2}$, as required.

This completes the argument that $\mathrm{M} \vDash \operatorname{Comp}(\mathrm{u}, \mathrm{a})$. We now move on to show that $M=\operatorname{MinComp}(u, a)$.

Assume now that $\mathrm{M} \vDash \operatorname{Comp}(\mathrm{v}, \mathrm{a})$.
Then $\mathrm{M} \vDash \exists \mathrm{w}_{1} \subseteq_{\mathrm{p}} \mathrm{v}\left(\operatorname{Pair}\left[\mathrm{a}, \mathrm{p}, \mathrm{w}_{1}\right] \& \mathrm{w}_{1} \varepsilon \mathrm{v}\right)$.
$\Rightarrow M \neq \operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{w}] \& \operatorname{Pair}\left[\mathrm{a}, \mathrm{p}, \mathrm{w}_{1}\right]$,
$\Rightarrow$ by the Pairing Lemma, $\mathrm{M}=\mathrm{w}=\mathrm{w}_{1}$,
$\Rightarrow \mathrm{M} F \mathrm{w} \varepsilon \mathrm{v}$.
Assume now that $\mathrm{M} \vDash \mathrm{y} \varepsilon \mathrm{u}$.
$\Rightarrow$ from $\mathrm{M} \vDash \forall \mathrm{z}(\mathrm{z} \varepsilon \mathrm{u} \leftrightarrow \mathrm{z}=\mathrm{w}), \mathrm{M} \vDash \mathrm{y}=\mathrm{w}$,
$\Rightarrow \mathrm{M} \boldsymbol{\mathrm { y }} \mathrm{y} \varepsilon \mathrm{v}$.
Thus we proved that $M \vDash \operatorname{Comp}(\mathrm{v}, \mathrm{a}) \rightarrow \forall \mathrm{y}(\mathrm{y} \varepsilon \mathrm{u} \rightarrow \mathrm{y} \varepsilon \mathrm{v})$.
To complete the argument that $\mathrm{M} \vDash \operatorname{MinComp}(\mathrm{u}, \mathrm{a})$, assume that

$$
\mathrm{M} \neq \operatorname{Pair}\left[\mathrm{z}_{1}, \mathrm{v}_{1}, \mathrm{~W}_{1}\right] \& \mathrm{~W}_{1} \varepsilon \mathrm{u} .
$$

$\Rightarrow$ from $\mathrm{M} \vDash \forall \mathrm{z}(\mathrm{z} \varepsilon \mathrm{u} \leftrightarrow \mathrm{z}=\mathrm{w}), \quad \mathrm{M} \vDash \mathrm{w}_{1}=\mathrm{w}$,
$\Rightarrow \mathrm{M} \neq \operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{w}] \& \operatorname{Pair}\left[\mathrm{z}_{1}, \mathrm{v}_{1}, \mathrm{w}\right]$,
$\Rightarrow$ by the Pairing Lemma, $M \vDash \mathrm{z}_{1}=\mathrm{a} \& \mathrm{v}_{1}=\mathrm{p}$.
Therefore we also have
$\mathrm{M} \vDash \forall \mathrm{z}_{1}, \mathrm{~V}_{1}, \mathrm{~W}_{1}\left(\operatorname{Pair}\left[\mathrm{z}_{1}, \mathrm{v}_{1}, \mathrm{~W}_{1}\right] \& \mathrm{w}_{1} \varepsilon \mathrm{u} \rightarrow\left(\mathrm{a}=\mathrm{a} \& \mathrm{z}_{1}=\mathrm{a}\right) \mathrm{v}\left(\mathrm{a}=\mathrm{b} \& \mathrm{z}_{1}=\mathrm{b}\right) \mathrm{v}\right.$

$$
\left.\mathrm{v} \exists \mathrm{n}<\mathrm{a}\left(\left(\mathrm{z}_{1} \leq \mathrm{na} \mathrm{v} \mathrm{z}_{1} \leq \mathrm{nb}\right)\right)\right) .
$$

So we finally have that $M=\operatorname{MinComp}(u, a)$.
In fact, we obtained
$M \vDash \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, a) \& \operatorname{Pair}[a, y, w] \& w \varepsilon u)$.

So $M \neq J(a)$.

Claim 2: $\mathrm{QT}^{+} \vdash \mathrm{J}(\mathrm{b})$.
From the proof of $\mathrm{QT}^{+} \vdash \mathrm{J}(\mathrm{a})$ we have that $M \vDash \exists!w_{1} \in I\left(\operatorname{Pair}\left[a, p, w_{1}\right] \& \exists!t_{1} \in I \operatorname{MinMax}+T_{b}\left(t_{1}, a_{1} a\right)\right)$.

Arguing exactly analogously, we obtain that $\mathrm{M} \vDash \exists!\mathrm{w}_{2} \in \mathrm{I}\left(\operatorname{Pair}\left[\mathrm{b}, \mathrm{q}, \mathrm{w}_{2}\right] \& \exists!\mathrm{t}_{2} \in \mathrm{I} \operatorname{MinMax}+\mathrm{T}_{\mathrm{b}}\left(\mathrm{t}_{2}, \mathrm{aw}_{2} \mathrm{a}\right)\right)$.
$\Rightarrow$ since $\mathrm{QT}^{+} \vdash \mathrm{a} \neq \mathrm{b}$, from $\mathrm{M} \vDash$ Pair $\left[\mathrm{a}, \mathrm{p}, \mathrm{w}_{1}\right]$ \& Pair $\left[\mathrm{b}, \mathrm{q}, \mathrm{w}_{2}\right]$, by the Pairing
Lemma, $\quad \mathrm{M} \vDash \mathrm{W}_{1} \neq \mathrm{W}_{2}$.
Let $u^{\prime}=t_{1} a w_{1} a\left(t_{1} t_{2}\right) a w_{2} a\left(t_{1} t_{2}\right)$. Then $M \neq I\left(u^{\prime}\right)$.
$\Rightarrow$ by the proof of Doubleton Lemma,

$$
\mathrm{M} \vDash \operatorname{Env}\left(\mathrm{t}_{1} \mathrm{t}_{2}, \mathrm{u}^{\prime}\right) \& \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u}^{\prime} \leftrightarrow \mathrm{w}=\mathrm{w}_{1} \mathrm{v} w=\mathrm{w}_{2}\right)
$$

On the other hand, by the principal hypothesis,

$$
\mathrm{M} \vDash \exists!\mathrm{u}_{3} \in \mathrm{I} \mathrm{~F}_{1}\left(\mathrm{a}, \mathrm{u}_{1}, \mathrm{u}_{3}\right) \text { and } \mathrm{M} \vDash \exists!\mathrm{u}_{4} \in \mathrm{I} \mathrm{~F}_{2}\left(\mathrm{a}, \mathrm{u}_{1}, \mathrm{u}_{4}\right)
$$

Just as above, we then obtain

$$
M \vDash \exists!w_{3} \in I\left(\operatorname{Pair}\left[\mathrm{aa}_{3} \mathrm{u}_{3}, \mathrm{w}_{3}\right] \& \exists!\mathrm{t}_{3} \in \mathrm{I} \operatorname{MinMax}+\mathrm{T}_{\mathrm{b}}\left(\mathrm{t}_{3}, \mathrm{aw}_{3} \mathrm{a}\right)\right)
$$

and $\quad M \vDash \exists!w_{4} \in I\left(\operatorname{Pair}\left[a b, u_{4}, w_{4}\right] \& \exists!t_{4} \in I M i n M a x+T_{b}\left(t_{4}, a_{4} a\right)\right)$.
Then, just as above, we again have that $\mathrm{M} \vDash \mathrm{w}_{3} \neq \mathrm{w}_{4}$, and, further, that $\mathrm{M} \vDash \mathrm{W}_{1} \neq \mathrm{W}_{3} \& \mathrm{~W}_{1} \neq \mathrm{W}_{4} \& \mathrm{~W}_{2} \neq \mathrm{W}_{3} \& \mathrm{~W}_{2} \neq \mathrm{W}_{4}$.

Letting $u^{\prime \prime}=\left(t_{1} t_{2} t_{3}\right) a_{3} a\left(t_{1} t_{2} t_{3} t_{4}\right) a_{4} a\left(t_{1} t_{2} t_{3} t_{4}\right)$, we likewise have that $\mathrm{M} \vDash \mathrm{I}\left(\mathrm{u}^{\prime \prime}\right)$ and $\mathrm{M} \vDash \operatorname{Env}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3} \mathrm{t}_{4}, \mathrm{u}^{\prime \prime}\right) \& \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u}^{\prime \prime} \leftrightarrow \mathrm{w}=\mathrm{w}_{3} \mathrm{v} \mathrm{w}=\mathrm{w}_{4}\right)$.

Since
$M=\operatorname{Env}\left(\mathrm{t}_{1} \mathrm{t}_{2}, \mathrm{u}^{\prime}\right) \& \operatorname{Env}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3} \mathrm{t}_{4}, \mathrm{u}^{\prime \prime}\right) \&\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3} \mathrm{a}\right) \mathrm{Bu} \mathrm{m}^{\prime \prime}$ \&

$$
\& \operatorname{Tallyb}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3}\right) \& \neg \exists \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u}^{\prime} \& \mathrm{w} \varepsilon \mathrm{u}^{\prime \prime}\right),
$$

it follows by the proof of Appending Lemma, that for

$$
\mathrm{u}=\mathrm{t}_{1} \mathrm{aw}_{1} \mathrm{a}\left(\mathrm{t}_{1} \mathrm{t}_{2}\right) \mathrm{aw}_{2} \mathrm{a}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3}\right) \mathrm{aw}_{3} \mathrm{a}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3} \mathrm{t}_{4}\right) \mathrm{aw}_{4} \mathrm{a}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3} \mathrm{t}_{4}\right),
$$

we have $M \vDash \operatorname{Env}\left(\mathrm{t}_{1} \mathrm{t}_{2} \mathrm{t}_{3} \mathrm{t}_{4}, \mathrm{u}\right) \& \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u} \leftrightarrow \mathrm{w} \varepsilon \mathrm{u}^{\prime} v \mathrm{w} \varepsilon \mathrm{u}^{\prime \prime}\right)$.
Hence

$$
M \vDash \operatorname{Set}(u) \& \forall w\left(w \varepsilon u \leftrightarrow w=w_{1} v w=w_{2} v w=w_{3} v w=w_{4}\right) .
$$

So (C1) holds.
Now, we have that
$M \vDash \exists w_{1} \subseteq_{\mathrm{p}} \mathrm{u}\left(\right.$ Pair $\left.\left[\mathrm{a}, \mathrm{p}, \mathrm{w}_{1}\right] \& \mathrm{w}_{1} \varepsilon \mathrm{u}\right)$
and $\quad M \vDash \exists w_{2} \subseteq_{p} u\left(\operatorname{Pair}\left[b, q, w_{2}\right] \& w_{2} \varepsilon u\right)$.
Since $\mathrm{QT}^{+} \vdash \mathrm{a} \leq \mathrm{b}$, this suffices to establish (C2) and (C3) of $\mathrm{M} \vDash \operatorname{Comp}(\mathrm{u}, \mathrm{b})$.
Since $\mathrm{QT}^{+} \vdash \forall \mathrm{z}(\mathrm{z}<\mathrm{b} \rightarrow \mathrm{z}=\mathrm{a})$, and $\mathrm{M} \vDash \mathrm{W}_{3} \varepsilon \mathrm{u} \& \mathrm{~W}_{4} \varepsilon \mathrm{u}$, we have from the choices of $\mathrm{w}_{3}$ and $\mathrm{w}_{4}$, that (C4) and (C5) of $\mathrm{M} \vDash \operatorname{Comp}(\mathrm{u}, \mathrm{b})$ also hold. For (C6), assume that

$$
\mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{z}, \mathrm{~s}_{1}, \mathrm{v}_{1}\right] \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{~s}_{2}, \mathrm{v}_{2}\right] \& \mathrm{v}_{1} \varepsilon \mathrm{u} \& \mathrm{v}_{2} \varepsilon \mathrm{u} .
$$

$\Rightarrow \mathrm{M} F\left(\mathrm{v}_{1}=\mathrm{w}_{1} \mathrm{v}_{1}=\mathrm{w}_{2} \mathrm{v}_{1}=\mathrm{w}_{3} \mathrm{v} \mathrm{v}_{1}=\mathrm{w}_{4}\right) \&$

$$
\&\left(\mathrm{v}_{2}=\mathrm{w}_{1} \mathrm{v} \mathrm{v}_{2}=\mathrm{w}_{2} \mathrm{v} \mathrm{v}_{2}=\mathrm{w}_{3} \mathrm{v} \mathrm{v}_{2}=\mathrm{w}_{4}\right) .
$$

Suppose, for a reductio that $M \vDash \mathrm{v}_{1} \neq \mathrm{V}_{2}$, say $\mathrm{M} \vDash \mathrm{v}_{1}=\mathrm{w}_{1} \& \mathrm{v}_{2}=\mathrm{w}_{2}$.
$\Rightarrow \mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{z}, \mathrm{s}_{1}, \mathrm{w}_{1}\right] \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{s}_{2}, \mathrm{w}_{2}\right]$.
But we have that $M \neq \operatorname{Pair}\left[a, p, w_{1}\right]$ \& $\operatorname{Pair}\left[b, q, w_{2}\right]$.
$\Rightarrow$ by the Pairing Lemma, $\mathrm{M} \vDash \mathrm{z}=\mathrm{a} \& \mathrm{z}=\mathrm{b}$, a contradiction.
Similarly for the other choice of for $v_{1}$ and $v_{2}$ from $u$.
Therefore $M \vDash v_{1}=v_{2}$, and (C6) of $M \vDash \operatorname{Comp}(u, b)$ also holds.
Assume now that $\mathrm{M} \vDash \operatorname{Comp}(\mathrm{v}, \mathrm{b})$.
Then

$$
\begin{aligned}
& M \neq \exists \mathrm{p}_{1} \subseteq_{\mathrm{p}} \mathrm{~V}\left(\operatorname{Pair}\left[\mathrm{a}, \mathrm{p}, \mathrm{p}_{1}\right] \& \mathrm{p}_{1} \varepsilon \mathrm{v}\right), \text { and } \\
& \mathrm{M} \vDash \exists \mathrm{p}_{2} \subseteq_{\mathrm{p}} \mathrm{v}\left(\operatorname{Pair}\left[\mathrm{~b}, \mathrm{q}, \mathrm{p}_{2}\right] \& \mathrm{p}_{2} \varepsilon \mathrm{v}\right), \text { and } \\
& \mathrm{M} \vDash \exists \mathrm{v}_{3}, \mathrm{p}_{3} \subseteq_{\mathrm{p}} \mathrm{v}\left(\mathrm{~F}_{1}\left(\mathrm{a}, \mathrm{v}_{1}, \mathrm{~V}_{3}\right) \& \mathrm{I}\left(\mathrm{v}_{3}\right) \& \operatorname{Pair}\left[\mathrm{aa}, \mathrm{v}_{3}, \mathrm{p}_{3}\right] \& \mathrm{p}_{3} \varepsilon \mathrm{v}\right), \text { and } \\
& \mathrm{M} \vDash \exists \mathrm{v}_{4}, \mathrm{p}_{4} \subseteq_{\mathrm{p}} \mathrm{v}\left(\mathrm{~F}_{2}\left(\mathrm{a}, \mathrm{v}_{1}, \mathrm{v}_{4}\right) \& \mathrm{I}\left(\mathrm{v}_{4}\right) \& \operatorname{Pair}\left[\mathrm{ab}, \mathrm{v}_{4}, \mathrm{p}_{4}\right] \& \mathrm{p}_{4} \varepsilon \mathrm{v}\right) .
\end{aligned}
$$

From the principal hypothesis, it follows that

$$
M \vDash v_{3}=u_{3} \& v_{4}=u_{4} .
$$

$\Rightarrow M \neq \operatorname{Pair}\left[a, p, w_{1}\right] \& \operatorname{Pair}\left[a, p, p_{1}\right]$ and $M \neq \operatorname{Pair}\left[b, q, w_{2}\right]$ \& $\operatorname{Pair}\left[b, q, p_{2}\right]$
and $M \neq \operatorname{Pair}\left[\mathrm{aa}_{,} \mathrm{u}_{3}, \mathrm{w}_{3}\right]$ \& Pair $\left[\mathrm{aa}, \mathrm{u}_{3}, \mathrm{p}_{3}\right]$ and $\mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{ab}, \mathrm{u}_{4}, \mathrm{w}_{4}\right]$ \& Pair $\left[\mathrm{ab}, \mathrm{u}_{4}, \mathrm{p}_{4}\right]$,
$\Rightarrow$ by the Pairing Lemma, $M \neq w_{1}=p_{1} \& w_{2}=p_{2} \& w_{3}=p_{3} \& w_{4}=p_{4}$,
$\Rightarrow$ from $\mathrm{MF}_{1} \varepsilon \mathrm{~W}_{1} \& \mathrm{~W}_{2} \varepsilon \mathrm{u}_{2} \mathrm{~W}_{3} \varepsilon \mathrm{u} \& \mathrm{~W}_{4} \varepsilon \mathrm{u}$, $\mathrm{M} F \mathrm{~W}_{1} \varepsilon \mathrm{~V} \& \mathrm{~W}_{2} \varepsilon \mathrm{~V} \& \mathrm{~W}_{3} \varepsilon \mathrm{~V} \& \mathrm{~W}_{4} \varepsilon \mathrm{~V}$.

Assume now that $\mathrm{M} \vDash \mathrm{y} \varepsilon \mathrm{u}$.
$\Rightarrow \mathrm{M} \vDash \mathrm{y}=\mathrm{w}_{1} \mathrm{v} \mathrm{y}=\mathrm{w}_{2} \mathrm{v} \mathrm{y}=\mathrm{w}_{3} \mathrm{v} \mathrm{y}=\mathrm{w}_{4}$,
$\Rightarrow \mathrm{M} \vDash \mathrm{y} \varepsilon \mathrm{v}$.
Thus $M \vDash \operatorname{Comp}(v, b) \rightarrow \forall y(y \varepsilon u \rightarrow y \varepsilon v)$.
Finally, assume that

$$
\mathrm{M} \vDash \operatorname{Pair}[\mathrm{z}, \mathrm{~s}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u} .
$$

$\Rightarrow \mathrm{MF} \mathrm{W}=\mathrm{W}_{1} \vee \mathrm{~V}=\mathrm{w}_{2} \vee \mathrm{~W}=\mathrm{w}_{3} \vee \mathrm{~W}=\mathrm{w}_{4}$,
$\Rightarrow M \vDash$ Pair $\left[\mathrm{z}, \mathrm{s}, \mathrm{w}_{1}\right]$ v Pair $\left[\mathrm{z}, \mathrm{s}, \mathrm{w}_{2}\right]$ v Pair[ $\left.\mathrm{z}, \mathrm{s}, \mathrm{w}_{3}\right]$ v Pair[ $\left.\mathrm{z}, \mathrm{s}, \mathrm{w}_{4}\right]$.
But we have M F Pair[a,p,w $\left.w_{1}\right]$ \& Pair[b,q, $\left.w_{2}\right]$ \& Pair[aa, $\left.u_{3}, w_{3}\right]$ \& Pair[ab, $\left.u_{4}, w_{4}\right]$.
$\Rightarrow$ by the Pairing Lemma, $\mathrm{M} \vDash \mathrm{z}=\mathrm{av} \mathrm{z}=\mathrm{b} v \mathrm{z}=\mathrm{aa} \vee \mathrm{z}=\mathrm{ab}$.
We have that $M \neq a<b$.
Hence, from $M \vDash m=b$, since $M \vDash a \leq a b$ \& $a a \leq a a \& a b \leq a b$, it follows that

$$
\mathrm{M} \vDash(\mathrm{~m}=\mathrm{b} \& \mathrm{z}=\mathrm{b}) \mathrm{v} \exists \mathrm{n}<\mathrm{b}(\mathrm{z} \leq \mathrm{na} \mathrm{v} \mathrm{z} \leq \mathrm{nb})
$$

whence $M$ F $(m=a \& z=a) \&(m=b \& z=b) v \exists n<b(z \leq n a v z \leq n b)$, as required.

This completes the argument that $\quad M \vDash \operatorname{MinComp}(u, b)$.

Claim 3: $\mathrm{QT}^{+} \vdash \forall \mathrm{x}(\mathrm{J}(\mathrm{x}) \rightarrow \mathrm{J}(\mathrm{Sx}))$.
Assume that $M \vDash J(m)$.
$\Rightarrow \mathrm{M}=\mathrm{I}(\mathrm{m})$,
$\Rightarrow$ since I is a string form, $\mathrm{M} \vDash \mathrm{I}(\mathrm{Sm})$.
We need to show that

$$
M \vDash \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, S m) \& \operatorname{Pair}[\operatorname{Sm}, y, w] \& w \varepsilon u)
$$

If $\mathrm{M} \vDash \mathrm{Sm}=\mathrm{b}$, what we need was proved in Claim 2.
So we may assume $M \vDash \neg S m=b$. Then $M \vDash \neg m=a$.
From the hypothesis $M \vDash J(m)$ we have that

$$
M \vDash \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, m) \& \operatorname{Pair}[m, y, w] \& w \varepsilon u)
$$

Let $u_{0}$ be $a \operatorname{u}$ in $M$ such that $M \vDash I(u) \& \operatorname{Set}(u) \& \operatorname{MinComp}(u, m, x)$.
Let $y_{0}$ be the unique $y$ in $M$ such that

$$
\mathrm{M} \vDash \exists \mathrm{w} \subseteq_{\mathrm{p}} \mathrm{u}_{0}\left(\operatorname{Pair}[\mathrm{~m}, \mathrm{y}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u}_{0}\right)
$$

$\Rightarrow$ since $M \neq I(m) \& I\left(y_{0}\right)$, by the Pairing Lemma, $\quad M \vDash \exists!w \subseteq_{p} u_{0}$ Pair $[m, y, w]$.
Let $\mathrm{w}_{0}$ be the unique such w in M .
From $M=\operatorname{MinComp}\left(\mathrm{u}_{0}, \mathrm{~m}\right), \quad \mathrm{M}=\operatorname{Comp}\left(\mathrm{u}_{0}, \mathrm{~m}\right)$.
$\Rightarrow \mathrm{M} \vDash \operatorname{Set}\left(\mathrm{u}_{0}\right)$,
$\Rightarrow$ since $M \vDash \mathrm{w}_{0} \varepsilon \mathrm{u}_{0}, \mathrm{M} \vDash \exists \mathrm{t} \subseteq_{\mathrm{p}} \mathrm{u}_{0} \operatorname{Env}\left(\mathrm{t}, \mathrm{u}_{0}\right)$.
Here $t$ uniquely depends on $u_{0}$. (See [2], (4.24b).) Since the string form I is downward closed w.r. to $\subseteq_{p}$, from $M \vDash I\left(u_{0}\right)$ we have that $M \vDash I(t)$.

From the principal hypothesis of the Theorem we have that
( $\dagger$ ) $\quad M \vDash \exists!v_{1} \in I F_{1}\left(m, y_{0}, v_{1}\right)$ and $M \vDash \exists!v_{2} \in I F_{2}\left(m, y_{0}, v_{2}\right)$.
$\Rightarrow$ by the Pairing Lemma,

$$
\begin{gathered}
M \vDash \exists!w_{1} \in I\left(\operatorname{Pair}\left[\mathrm{ma}_{\mathrm{v}}^{1}, \mathrm{w}_{1}\right] \& \exists!\mathrm{t}_{1} \in \mathrm{I} \operatorname{MinMax}+\mathrm{T}_{\mathrm{b}}\left(\mathrm{t}_{1}, \mathrm{aw}_{1} \mathrm{a}\right)\right) \text { and } \\
\mathrm{M} \vDash \exists!\mathrm{w}_{2} \in \mathrm{I}\left(\operatorname{Pair}\left[\mathrm{mb}, \mathrm{v}_{2}, \mathrm{w}_{2}\right] \& \exists!\mathrm{t}_{2} \in \mathrm{I} \operatorname{MinMax}{ }^{+} \mathrm{T}_{\mathrm{b}}\left(\mathrm{t}_{2}, \mathrm{aw}_{2} \mathrm{a}\right)\right) .
\end{gathered}
$$

Then, analogously to the proof of $\mathrm{QT}^{+} \vdash \mathrm{J}(\mathrm{b})$ above, we obtain, for

$$
\mathrm{u}^{\prime}=\mathrm{tt}_{1} \mathrm{aw}_{1} \mathrm{a}\left(\mathrm{tt}_{1} \mathrm{t}_{2}\right) \mathrm{aw}_{2} \mathrm{a}\left(\mathrm{tt}_{1} \mathrm{t}_{2}\right),
$$

that

$$
\mathrm{M} \vDash \mathrm{I}\left(\mathrm{u}^{\prime}\right) \& \operatorname{Env}\left(\mathrm{tt}_{1} \mathrm{t}_{2}, \mathrm{u}^{\prime}\right) \& \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u}^{\prime} \leftrightarrow \mathrm{w}=\mathrm{w}_{1} \mathrm{v} \mathrm{w}=\mathrm{w}_{2}\right) .
$$

From $M \neq \operatorname{MinComp}\left(u_{0}, m\right)$, we readily verify that

$$
\mathrm{M} \vDash \neg\left(\mathrm{w}_{1} \varepsilon \mathrm{u}_{0}\right) \& \neg\left(\mathrm{w}_{2} \varepsilon \mathrm{u}_{0}\right) .
$$

Then, since
$\mathrm{M}=\operatorname{Env}\left(\mathrm{t}, \mathrm{u}_{0}\right) \& \operatorname{Env}\left(\mathrm{tt}_{1} \mathrm{t}_{2}, \mathrm{u}^{\prime}\right) \&\left(\mathrm{tt}_{1} \mathrm{t}_{2} \mathrm{a}\right) \mathrm{Bu}^{\prime}$ \& Tallyb$\left(\mathrm{tt}_{1} \mathrm{t}_{2}\right)$ \&

$$
\& \neg \exists \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u}_{0} \& \mathrm{w} \varepsilon \mathrm{u}^{\prime}\right),
$$

by the proof of Appending Lemma, for

$$
\mathrm{u}=\mathrm{u}_{0} \mathrm{t}_{1} \mathrm{aw}_{1} \mathrm{a}\left(\mathrm{tt}_{1} \mathrm{t}_{2}\right) \mathrm{aw}_{2} \mathrm{a}\left(\mathrm{tt}_{1} \mathrm{t}_{2} \mathrm{t}_{3}\right),
$$

we have $M \vDash \operatorname{Env}\left(\mathrm{tt}_{1} \mathrm{t}_{2}\right) \& \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u} \leftrightarrow \mathrm{w} \varepsilon \mathrm{u}_{0} \mathrm{vw} \varepsilon \mathrm{u}^{\prime}\right)$.
Hence
$\mathrm{M} \vDash \operatorname{Set}(\mathrm{u}) \& \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u} \leftrightarrow\left(\mathrm{w} \varepsilon \mathrm{u}_{0} \mathrm{v} \mathrm{w}=\mathrm{w}_{1} \mathrm{v} \mathrm{w}=\mathrm{w}_{2}\right)\right)$.
So (C1) holds.
Note that, since the string form I is closed under *, from

$$
\mathrm{M} \vDash \mathrm{I}\left(\mathrm{u}_{0}\right) \& \mathrm{I}\left(\mathrm{t}_{1}\right) \& \mathrm{I}\left(\mathrm{w}_{1}\right) \& \mathrm{I}(\mathrm{t}) \& \mathrm{I}\left(\mathrm{t}_{2}\right) \& \mathrm{I}\left(\mathrm{w}_{2}\right)
$$

we have $M \vDash I(u)$.
We now proceed to argue that $M \vDash \operatorname{Comp}(u, S m)$.
It is straightforward to verify from $\mathrm{M} \vDash \operatorname{Comp}\left(\mathrm{u}_{0}, \mathrm{~m}\right)$ and the choice of $u$ that

$$
\mathrm{M} \vDash \exists \mathrm{q}_{1} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{a}, \mathrm{p}, \mathrm{q}_{1}\right] \& \mathrm{q}_{1} \varepsilon \mathrm{u}\right) \text { and } \mathrm{M} \vDash \exists \mathrm{q}_{2} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{~b}, \mathrm{q}, \mathrm{q}_{2}\right] \& \mathrm{q}_{2} \varepsilon \mathrm{u}\right)
$$

so that (C2) and (C3) of $\mathrm{M}=\operatorname{Comp}(\mathrm{u}, \mathrm{Sm})$ both hold.
For (C4), let $\mathrm{M} \vDash \mathrm{z}<\operatorname{Sm} \& \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{~V}_{3}\right] \& \mathrm{v}_{3} \varepsilon \mathrm{u} \& \mathrm{~F}_{1}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right)$ where $\mathrm{M} \vDash \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{~V}_{3} \subseteq_{\mathrm{p}} \mathrm{u}$.

We need to show that $\mathrm{M} \vDash \exists \mathrm{v} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{a}, \mathrm{u}, \mathrm{v}\right]\right.$ \& $\left.\mathrm{v} \varepsilon \mathrm{u}\right)$.
From $\mathrm{M} \vDash \mathrm{z}<\mathrm{Sm}, \quad \mathrm{M}=\mathrm{z}<\mathrm{mvz}=\mathrm{m}$.
Suppose $\mathrm{M} \vDash \mathrm{z}<\mathrm{m}$.
We have that $M \vDash v_{3} \varepsilon u$.
Using the Pairing Lemma and the definition of $<$, we verify that

$$
M \vDash v_{3} \neq W_{1} \text { and } M \vDash v_{3} \neq W_{2} .
$$

$\Rightarrow$ from $M \vDash \mathrm{~V}_{3} \varepsilon \mathrm{u}, \quad \mathrm{M} \vDash \mathrm{v}_{3} \varepsilon \mathrm{u}_{0}$,
$\Rightarrow \mathrm{MF} \mathrm{u}_{1} \subseteq_{\mathrm{p}} \mathrm{u}_{0}$.
Then, from $M \neq \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{3}\right] \& \mathrm{v}_{3} \varepsilon \mathrm{u}_{0} \& \mathrm{~F}_{1}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right)$ and $\mathrm{M}=\operatorname{Comp}\left(\mathrm{u}_{0}, \mathrm{~m}\right)$, we have that $\quad \mathrm{M} \vDash \exists \mathrm{v} \subseteq_{\mathrm{p}} \mathrm{u}_{0}$ ( $\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{a}, \mathrm{u}_{2}, \mathrm{v}\right] \& v \varepsilon \mathrm{u}_{0}$ ),
whence $\quad \mathrm{M} \vDash \exists \mathrm{v} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{Pair}\left[\mathrm{z}^{*} \mathrm{a}_{\mathrm{u}} \mathrm{u}_{2}, \mathrm{v}\right] \& \mathrm{v} \varepsilon \mathrm{u}\right)$, as required.
Suppose M F $\mathrm{z}=\mathrm{m}$.
Again, we are assuming that $M \vDash \operatorname{Pair}\left[\mathrm{z}, \mathrm{u}_{1}, \mathrm{v}_{3}\right] \& \mathrm{v}_{3} \varepsilon \mathrm{u} \& \mathrm{~F}_{1}\left(\mathrm{z}, \mathrm{u}_{1}, \mathrm{u}_{2}\right)$ where $M \vDash \mathrm{u}_{1}, \mathrm{u}_{2}, \mathrm{~V}_{3} \subseteq_{\mathrm{p}} \mathrm{u}$. Hence $\mathrm{M} \vDash \mathrm{I}\left(\mathrm{u}_{2}\right)$.

Just as above, $\quad \mathrm{M} \vDash \mathrm{V}_{3} \varepsilon \mathrm{u}_{0}$.
On the other hand, we also have that $M \vDash \exists w \subseteq_{p} u\left(\operatorname{Pair}\left[m, y_{0}, w\right] \& w \varepsilon u_{0}\right)$,
$\Rightarrow$ from $M \vDash \operatorname{Pair}\left[m, u_{1}, v_{3}\right] \& v_{3} \varepsilon u_{0}$ and clause (C6) of $M \vDash \operatorname{Comp}\left(u_{0}, m\right)$,

$$
\mathrm{M} \vDash \mathrm{y}_{0}=\mathrm{u}_{1} \& \mathrm{v}_{3}=\mathrm{w},
$$

$\Rightarrow$ from $M \vDash F_{1}\left(m, u_{1}, u_{2}\right) \& I\left(u_{2}\right)$ and $(\dagger), \quad M \vDash v_{1}=u_{2}$.
But then, from $M \neq \operatorname{Pair}\left[m a, v_{1}, w_{1}\right]$, we have

$$
\mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{ma}, \mathrm{u}_{2}, \mathrm{w}_{1}\right] \& \mathrm{w}_{1} \varepsilon \mathrm{u},
$$

where $\mathrm{M} \vDash \mathrm{w}_{1} \subseteq_{\mathrm{p}} \mathrm{u}$, as required.
Hence (C4) of $M \neq \operatorname{Comp}(u, S m)$ also holds.
For (C5), we argue in exactly the same way, except that references to $\mathrm{F}_{1}, \mathrm{z}^{*}$ a and $\mathrm{w}_{1}$ are replaced by $\mathrm{F}_{2}, \mathrm{z}^{*} \mathrm{~b}$ and $\mathrm{w}_{2}$.

Condition (C6) is verified using the corresponding condition from
$\mathrm{M} \vDash \operatorname{Comp}\left(\mathrm{u}_{0}, \mathrm{~m}\right)$ and the Pairing Lemma.
We now proceed to show that in fact $M \vDash \operatorname{MinComp}(u, S m)$.

Suppose that $M \neq \operatorname{Comp}\left(v^{\prime}, S m\right)$.
First, we want to show that $M \vDash \forall y\left(y \varepsilon v^{\prime} \rightarrow y \varepsilon u\right)$.
From $M \neq \operatorname{Comp}\left(v^{\prime}, S m\right)$ we have that $M \vDash \operatorname{Comp}\left(v^{\prime}, m\right)$.
From the hypothesis $M \vDash J(m)$ we have that $M \vDash \operatorname{MinComp}\left(u_{0}, m\right)$.
$\Rightarrow \mathrm{M} \vDash \forall \mathrm{y}\left(\mathrm{y} \varepsilon \mathrm{u}_{0} \rightarrow \mathrm{y} \varepsilon \mathrm{v}^{\prime}\right)$.
From $M \vDash J(m)$ we also have that $M \vDash \operatorname{Pair}\left[m, y_{0}, w_{0}\right] \& W_{0} \varepsilon u_{0}$.
$\Rightarrow \mathrm{M}=\mathrm{W}_{0} \varepsilon \mathrm{v}^{\prime}$.
But then, since, by ( $\dagger$ ), $\quad \mathrm{M} \vDash \mathrm{F}_{1}\left(\mathrm{~m}, \mathrm{y}_{0}, \mathrm{v}_{1}\right) \& \mathrm{~F}_{2}\left(\mathrm{~m}, \mathrm{y}_{0}, \mathrm{v}_{2}\right)$,
we have, from (C4) and (C5) of $\mathrm{M} \vDash \operatorname{Comp}\left(\mathrm{v}^{\prime}, \mathrm{Sm}\right)$ that

$$
\mathrm{M} \vDash \mathrm{~W}_{1} \varepsilon \mathrm{v}^{\prime} \& \mathrm{~W}_{2} \varepsilon \mathrm{v}^{\prime}
$$

where $M \neq \operatorname{Pair}\left[\mathrm{m}^{*} \mathrm{a}_{\mathrm{a}} \mathrm{v}_{1}, \mathrm{w}_{1}\right]$ \& $\operatorname{Pair}\left[\mathrm{m}^{*} \mathrm{~b}, \mathrm{v}_{2}, \mathrm{w}_{2}\right]$.
So we have that $\mathrm{M} \vDash \forall \mathrm{y}\left(\mathrm{y} \varepsilon \mathrm{u}_{0} \rightarrow \mathrm{y} \varepsilon \mathrm{v}^{\prime}\right) \& \mathrm{~W}_{1} \varepsilon \mathrm{v}^{\prime} \& \mathrm{~W}_{2} \varepsilon \mathrm{v}^{\prime}$.
But then from, the choice of $u$, it follows that

$$
\mathrm{M} \vDash \forall \mathrm{y}\left(\mathrm{y} \varepsilon \mathrm{u} \rightarrow \mathrm{y} \varepsilon \mathrm{v}^{\prime}\right),
$$

as required.
Suppose now that $M \neq \operatorname{Pair}[\mathrm{z}, \mathrm{v}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u}$.
$\Rightarrow \mathrm{M} \vDash \mathrm{w} \varepsilon \mathrm{u}_{0} \mathrm{v} \mathrm{w}=\mathrm{w}_{1} \mathrm{v} \mathrm{w}=\mathrm{w}_{2}$.
If $M \vDash w \in u_{0}$, then from $M \vDash \operatorname{MinComp}\left(u_{0}, m\right)$, we have that

$$
M \vDash(m=a \& z=a) \&(m=b \& z=b) v \exists n<m(z \leq n a v z \leq n b) .
$$

But then $M \vDash(m=a \& z=a) \&(m=b \& z=b) v \exists n<\operatorname{Sm}(z \leq n a v z \leq n b)$.
If $\mathrm{M} \vDash \mathrm{w}=\mathrm{w}_{1}$, we have that $\mathrm{M} \vDash \mathrm{z}=\mathrm{ma}$, whence

$$
\mathrm{M} \vDash \exists \mathrm{n}<\operatorname{Sm} \mathrm{z} \leq \mathrm{na} .
$$

Hence $M=(m=a \& z=a) \&(m=b \& z=b) v \exists n<\operatorname{Sm}(z \leq n a v z \leq n b)$, as required.

An analogous argument applies if $\mathrm{M}=\mathrm{w}=\mathrm{w}_{2}$.
This suffices to establish $\mathrm{M}=\operatorname{MinComp}(\mathrm{u}, \mathrm{Sm})$.
Now, we have that
$\mathrm{M} \vDash \exists!_{\mathrm{w}_{2} \in \mathrm{I}}\left(\operatorname{MinComp}(\mathrm{u}, \mathrm{Sm}) \& \operatorname{Pair}\left[\mathrm{Sm}, \mathrm{v}_{2}, \mathrm{w}_{2}\right] \& \mathrm{w}_{2} \varepsilon \mathrm{u}\right)$.
Suppose that $M \neq \operatorname{MinComp}(u, S m) \& \operatorname{Pair}\left[S m, y, w_{2}\right] \& w_{2} \varepsilon u$ where $M \vDash I(y)$.
$\Rightarrow$ from $\mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{Sm}_{\mathrm{v}} \mathrm{v}_{2}, \mathrm{w}_{2}\right] \& \mathrm{I}\left(\mathrm{w}_{2}\right)$, we have, by the Pairing Lemma, that

$$
\mathrm{M} \vDash \mathrm{v}_{2}=\mathrm{y} .
$$

So we have actually established that
$M \vDash \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, S m) \& \operatorname{Pair}[S m, y, w] \& w \varepsilon u)$,
and hence that $\mathrm{M} \vDash \mathrm{J}(\mathrm{Sm})$.
This completes the proof of Claim 3.

Claim 4: $\mathrm{QT}^{+} \vdash \forall \mathrm{x}\left(\mathrm{J}(\mathrm{x}) \rightarrow \mathrm{J}\left(\mathrm{x}^{*} \mathrm{a}\right)\right)$.
Exactly analogous to the proof of Claim 3.
Claims 1-4 establish that J is a string form.

Claim 5: $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{a}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{p})$.
Let $M \vDash I(y)$.
Assume M F $\mathrm{y}=\mathrm{p}$.
As shown in the proof of Claim 1, in $M$ there is a $u$, namely, $u=$ tawat, such that

$$
\mathrm{M} \vDash \operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u} .
$$

Then, again as shown in the proof of Claim 1, we have that

$$
\mathrm{M} \vDash \operatorname{MinComp}(\mathrm{u}, \mathrm{a}) \& \operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u}),
$$

whence $M \neq H(a, p)$, so $M \neq H(a, y)$.
Thus, $\quad \mathrm{M} \vDash \mathrm{y}=\mathrm{p} \rightarrow \mathrm{H}(\mathrm{x}, \mathrm{a}, \mathrm{y})$.
Conversely, let $\quad M \neq H(a, y)$.
$\Rightarrow$ by definition of $H, \quad M \neq \exists u, w(\operatorname{MinComp}(u, a) \& \operatorname{Pair}[a, y, w] \& w \varepsilon u)$,
$\Rightarrow \mathrm{M} \vDash \operatorname{Comp}(\mathrm{u}, \mathrm{a})$,
$\Rightarrow$ from (C2), $\quad \mathrm{M} \vDash \exists \mathrm{v}(\operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{v}] \& \mathrm{v} \varepsilon \mathrm{u})$,
$\Rightarrow$ from $\mathrm{M} \vDash \operatorname{Pair}[\mathrm{a}, \mathrm{y}, \mathrm{w}]$ \& $\operatorname{Pair}[\mathrm{a}, \mathrm{p}, \mathrm{v}] \& \mathrm{w} \varepsilon \mathrm{u} \& \mathrm{v} \varepsilon \mathrm{u}$, and clause (C6) of $M \vDash \operatorname{Comp}(u, a), \quad M \vDash y=p$.

Hence also $M \vDash H(a, y) \rightarrow y=p)$.
This completes the proof of Claim 5.

Claim 6: QT $^{+}+\forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{b}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{q})$.
Let $M \neq I(y)$.
Assume M F $\mathrm{y}=\mathrm{q}$.
We follow the proof of Claim 2 to obtain a $u$ in $M$ such that

$$
\mathrm{M} \vDash \operatorname{Pair}\left[\mathrm{~b}, \mathrm{u}_{2}, \mathrm{w}\right] \& \mathrm{w} \varepsilon \mathrm{u},
$$

where $\left.M \equiv \operatorname{MinComp}(u, b) \& \operatorname{Pair}\left[b, u_{2}, w\right] \& w \varepsilon u\right)$.

Then $M \neq H(b, y)$.

This shows that $\quad M \vDash y=q \rightarrow H(b, y)$.
To establish the converse, that $\mathrm{M} \vDash \mathrm{H}(\mathrm{b}, \mathrm{y}) \rightarrow \mathrm{y}=\mathrm{q}$, we argue analogously to the proof in Claim 5.

Claim 7: QT $^{+} \vdash \forall y \in J \forall v, z \in I\left(H(y, v) \rightarrow\left(F_{1}(y, v, z) \rightarrow H\left(y^{*} a, z\right)\right)\right)$.

Let $M \neq J(y)$ and $M \neq I(v) \& I(z)$.
Suppose $M \neq H(y, v) \& F_{1}(y, v, z)$.
$\Rightarrow$ from $\mathrm{M} \vDash \mathrm{J}(\mathrm{y})$,

$$
\mathrm{M} \vDash \exists \mathrm{u}_{0} \in \mathrm{I} \exists \mathrm{w} \subseteq_{\mathrm{p}} \mathrm{u}\left(\operatorname{MinComp}\left(\mathrm{u}_{0}, \mathrm{y}\right) \& \operatorname{Pair}[\mathrm{y}, \mathrm{v}, \mathrm{w}] \& \mathrm{w} \varepsilon \mathrm{u}_{0}\right) .
$$

We then obtain, exactly analogously to the proof of Claim 3, a u in M such that $\mathrm{M} \vDash \exists \mathrm{w}_{1}\left(\operatorname{MinComp}\left(\mathrm{u}, \mathrm{y}^{*} \mathrm{a}\right) \& \operatorname{Pair}\left[\mathrm{y}^{*} \mathrm{a}, \mathrm{z}, \mathrm{w}_{1}\right] \& \mathrm{w}_{1} \varepsilon \mathrm{u}\right)$, whence $M \vDash H\left(y^{*} a, z\right)$ follows.

This completes the argument for Claim 7.

Claim 8: $\quad Q^{+} \vdash \forall y \in J \forall v, z \in I\left(H(y, v) \& H\left(y^{*} a, z\right) \rightarrow F_{1}(y, v, z)\right)$.

Assume that $M \vDash H(y, v) \& H\left(y^{*} a, z\right) \quad$ where $M \vDash J(y)$ and $M \vDash I(v) \& I(z)$.
From the hypothesis $M \neq H(y, v)$ we have that

$$
\mathrm{M} \vDash \exists \mathrm{u}_{0}, \mathrm{~W}_{0}\left(\operatorname{MinComp}\left(\mathrm{u}_{0}, \mathrm{y}\right) \& \operatorname{Pair}\left[\mathrm{y}, \mathrm{v}, \mathrm{w}_{0}\right] \& \mathrm{w}_{0} \varepsilon \mathrm{u}_{0}\right) .
$$

From the principal hypothesis of the Theorem we have

$$
\mathrm{QT}^{+} \vdash \exists!\mathrm{z}^{\prime} \in \mathrm{I} \mathrm{~F}_{1}\left(\mathrm{y}, \mathrm{v}, \mathrm{z}^{\prime}\right) .
$$

We then obtain, exactly analogously to the proof of Claim 3, a u in $M$ such that

$$
M \vDash \exists w_{1}\left(\operatorname{MinComp}\left(u, y^{*} a\right) \& \operatorname{Pair}\left[y^{*} a, z^{\prime}, w_{1}\right] \& w_{1} \varepsilon u\right) .
$$

On the other hand, from the hypothesis $M \vDash H\left(y^{*} a, z\right)$, we have that

$$
M \neq \exists u^{\prime}, w^{\prime}\left(\operatorname{MinComp}\left(u^{\prime}, y^{*} a\right) \& \operatorname{Pair}\left[y^{*} a, z^{\prime}, w^{\prime}\right] \& w^{\prime} \varepsilon u^{\prime}\right) .
$$

Now, we have that, in general

$$
\mathrm{QT}^{+} \vdash \operatorname{MinComp}\left(\mathrm{u}_{1}, \mathrm{~m}\right) \& \operatorname{MinComp}\left(\mathrm{u}_{2}, \mathrm{~m}\right) \rightarrow \forall \mathrm{w}\left(\mathrm{w} \varepsilon \mathrm{u}_{1} \leftrightarrow \mathrm{w} \varepsilon \mathrm{u}_{2}\right) .
$$

From $M \neq \operatorname{MinComp}\left(u, y^{*} a\right) \& \operatorname{MinComp}\left(u^{\prime}, y^{*} a\right) \& w^{\prime} \varepsilon u^{\prime} \quad, M \neq w^{\prime} \varepsilon u$.
$\Rightarrow$ from $M \vDash \operatorname{Comp}\left(u, y^{*} a\right) \& \operatorname{Pair}\left[y^{*} a, z, w^{\prime}\right] \& \operatorname{Pair}\left[y^{*} a, z^{\prime}, w_{1}\right] \& W_{1} \varepsilon u$,

$$
\mathrm{M} \neq \mathrm{z}=\mathrm{z}^{\prime},
$$

$\Rightarrow M \vDash F_{1}(y, v, z)$, as required.
This completes the proof of Claim 8.

Claim 9: $Q^{+}+\vdash \forall y \in J \forall v, z \in I\left(H(y, v) \rightarrow\left(F_{2}(y, v, z) \rightarrow H(y * b, z)\right)\right)$.

Claim 10: $\quad Q^{+}+\vdash \forall y \in J \forall v, z \in I\left(H(y, v) \& H(y * b, z) \rightarrow F_{2}(y, v, z)\right)$.

These two claims are proved exactly analogously to Claims 8 and 9.
From the definition of the string form J we have
$M \vDash \forall m \in J \exists!y \in I \exists u \in I \exists w \subseteq_{p} u(\operatorname{MinComp}(u, m) \& \operatorname{Pair}[m, y, w] \& w \varepsilon u)$.
So from the definition of H we have

$$
\begin{equation*}
\mathrm{QT}^{+} \vdash \forall \mathrm{m} \in \mathrm{~J} \exists!\mathrm{y} \in \mathrm{I} \mathrm{H}(\mathrm{~m}, \mathrm{y}) . \tag{i}
\end{equation*}
$$

From Claims 5 and 6 we have
(iia)

$$
\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{a}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{p}),
$$

and

$$
\begin{equation*}
\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{I}(\mathrm{H}(\mathrm{~b}, \mathrm{y}) \leftrightarrow \mathrm{y}=\mathrm{q}) . \tag{iib}
\end{equation*}
$$

From Claim 7 and 8 we have
(iiia) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{J} \forall \mathrm{v}, \mathrm{z} \in \mathrm{I}\left(\mathrm{H}(\mathrm{y}, \mathrm{v}) \rightarrow\left(\mathrm{H}\left(\mathrm{y}^{*} \mathrm{a}, \mathrm{z}\right) \rightarrow \mathrm{F}_{1}(\mathrm{y}, \mathrm{v}, \mathrm{z})\right)\right)$,
and, from Claims 9 and 10, we obtain
(iiib) $\mathrm{QT}^{+} \vdash \forall \mathrm{y} \in \mathrm{J} \forall \mathrm{v}, \mathrm{z} \in \mathrm{I}\left(\mathrm{H}(\mathrm{y}, \mathrm{v}) \rightarrow\left(\mathrm{H}\left(\mathrm{y}^{*} \mathrm{~b}, \mathrm{z}\right) \rightarrow \mathrm{F}_{2}(\mathrm{y}, \mathrm{v}, \mathrm{z})\right)\right)$.

This concludes the proof of the Theorem.
5.2(a) For any string form $I \subseteq I_{\alpha}$ and $I \subseteq I_{\text {Add }}$ there is a string form $\mathrm{J} \equiv \mathrm{I}_{\text {Add } \alpha} \subseteq \mathrm{I}$ such that
$Q^{+}+\forall x, y \in J \forall u, v, w\left(A^{\#}(x, u) \& A^{\#}(y, v) \& \operatorname{AddTally}(u, v, w) \rightarrow A^{\#}\left(x^{*} y, w\right)\right)$.

Proof: Let $J(y)$ abbreviate

$$
I(y) \& \forall x \in I \forall u, v, w\left(A^{\#}(x, u) \& A^{\#}(y, v) \& \operatorname{AddTally}(u, v, w) \rightarrow A^{\#}\left(x^{*} y, w\right)\right)
$$

Since I may be assumed to be closed under * and downward closed under $\leq$, we may assume that I is closed under AddTally, $\alpha$ and $\beta$.

We argue that J is a string form.

For $y=a$, we have that $M \vDash I(a)$.
Assume $M \neq A^{\#}(x, u) \& A^{\#}(y, v) \& \operatorname{AddTally}(u, v, w) \quad$ where $M \vDash I(x)$.
Then $M \neq A^{\#}(a, v)$, whence, by (iia $\left.{ }^{\alpha}\right), M \vDash v=b b$.
From $M \neq A^{\#}(x, u), M \vDash \operatorname{Tallyb}(u)$.
Then $\mathrm{M} \vDash$ AddTally( $u, b b, w$ ), and by 3.4(d), $\mathrm{M} \vDash$ AddTally( $u, b b, S u)$.
By single-valuedness of AddTally, M ह w=Su.
On the other hand, by ( $\mathrm{i}^{\alpha}$ ), $\mathrm{M} \vDash \exists!\mathrm{w}^{\prime} \in \mathrm{I} \mathrm{A}^{\#}\left(\mathrm{x}^{*} \mathrm{a}, \mathrm{w}^{\prime}\right)$.
From $M \neq A^{\#}(x, u)$, by (iiia ${ }^{\alpha}$ ), $M \vDash w^{\prime}=u^{*} b$, and from $M \vDash \operatorname{Tallyb}(u)$,
$M \vDash w^{\prime}=S u$. Hence $M \vDash A^{\#}\left(x^{*} a, S u\right)$.
Then from $M \neq w=S u, M \vDash A^{\#}\left(x^{*} a, w\right)$, as required.
For $\mathrm{y}=\mathrm{b}$, again we have $\mathrm{M} \vDash \mathrm{I}(\mathrm{b})$.
Assume $M \neq A^{\#}(x, u) \& A^{\#}(y, v) \& A d d T a l l y(u, v, w)$ where $M F I(x)$.
Then $M \neq A^{\#}(b, v)$. By (iib ${ }^{\alpha}$ ), $M \vDash v=b$, so $M \vDash \operatorname{AddTally}(u, b, w)$.
By definition of AddTally, $\mathrm{M} \vDash \mathrm{w}=\mathrm{u}$.
By ( $\mathrm{i}^{\alpha}$ ), M $=\exists!w^{\prime} \in I A^{\#}\left(\mathrm{x}^{*} \mathrm{~b}, w^{\prime}\right)$.
Hence from $M \neq A^{\#}(x, u)$, by (iiib $)^{\alpha}, M \vDash w^{\prime}=u$. Thus $M \vDash A^{\#}\left(x^{*} b, u\right)$.
But then from $M \vDash w=u, M \neq A^{\#}\left(x^{*} b, w\right)$, as required.
Suppose now that $M \neq J(y)$.
Then $M \neq I(y)$, whence $M \neq I\left(y^{*} a\right)$ because $I$ is a string concept.
Assume now that $\mathrm{M} \vDash \mathrm{A}^{\#}(\mathrm{x}, \mathrm{u}) \& \mathrm{~A}^{\#}\left(\mathrm{y}^{*} \mathrm{a}, \mathrm{v}\right)$ \& AddTally $(\mathrm{u}, \mathrm{v}, \mathrm{w})$ where $\mathrm{M} \vDash \mathrm{I}(\mathrm{x})$.

Then $M \neq \operatorname{Tallyb}(u) . \operatorname{By}\left(i^{\alpha}\right), \quad M \vDash \exists!v_{0} \in I\left(\operatorname{Tallyb}\left(v_{0}\right) \& A^{\#}\left(y, v_{0}\right)\right)$.

From $M \neq A^{\#}\left(y^{*} a, v\right)$, by (iiia $\left.{ }^{\alpha}\right), M \neq v=v_{0}{ }^{*} b$.
By 3.5(a), M $\neq \exists!w_{0} \in I \operatorname{AddTallyb}\left(u, v_{0}, w_{0}\right)$.
Hence from $M \neq A^{\#}(x, u)$ and hypothesis $M \vDash J(y), M \vDash A^{\#}\left(x^{*} y, w_{0}\right)$.
From $M$ F AddTally(u,v,w), MFAddTally(u,vo*b,w).
But since $M \neq \operatorname{Tallyb}_{\mathrm{b}}(\mathrm{u})$ \& Tallyb $\left(\mathrm{v}_{0}\right)$, from $\mathrm{M} \vDash \operatorname{AddTallyb}_{\mathrm{b}}\left(\mathrm{u}, \mathrm{v}_{0}, \mathrm{w}_{0}\right)$, by 3.4(e),

$$
\mathrm{M} \vDash \operatorname{AddTallyb}\left(\mathrm{u}, \mathrm{v}_{0} * \mathrm{~b}, \mathrm{w}_{0} * \mathrm{~b}\right) .
$$

Then by single-valuedness of AddTally, $\mathrm{M} \vDash \mathrm{w}=\mathrm{w}_{0}{ }^{*} \mathrm{~b}$.
Since $M \neq I\left(y^{*} a\right)$, we have from $M \vDash I(x)$, by ( $i^{\alpha}$ ), that

$$
M \vDash \exists!w^{\prime} \in I A^{\#}\left(x^{*}\left(y^{*} a\right), w^{\prime}\right) .
$$

But $M \vDash x^{*}\left(y^{*} a\right)=\left(x^{*} y\right)^{*}$. Hence $M \vDash A^{\#}\left(\left(x^{*} y\right)^{*} a, w^{\prime}\right)$.
From $M \vDash A^{\#}\left(x^{*} y, w_{0}\right)$, by (iiia $\left.{ }^{\alpha}\right), M \vDash w^{\prime}=w_{0}{ }^{*} b$, and from $M \vDash w=w_{0}{ }^{*} b=w^{\prime}$,

$$
\mathrm{M} \vDash \mathrm{w}=\mathrm{w}^{\prime} .
$$

But then from $M \neq A^{\#}\left(x^{*}\left(y^{*} a\right), w^{\prime}\right), \quad M \vDash A^{\#}\left(x^{*}\left(y^{*} a\right), w\right)$, as required.
Therefore, $\mathrm{M} \vDash \mathrm{J}\left(\mathrm{y}^{*} \mathrm{a}\right)$.
On the other hand, for $y b$, we again have, from $M \vDash I(y)$, that $M \vDash I\left(y^{*} b\right)$.
Assume that $M \neq A^{\#}(x, u) \& A^{\#}\left(y^{*} b, v\right)$ \& $\operatorname{AddTally}(u, v, w)$ where $M \neq I(x)$.
By ( $\mathrm{i}^{\alpha}$ ), $\mathrm{M} \vDash \exists!\mathrm{v}_{0} \in \mathrm{I} \mathrm{A}^{\#}\left(\mathrm{y}, \mathrm{v}_{0}\right)$.
Then from $M \neq A^{\#}\left(y^{*} b, v\right)$, by (iiib $\left.{ }^{\alpha}\right), M \vDash v=v_{0}$.
By 3.5(a), $M \neq \exists!w_{0} \in I$ AddTally $\left(u, v_{0}, w_{0}\right)$. So $M \neq \operatorname{AddTally}\left(u, v_{,} w_{0}\right)$.
Then from $M \neq \operatorname{AddTally}(\mathrm{u}, \mathrm{v}, \mathrm{w})$, by single-valuedness of AddTally,

$$
\mathrm{M} \vDash \mathrm{w}=\mathrm{w}_{0} .
$$

From hypothesis $M \vDash J(y), M \neq A^{\#}\left(x^{*} y, w_{0}\right)$.

Since $M \neq I\left(y^{*} b\right)$, we have from $M \vDash I(x)$, by ( $i^{\alpha}$ ), that

$$
M \neq \exists!w^{\prime} \in I A^{\#}\left(x^{*}\left(y^{*} b\right), w^{\prime}\right) .
$$

But $M \vDash x^{*}\left(y^{*} b\right)=\left(x^{*} y\right)^{*} b$. So $M \vDash A^{\#}\left(\left(x^{*} y\right)^{*} b, w^{\prime}\right)$.
But since $M \neq I\left(x^{*} y\right)$, from $M \vDash A^{\#}\left(x^{*} y, w_{0}\right)$, by (iiib $\left.{ }^{\alpha}\right), M \vDash w^{\prime}=w_{0}{ }^{*}$.
So from $M \vDash w^{\prime}=w_{0}=w, M \vDash w^{\prime}=w$.
But then from $M \neq A^{\#}\left(x^{*}\left(y^{*} b\right), w^{\prime}\right), \quad M \vDash A^{\#}\left(x^{*}\left(y^{*} b\right), w\right)$, as required.
Therefore, $\mathrm{M} \neq \mathrm{J}\left(\mathrm{y}^{*} \mathrm{~b}\right)$, which completes the argument that J is a string form. Then the claim follows immediately.
6.1(a) $\quad \mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{x}_{2} \operatorname{Ex} \rightarrow \forall \mathrm{u}, \mathrm{v}\left(\mathrm{A}^{\#}\left(\mathrm{x}_{2}, \mathrm{u}\right) \& \mathrm{~B}^{\#}\left(\mathrm{x}_{2}, \mathrm{v}\right) \rightarrow \mathrm{Sv} \leq \mathrm{u}\right)$.
(b) $\quad Q T^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{y}) \& \mathrm{z}=\mathrm{bxy} \rightarrow \mathrm{I}^{*}(\mathrm{z})$.
(c) $\mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{u}) \&$ bxy $=$ buv $\rightarrow \mathrm{x}=\mathrm{u} \& \mathrm{y}=\mathrm{v}$.
(d) $\mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \rightarrow\left(\mathrm{x} \subseteq_{\mathrm{p}} \mathrm{a} \leftrightarrow \mathrm{x}=\mathrm{a}\right)$.
(e) $\quad \mathrm{QT}^{+} \vdash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{y}) \& \mathrm{I}^{*}(\mathrm{z}) \rightarrow\left(\mathrm{x} \subseteq_{\mathrm{p}} \mathrm{byz} \leftrightarrow \mathrm{x}=\mathrm{byz} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{y} \mathrm{v} \mathrm{x} \subseteq_{\mathrm{p}} \mathrm{z}\right)$.

Proof: (a) Assume $M \neq A^{\#}\left(x_{2}, u\right) \& B^{\#}\left(x_{2}, v\right)$ where $M \vDash x_{2} E x$ and $M \vDash I^{*}(x)$. Then $M \vDash \exists x_{1} x=x_{1} x_{2} \& x \neq a$, that is, $M \vDash x_{1} B x$.

From $M \neq I^{*}(x), M \vDash J^{*}(x) \& I^{*}\left(x_{1}\right) \& Æ(x)$, and also $M \vDash J^{*}\left(x_{1}\right)$.
$B y\left(i^{\alpha}\right)$ and $\left(\mathrm{i}^{\beta}\right), \quad \mathrm{M} \vDash \exists!\mathrm{v}_{1} \in \mathrm{~J}^{*} \mathrm{~A}^{\#}\left(\mathrm{x}_{1}, \mathrm{v}_{1}\right) \& \exists!\mathrm{w}_{1} \in \mathrm{~J}^{*} \mathrm{~B}^{\#}\left(\mathrm{x}_{1}, \mathrm{w}_{1}\right)$.
Now, from $M \neq Æ(x), M \neq v_{1} \leq W_{1}$.
Also from ( $\mathrm{i}^{\alpha}$ ), $\mathrm{M} \vDash \exists!\mathrm{y} \in \mathrm{J}^{*} \mathrm{~A}^{\#}(\mathrm{x}, \mathrm{y})$, and from ( $\left.\mathrm{i}^{\beta}\right), \mathrm{M} \vDash \exists!\mathrm{z} \in \mathrm{J}^{*} \mathrm{~B}^{\#}(\mathrm{x}, \mathrm{z})$, and we have that $M \neq A^{\#}\left(x_{1}{ }^{*} x_{2}, y\right) \& B^{\#}\left(x_{1}{ }^{*} x_{2}, z\right)$.

On the other hand, since also $\mathrm{M} \vDash \mathrm{J}^{*}\left(\mathrm{x}_{2}\right)$, again by ( $\mathrm{i}^{\alpha}$ ) and ( $\mathrm{i}^{\beta}$ ) we have

$$
\mathrm{M} \vDash \exists!\mathrm{v}_{2} \in \mathrm{~J}^{*} \mathrm{~A}^{\#}\left(\mathrm{x}_{2}, \mathrm{v}_{2}\right) \& \exists!\mathrm{w}_{2} \in \mathrm{~J}^{*} \mathrm{~B}^{\#}\left(\mathrm{x}_{2}, \mathrm{w}_{2}\right) .
$$

By 3.5(a), M $\vDash \exists!\mathrm{z}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{z}_{1}\right)\right.$ \& AddTally $\left.\left(\mathrm{v}_{1}, \mathrm{~V}_{2}, \mathrm{z}_{1}\right)\right)$
and $\mathrm{M} \vDash \exists \mathrm{I}_{\mathrm{z}_{2} \in \mathrm{~J}} \mathrm{~J}^{*}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{z}_{2}\right)\right.$ \& AddTally $\left.\left(\mathrm{w}_{1}, \mathrm{w}_{2}, \mathrm{z}_{2}\right)\right)$.
Now, from $M \neq A^{\#}\left(x_{1}, v_{1}\right) \& A^{\#}\left(x_{2}, v_{2}\right)$, by $5.2(a)$,

$$
\mathrm{M} \neq \mathrm{A}^{\#}\left(\mathrm{x}_{1}{ }^{*} \mathrm{x}_{2}, \mathrm{Z}_{1}\right),
$$

and from $M \neq B^{\#}\left(x_{1}, W_{1}\right) \& B^{\#}\left(x_{2}, W_{2}\right)$, by $5.2(b)$,

$$
\mathrm{M} \vDash \mathrm{~B}^{\#}\left(\mathrm{x}_{1}{ }^{*} \mathrm{x}_{2}, \mathrm{z}_{2}\right) .
$$

On the other hand, from $M \neq Æ(x) \& A^{\#}(x, y) \& B^{\#}(x, z), \quad M \vDash y=S z$.
So from $M \neq x=x_{1} x_{2}, \quad M \neq A^{\#}\left(x, z_{1}\right) \& B^{\#}\left(x, z_{2}\right)$.
Then since $M \neq J^{*}\left(z_{1}\right) \& J^{*}\left(z_{2}\right), \quad M \vDash z_{1}=y \& z_{2}=z$, and from $M \vDash y=S z$,

$$
\mathrm{M} \neq \mathrm{z}_{1}=\mathrm{S} \mathrm{z}_{2} .
$$

Hence from $\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{v}_{1}, \mathrm{v}_{2}, \mathrm{z}_{1}\right)$ \& $\left.\operatorname{AddTally}\left(\mathrm{w}_{1}, \mathrm{w}_{2}, \mathrm{z}_{2}\right)\right) \& \mathrm{v}_{1} \leq \mathrm{w}_{1}$, by 3.5(i), $\mathrm{M} \vDash \mathrm{Sw}_{2} \leq \mathrm{V}_{2}$.

From the uniqueness of $\mathrm{v}_{2}, \mathrm{~W}_{2}$ we have that $\mathrm{M} \vDash \mathrm{Sv} \leq \mathrm{u}$, as required.
(b) Assume $\mathrm{M} \vDash \mathrm{z}=\mathrm{bxy}$ where $\mathrm{M} \vDash \mathrm{I}^{*}(\mathrm{x}) \& \mathrm{I}^{*}(\mathrm{y})$.

Then $M \neq J^{*}(x) \& J^{*}(y)$, and since $J^{*}$ is a string form closed under *, $M \vDash J^{*}(b x y)$.

From $M \vDash I^{*}(x) \& I^{*}(y)$, we have that $M \vDash \notin(x) \& \mathbb{E}(y)$. It suffices to show that $M \neq Æ(z)$.

We proceed to show conditions (c1) and (c2) hold.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), \quad M \vDash \exists!v_{1} \in J^{*} A^{\#}\left(x, v_{1}\right) \& \exists!\mathrm{v}_{2} \in J^{*} \mathrm{~B}^{\#}\left(\mathrm{x}, \mathrm{v}_{2}\right)$
and

$$
M \vDash \exists!w_{1} \in J^{*} A^{\#}\left(y, w_{1}\right) \& \exists!w_{2} \in J^{*} B^{\#}\left(y, w_{2}\right) .
$$

From $M \vDash \notin(x) \& \mathbb{E}(y)$ we have $M \vDash v_{1}=\operatorname{Sv}_{2} \& w_{1}=S w_{2}$.
Again by ( $\mathrm{i}^{\alpha}$ ) and ( $\mathrm{i}^{\beta}$ ) we have

$$
M \vDash \exists!u_{1} \in J^{*} A^{\#}\left(z, u_{1}\right) \& \exists!u_{2} \in J^{*} B^{\#}\left(z, u_{2}\right) .
$$

Then $M \neq A^{\#}\left(b(x y), u_{1}\right) \& B^{\#}\left(b(x y), u_{2}\right)$.
By (iib ${ }^{\alpha}$ ) and (iib ${ }^{\beta}$ ) we have $\quad M \neq A^{\#}(b, b) \& B^{\#}(b, b b)$.
Once again by ( $\mathrm{i}^{\alpha}$ ) and ( $\mathrm{i}^{\mathrm{\beta}}$ ),

$$
M \vDash \exists!v_{3} \in J^{*} A^{\#}\left(x y, v_{3}\right) \& \exists!v_{4} \in J^{*} B^{\#}\left(x y, v_{4}\right) .
$$

By 3.5(a), M $=\exists!\mathrm{p}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tally}_{\mathrm{b}}\left(\mathrm{p}_{1}\right)\right.$ \& AddTally $\left.\left(\mathrm{b}, \mathrm{v}_{3}, \mathrm{p}_{1}\right)\right)$
and $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{2} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{p}_{2}\right)\right.$ \& AddTally $\left.\left(\mathrm{bb}, \mathrm{v}_{4}, \mathrm{p}_{2}\right)\right)$.
Then from $M \neq A^{\#}(b, b)$, by 5.2(a), $\quad M \neq A^{\#}\left(b(x y), p_{1}\right)$,
and from $M \neq B^{\#}(b, b b)$, by $5.2(b), \quad M \neq B^{\#}\left(b(x y), p_{2}\right)$.
By 3.4(c), $\mathrm{M} \neq \operatorname{AddTally}\left(\mathrm{b}, \mathrm{v}_{3}, \mathrm{v}_{3}\right)$, whence from $\mathrm{M}=\operatorname{AddTally}\left(\mathrm{b}, \mathrm{v}_{3}, \mathrm{p}_{1}\right)$, by single-valuedness of Addtally,

$$
\mathrm{M} \neq \mathrm{p}_{1}=\mathrm{v}_{3} .
$$

By 3.5(c), $\mathrm{M} \vDash$ AddTally $\left(\mathrm{bb}, \mathrm{v}_{4}, \mathrm{~Sv}_{4}\right)$, hence from $\mathrm{M} \vDash \operatorname{AddTally(bb,\mathrm {v}_{4},\mathrm {p}_{2})\text {,by}}$ single-valuedness of Addtally,

$$
\mathrm{M} \neq \mathrm{p}_{2}=\mathrm{Sv}_{4}
$$

By 3.5(a), $\mathrm{M} \vDash \exists!\mathrm{q}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tally} \mathrm{~b}_{\mathrm{b}}\left(\mathrm{q}_{1}\right)\right.$ \& AddTally $\left.\left(\mathrm{v}_{1}, \mathrm{w}_{1}, \mathrm{q}_{1}\right)\right)$
and $\quad \mathrm{M} \vDash \exists!\mathrm{q}_{2} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}\left(\mathrm{q}_{2}\right) \& \operatorname{AddTally}\left(\mathrm{v}_{2}, \mathrm{w}_{2}, \mathrm{q}_{2}\right)\right)$.
Then from $M \neq A^{\#}\left(x, v_{1}\right) \& A^{\#}\left(y, w_{1}\right)$, by 5.2(a),

$$
M \neq A^{\#}\left(x y, q_{1}\right),
$$

and from $M \neq B^{\#}\left(x, v_{2}\right) \& B^{\#}\left(y, w_{2}\right)$, by $5.2(b)$,

$$
\mathrm{M} \neq \mathrm{B}^{\#}\left(\mathrm{xy}, \mathrm{p}_{2}\right),
$$

Hence from $M \neq A^{\#}\left(x y, v_{3}\right) \& B^{\#}\left(x y, v_{4}\right)$, by single-valuedness of $A^{\#}$ and $B^{\#}$,

$$
M \neq q_{1}=v_{3} \& q_{2}=v_{4} .
$$

But from $M \neq A^{\#}\left(b(x y), u_{1}\right) \& A^{\#}\left(b(x y), p_{1}\right)$, by single-valuedness of $A^{\#}$,

$$
\mathrm{M} \vDash \mathrm{u}_{1}=\mathrm{p}_{1},
$$

and from $\mathrm{M} \vDash \mathrm{z}=\mathrm{b}(\mathrm{xy}) \& \mathrm{u}_{1}=\mathrm{p}_{1}=\mathrm{v}_{3}=\mathrm{q}_{1}, \quad \mathrm{M} \vDash \mathrm{A}^{\#}\left(\mathrm{z}, \mathrm{q}_{1}\right)$.
Also, from $M \neq B^{\#}\left(b(x y), u_{2}\right) \& B^{\#}\left(b(x y), p_{2}\right)$, by single-valuedness of $B^{\#}$,

$$
\mathrm{M} \vDash \mathrm{u}_{2}=\mathrm{p}_{2} .
$$

Then from $\mathrm{M} \vDash \mathrm{z}=\mathrm{b}(\mathrm{xy}) \& \mathrm{u}_{2}=\mathrm{p}_{2}=\operatorname{Sv}_{4}=\operatorname{Sq}_{2}, \quad \mathrm{M} \vDash \mathrm{B}^{\#}\left(\mathrm{z}, \mathrm{Sq}_{2}\right)$.
Now, from $M \neq \operatorname{AddTally}\left(\mathrm{v}_{1}, \mathrm{w}_{1}, \mathrm{q}_{1}\right) \& \mathrm{v}_{1}=\mathrm{Sv}_{2} \& \mathrm{w}_{1}=\mathrm{Sw}_{2} \quad$ we have

$$
\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{~Sv}_{2}, \mathrm{Sw}_{2}, \mathrm{q}_{1}\right) .
$$

On the other hand, from $\mathrm{M}=\operatorname{AddTally}\left(\mathrm{v}_{2}, \mathrm{w}_{2}, \mathrm{q}_{2}\right)$, by $3.5(\mathrm{~d})$,

$$
\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{~Sv}_{2}, \mathrm{~W}_{2}, \mathrm{Sq}_{2}\right) .
$$

By 3.4(e), M = AddTally ( $\mathrm{Sv}_{2}, \mathrm{Sw}_{2}, \mathrm{SSq}_{2}$ ).
But then from $\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{Sv}_{2}, \mathrm{Sw}_{2}, \mathrm{q}_{1}\right)$ by single-valuedness of Addtally,

$$
\mathrm{M} \vDash \mathrm{q}_{1}=\mathrm{SSq}_{2} .
$$

Since $M \vDash A^{\#}\left(z, q_{1}\right) \& B^{\#}\left(z, S q_{2}\right)$, this suffices to establish (c1).

For (c2), assume $M \vDash u B z \& A^{\#}\left(u, v_{1}\right) \& B^{\#}\left(u, v_{2}\right)$.
Then $\mathrm{M} \vDash \mathrm{uBb}(\mathrm{xy})$, and by 3.7(c),

$$
M \vDash u=b v u B b x v u=b x \vee \exists y_{1}\left(y_{1} B u \& u=b x y_{1}\right) .
$$

(1) $\mathrm{M} \vDash \mathrm{u}=\mathrm{b}$.
$B y\left(i b^{\alpha}\right)$ and (iib $\left.{ }^{\beta}\right), M \neq A^{\#}(b, b) \& B^{\#}(b, b b)$.
Then $M \neq A^{\#}(u, b) \& B^{\#}(u, b b)$, and by single-valuedness of $A^{\#}$ and $B^{\#}$,

$$
\mathrm{M} \vDash \mathrm{v}_{1}=\mathrm{b} \& \mathrm{v}_{2}=\mathrm{bb} .
$$

But then $M \vDash \mathrm{v}_{1} \leq \mathrm{v}_{2}$, as required.
(2) M F uBbx.

Then $M \vDash \exists x_{1}\left(x_{1} B x \& u=b x_{1}\right)$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), \quad M \vDash \exists!u_{1} \in J^{*} A^{\#}\left(\mathrm{x}_{1}, \mathrm{u}_{1}\right) \& \exists!\mathrm{u}_{2} \in \mathrm{~J}^{*} \mathrm{~B}^{\#}\left(\mathrm{x}_{1}, \mathrm{u}_{2}\right)$.
From $M \neq \mathbb{E}(x), \quad M \neq u_{1} \leq u_{2}$.
By (iib ${ }^{\alpha}$ ) and ( $\mathrm{iib}{ }^{\beta}$ ), $\quad \mathrm{M} \neq \mathrm{A}^{\#}(\mathrm{~b}, \mathrm{~b}) \& \mathrm{~B}^{\#}(\mathrm{~b}, \mathrm{bb})$.
By 3.5(a), $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{p}_{1}\right)\right.$ \& AddTally $\left.\left(\mathrm{b}, \mathrm{u}_{1}, \mathrm{p}_{1}\right)\right)$
and $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{2} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}_{\mathrm{b}}\left(\mathrm{p}_{2}\right)\right.$ \& AddTally $\left.\left(\mathrm{bb}, \mathrm{u}_{2}, \mathrm{p}_{2}\right)\right)$,
Then by $5.2(a), \quad M \neq A^{\#}\left(b x_{1}, p_{1}\right)$, and by 5.2(b), $\quad M \neq B^{\#}\left(\mathrm{bx}_{1}, \mathrm{p}_{2}\right)$.
By 3.4(c), $\mathrm{M} \vDash$ AddTally $\left(\mathrm{b}, \mathrm{u}_{1}, \mathrm{u}_{1}\right)$,
By 3.5(c), M \& AddTally(bb, $\mathrm{u}_{2}, \mathrm{Su}_{2}$ ).
Hence from $\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{b}, \mathrm{u}_{1}, \mathrm{p}_{1}\right) \& \operatorname{AddTally}\left(\mathrm{bb}, \mathrm{u}_{2}, \mathrm{p}_{2}\right)$,
by single-valuedness of Addtally, $\quad \mathrm{M} \vDash \mathrm{p}_{1}=\mathrm{u}_{1} \& \mathrm{p}_{2}=\mathrm{Su}_{2}$.
Now, from $M \vDash u=b x_{1} \& A^{\#}\left(u, v_{1}\right) \& B^{\#}\left(u, v_{2}\right)$, we have

$$
M \neq A^{\#}\left(b x_{1}, v_{1}\right) \& B^{\#}\left(b x_{1}, v_{2}\right) .
$$

Then from $M \neq A^{\#}\left(b x_{1}, p_{1}\right) \& B^{\#}\left(b x_{1}, p_{2}\right)$, by single-valuedness of $A^{\#}$ and $B^{\#}$,

$$
\mathrm{M} \vDash \mathrm{~V}_{1}=\mathrm{p}_{1} \& \mathrm{~V}_{2}=\mathrm{p}_{2},
$$

whence $\mathrm{M} \vDash \mathrm{v}_{1}=\mathrm{u}_{1} \& \mathrm{v}_{2}=\mathrm{Su}_{2}$.
But then from $M \vDash u_{1} \leq u_{2}$ we have that $M \vDash v_{1}=u_{1}<S u_{2}=v_{2}$.
By single-valuedness of $\mathrm{A}^{\#}$ and $\mathrm{B}^{\#}$, this suffices to establish (c2) in this case.
(3) $M \vDash u=b x$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), \quad M \vDash \exists!u_{1} \in J^{*} A^{\#}\left(x, u_{1}\right) \& \exists!u_{2} \in J^{*} B^{\#}\left(x, u_{2}\right)$.
From $M \neq Æ(x), \quad M \neq u_{1}=S_{2}$.
On the other hand, by ( $\mathrm{iib}^{\alpha}$ ) and ( $\mathrm{iib}^{\beta}$ ),

$$
M \neq A^{\#}(b, b) \& B^{\#}(b, b b) .
$$

By 3.5(a), $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tally} \mathrm{~b}_{\mathrm{b}}\left(\mathrm{p}_{1}\right)\right.$ \& AddTally $\left.\left(\mathrm{b}, \mathrm{u}_{1}, \mathrm{p}_{1}\right)\right)$
and $\quad \mathrm{M} \vDash \exists!\mathrm{p}_{2} \in \mathrm{~J}^{*}\left(\operatorname{Tally}_{\mathrm{b}}\left(\mathrm{p}_{2}\right)\right.$ \& AddTally $\left.\left(\mathrm{bb}, \mathrm{u}_{2}, \mathrm{p}_{2}\right)\right)$.
Reasoning exactly as in (2) with bx in place of $\mathrm{bx}_{1}$ we obtain

$$
\mathrm{M} \vDash \mathrm{v}_{1}=\mathrm{u}_{1}=\mathrm{Su}_{2}=\mathrm{v}_{2} .
$$

By single-valuedness of $A^{\#}$ and $B^{\#}$, this suffices.
(4) $\mathrm{M} \vDash \exists \mathrm{y}_{1}\left(\mathrm{y}_{1} \mathrm{By} \& \mathrm{u}=\mathrm{bxy}_{1}\right)$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), \quad M \neq \exists!w_{1} \in J^{*} A^{\#}\left(y_{1}, w_{1}\right) \& \exists!w_{2} \in J^{*} B^{\#}\left(y_{1}, w_{2}\right)$.
From $M \neq Æ(y), \quad M \neq w_{1} \leq w_{2}$.
Also by ( $\mathrm{i}^{\alpha}$ ) and ( $\mathrm{i}^{\beta}$ ), $\mathrm{M} \vDash \exists!\mathrm{u}_{1} \in \mathrm{~J}^{*} \mathrm{~A}^{\#}\left(\mathrm{x}, \mathrm{u}_{1}\right) \& \exists!\mathrm{u}_{2} \in \mathrm{~J}^{*} \mathrm{~B}^{\#}\left(\mathrm{x}, \mathrm{u}_{2}\right)$.
From $M \neq Æ(x), \quad M \neq u_{1}=S u_{2}$.
By 3.5(a), $\quad \mathrm{M} \vDash \exists!\mathrm{q}_{1} \in \mathrm{~J}^{*}\left(\operatorname{Tallyb}\left(\mathrm{q}_{1}\right) \& \operatorname{AddTally}\left(\mathrm{u}_{1}, \mathrm{w}_{1}, \mathrm{q}_{1}\right)\right)$
and $\quad \mathrm{M} \vDash \exists!\mathrm{q}_{2} \in \mathrm{~J}^{*}\left(\operatorname{Tally} \mathrm{~b}_{\mathrm{b}}\left(\mathrm{q}_{2}\right)\right.$ \& AddTally $\left(\mathrm{u}_{2}, \mathrm{w}_{2}, \mathrm{q}_{2}\right)$ ),
We then reason as in (1) with $u$ in place of $z$ and $y_{1}$ in place of $y$ that

$$
M \vDash A^{\#}\left(u, q_{1}\right) \& B^{\#}\left(u, S q_{2}\right) .
$$

From $M \neq \operatorname{AddTally}\left(\mathrm{u}_{1}, \mathrm{w}_{1}, \mathrm{q}_{1}\right) \& \mathrm{u}_{1}=\mathrm{Su}_{2}$,

$$
\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{Su}_{2}, \mathrm{w}_{1}, \mathrm{q}_{1}\right) .
$$


whence from $M \neq \mathrm{w}_{1} \leq \mathrm{w}_{2}$, by $3.5(\mathrm{~b}), \mathrm{M} \vDash \mathrm{q}_{1} \leq \mathrm{q}_{3}$.
From $\mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{u}_{2}, \mathrm{w}_{2}, \mathrm{q}_{2}\right)$ ), by $3.5(\mathrm{~d}), \mathrm{M} \vDash \operatorname{AddTally}\left(\mathrm{Su}_{2}, \mathrm{w}_{2}, \mathrm{Sq}_{2}\right)$.
By single-valuedness of Addtally, $\quad \mathrm{M} \vDash \mathrm{q}_{3}=\mathrm{Sq}_{2}$.
Hence $\mathrm{M}=\mathrm{q}_{1} \leq \mathrm{Sq}_{2}$.
Since $M \neq A^{\#}\left(u, q_{1}\right) \& B^{\#}\left(u, S q_{2}\right)$, this suffices to establish (c2) given single-valuedness of $\mathrm{A}^{\#}$ and $\mathrm{B}^{\#}$. This completes the argument for $\mathrm{M} \vDash \notin(z)$.
(c) Assume $M \neq b x y=b u v$ where $M \vDash I^{*}(x) \& I^{*}(u)$.

Then $M \neq J^{*}(x) \& J^{*}(u) \& Æ(x) \& Æ(u)$. By (QT3), $M \vDash x y=u v$.
So $M \neq x B(x y) \& u B(x y)$, and by 3.7(a),

$$
M \neq(x=u \& y=v) v x B u v u B x .
$$

Suppose that
(1) $M \vDash x B u$.
$B y\left(i^{\alpha}\right)$ and $\left(i^{\beta}\right), M \vDash \exists!x_{1} \in J^{*} A^{\#}\left(x, x_{1}\right) \& \exists!x_{2} \in J^{*} B^{\#}\left(x, x_{2}\right)$.
From $M \neq Æ(u), M \neq x_{1} \leq x_{2}$.
On the other hand, from $M \neq Æ(x), \quad M \neq x_{1}=S x_{2}$.
But then $\mathrm{M} \vDash \mathrm{x}_{1} \leq \mathrm{x}_{2}<\mathrm{Sx}_{2}=\mathrm{x}_{1}$, contradicting $\mathrm{M} \vDash \mathrm{I}_{0}\left(\mathrm{x}_{1}\right)$.
Hence (1) is ruled out.
(2) $\mathrm{M} \vDash \mathrm{uBx}$.

Ruled out exactly analogously to (a).

Therefore, $\mathrm{M} \vDash \mathrm{x}=\mathrm{u}$ \& $\mathrm{y}=\mathrm{v}$, as required.
(d) is immediate from the definition of $\subseteq_{\mathrm{p}}$ by (QT2). .
7.3. $\mathrm{WQT}^{*}$ is locally finitely satisfiable.

Proof: Let $S$ be a finite set of axioms of $W_{Q T}{ }^{*}$.

For variable-free terms $s, t$ of $\mathcal{L}_{\text {QT, }} \mathrm{E}^{*}$, let $\mathrm{s} \sim \mathrm{t} \Leftrightarrow \operatorname{val}(\mathrm{s})=\operatorname{val}(\mathrm{t})$, that is, if $\mathrm{s}, \mathrm{t}$ represent the same string. E.g.,

$$
a^{*}\left(b^{*}\left(a^{*} b\right)\right) \sim a^{*}\left(\left(b^{*} a\right)^{*} b\right) \sim\left(a^{*} b\right)^{*}\left(a^{*} b\right) \sim\left(\left(a^{*} b\right)^{*} a\right)^{*} b \sim\left(a^{*}\left(b^{*} a\right)\right)^{*} b
$$

$\sim$ being an equivalence relation between terms, we let $[\mathrm{t}]=\{\mathrm{s} \mid \mathrm{t} \sim \mathrm{s}\}$.

Now, let $D=\left\{a, b, t_{1}, \ldots, t_{n}\right\}$, where $t_{1}, \ldots, t_{n}$ are all variable-free terms occurring in $S$. We let $D^{*}=\{[t] \mid t \in D\}$. Since the equivalence classes of terms with respect to $\sim$ can be identified with strings, $D^{*}$ consists of $a, b$ and the strings represented by terms occurring in S . We take $\mathrm{D}^{*}$ to be the domain of the model, $M$, and let the letters $\mathrm{a}, \mathrm{b}$ denote [a] and [b], resp. .

Let $\mathrm{f}^{m}: \mathrm{D}^{*} \times \mathrm{D}^{*} \rightarrow \mathrm{D}^{*}$, where, for any $[\mathrm{u}],[\mathrm{v}] \in \mathrm{D}^{*}$,
$\mathrm{f}^{M}([\mathrm{u}],[\mathrm{v}])=[\mathrm{t}]$ if for some $\mathrm{t} \in \mathrm{D}, \mathrm{t} \sim\left(\mathrm{u}^{*} \mathrm{v}\right)$, and $\mathrm{f}^{M}([\mathrm{u}],[\mathrm{v}])=\mathrm{b}$ otherwise, interpret the binary operation *.

Let $\mathrm{R}^{M} \subseteq \mathrm{D}^{*} \times \mathrm{D}^{*}$, where, for any $[\mathrm{u}],[\mathrm{v}] \in \mathrm{D}^{*}$,

$$
\mathrm{R}^{M}([\mathrm{u}],[\mathrm{v}]) \Leftrightarrow \text { for some } \mathrm{s}, \mathrm{t} \in \Sigma^{\mathrm{t}}, \mathrm{u} \sim \mathrm{~s} \text { and } \mathrm{v} \sim \mathrm{t} \text { and } \mathrm{s} \text { is a subterm of } \mathrm{t} \text {, }
$$ interpret the relational symbol ㄷ..

Suppose now that $\left[\mathrm{s}_{1}\right]=\left[\mathrm{t}_{1}\right]$ and $\left[\mathrm{s}_{2}\right]=\left[\mathrm{t}_{2}\right]$, where $\left.\left[\mathrm{s}_{1}\right],\left[\mathrm{s}_{2}\right],\left[\mathrm{t}_{1}\right], \mathrm{t}_{2}\right] \in \mathrm{D}^{*}$. Then $s_{1} \sim t_{1}$ and $s_{2} \sim t_{2}$, whence $\left(s_{1}{ }^{*} s_{2}\right) \sim\left(t_{1}{ }^{*} t_{2}\right)$. Suppose further that for some term $\mathrm{s} \in \mathrm{D}, \mathrm{s} \sim\left(\mathrm{s}_{1}{ }^{*} \mathrm{~s}_{2}\right)$. Then $\mathrm{f}^{M}\left(\left[\mathrm{~s}_{1}\right],\left[\mathrm{s}_{2}\right]\right)=[\mathrm{s}]$. But $\mathrm{s} \sim\left(\mathrm{s}_{1}{ }^{*} \mathrm{~s}_{2}\right) \sim\left(\mathrm{t}_{1}{ }^{*} \mathrm{t}_{2}\right)$. Hence $\mathrm{f}^{M}\left(\left[\mathrm{t}_{1}\right],\left[\mathrm{t}_{2}\right]\right)=[\mathrm{s}]$, and we have $\mathrm{f}^{M}\left(\left[\mathrm{~s}_{1}\right],\left[\mathrm{s}_{2}\right]\right)=\mathrm{f}^{M}\left(\left[\mathrm{t}_{1}\right],\left[\mathrm{t}_{2}\right]\right)$. Suppose, on the other hand, that for no term $\mathrm{s} \in \mathrm{D}, \mathrm{s} \sim\left(\mathrm{s}_{1}{ }^{*} \mathrm{~s}_{2}\right)$. Then $\mathrm{f}^{M}\left(\left[\mathrm{~s}_{1}\right],\left[\mathrm{s}_{2}\right]\right)=[\mathrm{b}]$. But $\left(s_{1}{ }^{*} s_{2}\right) \sim\left(t_{1}{ }^{*} t_{2}\right)$, so for no term $s \in D, s \sim\left(t_{1}{ }^{*} t_{2}\right)$. Hence $f^{M}\left(\left[t_{1}\right],\left[t_{2}\right]\right)=[b]$, and again $\mathrm{f}^{M}\left(\left[\mathrm{~s}_{1}\right],\left[\mathrm{s}_{2}\right]\right)=\mathrm{f}^{M}\left(\left[\mathrm{t}_{1}\right],\left[\mathrm{t}_{2}\right]\right)$. Under the same hypothesis $\left[\mathrm{s}_{1}\right]=\left[\mathrm{t}_{1}\right]$ and $\left[\mathrm{s}_{2}\right]=\left[\mathrm{t}_{2}\right]$, we have that
$\mathrm{R}^{M}\left(\left[\mathrm{~s}_{1}\right],\left[\mathrm{s}_{2}\right]\right) \Leftrightarrow$ for some terms $\mathrm{u}_{1}, \mathrm{u}_{2} \in \Sigma^{\mathrm{\tau}}, \mathrm{~s}_{1} \sim \mathrm{u}_{1}$ and $\mathrm{s}_{2} \sim \mathrm{u}_{2}$ and
$u_{1}$ is a subterm of $u_{2} \Leftrightarrow$ for some terms $u_{1}, u_{2} \in \Sigma^{\tau}, t_{1} \sim u_{1}$ and $t_{2} \sim u_{2}$ and
$\mathrm{u}_{1}$ is a subterm of $\mathrm{u}_{2} \Leftrightarrow \mathrm{R}^{M}\left(\left[\mathrm{t}_{1}\right],\left[\mathrm{t}_{2}\right]\right)$.

Thus the definitions of $\mathrm{f}^{M}$ and $\mathrm{R}^{M}$ do not depend on the choice of terms $\mathrm{s}, \mathrm{t}$.

A straightforward induction on the complexity of $\mathcal{L}_{\mathrm{C}, \varsigma^{*}-\text {-terms }}$ shows that if t is among the terms in D , then its interpretation $\mathrm{t}^{M}$ is [ t$]$. It is then immediate that the resulting model $M$ satisfies all of the axioms in the finite set S . $\quad$

