DETERMINACY OF SCHMIDT'S GAME AND OTHER INTERSECTION GAMES

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Schmidt's game, and other similar intersection games have played an important role in recent years in applications to number theory, dynamics, and Diophantine approximation theory. These games are real games, that is, games in which the players make moves from a complete separable metric space. The determinacy of these games trivially follows from the axiom of determinacy for real games, ADR, which is a much stronger axiom than that asserting all integer games are determined, AD. One of our main results is a general theorem which under the hypothesis AD implies the determinacy of intersection games which have a property allowing strategies to be simplified. In particular, we show that Schmidt's (α, β, ρ) game on R is determined from AD alone, but on Rn for $n \geq 3$ we show that AD does not imply the determinacy of this game. We then give an application of simple strategies and prove that the winning player in Schmidt's (α, β, ρ) game on R has a winning positional strategy, without appealing to the axiom of choice. We also prove several other results specifically related to the determinacy of Schmidt's game. These results highlight the obstacles in obtaining the determinacy of Schmidt's game from AD.

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CHAPTER 1

GAMES

1.1. Background

In this section, we give some historical context and background for the study of mathematical games and their applications.

There are many ways to formalize infinite games and strategies as mathematical objects, and theorems about games and strategies can, of course, be translated into language using only primitive concepts like functions or sets. Somewhat surprisingly, using the intuition from viewing these objects as games and strategies can be useful in proving theorems. For example, the periodicity theorems of descriptive set theory (see [19]). One of the most useful application of the study of games and their determinacy to mathematics in general is to prove dichotomy theorems. If one can prove that player $\boldsymbol{I}$ having a winning strategy implies that some statement A holds, while player $\boldsymbol{II}$ having a winning strategy implies that B holds, then, provided the game is determined, there is a dichotomy between whether A holds or B holds. A prime example of this is Gowers' use in [7] of a game to prove a dichotomy for Banach spaces between containing an unconditional basic sequence or a hereditarily indecomposable subspace.

The first instance of an infinite game appeared in the Scottish Book in the early twentieth century (see [27] for a description of the Scottish Book and its rather interesting historical context) as problem 43. This game is now known as the Banach-Mazur game, and is played as follows: A target set $A \subseteq \mathbb{R}$ is dealt, and two players (named $\boldsymbol{I}$ and $\boldsymbol{II}$) alternate in specifying nontrivial closed intervals in $\mathbb{R}$ which must be nested. After an infinite number of moves have been played, the intersection of these intervals is computed, and either this intersection meets A and player $\boldsymbol{I}$ is declared the winner, or is contained in A^c, in which case player $\boldsymbol{II}$ is declared the winner. Mazur proved that both players can avoid any meager subset of $\mathbb{R}$. By this we mean that if A is meager, player $\boldsymbol{II}$ has a winning strategy, and if A^c is meager then player $\boldsymbol{I}$ has a winning strategy. The problem posed in the Scottish book was essentially the converse: Mazur conjectured that if player ***I*** has a winning strategy, then A must be comeager in some interval, and likewise if player ***II*** has a winning strategy, then A^c must be comeager in $\mathbb{R}$. In 1935, Banach proved Mazur's conjecture (see [23]). This Banach-Mazur game can be generalized to arbitrary complete metric spaces by replacing the interval moves with closed balls.

Variations of this game were also included in the Scottish Book. Ulam presented a game in which the players alternate playing only zeroes or ones, building the binary expansion of a real number; the winner was decided by whether the real number built was in a fixed target set. Ulam's game turns out to be remarkably general, as should be apparent by the definition we give for a general game in the following paragraphs. Included also was another game defined by Banach in which players alternate specifying positive real numbers, building a decreasing sequence. After an infinite sequence has been played, the sum is computed and the winner is decided by whether this sum is in a fixed target set (player ***II*** wins by default if the sum diverges). This *Banach game* was studied in detail by Freiling [4] and Becker [2], and much of the work presented here is built upon those ideas. There are many interesting questions regarding these and other games, including both questions about the games themselves, and about their applications to proving theorems about the underlying target sets.

We will give the definition for a game in Definition 3, but first we fix some notation and definitions upon which it relies. This definition will encompass all of the games described above and will be our working definition. We will, in Chapter 4 restrict to a particular class of games to which our main results apply.

We let $\omega = \mathbb{N} = \{0, 1, 2, \dots\}$ denote the natural numbers.

Definition 1. Let X be a set. Let $X^{<\omega}$ denote the set of all finite sequences from X, and X^ω denote the set of all infinite sequences from X. We use $\subseteq$ to denote the initial segment relation on $X^{<\omega} \cup X^\omega$. In other words for $u, v \in X^{<\omega}$ and $x \in X^\omega$, by $u \subseteq v$ we mean that u is an initial segment of v and by $u \subseteq x$ we mean u is an initial segment of x.

For $u \in X^{<\omega}$, we denote the length of u as $|u|$. We use $\restriction$ to denote restriction, so that $u\restriction n$ is the initial segment of u with length n (or just u if $|u| \leq n$). Finally, we use the symbol $^\frown$ to denote concatenation: More specifically, for $u, v \in X^{<\omega}$, $m \in X$, and $x \in X^\omega$, we let

- $u^\frown m = (u_0, \dots u_{|u|-1}, m)$
- $u^\frown v = (u_0, \dots u_{|u|-1}, v_0, \dots v_{|u|-1})$
- and $u^\frown x = (u_0, \dots, u_{|u|-1}, x_0, \dots)$.

For $u \in X^{<\omega}$, we let

$$N_u = \{x \in X^\omega \colon u \subseteq x\}$$

and for $x \in X^\omega$ let $\overline{x} \in X^\omega$ be defined by $(\overline{x})(n) = x(n+1)$

Definition 2. We call $T \subseteq X^{<\omega}$ a *tree on* X if T is nonempty and closed under initial segments, that is for any $u \in T$ and $v \in X^{<\omega}$, if $v \subseteq u$ then $v \in T$. For a tree T, let $[T]$ denote the *body* of T (the set of *branches through* T), in other words, we let

$$[T] = \{x \in X^\omega \colon \forall n \in \omega,\ x\restriction n \in T\}.$$

For $p \in X^{<\omega}$, we denote by T_p the tree

$$T_p = \{q \in T \colon q \subseteq p \vee p \subseteq q\}$$

A node u of T is *terminal* if there is no $v \in T$ which properly extends it.

Definition 3. Let X be a set, let $T \subseteq X^{<\omega}$ be a tree consisting of finite sequences of elements of X, and let $A \subseteq X^\omega$ be any subset of the infinite sequences from X. We define the *game* $\mathcal{G}(A, T)$ of two players with *rules* T and *payoff* A. The players alternate playing *moves* from X to build an infinite sequence. We call each sequence $p \in T$ of moves a *position*. If any player makes a move to a position outside of T, then that player immediately loses. If both players follow the rules and produce a sequence which is a branch through T, then player ***I*** is declared to be the winner if the sequence is in A, and otherwise ***II*** is declared the winner. Formally, we identify $\mathcal{G}(A, T)$ with the set of infinite sequences which are wins for player $\boldsymbol{I}$:

$$\mathcal{G}(A,T) = \{x \in X^\omega \colon x \in [T] \cap A \vee (\exists n \in \omega\ \forall i < 2n\ x{\restriction}i \in T \wedge x{\restriction}2n \notin T)\}$$

We often say that $\mathcal{G}(A,T)$ is a *game on* X or a *game on* T.

A *strategy for player* $\mathbf{P}$ for a game with rules T is a subtree $\sigma \subseteq T$ so that

(1) for every position $p \in \sigma$ which is not terminal in T at which it is $\mathbf{P}$'s turn to move, there is exactly one move m so that $p^\frown m \in \sigma$

(2) and for every position $p \in \sigma$ at which it is not $\mathbf{P}$'s turn to move, then $p^\frown m \in \sigma \Leftrightarrow p^\frown m \in T$.

For σ a strategy for player $\mathbf{P}$, and p a position with $\mathbf{P}$ to move, we denote by $\sigma(p)$ the unique extension of p in σ.

If σ is a strategy for $\boldsymbol{I}$, and $\vec{z} = (x_1, x_3, \dots)$ is a sequence of moves for $\boldsymbol{II}$, we write $\sigma * \vec{z}$ to denote the corresponding run $(x_0, x_1, x_2, x_3, \dots)$ where $x_{2n} = \sigma(x{\restriction}2n)$. We likewise define $\vec{z} * \tau$ for τ a strategy for $\boldsymbol{II}$ and $\vec{z} = (x_0, x_2, \dots)$ a sequence of moves for $\boldsymbol{I}$. If σ, τ are strategies for $\boldsymbol{I}$ and $\boldsymbol{II}$ respectively, then we let $\sigma * \tau$ denote the run $(x_0, x_1, \dots)$ where $x_{2n} = \sigma(x{\restriction}2n)$ and $x_{2n+1} = \tau(x{\restriction}2n+1)$ for all n.

Given a position p in T, and a strategy σ for a player, we say that p is *consistent with* σ simply if $p \in \sigma$. Likewise if $x \in [T]$, we say x is consistent with σ if $x \in [\sigma]$. A strategy is said to be a *winning strategy* for $\mathbf{P}$ if whenever their play is always consistent with the strategy, then they must win.

A game $\mathcal{G}(A,T)$ is said to be *determined* if one of the players has a winning strategy.

Remark 1. If $G \subseteq X^\omega$, then $G = \mathcal{G}(G, X^{<\omega})$, and so we will call such a G a game as well. In fact, we will make no distinction (except in notation) between sets of infinite sequences and the games with rules defined in Definition 3, since $\mathcal{G}(A,T) = \mathcal{G}(\mathcal{G}(A,T), X^{<\omega})$

1.2. Determinacy Theorems

Infinite games were studied in the generality of Definition 3 first by Gale and Stewart in 1953 [6]. They proved the following theorem.

Theorem (Gale and Stewart [6]). *Let $\mathcal{G}(A,T)$ be a game on a set X. If A is closed in the topology on X^ω which is the product of the discrete topology on X, then $\mathcal{G}(A,T)$ is determined.*

One can show that the determinacy of all closed games implies the determinacy of all open games. Closed and open games are characterized by the property that one player wins only if they win in a finite number of moves, while the other player wins if and only if they avoid losing at every finite position. This initiated a program to prove determinacy for abstract games based on their topological complexity, and some progress was made in the subsequent years, for instance:

Theorem (Wolfe [28]). *Let $\mathcal{G}(A,T)$ be a game on a set X. If A is the countable union of closed sets in X^ω then $\mathcal{G}(A,T)$ is determined.*

Determinacy was proved for all games which are the countable intersection of countable unions of closed games ($\mathbf{\Pi}^0_3$ games) by Davis [3] in 1964, although the proofs required more complicated arguments.

Connections with set theory became apparent when in 1970, Martin [10] proved that if a measurable cardinal exists, then all *analytic* games on ω are determined, that is games $G \subseteq \omega^\omega$ which are the continuous image of a closed set in ω^ω. Using the ideas of Martin, the a new proof of determinacy theorem of Davis [3] was found, and Paris [24] was able to prove the determinacy of $\mathbf{\Sigma}^0_4$ games.

In [5], Friedman proved that to extend these determinacy results to games of higher topological complexity, one must make necessary use of set theory, in particular one must iterate the power set operation a number of times corresponding to the level of the Borel hierarchy (see Chapter 2) for which determinacy is being proved. In 1975, Martin proved the remarkable result

Theorem (Martin [11], [12]). *Let $\mathcal{G}(A,T)$ be a game on a set X. If A is a Borel subset of X^ω, then $\mathcal{G}(A,T)$ is determined.*

There are, however, many games which are not determined. Gale and Stewart, also in

[6], proved that there exist nondetermined games on $2 = \{0, 1\}$, by diagonalizing against all possible strategies. In fact, for any tree T of rules in which each player must infinitely often have at least two legal moves, if $B \subseteq X^\omega$ is a Bernstein set (a set which has the property that neither B nor its compliment contain an uncountable closed set, see [23]), then $\mathcal{G}(B, T)$ is not determined.

It should be noted that all the determinacy results just discussed require the axiom of choice, AC for their full generality (for games on all sets X). In fact, the theorem of Gale and Stewart above is equivalent to AC. To see this, consider the one round game in which player $\boldsymbol{I}$ plays a nonempty set from some collection and player $\boldsymbol{II}$ must pick an element from the set. This is a closed (in fact, closed *and* open) game, and clearly player $\boldsymbol{I}$ can't have a winning strategy, but a winning strategy for player $\boldsymbol{II}$ is a choice function. Weaker theorems giving the *quasideterminacy* (essentially determinacy with multi-valued strategies) were proved in 1994 in the absence of AC by Hurkens [8].

CHAPTER 2

POINTCLASSES AND SUSLIN SETS

In this chapter, we explore the theory of pointclasses and the background of descriptive set theory which is necessary to state our main results. Here we give statements of several important classical theorems from the subject, and also present a proof of one of these facts which we will make use of later in Chapter 4 as this is one of the most important examples of an application of games to descriptive set theory.

Definition 4. A pointclass $\mathbf{\Gamma}$ is a class of subsets of Polish spaces so that for any continuous function $f\colon X \to Y$, if $B \in \mathbf{\Gamma}\restriction Y$ then $f^{-1}(B) \in \mathbf{\Gamma}\restriction X$.

Note that since $\mathbf{\Gamma}$ needs to be closed under continuous preimages under functions from arbitrary Polish spaces, a pointclass $\mathbf{\Gamma}$ is not a set, but is a proper class binary relation, with $(A, X) \in \mathbf{\Gamma} \Leftrightarrow A \in \mathbf{\Gamma}\restriction X$

Example 1 (The Borel hierarchy). Let $\mathbf{\Sigma}^0_1\restriction X$ denote the open sets in X and $\mathbf{\Pi}^0_1\restriction X$ denote the closed sets in X. For each $1 < \alpha < \omega_1$, define

$$\mathbf{\Pi}^0_\alpha\restriction X = \left\{X \setminus A\colon A \in \mathbf{\Sigma}^0_\alpha\restriction X\right\}$$

$$\mathbf{\Sigma}^0_\alpha\restriction X = \left\{\bigcup_{n\in\omega} A_n\colon \forall n\ \exists\beta_n < \alpha\ A_n \in \mathbf{\Pi}^0_{\beta_n}\right\}$$

and for each $\alpha < \omega_1$, define

$$\mathbf{\Delta}^0_\alpha\restriction X = \mathbf{\Sigma}^0_\alpha\restriction X \cap \mathbf{\Pi}^0_\alpha\restriction X$$

One can show by induction that for each α, $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$ are pointclasses. It is also easy to see that the Borel subsets of X coincide with those sets which lie in $\mathbf{\Sigma}^0_\alpha$ for some α

2.1. Suslin Sets

Definition 5. A set $A \subseteq \omega^\omega$ is *κ-Suslin* if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ so that

$$A = \mathfrak{p}[T] = \{x \in \omega^\omega\colon \exists f \in \kappa^\omega\ (x, f) \in [T]\}.$$

We say T is a *Suslin representation* for A. If A is κ-Suslin, then we say $X \setminus A$ is *co-κ-Suslin.* We say A is *Suslin* if it is κ-Suslin for some κ, and *co-Suslin* if it is co-κ-Suslin for some κ.

2.2. Periodicity and Uniformization

In this section, we state the definitions and theorems about Suslin sets which we make use of in Chapter 4.

Definition 6. If $\mathbf{\Gamma}$ is a pointclass, then define

$$\check{\mathbf{\Gamma}}\restriction X = \{X \setminus A \colon A \in \mathbf{\Gamma}\}$$

$$\exists^{\omega^\omega}\mathbf{\Gamma}\restriction X = \left\{\exists^{\omega^\omega} A \colon A \in \mathbf{\Gamma}\restriction X \times \omega^\omega\right\}$$

$$\forall^{\omega^\omega}\mathbf{\Gamma}\restriction X = \left\{\forall^{\omega^\omega} A \colon A \in \mathbf{\Gamma}\restriction X \times \omega^\omega\right\}$$

where

$$x \in \left(\exists^{\omega^\omega} A\right) \Leftrightarrow \exists y \ (x, y) \in A$$

and likewise for $\forall$.

Theorem (Moschovakis, see [19] Theorem 6C.2). *If every set in $\mathbf{\Gamma}$ is Suslin, then every set in $\exists^{\omega^\omega}\mathbf{\Gamma}$ is Suslin.*

The next theorem is a fundamental result in descriptive set theory. We state it in the form in which we will make use of it. We give a proof of this theorem, both for the sake of completeness and as an example of the applications of games and determinacy.

Theorem (Moschovakis, see [19] Theorem 6C.3). *If every set in $\mathbf{\Gamma}$ is Suslin, then every set in $\forall^{\omega^\omega}\mathbf{\Gamma}$ is Suslin, provided certain games (the $G_p(x, x')$ defined during the proof) are determined.*

Proof. Suppose $A \in \forall^{\omega^\omega}\mathbf{\Gamma}$, say with $B \in \mathbf{\Gamma}$ such that

$$x \in A \Leftrightarrow \forall y \ (x, y) \in B$$

and using the fact that B is Suslin:

$$(x, y) \in B \Leftrightarrow \exists z \in \kappa^\omega \colon (x, y, z) \in [T]$$

For each pair $x_{\boldsymbol{I}}, x_{\boldsymbol{II}} \in \omega^\omega$ and position $p \in \omega^{<\omega}$ consider the game $G_p(x_{\boldsymbol{II}}, x_{\boldsymbol{I}})$ in which players alternate moves to produce two reals $y_{\boldsymbol{I}}$ and $y_{\boldsymbol{II}}$. in the manner the following diagram:

$$G_p(x_{\boldsymbol{II}}, x_{\boldsymbol{I}}) \quad \begin{array}{lcccccccccc} \boldsymbol{I} & y_{\boldsymbol{II}}(0) & & y_{\boldsymbol{II}}(1) & & y_{\boldsymbol{II}}(2) & & y_{\boldsymbol{II}}(3) & & & \\ \boldsymbol{II} & & y_{\boldsymbol{I}}(0) & & y_{\boldsymbol{I}}(1) & & y_{\boldsymbol{I}}(2) & & y_{\boldsymbol{I}}(3) & & \ldots \end{array}$$

The players are playing offensive moves, attacking their opponent's x, and both are forced to start from the position p. We view the pair $(x_{\boldsymbol{I}}, y_{\boldsymbol{I}})$ as belonging to player $\boldsymbol{I}$, and $(x_{\boldsymbol{II}}, y_{\boldsymbol{II}})$ as belonging to player $\boldsymbol{II}$. If $(x_{\boldsymbol{I}}, p^\frown y_{\boldsymbol{I}}) \notin B$, player $\boldsymbol{II}$ wins, otherwise if $(x_{\boldsymbol{II}}, p^\frown y_{\boldsymbol{II}}) \notin B$, then player $\boldsymbol{I}$ wins. If both $(x_{\boldsymbol{I}}, p^\frown y_{\boldsymbol{I}})$ and $(x_{\boldsymbol{II}}, p^\frown y_{\boldsymbol{II}})$ are in B, then let $z_{\boldsymbol{I}}$ be the leftmost branch of $T_{x_{\boldsymbol{I}}, p^\frown y_{\boldsymbol{I}}}$ and $z_{\boldsymbol{II}}$ be the leftmost branch of $T_{x_{\boldsymbol{II}}, p^\frown y_{\boldsymbol{II}}}$. We declare $\boldsymbol{II}$ the winner of $G_p(x_{\boldsymbol{II}}, x_{\boldsymbol{I}})$ if $z_{\boldsymbol{II}} \restriction n \leq_{\text{lex}} z_{\boldsymbol{I}} \restriction n$, where n is the length of p. Equivalently, we can simply say $\boldsymbol{II}$ wins iff

$$\exists z \in [T_{x_{\boldsymbol{I}}, p^\frown y_{\boldsymbol{I}}}] \Rightarrow \left[\exists z_{\boldsymbol{II}} \in [T_{x_{\boldsymbol{II}}, p^\frown y_{\boldsymbol{II}}}] \, \forall z_{\boldsymbol{I}} \in [T_{x_{\boldsymbol{I}}, p^\frown y_{\boldsymbol{I}}}] \, \left(z_{\boldsymbol{II}} \restriction n \leq_{\text{lex}} z_{\boldsymbol{I}} \restriction n \right) \right]$$

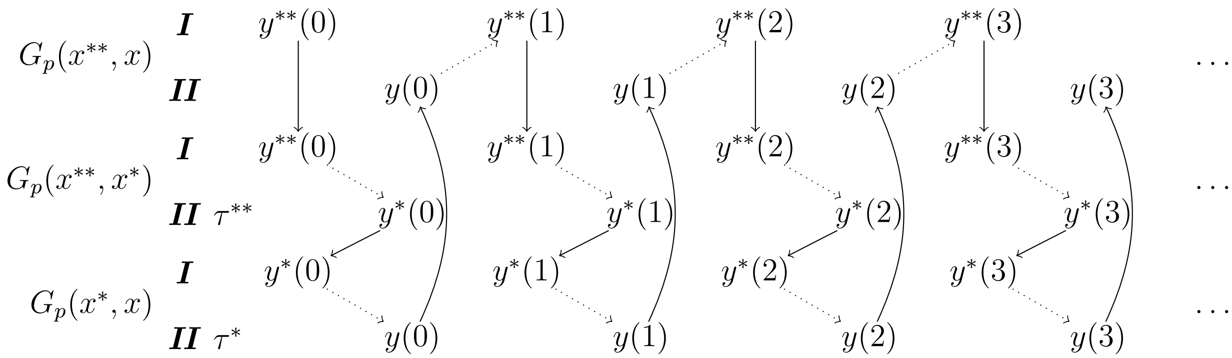
We define for each position p a binary relation $\preceq_p$ on ω^ω by

$$x \preceq_p x' \Leftrightarrow \boldsymbol{II} \text{ has a winning strategy in } G_p(x, x')$$

We claim that if the games $G_p(x, x')$ are all determined, then $\preceq_p$ is a prewellordering. for now a position p of length n.

Claim 1. *$\preceq_p$ is transitive on A*

Proof. Suppose x^{**}, x^*, x are in A with $x^{**} \preceq_p x^* \preceq_p x$, and let τ^{**} and τ^* be winning strategies witnessing this. We demonstrate how to win $G_p(x^{**}, x)$ as $\boldsymbol{II}$, with solid arrows indicating copying of moves, and dotted arrows indicating obtaining moves from the strategies or from our opponent



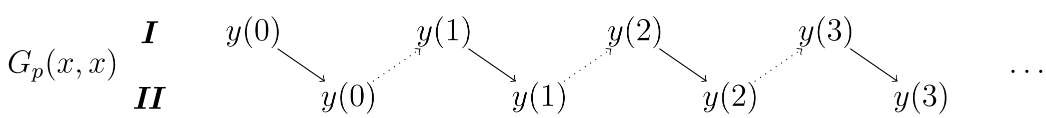
Now given z^{**}, z^*, and z leftmost branches of $T_{x^{**},p^\frown y^{**}}$ $T_{x^*,p^\frown y^*}$ and $T_{x,p^\frown y}$ respectively, we have that since τ^{**} is winning for ***II*** in $G_p(x^{**},x^*)$ and τ^* is winning for ***II*** in $G_p(x^*,x)$, we have

$$z^{**}\restriction n \leq_{\text{lex}} z^*\restriction n \leq_{\text{lex}} z\restriction n$$

and so this result is a win for ***II*** in $G_p(x^{**},x)$, so that $x^{**} \preceq_p x$. ■

Claim 2. *$\preceq_p$ is reflexive*

Proof. We will show ***II*** wins $G_p(x,x)$, again with solid lines indicating move copying, and dotted lines indicating our opponent's movement.



If $(x,y) \notin B$, then our opponent has lost, as ***I*** has the first burden of that rule. If it is, then for the leftmost branch z, clearly $z\restriction n \leq_{\text{lex}} z\restriction n$. ■

Claim 3. *$\preceq_p$ is connected.*

Proof. Suppose that for some x, x^*, we had both $x \not\preceq_p x^*$ and $x^* \not\preceq_p x$. If the games $G_p(x^*,x)$ and $G_p(x,x^*)$ are both determined, then we have winning strategies σ and σ^* for ***I*** in each game. We show this is absurd, with solid lines indicating move copying and dotted lines indicating obtaining moves from strategies.

$$
\begin{array}{llccccccccc}
 & \boldsymbol{I}\ \sigma & y^*(0) & & y^*(1) & & y^*(2) & & y^*(3) & & \\
G_p(x^*, x) & & & & & & & & & & \dots \\
 & \boldsymbol{II} & & y(0) & & y(1) & & y(2) & & y(3) & \\
 & \boldsymbol{I}\ \sigma^* & y(0) & & y(1) & & y(2) & & y(3) & & \\
G_p(x, x^*) & & & & & & & & & & \dots \\
 & \boldsymbol{II} & & y^*(0) & & y^*(1) & & y^*(2) & & y^*(3) &
\end{array}
$$

It must be that (x^*, y^*) and (x, y) are both in B, since otherwise these would be losses for σ^*, σ respectively. If this results in leftmost branches z^* and z, then we have $z^* \restriction n \not\leq_{\text{lex}} z \restriction n$ and $z \restriction n \not\leq_{\text{lex}} z^* \restriction n$, a contradiction. ■

Claim 4. $\preceq_p$ *is wellfounded*

Proof. Suppose it wasn't wellfounded, and use DC to obtain a descending chain

$$x_0 \not\preceq_p x_1 \not\preceq_p x_2 \not\preceq_p \dots$$

and if each of the games $G_p(x_k, x_{k+1})$ are determined, obtain a sequence of strategies σ_k winning for $\boldsymbol{I}$ in $G_p(x_k, x_{k+1})$. We will show this is absurd, with solid lines indicating copying and dotted lines indicating obtaining moves from the strategies.

$$
\begin{array}{llccccccccc}
 & \boldsymbol{I}\ \sigma_3 & y_3(0) & & y_3(1) & & y_3(2) & & y_3(3) & & \\
G_p(x_3, x_4) & & & & & & & & & & \dots \\
 & \boldsymbol{II} & & y_4(0) & & y_4(1) & & y_4(2) & & y_4(3) & \\
 & \boldsymbol{I}\ \sigma_2 & y_2(0) & & y_2(1) & & y_2(2) & & y_2(3) & & \\
G_p(x_2, x_3) & & & & & & & & & & \dots \\
 & \boldsymbol{II} & & y_3(0) & & y_3(1) & & y_3(2) & & y_3(3) & \\
 & \boldsymbol{I}\ \sigma_1 & y_1(0) & & y_1(1) & & y_1(2) & & y_1(3) & & \\
G_p(x_1, x_2) & & & & & & & & & & \dots \\
 & \boldsymbol{II} & & y_2(0) & & y_2(1) & & y_2(2) & & y_2(3) & \\
 & \boldsymbol{I}\ \sigma_0 & y_0(0) & & y_0(1) & & y_0(2) & & y_0(3) & & \\
G_p(x_0, x_1) & & & & & & & & & & \dots \\
 & \boldsymbol{II} & & y_1(0) & & y_1(1) & & y_1(2) & & y_1(3) &
\end{array}
$$

Note that for each k, we must have $(x_{k+1}, y_{k+1}) \in B$, since otherwise we would have that σ_k has lost. Given the resulting leftmost branches $z_{k+1} \in [T_{x_{k+1}, p^\frown y_{k+1}}]$, since each σ_k is winning, we have an infinite descending chain

$$z_1 \not\leq_{\text{lex}} z_2 \not\leq_{\text{lex}} z_3 \not\leq_{\text{lex}} \cdots$$

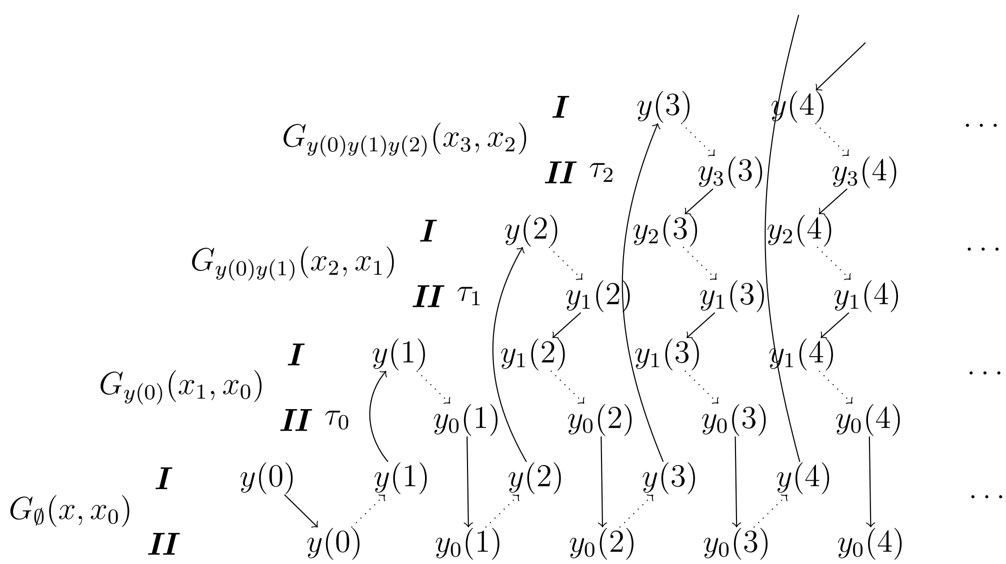
which is a contradiction. ■

Now for each position p, let $\varphi_p(x)$ be the ordinal rank of x in the prewellorder $\preceq_p$. Let $\{p_i : i \in \omega\}$ enumerate $\omega^{<\omega}$, and define a tree S by

$$(s,t) \in S \leftrightarrow \exists x \in A \left[s \subseteq x \wedge t = \varphi_{p_0}(x)^\frown \cdots {}^\frown \varphi_{p_{|s|-1}}(x) \right]$$

We claim that $A = \mathfrak{p}[S]$. If $x \in A$, then clearly $(x,f) \in [S]$ where $f(n) = \varphi_{p_n}(x)$. Suppose next that $(x,f) \in [S]$. Then we have for every n some $x_n \in A$ so that $x \restriction n = x_n$ and $f \restriction n = \varphi_{p_0}(x_n)^\frown \cdots {}^\frown \varphi_{p_{n-1}}(x_n)$.

We aim to show that for any given y that $(x,y) \in B$ so that $x \in A$. Given y, fix $p = y(0)^\frown \cdots {}^\frown y(n-1)$. Let i be such that $p = p_i$ and consider the game $G_p(x_n, x_{n+1})$ for any $n > i$. By choice of x_n, x_{n+1}, we have that $\varphi_p(x_n) = f(i) = \varphi_p(x_{n+1})$, and so we have that ***II*** wins the game $G_p(x_{n+1}, x_n)$. So now we replace the sequence $\{x_n\}$ by a subsequence so that for every n, ***II*** wins $G_{y(0)^\frown \cdots {}^\frown y(n)}(x_{n+1}, x_n)$, and fix τ_n winning in this game. We will show two things: First that $x \in A$, by describing a strategy in $G_\emptyset(x, x_0)$. Then we will show that in fact this strategy wins $G_\emptyset(x, x_0)$ and so $x \preceq_\emptyset x_0$. We will play $G_\emptyset(x, x_0)$ as ***II***, filling in moves as in the following diagram.



Now since each x_n is in A, we have that $(x_n, y) \in B$ for any y, in particular we have $(x_n, y_n) \in B$ (here $y_n = y(0)^\frown \cdots {}^\frown y(n)^\frown y_n(n+1)^\frown \cdots$ is the real played by τ_n, including the forced initial segment). Now since τ_n is supposed to have won, we have that if z_n is leftmost in $[T_{x_n,y_n}]$, then

$$z_{n+1} \restriction n+1 \leq_{\text{lex}} z_n \restriction n+1$$

for every n. Now we note that $z_n(0)$ may only decrease finitely often, and so must eventually stabilize, likewise with $z_n(i)$ for any i. So we can define

$$z(k) = \liminf_n z_n(k) = \lim_n z_n(k)$$

Finally, we see that $(x_n, y_n, z_n) \to (x, y, z)$ and so then since $[T]$ is closed, we have that $(x, y, z) \in [T]$, and thus we have shown

$$\forall y \; \exists z \in [T_{x,y}]$$

and thus $x \in A$.

We also have, however, that the leftmost branch of $T_{x,y}$ is $\leq_{\text{lex}}$-left of z, and $z \restriction k \leq_{\text{lex}} z_n \restriction k$ for every n, so in fact this run results in a win of the game $G_\emptyset(x, x_0)$. With a mild modification, we could have won instead the game $G_\emptyset(x, x_n)$ for any n. □

Having a Suslin representation is a powerful property, and while we do not need here definability estimates for the Suslin representation, these can be obtained by a more careful proof than the one we gave. Such a proof would proceed by using the very important notion of a *scale*, which we give the definition of next for the sake of completeness.

Definition 7. Suppose $\{\varphi_n\}_n$ is a sequence of functions $\varphi_n \colon A \to \kappa$ with the property that if

(1) If $x_i \to x$ with $\{x_i\} \subseteq A$
(2) and if $\varphi_n(x_i)$ is eventually constant for every n.

then

(1) $x \in A$

(2) and for every n, $\varphi_n(x) \leq \lim_{i\to\infty} \varphi_n(x_i)$

then $\{\varphi_n\}_n$ is a *scale on A*.

One can perhaps see the shadows of this definition in the proof we presented. The main advantage of using scales over the Suslin representations themselves is that we can make sense of how complicated a scale is by examining the prewellorderings $\preceq_n$ that φ_n induces on A. Since most of our results appeal simply to AD, we refer the reader to [19] for significantly more exploration of this topic.

We will make use of only a few of the many applications of Suslin representations, which we introduce next.

Definition 8. Let $R \subseteq X \times Y$ so that for every $x \in X$, there is some $y \in Y$ so that $(x, y) \in R$. A function $f\colon X \to Y$ is a *uniformization* of R if for every $x \in X$, $(x, f(x)) \in R$.

The next result is a fundamental consequence of sets having Suslin representations, which has been proven in many different forms and levels of generality. We state the version which we make use of here.

Theorem (Novikov-Kondo-Addison see [19]). *Every relation $R \subseteq \omega^\omega \times \omega^\omega$ which is Suslin has a uniformization*

CHAPTER 3

DETERMINACY

The proof of the existence of a nondetermined game on the set $X = \{0, 1\}$ by Gale and Stewart required the use of the axiom of choice, in order to obtain a well-orderering of the collection of all possible strategies to carry out a diagonalization through transfinite induction. One can exhibit explicit games on large sets X which are not determined, but if we restrict our choice of X, then we are able to study the consequences of assuming games on X are determined.

Definition 9 (Mycielski Steinhaus [22]). Let X be a set. We will let AD_X abbreviate the statement that all games on X are determined.

These axioms were first introduced by Mycielski and Steinhaus in [22] in 1962. Of particular interest is the *axiom of determinacy*, which we will denote by $\mathsf{AD} = \mathsf{AD}_\omega$ for games played on countable sets. One can show by an encoding argument that AD is equivalent to AD_F for F any finite set. One can also consider determinacy axioms for restricted classes of games with specific sets of rules, for instance, the assumption that all Banach-Mazur games, or all Banach games are determined. Banach game determinacy was investigated in detail by Freiling in [4] and Becker in [2]. Freiling proved that the determinacy of Banach games implies AD while Becker proved the converse.

It may be important to note here that while Gale and Stewart showed that $\neg\mathsf{AD}_2$ is a theorem of ZFC, the axiom AD_2 does not contradict the *countable* axiom of choice (and in fact implies it, for subsets of $\mathbb{R}$), nor the stronger axiom of dependent choices. Much of mathematics remains intact under if one eschews AC and replaces it with AD_2 or $\mathsf{AD} = \mathsf{AD}_\omega$..

The consequences of AD were investigated further by Mycielski in [20] and [21], and by many other authors as well. One can show that under the hypothesis AD, all subsets of the real line are Lebesgue measurable, have the Baire property, and have the perfect set property. These desirable regularity properties, which hold for all analytic sets in any model of ZF, give us more motivation to investigate the consequences of AD, as it seems able to extend the theory of analytic sets to all subsets of the reals. Another motivation to the investigation of AD is that many restricted forms of determinacy hypothesis are consistent with AC (relative to large cardinals), for instance, PD the axiom of projective determinacy: that all games on ω with projective payoff are determined. The projective subsets contain the Borel sets and analytic sets, as well as a hierarchy of more complicated sets, and so PD can be seen as an extension of Martin's Borel determinacy theorem to a larger class of sets.

The investigation into the consequences of determinacy axioms was eventually justified by connections to large cardinal axioms of set theory. Martin [10] showed that if a measurable cardinal exists, then analytic games on ω are determined. Soon after, more determinacy was shown to hold from the assumption that stronger large cardinals exist. In 1988, Martin and Steel [15] proved that PD is true provided infinitely many *Woodin cardinals* exist, and, also in 1988, Woodin showed that a slightly stronger hypothesis implies that every game in $L(\mathbb{R})$ is determined. This means that if certain large cardinals exist, there is a model of the theory $\mathsf{ZF} + \mathsf{AD}$ with contains all of $\mathbb{R}$. This model necessarily doesn't satisfy AC, but nevertheless contains almost every object studied within the normal course of mathematics, with certain exceptions.

Another determinacy principle which will be relevant to us here is the axiom $\mathsf{AD}_{\mathbb{R}}$, the assertion that all real games are determined. Both AD and $\mathsf{AD}_{\mathbb{R}}$ play an important role in modern descriptive set theory. although both axioms contradict the axiom of choice, AC, and thus are not adopted as axioms for the true universe V of set theory, they play a critical role in developing the theory of natural models such as $L(\mathbb{R})$ containing "definable" sets of reals. It is known that $\mathsf{AD}_{\mathbb{R}}$ is a much stronger assertion than AD (see Theorem 4.4 of [26]).

Sitting between AD and $\mathsf{AD}_{\mathbb{R}}$ is the determinacy of another class of games called $\frac{1}{2}\mathbb{R}$ games, in which one of the players plays reals and the other plays integers. The proof of one of our theorems will require the use of $\frac{1}{2}\mathbb{R}$ games. The axiom $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ that all $\frac{1}{2}\mathbb{R}$ games are determined is known to be equivalent to $\mathsf{AD}_{\mathbb{R}}$ ($\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$ immediately implies Unif, see Theorem 3.1 below). However, AD suffices to obtain the determinacy of $\frac{1}{2}\mathbb{R}$ games with

Suslin, co-Suslin payoff (a result of Woodin, see [9]). As in [2], this fact will play an important role in one of our theorems.

We briefly review some of the terminology and results related to the determinacy of games and some associated notions concerning pointclasses which we will need for the proofs of some of our results.

We will introduce several axioms which assert the determinacy of certain types of games. To do this, we first specify the types of games under consideration

DEFINITION 10. If $G \subseteq \omega^\omega$ is a game, we call G an *integer game*. If $T \subseteq (\omega \cup \mathbb{R})^{<\omega}$ is a tree of rules such that one of the players legal moves are exclusively in ω, while the other player's moves are from $\mathbb{R}$, then any game G on T is called a *half-$\mathbb{R}$ game*. If G is a game so that $G \subseteq \mathbb{R}^\omega$, we call G a *real game*.

We now state the result of Woodin which is critical to our work in Chapter 4

THEOREM (Woodin, see [9]). *Assume* AD. *Suppose G is a half-$\mathbb{R}$ game which is Suslin and co-Suslin. Then G is determined, and if the integer player has a winning strategy, then there is a winning strategy projective over G for that player.*

We have introduced above the axioms AD, $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$, and $\mathsf{AD}_\mathbb{R}$ which assert the determinacy of integer games, half-real games, and real games respectively. We trivially have $\mathsf{AD}_\mathbb{R} \Rightarrow \mathsf{AD}_{\frac{1}{2}\mathbb{R}} \Rightarrow \mathsf{AD}$. All three of these axioms contradict AC, the axiom of choice. They are consistent, however, with DC, the axiom of dependent choice, which asserts that if T is a non-empty *pruned* tree (i.e., if $(x_0, \ldots, x_n) \in T$ then $\exists x_{n+1}\ (x_0, \ldots, x_n, x_{n+1}) \in T$) then there is a branch f through T (i.e., $\forall n\ (f(0), \ldots, f(n)) \in T$). DC is a slight strengthening of the axiom of countable choice. On the one hand, DC holds in the minimal model $L(\mathbb{R})$ of AD, while on the other hand even $\mathsf{AD}_\mathbb{R}$ does not imply DC. Throughout, we will be working in the background theory of $\mathsf{ZF} + \mathsf{DC}$.

The axiom $\mathsf{AD}_\mathbb{R}$ is strictly stronger than AD (see [26]), and in fact it is known that $\mathsf{AD}_\mathbb{R}$ is equivalent to AD + Unif, where Unif is the axiom that every $R \subseteq \mathbb{R} \times \mathbb{R}$ has a uniformization. This equivalence will be important for our argument in Theorem 4.5 that

AD does not suffice for the determinacy of Schmidt's game in $\mathbb{R}^n$ for $n \geq 3$. The notion of uniformization is closely connected with the descriptive set theoretic notion of a *scale*. If a set $R \subseteq X \times Y$ (where X, Y are Polish spaces) has a scale, then it has a uniformization. The only property of scales which we use is the existence of uniformizations, so we will not give the definition, which is rather technical, here.

We recall that a *pointclass* $\mathbf{\Gamma}$ is a collection of subsets of Polish spaces closed under continuous preimages, that is, if $f \colon X \to Y$ is continuous and $A \subseteq Y$ is in $\mathbf{\Gamma}$, then $f^{-1}(A)$ is also in $\mathbf{\Gamma}$. We say $\mathbf{\Gamma}$ is selfdual if $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$ where $\check{\mathbf{\Gamma}} = \{X - A \colon A \in \mathbf{\Gamma}\}$ is the dual pointclass of $\mathbf{\Gamma}$. We say $\mathbf{\Gamma}$ is non-selfdual if $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$. A set $U \subseteq \omega^\omega \times X$ is *universal* for the $\mathbf{\Gamma}$ subsets of X if $U \in \mathbf{\Gamma}$ and for every $A \subseteq X$ with $A \in \mathbf{\Gamma}$ there is an $x \in \omega^\omega$ with $A = U_x = \{y \colon (x, y) \in U\}$. It is a consequence of AD that every non-selfdual pointclass has a universal set.

Recall from Definition 5 that for κ an ordinal number we say a set $A \subseteq \omega^\omega$ is κ-Suslin if there is a tree T on $\omega \times \kappa$ such that $A = p[T]$, where $p[T] = \{x \in \omega^\omega \colon \exists f \in \kappa^\omega \ (x, f) \in [T]\}$ denotes the projection of the body of the tree T. We say A is Suslin if it is κ-Suslin for some κ. We say A is co-Suslin if $\omega^\omega \setminus A$ is Suslin. For a general Polish space X, we say $A \subseteq X$ is Suslin if for some continuous surjection $\varphi \colon \omega^\omega \to X$ we have that $\varphi^{-1}(A)$ is Suslin (this does not depend on the choice of φ). Scales are essentially the same thing as Suslin representations, in particular a set $A \subseteq Y$ is Suslin iff it has a scale, thus relations which are Suslin have uniformizations. If $\mathbf{\Gamma}$ is a pointclass, then we say a set A is *projective over* $\mathbf{\Gamma}$ if it is in the smallest pointclass $\mathbf{\Gamma}'$ containing $\mathbf{\Gamma}$ and closed under complements and existential and universal quantification over $\mathbb{R}$. Assuming AD, if $\mathbf{\Gamma}$ is contained in the class of Suslin, co-Suslin sets, then every set projective over $\mathbf{\Gamma}$ is also Suslin and co-Suslin. For this result, more background on these general concepts, as well as the precise definitions of scale and the scale property, the reader can refer to [19].

Results of Martin and Woodin (see [16] and [13]) show that assuming $\mathsf{AD} + \mathsf{DC}$, the axioms $\mathsf{AD}_\mathbb{R}$, Unif, and scales are all equivalent. More precisely we have the following.

Theorem 3.1 (Martin, Woodin). *Assume* $\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$. *Then the following are equivalent:*

(1) $\mathsf{AD}_\mathbb{R}$

(2) *Unif*

(3) *Every $A \subseteq \mathbb{R}$ is Suslin.*

Scales and Suslin representations are also important as it follows from AD that ordinal games where the payoff set is Susin and co-Suslin (the notion of Suslin extends naturally to sets $A \subseteq \lambda^\omega$ for λ an ordinal number) are determined (one proof of this is due to Moschovakis, Theorem 2.2 of [18], another due to Steel can be found in the proof of Theorem 2 of [14]), although we will not need this result for our proofs here.

A strengthening of AD, due to Woodin, is the axiom AD^+. This axiom has been very useful as it allows the development of a structural theory which has been used to obtain a number of results. It is not currently known if AD^+ is strictly stronger than AD, but it holds in all the natural models of AD obtained from large cardinal axioms (it holds, in particular, in the model $L(\mathbb{R})$, so AD^+ is strictly weaker that $\mathsf{AD}_{\mathbb{R}}$). In our Theorem 4.5 we in fact show that AD^+ does not suffice to get the determinacy of Schmidt's (α, β, ρ) game in $\mathbb{R}^n$ for $n \geq 3$.

CHAPTER 4

MAIN RESULTS

4.1. Schmidt's Game

In 1966, Schmidt [25] introduced a two-player game, now known as Schmidt's game. Schmidt invented the game as a tool for primarily studying certain sets which arise in number theory and Diophantine approximation theory. Schmidt's game, and other similar games, have since become an important tool in dynamics, number theory and related areas.

Questions regarding which player, if any, has a winning strategy in various games have been systematically studied over the last century. Schmidt's game and these other related games are real games, as in Definition 10. The axiom of determinacy for real games, $\mathsf{AD}_{\mathbb{R}}$, would easily imply the determinacy of Schmidt's game, but it is a much stronger hypothesis than AD (see Chapter 3 for a more thorough discussion). A natural question is how strong of a determinacy assumption is necessary to obtain the determinacy of Schmidt's game. In particular, can one obtain the determinacy of this game from AD, or does one need the full strength of $\mathsf{AD}_{\mathbb{R}}$?

Consider the case of the Banach-Mazur game on a Polish space (X, d) with target set $A \subseteq X$. Here the players $\boldsymbol{I}$ and $\boldsymbol{II}$ at each turn n plays somehow a real which codes a closed ball $B(x_n, \rho_n) = \{y \in X : d(x_n, y) \leq \rho_n\}$. The only "rule" of the game is that the balls played must constitute a decreasing sequence. If both players follow this rule, then $\boldsymbol{II}$ wins iff $\bigcap_n B(x_n, \rho_n) \cap A \neq \emptyset$. Although this is a real game, this game is determined for any $A \subseteq X$ just from AD. One can prove this by showing first that AD implies all sets have the Baire property, after sufficient analysis of the game itself. Alternatively it follows from the easy fact that the Banach-Mazur game is equivalent to the integer game in which both players play closed balls with "rational centers" (i.e., from a fixed countable dense set) and rational radii.

For Schmidt's game on a Polish space (X, d) with target set $A \subseteq X$, we have in addition fixed parameters $\alpha, \beta \in (0, 1)$. In this game $\boldsymbol{I}$'s first move is a closed ball $B(x_0, \rho_0)$

as in the Banach-Mazur game. For the remainder of the game, the players play a decreasing sequence of closed balls as in the Banach-Mazur game, but with an extra restriction on the radii. Namely, ***II*** must always multiply the previous radius by a factor of α, and ***I*** must do the same with β. So, at move $2n$, ***I*** plays a closed ball of radius $\rho_{2n} = (\alpha\beta)^n \rho_0$, and at move $2n+1$, ***II*** plays a closed ball of radius $\rho_{2n+1} = \alpha(\alpha\beta)^n\rho_0$. As in the Banach-Mazur game, if both players follow these rules, then ***II*** wins iff $x \in A$ where $\{x\} = \bigcap_n B(x_n, \rho_n)$. We call this game the (α, β) Schmidt's game for A. A variation of Schmidt's game which we will concentrate on here, first introduced by Akhunzhanov in [1], has an additional rule that the initial radius $\rho_0 = \rho$ of ***I***'s first move is fixed in advance. We call this the (α, β, ρ) Schmidt's game for T. In all practical applications of Schmidt's game we are aware of, the difference between these two versions is immaterial. However, in general, these games are not literally equivalent, as the following simple example demonstrates.

Example 2. Consider $\mathbb{R}$ with the usual metric and let the target set for ***II*** be $A = (-\infty, -1] \cup [1, \infty) \cup \mathbb{Q}$. Notice that this set is dense. It is easy to see that if $\rho \geq 2$ and $\alpha \leq \frac{1}{4}$ then for any β, ***II*** wins the (α, β, ρ)-game, simply by maximizing the distance from the center of her first move to the origin. But if ***I*** is allowed to choose any starting radius and $\beta < \frac{1}{2}$, then he is allowed to play, for instance, $(0, \frac{1}{2})$, and then on subsequent moves, simply avoid each rational one at a time, so that in fact ***I*** wins the (α, β)-game.

In the case of Schmidt's game (either variation) it is not immediately clear that the game is equivalent to an integer game, and thus it is not clear that AD suffices for the determinacy of these games. Our main work here is to investigate these implications regarding the determinacy of Schmidt's game.

Another class of games which to Schmidt's game are the so-called Banach games introduced in the Scottish Book [17]. These Banach games and their determinacy were investigated by Becker and Freiling [2] [4] (with an important result being obtained by Martin). Work of these authors has shown that the determinacy of these games is equivalent to AD. Methods similar to those used by Becker, Freiling, and Martin are instrumental in the proofs of our results as well.

In this chapter we prove our main results, and then use them to obtain results regarding the determinacy of Schmidt's game. We prove general results, Theorems 4.1, 4.2, which give some conditions under which certain real games are determined under AD alone. Roughly speaking, these results state that "intersection" games which admit strategies which are simple enough to be "coded by a real,", are determined from AD. We make these notions precise in Section 4.2. Schmidt's game, Banach-Mazur games, and other similar games are easily seen to be intersection games in this sense. The simple strategy condition, however, is a more subtle one to check, and depends heavily on the specifics of the game. For Schmidt's (α, β, ρ) game on $\mathbb{R}$, we show the simple strategy condition holds, and so this game is determined from AD alone. Moreover, for the (α, β) Schmidt's game on $\mathbb{R}$, this gives us that AD implies that either player $\boldsymbol{I}$ has a winning strategy or else for every ρ, $\boldsymbol{II}$ has a winning strategy in the (α, β, ρ) game (this does not immediately give a strategy for $\boldsymbol{II}$ in the (α, β) game from AD, as this would require us to uniformize the relation on pairs (ρ, τ) to choose, as a function of ρ, a winning strategy τ for $\boldsymbol{II}$ in the (α, β, ρ) game). For $\mathbb{R}^n$, $n \geq 2$, the simple strategy condition is not met. In fact, our Theorem 4.5 shows that for $n \geq 3$ we show that the determinacy of Schmidt's (α, β, ρ) games does not follow from AD. For $n = 2$, we do not know if AD is sufficient to obtain the determinacy of Schmidt's game.

We now give the more formal definition of Schmidt's game.

Definition 11. Given a Polish space (X, d) and $A \subseteq X$, we define the *Schmidt's* (α, β) *game with target set* A by the following:

Players $\boldsymbol{I}$ and $\boldsymbol{II}$ alternate playing pairs (x_i, ρ_i) in $Y = X \times \mathbb{R}_{>0}$. The tree of rules $R \subseteq Y^{<\omega}$ for this game is defined by the conditions

(1) $\rho_{i+1} + d(x_i, x_{i+1}) \leq \rho_i$

(2) $\rho_{i+1} = \begin{cases} \alpha\rho_i & \text{if } i \text{ is even} \\ \beta\rho_i & \text{if } i \text{ is odd} \end{cases}$.

The rules guarantee that the closed balls $B(x_i, \rho_i) = \{x \in \mathbb{R}^n \colon d(x, x_i) \leq \rho_i\}$ are nested.

Since the $\rho_i \to 0$, there is a unique point $z \in X$ such that $\{z\} = \bigcap_i B(x_i, \rho_i)$. For $\vec{x} \in [R]$, a run of the game following the rules, we let $f(\vec{x})$ be this corresponding point z. The payoff set $B \subseteq Y^\omega$ for player $\boldsymbol{I}$ is $\{\vec{x} \in Y^\omega \cap [R] \colon f(\vec{x}) \notin T\}$.

Formally, when we refer to Schmidt's (α, β) game with target set A, we are referring to the game $\mathcal{G}(B, R)$ with these sets B and R just described. The formal definition of *Schmidt's* (α, β, ρ) *game with target set* A is analogous.

4.2. Simple Strategies and Intersection Games

We now work towards proving a general result which states that certain real games are equivalent to $\frac{1}{2}\mathbb{R}$ games. The essential point is that real games which are intersection games (i.e., games where the payoff only depends on the intersection of the sets which the players are coding by their moves) with the property that if one of the players has a winning strategy in the real game, then that player has a simple strategy "coded by a real", then the game is equivalent to a $\frac{1}{2}\mathbb{R}$ game. In [2] a result, which is attributed there to Martin, is presented which showed that the determinacy of a certain class of real games, called Banach games, follows from $\mathsf{AD}_{\frac{1}{2}\mathbb{R}}$, the axiom which asserts the determinacy of $\frac{1}{2}\mathbb{R}$ games (that is, games in which one player plays reals, and the other plays integers). In Theorem 4.1 we use ideas similar to those of Martin to prove a general result which applies to intersection games satisfying a "simple strategy" hypothesis. This means that because many games with applications to number theory and dynamics are naturally intersection games, the main hypothesis to check, in practice, is the simple strategy hypothesis.

Definition 12. Let $\boldsymbol{\Gamma}$ be a pointclass. A simple one-round $\boldsymbol{\Gamma}$ strategy s for the Polish space X is a sequence $s = (M_n, y_n)_{n \in \omega}$ where $y_n \in X$, $M_n \in \boldsymbol{\Gamma}$, and the M_n are a partition of X. A simple $\boldsymbol{\Gamma}$ strategy τ for player $\boldsymbol{II}$ is a collection $\{s_u\}_{u \in \omega^{<\omega}}$ of simple one-round $\boldsymbol{\Gamma}$ strategies s_u. A simple $\boldsymbol{\Gamma}$ strategy σ for player $\boldsymbol{I}$ is a pair $\sigma = (\bar{y}, \tau)$ where $\bar{y} \in X$ is the first move and τ is a simple $\boldsymbol{\Gamma}$ strategy for player $\boldsymbol{II}$.

We will try to motivate the idea of a simple strategy here. First, a simple one-round strategy is that if the opponent moves in the set M_n, then the strategy will respond with

y_n. Thus there is only "countably much" information in the strategy; it is coded by a real in a simple manner. If $s = (M_n, y_n)$ is a simple one-round strategy, we will write $s(n) = y_n$ and also $s(x) = y_n$ for any $x \in M_n$. A full simple strategy produces after each round a new simple one-round strategy to follow for the subsequent round. For example, suppose σ is a simple strategy for $\boldsymbol{I}$. σ gives a first move $x_0 = \bar{y}$ and a simple one-round strategy $s_\emptyset$. If $\boldsymbol{II}$ plays x_1, then $x_2 = \sigma(x_0, x_1) = s_\emptyset(x_1)$ =the unique y_{n_0} such that $x_1 \in A_{n_0}$ where $s_\emptyset = (M_n, y_n)$. If $\boldsymbol{II}$ then plays x_3, then σ responds with $s_{n_0}(x_3)$. The play by σ continues in this manner. Formally, a general simple strategy is a sequence $(s_u)_{u \in \omega^{<\omega}}$ of simple one-round strategies, indexed by $u \in \omega^{<\omega}$.

If $\boldsymbol{\Gamma}$ is a pointclass with a universal set $U \subseteq \omega^\omega \times X$, then we can use U to code simple $\boldsymbol{\Gamma}$ strategies. Namely, the simple one-round $\boldsymbol{\Gamma}$ strategy $s = (M_n, y_n)$ is coded by $z \in \omega^\omega$ if z codes a sequence $(z)_n \in \omega^\omega$ and $U_{(z)_{2n}} = M_n$ and $(z)_{2n+1}$ codes the response $y_n \in X$ in some reasonable manner (e.g., via a continuous surjection from ω^ω to X).

Remark 2. For the remainder of this section, X and Y will denote Polish spaces.

Definition 13. Let $R \subseteq X^{<\omega}$ be a tree on X which we will view as the rules of some game on X. We say a simple one-round $\boldsymbol{\Gamma}$ strategy s *follows the rules* R at position $p \in R$ if for any $x \in X$, if $p^\frown x \in R$, then $p^\frown x^\frown s(x) \in R$.

Definition 14. Let $R \subseteq X^{<\omega}$ be a set of rules for a real game. Suppose $p \in X^{<\omega}$ is a position in R. Suppose $f \colon X \to X$ is such that for all $x \in X$, if $p^\frown x \in R$, then $p^\frown x^\frown f(x) \in R$ (i.e., f is a one-round strategy which follows the rules at p). A *simplification* of f at p is simple one-round strategy $s = (M_n, y_n)$ such that

(1) For every x in any M_n, if $p^\frown x \in R$, then $p^\frown x^\frown y_n \in R$.
(2) For every n, if there is an $x \in M_n$ such that $p^\frown x \in R$, then there is an $x' \in M_n$ with $p^\frown x' \in R$ and $f(x') = y_n$.

We say τ is a $\boldsymbol{\Gamma}$ simplification of f if all of the sets M_n are in $\boldsymbol{\Gamma}$.

Definition 15. We say a tree $R \subseteq X^{<\omega}$ is *positional* if for every pair of positions $p, q \in R$

of the same length and for every move $x \in X$, if $p^\frown x$, $q^\frown x$ are both in R then for all $r \in X^{<\omega}$, $p^\frown x^\frown r \in R$ iff $q^\frown x^\frown r \in R$.

THEOREM 4.1 ($\mathsf{ZF} + \mathsf{DC} + \mathsf{AD}$). *Let $\mathbf{\Gamma}$ be a pointclass with a universal set and suppose $\mathbf{\Gamma}$ lies within the Suslin, co-Suslin sets. Suppose $B \subseteq X^\omega$ and $R \subseteq X^{<\omega}$ is a positional tree of rules, and suppose both B and R are in $\mathbf{\Gamma}$. Let $G = \mathcal{G}(B, R)$ be the real game on X with payoff B and rules R. Suppose the following two conditions on G hold:*

(1) *(intersection condition) For any $\vec{x}, \vec{y} \in [R]$, if $x(2k) = y(2k)$ for all k, then $\vec{x} \in B$ iff $\vec{y} \in B$.*
(2) *(simple one-round strategy condition) If $p \in R$ has odd length, and $f\colon X \to X$ is a rule following one-round strategy at p, then there is a $\mathbf{\Gamma}$-simplification of f at p.*

Then G is equivalent to a Suslin and co-Suslin $\frac{1}{2}\mathbb{R}$ game G^* in the sense that whichever player (if any) has a winning strategy in G^*, also has a winning strategy in G.

PROOF. Consider the game G^* where $\boldsymbol{I}$ plays pairs (x_{2k}, s_{2k}) and $\boldsymbol{II}$ plays integers n_{2k+1}. We set the rules R^* of G^* to be that $\boldsymbol{I}$ must play at each round a real coding s_{2k} which is a simple one-round $\mathbf{\Gamma}$ strategy which follows the rules R relative to a position $p^\frown x_{2k}$ for any p of length $2k$ (this does not depend on the particular choice of p as R is positional). $\boldsymbol{I}$ must also play such that $x_{2k} = s_{2k-2}(n_{2k-1})$. We require that $\boldsymbol{II}$ play each n_{2k+1} so that there is a legal move $x_{2k+1} \in M^{s_{2k}}_{n_{2k+1}}$ with $p^\frown x_{2k}{}^\frown x_{2k+1} \in R$ (for any p of length $2k$).

If $\boldsymbol{I}$ and $\boldsymbol{II}$ have followed the rules R^*, to produce x_{2k}, s_{2k} and n_{2k+1}, the payoff condition for G^* is as follows. Since $\boldsymbol{II}$ has followed the rules, there is a sequence x_{2k+1} such that the play $(x_0, x_1, \dots) \in [R]$. $\boldsymbol{I}$ then wins the run of G^* iff $(x_0, x_1, \dots) \in B$. We note here that by the intersection condition, this is independent of the particular choice of the x_{2k+1}.

From the definition, G^* is a Suslin, co-Suslin game.

We show that G^* is equivalent to G. Suppose first that $\boldsymbol{I}$ wins G^* by σ^*. Then σ^* easily gives a strategy Σ for G. For example, let $\sigma^*(\emptyset) = (x_0, s_0)$. Then $\Sigma(\emptyset) = x_0$. If $\boldsymbol{II}$ plays x_1, then let n_1 be such that $x_1 \in M^{s_0}_{n_1}$. Then $\Sigma(x_0, x_1) = s_0(n_1)$. We continue in this manner to define Σ. If $(x_0, x_1, \dots)$ is a run of Σ, then there is a corresponding run $((x_0, s_0), n_1, \dots)$ of σ^*. As each s_{2k} follows the rules R, then as long as ***II***'s moves follow the rules R, ***I***'s moves by Σ also follow the rules R. If ***II*** has followed the rules R in the run of G, then the run $((x_0, s_0), n_1, \dots)$ of σ^* has followed the rules for G^* (***II*** has followed the rules of G^* since for each n_{2k+1}, x_{2k+1} witnesses that n_{2k+1} is a legal move). Since σ^* is winning for G^*, the sequence $(x_0, x'_1, x_2, x'_3, \dots) \in B \cap [R]$ for some x'_{2k+1}. By the intersection hypothesis, $(x_0, x_1, x_2, x_3, \dots) \in B$.

Now we use the assumption of AD. Assume now that ***II*** has winning strategy τ' in G^*. We first note that there is winning strategy τ^* for ***II*** in G^* such that τ^* is projective over $\mathbf{\Gamma}$. To see this, first note that the payoff set for G^* is projective over $\mathbf{\Gamma}$ by the hypothesis that R and B are in $\mathbf{\Gamma}$ and the definition of G^*. By a result of Woodin in [9] (since ***II*** is playing the integer moves in G^*) there is a winning strategy τ^* which is projective over $\mathbf{\Gamma}'$, and thus projective over $\mathbf{\Gamma}$. For the rest of the proof we fix a winning strategy τ^* for ***II*** in G^* which is projective over $\mathbf{\Gamma}$.

We will define a strategy Σ for ***II*** in G. We consider the first round of G. Suppose ***I*** moves with x_0 in G. We may assume that $(x_0) \in R$.

Claim 5. *There is an x_1 with $(x_0, x_1) \in R$ such that for all x_2 with $(x_0, x_1, x_2) \in R$, there is a simple one-round $\mathbf{\Gamma}$ strategy s_0 which follows the rules R from position x_0 (so (x_0, s_0) is a legal move for **I** in G^*) such that if $n_1 = \tau^*(x_0, s_0)$ then $x_1 \in M^{s_0}_{n_1}$ and $x_2 = s_0(x_1)$.*

Proof. Suppose not, then for every x_1 with $(x_0, x_1) \in R$ there is an x_2 with $(x_0, x_1, x_2) \in R$ which witnesses the failure of the claim. We define the relation $S(x_1, x_2)$ to hold iff $(x_0, x_1) \notin R$ or $(x_0, x_1, x_2) \in R$ and the claim fails, that is, for every simple one-round $\mathbf{\Gamma}$ strategy s which follows R, if we let $n_1 = \tau^*(x_0, s)$, then either $x_1 \notin M^s_{n_1}$ or $x_2 \neq s(x_1)$. Since τ^*, B, R are projective over $\mathbf{\Gamma}$, so is the relation S. By the assumption of the claim, $\mathrm{dom}(S) = \mathbb{R}$. Since S is projective over $\mathbf{\Gamma}$, it is within the Suslin sets, and thus there is a uniformization f for S. Note that by the definition of S, we have that f follows the rules R. By the simple one-round strategy hypothesis of Theorem 4.1, there is a $\mathbf{\Gamma}$-simplification

s_0 of f. Let $n_1 = \tau^*(x_0, s_0)$. Since τ^* follows the rules R^* for $\boldsymbol{II}$, there is an $x_1 \in M^{s_0}_{n_1}$ such that $(x_0, x_1) \in R$. Since s_0 is a simplification of f, there is an x_1' with $(x_0, x_1') \in R$ and $f(x_1') = s_0(n_1)$. Let $x_2 = f(x_1')$. From the definition of S we have that $(x_0, x_1', x_2) \in R$. Since $S(x_1', x_2)$, we can conlude that there does not exist an s (following the rules) such that $(x_1' \in M^s_{n_1}$ and $x_2 = s(x_1'))$ where $n_1 = \tau^*(x_0, s)$. But on the other hand, the s_0 we have produced does have this property, a contradiction. This proves the claim. ■

Now that we've proved this claim, we begin to define the strategy Σ. We would like to have $\Sigma(x_0)$ be any x_1 witnessing the claim. Now since the relation $A(x_0, x_1)$ which says that x_1 satisfies the claim relative to x_0 is projective over $\boldsymbol{\Gamma}$, it is also Suslin, and so we can uniformize it to produce the first round $x_1(x_0)$ of the strategy Σ.

Suppose $\boldsymbol{I}$ now moves x_2 in G. For each such x_2 such that $(x_0, x_1, x_2) \in R$, there is a rule-following simple one-round $\boldsymbol{\Gamma}$ strategy s_0 as in the claim for x_1 and x_2. The relation $A'(x_0, x_2, s_0)$ which says that s_0 satisfies the claim for $x_1 = x_1(x_0)$, x_2 is projective over $\boldsymbol{\Gamma}$, thus is Suslin and so has a uniformization $g(x_0, x_2)$. In the simulation of G^* we will have $\boldsymbol{I}$ play $(x_0, g(x_0, x_2))$. Note that $n_1 = \tau^*(x_0, s_0)$ is such that $x_1 \in M^{s_0}_{n_1}$, and $x_2 = s_0(x_1)$.

This completes the definition of the first round of Σ, and the proof that a one-round play according to Σ has a one-round simulation of G^* according to τ^*, which will guarantee that Σ wins. The definition of Σ for the general round is defined in exactly the same way, using DC to obtain all of the uniformizations required. The above argument also shows that a run of G following Σ has a corresponding run of G^* following τ^*. If $\boldsymbol{I}$ has followed the rules of G, then $\boldsymbol{I}$ has followed the rules of G^* in the associated run. Since τ^* is winning for $\boldsymbol{II}$ in G^*, there is no sequence x'_{2k+1} of moves for $\boldsymbol{II}$ such that $(x_0, x_1', x_2, x_3', \dots) \in B \cap [R]$. In particular, $(x_0, x_1, x_2, x_3, \dots) \notin B$ (since $(x_0, x_1, \dots) \in [R]$). Thus, $\boldsymbol{II}$ has won the run of G following Σ. □

If G is a real game on the Polish space X with rule set R, we will say that G is an *intersection game* if it satisfies the intersection condition of Theorem 4.1. Note that this is equivalent to saying that there is a function $f\colon X^\omega \to Y$ for some Polish space Y such that $f(\vec{x}) = f(\vec{y})$ if $x(2k) = y(2k)$ for all k, and the payoff set for G is of the form $f^{-1}(T)$ for some

$T \subseteq Y$. In many examples, including Schmidt's game, the rules R require the players to play decreasing closed sets with diameters going to 0 in some Polish space, and the function f is simply giving the unique point of intersection of these sets. If we have a fixed rule set R and a fixed function f, the *class of games* $G_{R,f}$ associated to R and f is the collection of games with rules R and payoffs of the form $f^{-1}(T)$ for $T \subseteq Y$. Thus, we will allow the payoff set T to vary, but the set of rules R and the "intersection function" f are fixed. In practice, R and f are usually quite simple, such as Borel relations/functions.

THEOREM 4.2 (AD). *Suppose $\boldsymbol{\Gamma}$ is pointclass within the Suslin, co-Suslin sets which has a universal set, and suppose $G_{R,f}$ is a class of intersection games on the Polish space X with R, $f \in \boldsymbol{\Gamma}$, and R is positional (as above $f\colon X^\omega \to Y$, where Y is a Polish space). Suppose that for every $T \subseteq Y$ which is Suslin and co-Suslin, if player $\boldsymbol{I}$ or $\boldsymbol{II}$ was a winning strategy in $G_{R,f}(T)$, then that player has a winning simple $\boldsymbol{\Gamma}$-strategy. Then for every $T \subseteq Y$, the game $G_{R,f}(T)$ is determined.*

PROOF. Fix the rule set R and function f in $\boldsymbol{\Gamma}$. Let $T \subseteq Y$, we show the real game $G_{R,f}(T)$ is determined. Following Becker, we consider the integer game G where $\boldsymbol{I}$ and $\boldsymbol{II}$ play out reals x and y which code trees (indexed by $\omega^{<\omega}$) of simple one-round $\boldsymbol{\Gamma}$ strategies. They will do this using a fixed universal set U for $\boldsymbol{\Gamma}$. The winning condition for $\boldsymbol{II}$ as follows. If exactly one of x, y fails to code simple $\boldsymbol{\Gamma}$-strategy, then the corresponding player loses. If both players fail to code simple $\boldsymbol{\Gamma}$-strategies, then $\boldsymbol{II}$ wins. If x codes a simple $\boldsymbol{\Gamma}$-strategy σ_x and y codes a simple $\boldsymbol{\Gamma}$-strategy τ_y, then we declare $\boldsymbol{II}$ the winner iff $\sigma_x * \tau_y \in G_{R,f}(T)$, where $\sigma * \tau$ denotes the unique sequence of reals obtained by playing σ and τ against each other. From AD, the game G is determined. Without loss of generality we may assume that $\boldsymbol{II}$ has a winning strategy w for G. Let $S_1 \subseteq \omega^\omega$ be the set of z such that z codes a simple $\boldsymbol{\Gamma}$-strategy for player $\boldsymbol{I}$ which follows the rules R. Likewise, we let S_2 be the set of z coding rule following $\boldsymbol{\Gamma}$-strategies τ_z for $\boldsymbol{II}$. Note that S_1, S_2 are projective over $\boldsymbol{\Gamma}$. Let

$$A = \{\vec{y} \in X^\omega \colon \exists z \in S_1\ \vec{y} = \sigma_z * \tau_{w(z)}\}.$$

Note that because w is a winning strategy for $\boldsymbol{II}$ in G, $A \subseteq X^\omega \setminus G_{R,f}(T)$, so

$f(A) \subseteq Y \setminus T$. Note that A is projective over $\mathbf{\Gamma}$ by the complexity assumption on R and the fact that S_1 is also projective over $\mathbf{\Gamma}$. Further, we claim that it suffices to show that ***II*** wins the real game $G_{R,f}(Y \setminus f(A))$. This is because if ***II*** wins $G_{R,f}(Y \setminus f(A))$ with run $\vec{y}$, i.e. $\vec{y} \notin G_{R,f}(Y \setminus f(A))$, then $f(\vec{y}) \in f(A) \subseteq Y \setminus T$, so $\vec{y} \notin G_{R,f}(T)$, thus $\vec{y}$ is a winning run for ***II*** in $G_{R,f}(T)$.

We see that $Y \setminus f(A)$ is projective over $\mathbf{\Gamma}$, and thus by Theorem 4.1, $G_{R,f}(Y \setminus f(A))$ is equivalent to a Suslin, co-Suslin $\frac{1}{2}\mathbb{R}$ game which is determined, and so $G_{R,f}(Y \setminus f(A))$ is determined. Now by the determinacy of the game, it suffices to show that ***I*** doesn't have a winning strategy in $G_{R,f}(Y \setminus f(A))$.

Suppose ***I*** had a winning strategy for $G_{R,f}(Y \setminus f(A))$. By hypothesis, ***I*** has a winning simple $\mathbf{\Gamma}$-strategy coded by some $z \in \omega^\omega$. Let $\vec{y} = \sigma_z * \tau_{w(z)}$, noting that since $z \in S_1$ we must have $w(z) \in S_2$, or else w wasn't winning for ***II***. Since σ_z is a winning strategy for ***I*** in $G_{R,f}(Y \setminus f(A))$, we have $f(\vec{y}) \in Y \setminus f(A)$. On the other hand, we defined A to simply collect all possible results of w, and so we have that $f(\vec{y}) \in f(A)$, a contradiction. $\square$

We now apply Theorem 4.2 to deduce the determinacy of Schmidt's (α, β, ρ) games in $\mathbb{R}$ from AD.

Theorem 4.3 (AD). *For any $\alpha, \beta \in (0,1)$, any $\rho \in \mathbb{R}_{>0}$, and any $T \subseteq \mathbb{R}$, the (α, β, ρ) Schmidt's game with target set T is determined.*

Proof. We let $\mathbf{\Gamma}$ be the pointclass $\mathbf{\Pi}^1_1$ of co-analytic sets. Let R be the tree described by the rules of the (α, β, ρ) Schmidt's game. R is clearly a closed set and is positional. The function f of Theorem 4.2 is given by $\{f((x_i, \rho_i)_i)\} = \bigcap_i B(x_i, \rho_i)$. This function f clearly satisfies the intersection condition. So we have that $G_{R,f}$ is a class of intersection games. Note also that f is continuous, so $f \in \mathbf{\Gamma}$.

It remains to verify the simple strategy condition of Theorem 4.2. The argument is symmetric between the two players, so we consider only the case of player ***II***. In fact we show that for *any* $T \subseteq \mathbb{R}$, if ***II*** has a winning strategy for the (α, β, ρ) Schmidt's game, then ***II*** has a simple Borel strategy. Let Σ be a winning strategy for ***II*** in this (real) game.

We consider Σ restricted to the first round of the game. Observe that for every first move $z_0 \in \mathbb{R}$, there is a half-open interval I_{z_0} of the form $[z_0, z_0 + \epsilon)$ or $(z_0 - \epsilon, z_0]$ such that for any $x_0 \in I_{z_0}$, we have that $((x_0, \rho_0), \Sigma(z_0, \rho_0)) \in R$. In other words, for any $x_0 \in I_{z_0}$ we have that Σ's response to (z_0, ρ_0) is still a legal response to the play (x_0, ρ_0). Let $\mathcal{C}$ be the collection of all intervals $I = [z, z + \epsilon)$ or $I = (z - \epsilon, z]$ having this property. So, $\mathcal{C}$ is a cover of $\mathbb{R}$ (or more generally, the legal moves for ***I*** at the current position) by half-open intervals. There is a countable subcollection $\mathcal{C}' \subseteq \mathcal{C}$ which covers $\mathbb{R}$. To see that this is the case, first get a countable $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\cup \mathcal{C}_0 \supseteq \bigcup_{I \in \mathcal{C}} \operatorname{int}(I)$. The remainder $\mathbb{R} \setminus \bigcup_{I \in \mathcal{C}} \operatorname{int}(I)$ must be countable, and so adding countably many sets of $\mathcal{C}$ to $\mathcal{C}_0$ will get $\mathcal{C}'$ as desired. Let $\mathcal{C}' = \{I_{z_n}\}_{n \in \omega}$. We define the first round of the simple Borel strategy τ is given by (M_n, y_n) where $M_n = \{(x_0, \rho_0) \colon x_0 \in I_{z_n} \setminus \bigcup_{m<n} I_{z_m}\}$ and $y_n = \Sigma(z_n, \rho_0)$. Clearly (M_n, y_n) is a simple one-round Borel strategy which follows the rules R of the (α, β, ρ) Schmidt's game. This defines the first round of τ. Using DC to obtain the necessary countable subcovers, we continue inductively to define each subsequent round of τ in a similar manner.

To see that τ is a winning strategy for ***II***, simply note that for any run of τ following the rules there is a run of Σ which produces the same point of intersection. □

From this theorem, we immediately obtain the following corollary about Schmidt's original (α, β) game.

Corollary 4.4 (AD). *For any $\alpha, \beta \in (0, 1)$, and any $T \subseteq \mathbb{R}$, exactly one of the following holds.*

(1) *Player **I** has a winning strategy in Schmidt's (α, β) game.*
(2) *Player **II** has a winning strategy in Schmidt's (α, β, ρ) game for every $\rho \in \mathbb{R}_{>0}$, .*

In contrast to these results, we next observe that the situation is dramatically different for $\mathbb{R}^n$, $n \geq 3$.

Theorem 4.5. AD^+ *does not imply that the (α, β, ρ) Schmidt's game for $T \subseteq \mathbb{R}^n$, $n \geq 3$ are determined.*

PROOF. We will show that if all of these games in $\mathbb{R}^3$ are determined, then all relations $R \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized. We focus on the case $n = 3$, since the proof for larger n is identical. Let $R \subseteq \mathbb{R} \times [0, 2\pi)$ such that $\forall x \in \mathbb{R}\ \exists \theta \in [0, 2\pi)\ (x, \theta) \in R$. Let $r = \rho - 2\rho\alpha(1-\beta)\sum_{n=0}^{\infty}(\alpha\beta)^n$. We define the target set for player ***II*** by the formula

$$T = \{(x, r\cos\theta, r\sin\theta) \colon (x, \theta) \in R\} \cup \{(x, y, z) \colon y^2 + z^2 > r\}.$$

The value r is simply the distance from the x-axis that would be obtained if ***I*** makes a first move $B((x_0, 0, 0), \rho)$ centered on the x-axis, and at each subsequent turn ***II*** always moves to maximize the distance from the x-axis and then ***I*** moves to minimize this distance. Note that these moves necessarily will have the same x-coordinate x_0. Thus the target set T codes the relation R to be uniformized along the boundary of the cylinder of radius r centered along the x-axis, with the inside of the cylinder belonging to player ***I*** and the outside belonging to player ***II***.

We claim that ***I*** cannot win the (α, β, ρ) Schmidt's game for this particular T. If ***I*** plays his center not on the x-axis, then ***II*** can win easily in finitely many moves by simply playing to maximize distance to the x-axis. This will win, since r was defined to be exactly the distance if the players opposed each other in minimizing/maximizing distance to the x-axis. Now we can suppose ***I*** plays his first move centered at $(x, 0, 0)$. Fix θ so that $R(x, \theta)$ holds. Then ***II*** can win by always playing tangent towards the point $(x, r\cos\theta, r\sin\theta)$, which will maximize distance to the x-axis. If ***I*** resists and continues to play optimally, minimizing distance to the x-axis, then the limit point will be in $\{(x, r\cos\theta, r\sin\theta) \colon (x, \theta) \in R\}$. If ***I*** ever deviates from this, then again ***II*** can again win after finitely many moves. This shows that ***I*** does not have a winning strategy, so if this game is determined then ***II*** has a winning strategy τ. By exactly the same reasoning of the previous argument, τ must maximize distance from the x-axis in response to optimal play by ***I***. But then we can define a uniformization f of R by

$$f(x) = \theta \iff \tau\Big(B\big((x, 0, 0), \rho\big)\Big) = B\Big((x, (\rho - \alpha\rho)\cos\theta, (\rho - \alpha\rho)\sin\theta), \alpha\rho\Big).$$

□

CHAPTER 5

OTHER RESULTS

In this chapter we give an interesting application of the simple strategy hypothesis for Schmidt's game on $\mathbb{R}$ to show that whichever player has a winning strategy must have a winning positional strategy i.e., a strategy which needs only the latest move to compute a response. Schmidt [25] proved this fact for general intersection games, but the proof heavily relies on the axiom of choice, which we are able to avoid here using simple strategies. We make all this precise as required throughout the chapter.

The fact that strategies can be simplified actually gives a fairly useful result about so-called *positional winning strategies* in Schmidt's game.

Definition 16. A *positional strategy* for a game on a set X is a function $f : X \to X$. A positional strategy is *winning* for a player if every run which follows the strategy on every move, i.e. $x_{n+1} = f(x_n)$ is a win for that player. For player ***I*** a positional strategy must include a special first move, separate from the instructions for responses.

This definition makes sense in the context of intersection games, in which the last move is generally the intersection of all moves up to that point. This definition can be made more general by specifying exactly to what extent information can be ignored by the strategy. For instance, when considering classes of games in which the rules are not quite positional, but vary based on also the current round of the game, it may be more appropriate to call a strategy positional if it considers only the latest move *and* what round of the game it is.

We can use the technology of simple strategies to give us the following theorem regarding the existence of positional strategies in Schmidt's game on $\mathbb{R}$.

Theorem 5.1 ($\mathsf{ZF} + \mathsf{DC}$). *Let $T \subseteq \mathbb{R}$ and $\alpha, \beta \in (0,1)$ and $\rho > 0$. Whichever player has a winning strategy in Schmidt's (α, β, ρ)-game with target set T has a winning positional strategy. If player **I** has a winning strategy in Schmidt's (α, β)-game, then player **I** has a positional winning strategy. If player **II** has a winning strategy in Schmidt's (α, β)-game* *and if any of the following hold*

- AC
- AD *and T is Suslin*
- *T is Borel*

then player ***II*** *has a winning positional strategy.*

REMARK 3. We note that the argument we are about to give is not particular to Schmidt's game, and the only use of DC, as opposed to countable choice, is to guarantee that a simple strategy exists for the winning player (see the proof of Theorem 4.3). The argument below works for any intersection game with positional rules which satisfies the simple strategy hypothesis.

PROOF. We will first prove the portion regarding the (α, β, ρ)-game. We've already proven that whichever player wins has a winning Borel simple strategy. It is worth noting that one can use the complexity of the simple strategy in the proof below to get a complexity bound on the positional strategies in all cases except for player ***II*** winning in the (α, β)-game, but we will not concern ourselves with that here.

Let $\tau = \{(A_{s^\frown i}, (x_{s^\frown i}, r_{s^\frown i})\}_{s \in \omega^{<\omega}, i \in \omega}$ be a simple winning strategy for player ***II*** in the (α, β, ρ)-game. We first define a simple choice function we will need in the proof. Let $s^\frown i \in \omega^{n+1}$. If $s = \emptyset$, let $z_i = z_{s^\frown i} \in A_i$. If $s \neq \emptyset$, let $z_{s^\frown i} \in A_{s^\frown i}$ and a legal response to ***II***'s play of x_s, if such a legal response exists. Otherwise let $z_{s^\frown i} \in A_{s^\frown i}$ be arbitrary. This makes sense, as the rules of the game are positional (see 15). We only use countable choice to define the $z_{s^\frown i}$.

We will define a positional strategy $\hat{\tau}$. Let (x, r) be some potential move by ***I*** for which we need to define a response. If r is not of the form $(\alpha\beta)^n\rho$ for some n, then this move is illegal, and so we may play anything, say $\hat{\tau}(x, r) = (x, \alpha r)$. Now if r is of the form $(\alpha\beta)^n\rho$ for some n, then let $s^\frown i \in \omega^{n+1}$ be lexicographically-least so that $x \in A_{s^\frown i}$ and so that the sequence $(z_{s \restriction 1}, z_{s \restriction 2}, \ldots, z_{s \restriction n}, x)$ of centers for moves by ***I*** is a legal sequence to play against τ. If no such $s^\frown i$ exists, again we play arbitrarily, say $\hat{\tau}(x, r) = (x, \alpha r)$. If we have such an

$s^\frown i$, then τ's responses to both $(z_{s\restriction 1}, z_{s\restriction 2}, \ldots, z_{s\restriction n}, x)$ and $(z_{s\restriction 1}, z_{s\restriction 2}, \ldots, z_{s\restriction n}, z_{s^\frown i})$ of centers will be the same, and so we play

$$\hat{\tau}(x, r) = \tau\left(\left\{\left(z_{(s^\frown i)\restriction j}, (\alpha\beta)^j \rho\right)\right\}_{1 \leq j \leq n+1}\right).$$

This completes the definition of $\hat{\tau}$. To see that it wins, let $\{y_j, r_j\}_{j\in\omega}$ be a run following $\hat{\tau}$. It is important to note that the only case in which we could have played arbitrarily is if our opponent broke the rules, since the $\{A_{s^\frown i}\}_{i\in\omega}$ form a partition of the possible moves by $\boldsymbol{I}$ and so by induction we have a previous $s^\frown i$ which is of the appropriate length which we can extend by any legal move our opponent plays.

By the definition of $\hat{\tau}$ we have, for each y_{2k+1}, some lexicographically least $s_k{}^\frown i_k$ and the corresponding sequence of centers $\left\{z_{(s_k{}^\frown i_k)\restriction j}\right\}_{j\leq k}$ so that

$$(y_{2k+1}, r_{2k+1}) = \tau\left(\left\{\left(z_{(s_k{}^\frown i_k)\restriction j}, (\alpha\beta)^j \rho\right)\right\}_{1 \leq j \leq k+1}\right).$$

By the lexicographical minimality of each $s_k{}^\frown i_k$, it must be that $s_{k+1}(0) \leq s_k(0)$ for all k, and this digit can only decrease finitely often, and so must stabilize. This means $z_{(s_k{}^\frown i_k)\restriction 1}$ and $\tau((z_{(s_k{}^\frown i_k)\restriction 1}, r_0))$ are both eventually constant. For any k large enough so that this has occurred, we must also have $s_{k+1}(1) \leq s_k(1)$, and so $z_{(s_k{}^\frown i_k)\restriction 2}$ and $\tau((z_{(s_k{}^\frown i_k)\restriction 1}, r_0), (z_{(s_k{}^\frown i_k)\restriction 2}, r_2))$ are also eventually constant. Continuing, we have that both the centers $z_{(s_k{}^\frown i_k)\restriction m}$ and the responses $\tau\left(\left\{\left(z_{(s_k{}^\frown i_k)\restriction j}, (\alpha\beta)^j \rho\right)\right\}_{1 \leq j \leq m}\right)$ are eventually constant (as $k \to \infty$) for every fixed $m \geq 1$.

The eventual constant values for these moves give us a sequence of positions which converges to a full run $\{z_0, z_1, \ldots\}$ which is consistent with τ, but this run may disagree with the original run $\{y_0, y_1, \ldots\}$. However, using a legality argument, it is not hard to show that the centers of these moves converge to the same limit point: Let $y_\infty = \lim_{n\to\infty} y_n$, and let $\epsilon > 0$. Let N_0 be large enough so that $(\alpha\beta)^{N_0}\rho < \epsilon/2$, and let M_0 be large enough so that for any $k \geq M_0$, $z_{(s_k{}^\frown i_k)\restriction j+1} = z_{2j}$ for all $j \leq N_0$. Then we have for any $k \geq M_0$ that

$$y_k = \tau\left((z_0, r_0), (z_2, r_2), \ldots (z_{2N_0}, (\alpha\beta)^{N_0}\rho), (z_{(s_k{}^\frown i_k)\restriction N_0+2}, r_{N_0+2}), \ldots (z_{(s_k{}^\frown i_k)\restriction k+1}, r_{k+1})\right).$$

Now since all the moves made are legal (by the choice of $z_{s^\frown i}$ and since τ is winning, and thus rule-following), we can conclude that for any $j \geq 2N_0$, both z_j and y_k are legal moves extending a position in which z_{2N_0} was played, and so they both must be in an interval around z_{2N_0} of radius $(\alpha\beta)^{N_0}\rho$. Thus

$$|z_j - y_k| \leq |z_j - z_{2N_0}| + |z_{2N_0} - y_k| < 2(\alpha\beta)^{N_0}\rho < \epsilon$$

And so we have $z_n \to y_\infty$ as well. Thus since τ is winning for player ***II***, $y_\infty \in T$ and so the run $\{y_j, r_j\}_{j\in\omega}$ is a win for ***II*** as well.

The case for player ***I*** is identical, as we must only include the first move as an extra instruction. Easily then, we also have that the (α, β)-game for player ***I*** is positional, since he is able to decide which ρ to play.

To see that player ***II*** has a positional strategy in the (α, β)-game we must consider the several cases. If AC holds, then the game is positional by an argument included in Schmidt [25], which resembles the argument we gave above, but well-orders all possible moves.

If AD holds and T is Suslin, then since we assume that since player ***II*** has a winning strategy in the (α, β) game, player ***II*** has a winning strategy in the (α, β, ρ)-game for each fixed ρ, which means player ***II*** has a simple Borel winning strategy, which can be thought of as coded by a real using a $\mathbf{\Pi}^1_1$ universal set. We consider the relation on pairs (ρ, τ) which says that τ codes a winning simple strategy in the (α, β, ρ) game. Since T is Suslin, this relation is also Suslin (assuming AD, the Suslin sets are closed under quantification over reals), and thus we can uniformize it to pick a simple winning strategy for each ρ. Now we will mimic the operation above to produce a positional strategy. We must now consider potential moves by ***I*** of the form (x, r) where r is arbitrary, and must choose some ρ uniformly from it to use the simple strategy corresponding to ρ to use the above argument. We simply pretend as though we are playing using the largest possible ρ which is less than or equal to some fixed constant (say 1 for instance) so that $r = (\alpha\beta)^n\rho$ for some n, if such a ρ exists. If not, we play arbitrarily, but legally, say by copying our opponent's center. Note that at some point in the game after enough legal moves, there must eventually be ρ satisfying $r = (\alpha\beta)^n\rho$ for some

n and ρ less than this constant. Once such a ρ exists, we choose the lexicographically-least $s^\frown i$ corresponding to the simple strategy assigned to ρ as before, and define our positional strategy exactly as in the first half of this proof relative to this simple strategy. To see that this wins, we simply observe that our choice of ρ can only increase finitely often, and so must be eventually constant, at which point the argument that we win reduces to the one given above.

In the case that we don't have AD but T is Borel, we note that the relation we uniformized above in this case is $\mathbf{\Pi}^1_1$, and so we can uniformize it with no extra hypotheses. It is important here that the simple strategies are Borel simple strategies. The rest of the argument is the same as in the case of AD and T is Suslin. $\square$

In Chapter 4 we showed that AD suffices to get the determinacy of the (α, β, ρ) Schmidt's game for any target set $T \subseteq \mathbb{R}$, but that for $T \subseteq \mathbb{R}^n$, $n \geq 3$, AD (or AD^+) is not sufficient. In the proof of the positive result for $\mathbb{R}$, we used a reduction of Schmidt's (α, β, ρ) game to a $\frac{1}{2}\mathbb{R}$ game. The negative result, that AD does not suffice for $T \subseteq \mathbb{R}^n$ with $n \geq 3$, shows that generally the (α, β, ρ) Schmidt's game is not equivalent to any integer game (for $T \subseteq \mathbb{R}$ it still may be possible that the game is equivalent to an integer game). The next question we address is to what extent we can reduce Schmidt's game to an integer game. In this chapter we prove two results concerning this question.

In the proof of Theorem 4.5 we defined the value $r = r(\alpha, \beta)$, which was calibrated to the particular values of α, β, and then we used this r to define our target set T. In other words, if we change the values of α, β to α', β', using the same target set, so that $r(\alpha'\beta') \neq r(\alpha, \beta)$, then the game can be shown easily to be determined. We will show in Theorem 5.3 a general result related to this phenomenon. Namely, we show that assuming AD, for a particular T (in any Polish space) and each possible value of $p \in (0, 1)$ there is at most one choice of α, β with $\alpha\beta = p$ which will give that (α, β) Schmidt's game with target set T is not determined. Thus the values of α, β, in fact, must be tuned precisely to have a possibility of the game being not determined from AD.

We note also that the proof of Theorem 4.5 uses critically the ability of each player to play *tangent*, in order to maximize or minimize the distance to the x-axis. In Theorem 5.4 below, we make this more precise by showing that if we modify Schmidt's (α, β, ρ) game by requiring the players to make non-tangent moves, then this modification is determined from AD alone. Thus, the ability of the players to play tangent at each move is a key obstacle in reducing Schmidt's game to an integer game.

In the case of the Banach-Mazur game, the modification to a rational game is fairly straightforward, i.e. the allowed moves for the players are just specifying of balls with centers coming from fixed countable dense subset of X and the radii are positive rationals, in Schmidt's game we make an analogous adjustment, but since α and β may be irrational, we need to allow more possible radii. We give the precise definition now.

Definition 17. For a Polish (X, d) and a fixed countable dense subset $D \subseteq X$ we define the *rational Schmidt* (α, β) game by modifying Schmidt's (α, β)-game to restrict the set of allowed moves for both players to balls $B(x_i, \rho_i)$ where $x_i \in D$ and $\rho_i \in \left(\bigcup_{n,m\in\mathbb{N}} \alpha^n \beta^m \mathbb{Q}_{>0}\right)$.

Theorem 5.2. *Let (X, d) be a Polish space. Let $0 < \alpha < \alpha' < 1$, $0 < \beta' < \beta < 1$, and $\alpha\beta = \alpha'\beta'$.*

1. *If **II** wins the rational Schmidt's (α', β') game for target set T then **II** wins Schmidt's (α, β) game for T.*
2. *If **I** wins the rational Schmidt's (α, β) game for target set T then **I** wins Schmidt's (α', β') game for T.*

Proof. We will prove only the first statement, as the proof of the second is similar. Fix the target set $T \subseteq X$. Let τ be a winning strategy for ***II*** in the rational Schmidt's (α', β') game. We will construct a strategy for ***II*** in Schmidt's (α, β) game by using τ.

Suppose ***I*** plays (x_0, ρ_0) as his first move in the (α, β) game. Let $\rho = \rho_0$ to conserve notation. Let $\rho' \in \left(\bigcup_{n,m\in\mathbb{N}} \alpha^n \beta^m \mathbb{Q}_{>0}\right)$ with

$$\rho \frac{\alpha}{\alpha'} \frac{1-\beta}{1-\beta'} < \rho' < \rho \frac{1-\alpha}{1-\alpha'} \tag{6}$$

We observe that such a choice of ρ' is possible, since $\frac{\alpha}{\alpha'}\frac{1-\beta}{1-\beta'} < 1$ and $\frac{1-\alpha}{1-\alpha'} > 1$ and $\bigcup_{n,m\in\mathbb{N}} \alpha^n\beta^m\mathbb{Q}_{>0}$ is dense in $\mathbb{R}^{>0}$.

Let $\epsilon_n = \min\{(\alpha\beta)^n(\rho(1-\alpha)-\rho'(1-\alpha')), (\alpha\beta)^{n-1}(\alpha'\rho'(1-\beta')-\alpha\rho(1-\beta))\}$.

Note that by Equation (6), we have $\epsilon_n > 0$. Now let $(x_1', \alpha'\rho') = \tau(x_0', \rho')$ where $x_0' \in D \cap B(x_0, \epsilon_0)$. Let $x_1 = x_1'$. By the definition of ϵ_0 and (6), $B(x_1, \alpha\rho) \subseteq B(x_0, \rho)$, thus $(x_1, \alpha\rho)$ is a legal response to (x_0, ρ) in Schmidt's (α, β) game.

Now given a position with centers $\{x_k : k \leq 2n\}$, continue by induction to generate x_{2n+1} by considering $(x'_{2n+1}, (\alpha'\beta')^n\alpha'\rho') = \tau(\{(x_k', r_k) \colon k \leq 2n\})$ where for each $1 \leq k \leq n$, x'_{2k-1} is the response by τ and $x'_{2k} \in D \cap B(x_{2k}, \epsilon_k)$. Again by the choice of ϵ_n and (6), $B(x_{2n+1}, (\alpha\beta)^n\alpha\rho) \subseteq B(x_{2n}, (\alpha\beta)^n\rho)$.

We have defined a strategy for ***II*** in Schmidt's (α, β) game which so that any run which is consistent with this strategy with centers $\{x_k \colon k \in \omega\}$ has a simulation consistent with τ with centers $\{x_k' \colon k \in \omega\}$ such that for all k, $x_{2k+1} = x'_{2k+1}$, so that $\lim_{n\to\infty} x_n' = \lim_{n\to\infty} x_n$ and so since τ was winning in the rational game with target set T, we have $\lim_{n\to\infty} x_n \in T$. So the strategy we have constructed is winning in Schmidt's (α, β) game with target set T. □

As an immediate consequence we have the following.

THEOREM 5.3 (AD). *Let (X, d) be a Polish space. Let $T \subseteq X$. Let $p \in (0,1)$, then there is at most one point $(\alpha, \beta) \in (0,1)^2$ with $\alpha\beta = p$ at which Schmidt's (α, β) game for T is not determined.*

PROOF. Suppose that Schmidt's (α, β) game is not determined with $\alpha\beta = p$. Let $\alpha_1 < \alpha < \alpha_2$ and $\beta_1 > \beta > \beta_2$ with $\alpha_1\beta_1 = \alpha\beta = \alpha_2\beta_2$. By Theorem 5.2 part (1), ***II*** cannot have a winning strategy in the rational Schmidt's (α_2, β_2) game, since ***II*** doesn't have a winning strategy in Schmidt's (α, β) game by assumption. Thus ***I*** must have a winning strategy in the rational Schmidt's (α_2, β_2) game for any such (α_2, β_2) (by AD) and so by Theorem 5.2 part (2), we have that ***I*** wins Schmidt's (γ, δ) game for any $(\gamma, \delta) \in (0,1)^2$ with $\gamma\delta = p$ and $\alpha < \gamma$. A symmetric argument shows that ***I*** has no winning strategy in the rational

Schmidt's (α_1, β_1) game, so ***II*** must have a winning strategy in Schmidt's (γ, δ) game for any $(\gamma, \delta) \in (0,1)^2$ with $\gamma\delta = p$ and $\gamma < \alpha$. $\square$

We now consider a modification of Schmidt's game where we restrict the players to making non-tangent moves. We consider a general Polish space (X, d).

DEFINITION 18. We say the ball $B(x_{n+1}, \rho_{n+1})$ is *tangent* to the ball $B(x_n, \rho_n)$ if $\rho_{n+1} + d(x_n, x_{n+1}) = \rho_n$.

In the *non-tangent* Schmidt's (α, β, ρ) game with target set $T \subseteq X$, the rules are that each player must play moves legal in Schmidt's (α, β, ρ) game, but with the additional restriction that no tangent moves are allowed. Observe that the non-tangent Schmidt's game is still an intersection game, and the rule set R is still Borel. We will show that the simple strategy hypothesis of Theorem 4.2 is also satisfied, thus the non-tangent Schmidt's game is determined from AD. It is clear that the rules of this game are positional, so it will suffice to check the other hypotheses of Theorem 4.2. The proof is quite similar to that of Theorem 4.3.

THEOREM 5.4 (AD). *Let (X, d) be a Polish space, and let $\alpha, \beta \in (0,1)$, $\rho \in \mathbb{R}_{>0}$, and $T \subseteq X$, the non-tangent (α, β, ρ) Schmidt's game with target set T is determined.*

PROOF. We will show that whichever player has a winning strategy in the non-tangent (α, β, ρ) Schmidt game also has a simple Borel winning strategy (in the sense of Definition 12). We will then appeal to Theorem 4.2 to obtain the result.

Without loss of generality, suppose ***II*** has a winning strategy Σ in the non-tangent (α, β, ρ) Schmidt's game. We define a simple Borel strategy τ for ***II*** from Σ. Suppose ***I*** makes first move $B(x_0, \rho)$, and Σ responds with $B(x_1, \alpha\rho)$, which by the rules must not be tangent to $B(x_0, \rho)$. Let $\epsilon = \rho(1-\alpha) - d(x_0, x_1) > 0$. For any x_0' with $d(x_0', x_0) < \epsilon$, if ***I*** had played $B(x_0', \rho)$, then $B(x_1, \alpha\rho)$ is still a legal response for for ***II***. In other words, for each x_0, there is an open ball $U(x_0)$ of some radius, for which any $x_0' \in U(x_0)$ is such that Σ's response to (x_0, ρ) is also legal following (x_0', ρ). Let $\mathcal{C}$ be the set of all such open balls $U(x_0)$. Then $\mathcal{C}$ is an open cover of X, and since X is a Polish space, it is Lindelöf, and thus we can find a countable subcover $\mathcal{C}'$ of $\mathcal{C}$. Let $\mathcal{C}' = \{U(z_n)\}_{n\in\omega}$, where we now choose the points z_n for each of the countably many U in $\mathcal{C}'$. We define the first round of our simple Borel strategy τ by (A_n, y_n) where $A_n = \{(x_0, \rho) \colon x_0 \in U_{z_n} \setminus \bigcup_{m<n} U_{z_m}\}$ and $y_n = \Sigma(z_n, \rho)$. Clearly we have that (A_n, y_n) is a simple one-round Borel strategy and it also follows the rules of the non-tangent (α, β, ρ) Schmidt's game. Using DC to get the countable subcovers, we continue inductively to define each subsequent round of τ in a similar manner.

To see that τ is winning for $\boldsymbol{II}$, we simply note that for any run of τ which follows the rules, we have a run of Σ producing the same point of intersection. $\square$

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