# A Model of Collaboration Network Formation with Heterogeneous Skills

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#### Abstract

Collaboration networks provide a method for examining the highly heterogeneous structure of collaborative communities. However, we still have limited theoretical understanding of how individual heterogeneity relates to network heterogeneity. The model presented here provides a framework linking an individual's skill set to her position in the collaboration network, and the distribution of skills in the population to the structure of the collaboration network as a whole. This model suggests that there is a non-trivial relationship between skills and network position: individuals with a useful combination of skills will have a disproportionate number of links in the network. Indeed, in some cases, an individual's degree is non-monotonic in the number of skills she has-an individual with very few skills may outperform an individual with many. Special cases of the model suggest that the degree distribution of the network will be skewed, even when the distribution of skills is uniform in the population. The degree distribution becomes more skewed as problems become more difficult, leading to a community dominated by a few high-degree superstars. This has striking implications for labor market outcomes in industries where production is largely the result of collaborative effort.

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Collaboration is an increasingly important part of economic activity. As our economy shifts away from manufacturing, towards more knowledge-based industries, the dominant methods of production have shifted away from assembly lines towards collaborative teambased production. As a result, knowledge-based firms look more and more like universities and national labs, where collaborative problem-solving is a vital part of work. It is generally accepted that collaboration is vital because it allows individuals with different skills to pool their efforts and solve more difficult problems than any of them could alone (Hong & Page, 2001, Polzer *et al.*, 2002, Thomas-Hunt *et al.*, 2003, Phillips *et al.*, 2004). As collaboration between heterogeneous workers becomes more central to economic production, it is increasingly important to consider not only the overall composition of the workforce, but also the *structure* of their interactions.

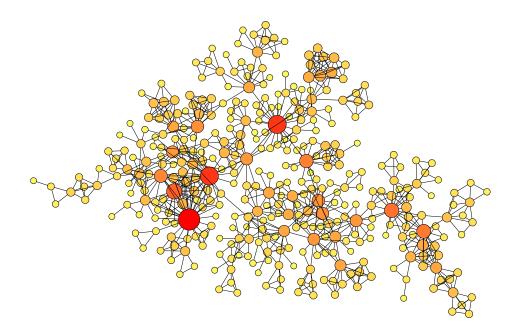


Figure 1: A network of coauthorship among network scientists.

Network theory provides a useful perspective when grappling with this complex set of interactions between collaborators, and a growing literature has examined the structure of collaboration networks, such as the network science coauthorship network shown in Figure 1 (data courtesy of Newman, 2006). One of the most striking features of collaboration networks are their extreme heterogeneity. Some individuals in this network have many links,

while most have very few. This skewed degree distribution can be found in most collaboration networks (Powell *et al.*, 1996, Barabási & Albert, 1999, Newman, 2001, Gleiser & Danon, 2003, Moody, 2004, Uzzi & Spiro, 2005, Acedo *et al.*, 2006, Goyal *et al.*, 2006, Iyer *et al.*, 2006), and suggests a similar disparity in both community engagement and productivity.

There is ample evidence that this network heterogeneity is important in the function of a collaborative community. On an individual level, a person's position in the network reflects her role in the collaborative community, and has an effect on her behavior and outcomes. Network position is correlated with influence (Menzel & Katz, 1955, DeMarzo *et al.*, 2003, Golub & Jackson, 2010, Banerjee *et al.*, 2011), access to knowledge, tools, and information (see Jackson, 2008 for a summary of this literature), and the probability of finding a job (Calvó-Armengol, 2004 and Calvó-Armengol & Jackson, 2004), not to mention that individuals with more collaborators tend to be more productive. On a more global level, the overall structure of the collaboration network reflects the nature of the problem solving community–for example, a workplace where a single individual drives most of the collaborative interaction will function differently from one where no individual is dominant. In particular, the structure of the collaboration network affects the flow of information and ideas, the speed of problem solving efforts, and the spread of effective technologies (see Jackson & Yariv, 2007, Jackson & Rogers, 2007 for examples).

This suggests two interesting classes of questions relating to collaboration networks:

- 1. What characteristics of an individual affect her position on the collaboration network?
- 2. What aspects of the collaborative process (eg: institutions, characteristics of problems and problem solving communities) affect the overall structure of the collaboration network?

A large empirical literature on heterophily in collaboration networks suggests that skill complementarity is an important driving factor behind decisions over collaborators (Moody (2004)Rivera *et al.* (2010)). Therefore, it seems obvious that an individual's skill set will govern their role in the collaboration network —that individuals occupy different places in the network, not solely due to luck, but rather due to their different combinations of skills and abilities—their human capital. However, despite the advances made in understanding network formation (described in detail below) we still have little theoretical understanding of

how this heterogeneity in human capital maps onto observed network heterogeneity. In this paper, I fill that gap with a model that links an individual's skill set to her position on the collaboration network. This allows me to also link the global structure of the collaboration network to the overall distribution of skills in the population.

I start with a population of individuals, each of whom is endowed with a set of discrete skills, drawn from a larger pool of skills that are relevant for solving problems.<sup>1</sup> That set of skills represents the human capital of the problem solver—the tools, techniques and knowledge that are useful inputs to knowledge-based production. When problems are difficult, the individuals collaborate with others who have complementary skills. The result is a collaboration network. The structure of this network and an individual's position in it will necessarily differ depending on the distribution of skills in the population. For example, if individuals in a community have only one skill apiece, then individuals with rare skills will have more collaborators. But when individuals potentially have multiple skills, the picture is not so straightforward. Through several examples, I will show that an individual with rare skills. Moreover, degree is sometimes non-monotonic in the size of one's skill set: individuals with more skills will sometimes have fewer collaborators than those with many skills. These examples illustrate the value of a model with a flexible notion of skill heterogeneity.

In the general model, where the population of problem solvers has an arbitrary distribution of skills, I show that an individual's degree on the collaboration network is a supermodular function of her set of skills. I then use a set of examples to illustrate how a more complex model of human capital allows for a more realistic picture of the relationship between skills and collaborative production. In particular, when individuals are allowed to have multiple skills, it is possible for the relationship between skills and collaboration to be not only nonlinear, but *non-monotonic*.

I then use a special case—called the Bernoulli Skills Model—to show how the process of collaboration can exaggerate small differences in skills to generate surprisingly large gaps in observed degree. Small initial differences in individual skill sets create large differences in

<sup>&</sup>lt;sup>1</sup>This idea of giving individuals multiple skills is rooted in an older labor literature, starting with Roy, 1951, which gives individuals an ability level in each of two different occupations (for example, Roy, 1951 gives an individual a skill level as a "hunter" and a "fisherman"). There has been recent work that explicitly starts to migrate towards a model where individuals can utilize both skills for tasks (rather than simply using one or the other). Examples include Lazear, 2004, 2005 and Astebro & Thompson, 2011. However, all of these works confine their analysis two skills. Here, I consider skill sets of arbitrary size. In a later section, I show that this distinction is significant.

the distribution of links, meaning that the distribution of links in the collaboration network may be skewed, even when the distribution of skills is not. Moreover, as problems become more difficult, the degree distribution of the network becomes more skewed, and the network becomes dominated by a few, high-degree superstars. Using a second special case, called the Ladder Model, I observe another pattern–as skills become increasingly hierarchical, the network becomes increasingly skewed, and superstars emerge.

There is already an extensive literature on the formation of collaboration networks, which I have outlined in more detail below. This model provides insights beyond those offered by these models. On an individual level, this model provides a useful connection between agent heterogeneity and outcomes. Preferential attachment provides a dynamic in which individuals who are already successful might become more successful. The model presented here connects link formation to skill complementarity, and thus provides insights into what happens before the preferential attachment dynamic takes hold. This is pertinent in collaborative communities, because people enter the system all the time. The question of which of those individuals will eventually become a star has not been completely explored in a theoretical framework. On a more global level, this paper provides several predictions about how collaboration networks will differ in different populations. The model predicts that degree distributions will become more skewed as problems become more difficult. There is some indication that this is true in academic fields. In particular, data on economists suggests that the degree distribution of the network has become more unequal over time. If we believe that research problems have also become more complex over that time period, then this theory suggests a possible explanation for that change. By linking the distribution of skills in the population to the structure of the collaboration network, this model provides insights into what generates the differences between communities.

The framework and results I present here also have implications for our understanding of heterogeneity in labor markets. As firms in knowledge-based industries shift towards team-based production, labor productivity becomes increasingly tied to collaborative effort. I show that when workers are allowed to have multiple skills, the resulting distribution of productivity is much different than when we assume that individuals have a "type" (e.g.: hunter or fisherman). Moreover, because skills are synergistic and work in combination, the relationship between an individual's skill set and her wages cannot be captured by a linear pricing schedule, where each skill is valued in isolation. This means that treating skills in combination may explain more wage variation than more traditional economic understanding of labor heterogeneity. I include a brief discussion of these issues below.

It is also worth noting the broader contribution of this model—it provides a general framework for understanding how individual heterogeneity affects both network structure and collaborative production. Particular specifications of the fundamentals of the model—the distribution of skills, the distribution of problems, the functional form of the production and payoff functions—will produce different outcomes. Thus, the framework introduced here has the potential to spawn additional work, exploring those relationships. I conclude with a discussion of some of those possible extensions.

### 1 Literature

This paper draws upon two distinct literatures. The first is the literature linking skills to productivity and labor market outcomes. This literature treats skills in one of several canonical ways. One branch of the literature treats skills in terms of occupation: an individual is categorized according to what job they perform, and is allowed only one category. For example, an employee might be a manager or a line worker, but not a combination of both. This model works well in the context of manufacturing, where each individual has a well-defined job, but works less well in the context of knowledge-based production. Another branch of the literature treats skill level as a one-dimensional quantity, ultimately placing individuals in either high or low skilled baskets. This includes the wealth of signaling literature, starting with Spence, 1973. This method for modeling skills is convenient, because it allows a partial ordering over individuals. However, it seems a better model for some skills (eg: piece work) than others (eg: problem solving). A third literature takes a kind of hybrid approach, allowing individuals to have an ability level in two different skills. This includes the vast literature stemming from Roy, 1951. Roy imposes a condition that individuals use only one of their skills at a time. More recent work has allowed individuals to use both skills at once. Examples include Lazear, 2004, 2005 and Astebro & Thompson, 2011

This paper is closest to the last category of models, but represents a significant generalization from this approach. In particular, individuals are not confined to two skills, and may have an arbitrarily complex basket of skills. This model of skill heterogeneity is appealing, because it better represents the rich, multi-faceted skill sets of workers in knowledge-based industries. Moreover, as I will show in Section 4, allowing individuals to have three or more skills has a significant impact on the distribution of productivity in the collaborative community, making it a crucial extension of previous labor models. Moreover, by altering the distribution of skills in the population, this model encompasses all three canonical models of labor heterogeneity, while also allowing for other, more complex types of heterogeneity.<sup>2</sup>

This more multi-dimensional model of skills also interfaces well with the empirical literature on collaborative linking decisions. When problems are hard, problem-solvers tend to seek out those who have skills different than their own (Rivera *et al.*, 2010). Moody, 2004 suggests that individuals facing difficult problems seek out those with complementary skills because it is easier to work with other researchers than it would be to obtain a new set of skills themselves. Here, I take this need for skill complementarity in collaboration as given.

This paper also contributes to the literature on social network formation. This literature is vast, and growing rapidly. One branch of the literature focuses on the role of individual decisions in network formation. For example, in the Connections Model (Jackson & Wolinsky, 1996) players gain a benefit for both direct and indirect links and pay a cost for each direct link made. In the Coauthor Model, Jackson & Wolinsky, 1996, players must allocate their effort across multiple projects where the payoff from a paper is inversely related to the number of links the two coauthors have. In Goyal & Moraga-González, 2001, firms choose a set of links and an effort level to put into research and development, given that such links general both perfect and imperfect spillovers. The resulting networks are useful because individuals in decision-based models respond to incentives, allowing us to see how changes in the community change the fundamentals of network structure, such as density.

Another branch of the literature allows links to form via a stochastic process. In the preferential attachment model, entering nodes link to existing nodes with a probability proportional to their current degree. Nodes who gain an early benefit from extra links tend to amass even more links over time. This model and its derivatives provide a glimpse into the mechanisms behind the observed skew in degree distribution (see, among others, Barabási & Albert, 1999, Jackson & Rogers, 2007 and Ramasco *et al.*, 2007). The incumbency model of network growth presented in Guimerà *et al.*, 2005 captures a different dynamic of collaborative network formation-the role of incumbency. Individuals create new links via

<sup>&</sup>lt;sup>2</sup>It is simple to see how this model subsumes the model with two discrete skills. However, it is flexible enough to subsume the other two models as well. In particular, by allowing for skills to build off of one another (as in Section 6), then it is possible to move away from a binary skill set to one where individuals have an ability ordering in one or more skill areas.

serial team formation. In every round, they form teams taking into account both whether a potential team member is new to the community, and any previous working relationships.

There has also been progress made in linking network heterogeneity to other kinds of individual heterogeneity: Jackson & Rogers, 2005 considers heterogeneity in the costs and benefits of link formation, Galeotti *et al.*, 2006 considers heterogeneity in link costs in Nash Networks, and Carayol & Roux, 2009 consider the effects of geographic inhomogeneity. However, none of these previous works consider a type of heterogeneity crucial to an economic understanding of collaborative production: labor heterogeneity.

The contributions of this paper to the literature on network formation take place on two different levels. On a global level, this model links differences in network structure to differences in the underlying population. This provides new insights into why different collaboration networks have different structures. On the individual level, allowing individuals to have heterogeneous skill sets allows us to explore the determinants of network position, and in particular, which individuals become stars.

In this paper, I combines these two literatures to create a powerful, yet flexible theoretical framework for understanding how labor heterogeneity shapes network formation. The field of labor economics provides insight into skill heterogeneity, giving us perspective on the role of skill skill sets in determining an individual's role in a collaboration networ, and conversely, considering collaboration networks can give greater understanding of how those skill sets are translated into labor market outcomes in the case of collaborative, team-based production.

# 2 A General Model of Skills, Problem Solving, and Collaboration Networks

#### 2.1 Inputs: Problem Solving Population and Problems

Let  $I = \{1, 2, ... N\}$  be the set of problem solvers (individuals).

Let  $S = \{a_1...a_M\}$  denote the (finite) set of all skills.

A individual *i*'s *skill set* is the subset of those skills she possesses,  $A_i \subseteq S$ . The frequency of skill set A in the population is given by  $\Psi(A)$ , a probability measure with support  $\Sigma(\Psi) \subseteq 2^S$ -that is,  $\Psi(A)$  is the fraction of the individuals in I who have the skill set  $A\subseteq S.^3$ 

Each individual is endowed with a copy of a problem requiring a subset of the skills available in the population,  $\omega \subseteq S.^4$ 

Thus, the inputs to the model are a set of skills used to solve problems (S) and a population of problem solvers  $(\Psi)$ .

#### 2.2 Collaboration and Problem Solving

A collaboration is a subset of the problem solvers,  $C \subset I$ . A collaboration can solve a problem if together they possess all of the required skills-that is, if the problem is solved if  $\omega \subseteq \bigcup_{i \in C_i} A_j$ .

The problem yields a payoff of 1 if solved. If a problem solver can solve her problem alone (that is, if  $\omega \subseteq A_i$ ) then she keeps the entire payoff. If she solves it with the help of others, she splits the payoff evenly with her collaborators, giving each a share of  $\frac{1}{|C_i|}$ .<sup>5</sup>Since others in the community face their own problems, a problem solver may be asked to help with other problems as well. Thus, individual *i*'s payoff is the sum the payoff she gets from solving her own problem, plus the payoffs she gets from collaborating with others on their problems:

$$u_{i} = \frac{1}{|C_{i}|} + \sum_{j \neq i \text{ st } i \in C_{j}} \frac{1}{|C_{j}|}$$

<sup>6</sup> An individual chooses her set of collaborators  $(C_i)$  from the pool of all possible collaborators  $(I)^7$  to maximize her utility. Note that her payoff to solving her own problem is always

<sup>&</sup>lt;sup>3</sup>Formally,  $\Psi$  is a *frequency* distribution, rather than a probability distribution–that is,  $\Psi$  is a *realized* distribution of skill sets across the population. The distinction between frequency and probability distributions disappears when N is large–using a frequency distribution allows me to also make statements about small N as well.

<sup>&</sup>lt;sup>4</sup>For simplicity, I assume that all individuals face the same problem. The results are identical if each individual faces a different problem drawn from a known distribution of problems,  $\Omega$ , with support  $2^S$ , and collaborators are chosen *ex ante*. If they choose their collaborators *ex post*, then the results are similar, if notationally more complex.

<sup>&</sup>lt;sup>5</sup>This particular distribution scheme has a number of points to recommend it: it does not require that agents have control over the way payoffs are split (advantageous when payoffs are non-monetary), and it gives each individual her Shapely value for the coalition. Many alternative splitting schemes will be behaviorally identical (see below).

<sup>&</sup>lt;sup>6</sup>It is important to highlight that this payoff is not equivalent to a market wage. A model producing equilibrium wages would require considerable additional machinery, including time constraints and search costs, and is therefore beyond the scope of the current work.

<sup>&</sup>lt;sup>7</sup>Note that this model assumes that an individual looking for collaborators knows the skill sets of all her colleagues, and thus does not address the issue of search. This assumption is not problematic when collaborative communities are small or tight-knit (eg: within firm or in research subfields). However, in other contexts-particularly ones where particular skill combinations are very rare-this assumption of global information may start to become unrealistic. A model incorporating local search may alter the results

positive, and thus it is always incentive compatible for her to solve the problem. Since each individual controls only her own collaborative decisions, the utility-maximizer chooses  $C_i$  to minimize the number of collaborators she must work with on her own problem-in other words, she chooses a minimal subcover of the set of skills she lacks- $A_i^c = \omega_i \setminus A_i$ . Let  $\mathbb{C}_i$  denote the set of all minimal subcovers of  $A_i^c$ . If there exist multiple minimal subcovers (ie: if  $|\mathbb{C}_i| > 1$ ) then I assume that the individual chooses  $C_i^* \in \mathbb{C}_i$  at random.<sup>8</sup>

It is worth noting that while I have chosen to model the motivate the decision to minimize the set of collaborators via an equal split of payoffs, the results that follow are unchanged under any distribution scheme which induces problem solvers to minimize the number of collaborators they use. So, for example, if collaborating with additional individuals is costly (eg: in terms of time or communication), then the results that follow will still hold with no modification. The results will not hold under any payoff structure where it is not individually rational to minimize the set of collaborators used to solve a problem.

#### 2.3 Complementary Skills Networks

For a given a set of collaborations,  $C = \{C_1...C_N\}$ , the collaboration network is represented by an adjacency matrix, g(C), where  $g_{ij}(C) = 1$  if  $j \in C_i$ . Note that the network is directed-since  $j \in C_i$  does not necessarily imply  $i \in C_j$ , it may be that  $g_{ij}(C) \neq g_{ji}(C)$ . However, the links are mutual, in the sense that j will never want to terminate a link (see Section 2.6.1 for further discussion). When all collaborators are chosen optimally (that is, when  $C_i \in \mathbb{C}_i \forall i$ ), I will call the result a *complementary skills network* (or *complementarity network* for short).

**Definition.** A network, g(C), is a complementary skills network (complementarity network) if each individual in the network chooses a minimal set of collaborators required to solve her problem–eg: if  $C_i \in \mathbb{C}_i \forall i$ .

Since the set of minimal subcovers each individual  $(\mathbb{C}_i)$ , depends on the distribution of skills in the population, I use  $\Gamma(\Psi)$  to denote the set of complementary skills networks for a particular distribution of skills,  $\Psi$ . In other words,  $\Gamma(\Psi)$  is a set of networks over which

presented here, and would be a fertile area for study.

<sup>&</sup>lt;sup>8</sup>Since the individual indifferent between minimal subcovers, this choice at random follows convention. The results that follow are not sensitive to this assumption. In particular, if an individual is biased towards choosing collaborators who are not as busy (eg: those with fewer links) then the results that follow hold precisely, rather than on average. If an individual is biased towards choosing collaborators who are popular (eg: have many links) than the results that follow are qualitatively similar, and if anything are more striking.

all individuals are maximizing utility. In all of the measures that follow, I will average over all networks in  $\Gamma(\Psi)$ .<sup>9</sup>

#### 2.4 Network Measures

Recall that I will be looking at two different questions: first, how an individual's position on the network depends on her set of skills, and second how the overall structure of the network depends on the distribution of skills in the population. For both of these questions, I will focus on degree in the network. While degree is clearly not the only network measure defining an individual's position in the collaborative community, an exploration of other measures is outside the scope of the current work.

First, I will look at how an individual's average in-degree on the networks in  $\Gamma(\Psi)$  depends on her skill set. I will denote individual *i*'s in-degree on a particular network as  $d_i$ . Her average in-degree will be  $E[d_i]$ , where the expectation is taken over all networks in  $\Gamma(\Psi)$ -that is, all networks that the individual is indifferent between. In Section 3, I will consider the mapping between an individual's skill set an her expected in-degree in a particular problem-solving population,  $E[d_i] = f(A_i, \Psi)$ .

I will also consider how the overall distribution of in-degree depends on the distribution of skills in the population. Let  $\Delta$  denote the distribution of expected in-degree. That is,  $\Delta(d)$  is the fraction of the individuals who have expected in-degree d, where the expectation is taken over all  $g(C) \in \Gamma(\Psi)$ .<sup>10</sup>

Before continuing, a brief word about network notation is in order. First, note that for ease of reading I will usually drop the argument of g(C). I will denote a link from i to jby ij. Using a slight abuse of notation, I will use g to refer to both the adjacency matrix (as above) and the set of links in the network-that is,  $ij \in g$  if i is connected to j in the network g. In a similar abuse of notation, I will use g - ij to represent the network that results when the link ij is removed from an existing network, g, and g + ij to represent the network that results when the link ij is added to the existing network, g. Finally, for clarity in the exposition, I will refer to in-degree simply as "degree". This last point should cause

 $<sup>^{9}</sup>$ Note that this averaging has the effect of dividing the space of all possible networks into equivalent classes of networks that are functionally equivalent from the perspective on individuals in the community. This is a useful coarse-graining technique that may have applications outside of the current model.

<sup>&</sup>lt;sup>10</sup>Alternatively, we might plot the distribution of degree across all networks  $g \in \Gamma(\Psi)$ . That is, we could set  $\Delta(d) = \sum_{g \in \Gamma(\Psi)} \delta_g(d)$  where  $\delta_g(d)$  is the fraction of nodes in network g with degree d. This choice does not affect the results.

no significant confusion–I use in-degree because it has a clear, empirical interpretation, but the results qualitatively similar if we consider an individual's degree in the directed network (the sum of his in-degree and out-degree), or use the degree of the individual in a network where directed links are projected into undirected links.

#### 2.5 Example

An example will help clarify the structure of this model. Suppose all of the individuals in the population face the same problem requiring three skills:  $\omega = \{a, b, c\} \forall i$  –for example, developing a web application might require programming skills, user interface design skills, and marketing skills. Suppose the distribution of skills is such that everyone in the population has at least one skill, but no one has all of the skills required. In other words, every individual in the population has something to add, but none of them can solve the problem on their own. Suppose further that each skill combination is equally likely–in other words,  $\Psi(A) = \frac{1}{6}$  for all  $A \in \{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}\}$  and  $\Psi(A) = 0$  otherwise (see Figure 2).

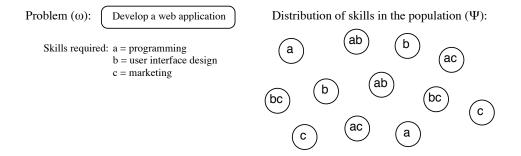


Figure 2: An example of a problem and population of problem solvers

In this particular population of problem solvers, each individual needs exactly one collaborator to solve the problem-that is, someone with skill set  $\{a\}$  must link to someone with the skill set  $\{b, c\}$ , someone with skill set  $\{b, c\}$  may choose from those with skill sets  $\{a\}, \{a, b\}$ , and  $\{a, c\}$ , and so on. The problem solver is indifferent between any two individuals who have the skills that she needs. By linking those who collaborate on problems, we obtain a collaboration network. There will be one such network for every set of optimal collaborations. Figure 3 shows two collaborators. The set of all such networks for a given population of problem solvers is denoted by  $\Gamma(\Psi)$ .

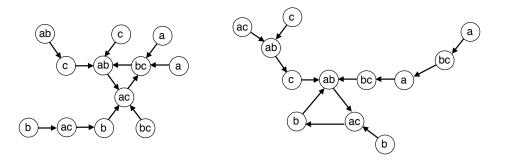


Figure 3: An example of two networks formed from population/problem in Figure 2

#### 2.6 Discussion

This model produces outcomes consistent with several empirical facts about collaboration and problem solving. First, the model predicts that collaboration will be more important when the average individual has a thinner slice of the total skills required to solve a problem– in other words, as problems become more difficult, collaboration networks will become more densely connected. This prediction is born out in the data–collaborative work has become increasingly common in a variety of academic fields (see Barabasi *et al.*, 2002 (physics), Grossman & Ion, 1995 (mathematics), Moody, 2004 (sociology), Acedo *et al.*, 2006 (management science), Laband & Tollison, 2000 and Goyal *et al.*, 2006 (economics)). Moreover, the literature supports a connection between this increased reliance on collaboration and the difficulty of problems faced, specifically the increasing complexity of required methodologies (Laband & Tollison, 2000and Moody, 2004). This highlights one of the advantages of using this model of collaboration network formation: it allows for a direct connection between problem difficulty and collaborative effort.

#### 2.6.1 A Note on Stability and Efficiency of the Complementarity Network

Before considering any specific questions about the complementarity network, it is worth considering it's stability and efficiency. Jackson & Wolinsky, 1996 introduce an equilibrium concept of network stability, called *pairwise stability*. Briefly, a network is pairwise stable if no individual would prefer to terminate an existing link, and if no pair of individuals would prefer to add a link (see Appendix A for a more formal definition in the case of a

directed network). Pairwise stability implies that links are mutual, because both individuals involved agree to maintain the link. Theorem 1 states that any complementary skills network is pairwise stable, implying that all links in the network are mutual. Because the equilibrium links are mutually beneficial, one could functionally think of a complementarity network as either directed (because that is how it is constructed) or undirected (because that it is how it functions). For ease of reading, I will usually omit the directional arrows from networks pictured in this paper.

Moreover, any complementary skills network is strongly efficient-that is, the population extracts the maximum possible value from the network. This result-that equilibrium networks are efficient-contrasts with two other models of social network formation-Jackson & Wolinsky, 1996 and Goyal & Moraga-González, 2001-in which pairwise stable networks tend to have more links than is efficient.

**Theorem 1.** Any complementary skills network,  $g \in \Gamma(\Psi)$ , is pairwise stable and strongly efficient.

Proof. See Appendix A

### 3 Skills and Degree in the Collaboration Network

In this section, I consider the relationship between an individual's skill set, and her degree in the collaboration network.

#### 3.1 Skills and Degree: An Example

Before presenting general results, it is useful to see an example. Consider the example from the previous section: each individual faces a problem requiring three skills,  $S = \{a, b, c\}$ , and they each have one or two of those skills, so that  $\Psi$  has support  $\{\{a\}, \{b\}, \{c\}, \{ab\}, \{ac\}, \{bc\}\}$ .

An individual's in-degree on the network will depend on the number of people who need her skills (the demand for a subset of her skills) and the number of other people who can provide those skills (the supply of that subset of skills). For example, consider someone with the skill set  $\{a\}$ . Her skills are in demand by anyone who needs skill *a*-in this example, anyone with skill set  $\{b, c\}$ . A person with skill set  $\{b, c\}$  may collaborate with anyone who has skill *a*, including those with skill sets  $\{a\}, \{a, b\}, \text{ or } \{a, c\}$ . So in this example, the

expected degree of an individual with skill set  $\{a\}$  is

$$E\left[d\left(\{a\}\right)\right] = \frac{\Psi\left(\{b,c\}\right)}{\Psi\left(\{a\}\right) + \Psi\left(\{a,b\}\right) + \Psi\left(\{a,c\}\right)}$$

Similarly, an individual with the skill set  $\{a, b\}$  can help anyone who needs skill a, anyone who needs skill b, and anyone who needs skills a and b, yielding expected degree

$$E\left[d\left(\{a,b\}\right)\right] = \frac{\Psi\left(\{b,c\}\right)}{\Psi\left(\{a\}\right) + \Psi\left(\{a,b\}\right) + \Psi\left(\{a,c\}\right)} + \frac{\Psi\left(\{a,c\}\right)}{\Psi\left(\{b\}\right) + \Psi\left(\{a,b\}\right) + \Psi\left(\{b,c\}\right)} + \frac{\Psi\left(\{c\}\right)}{\Psi\left(\{b,c\}\right)} + \frac{\Psi\left(\{b,c\}\right)}{\Psi\left(\{b,c\}\right)} + \frac{\Psi\left(\{b,c\}\right)}{\Psi\left(\{b,c\}\right)}$$

Note that an individual with skills a and b will have more links, on average, than an individual with skill a and an individual with skill b put together:  $E[d(\{a,b\})] > E[d(\{a\})] + E[d(\{b\})]$ . This is because an individual with both skills can help anyone who needs skill a, anyone who needs skill b, and anyone who needs both. This means that an individual with both of those skills will have, on average, more than twice the number of links than a person with either of those skills in isolation.

#### 3.2 Skills and Degree: General Results

A similar type of result holds more generally. Theorem 2 states that an individual's expected degree in a complementary skills network is a supermodular function of her set of skills. That means that regardless of the skill set required for the problem or the distribution of skills in the population, an individual with skill set  $A \cup B$  will have at least as many links as individuals with skill sets A and B put together. The intuition for this result is similar to the above example–the set of all problems that can be solved by someone with the skill set  $A \cup B$  includes those that can be solved by someone with skill set A, and those that can be solved by someone with skill sets.

**Theorem 2.** For any set of skills, S, and distribution of those skills,  $\Psi$ , an individual's expected degree over the networks in  $\Gamma(\Psi)$  is a supermodular function of her set of skills. That is,  $Ed(A \cup B) + Ed(A \cap B) \ge Ed(A) + Ed(B)$ .

*Proof.* Here, I will prove the result for the case where an individual needs only one collaborator to solve her problem. For the sake of clarity, also I consider the case where  $A \cap B = \emptyset$ . The proof for the general result is similar, and can be found in Appendix B. Since  $d(A \cap B) = d(\emptyset) = 0$ , we need to show that  $Ed(A \cup B) \ge Ed(A) + Ed(B)$ . Consider

 $d(A \cup B)$ . The fraction of the population needing  $C \subseteq A \cup B$  is  $\delta(C) = \Psi(S \setminus C)$ . The fraction who can supply the set C is  $\sigma(C) = \sum_{D \subseteq S \setminus C} \Psi(C \cup D)$ . Thus

$$E\left[d\left(A\cup B\right)\right] = \sum_{C\subseteq A\cup B} \frac{\Psi\left(S\backslash C\right)}{\sum_{D\subseteq S\backslash C} \Psi\left(C\cup D\right)} = \sum_{C\subseteq A\cup B} \frac{\delta\left(C\right)}{\sigma\left(C\right)}$$

We can partition  $C \subseteq A \cup B$  into three categories:

1)  $C \subseteq A$ 2)  $C \subseteq B$ 3)  $C \subseteq A \cup B, C \notin A, C \notin B.$ 

This division gives us the following:

$$E\left[d\left(A\cup B\right)\right] = \sum_{C\subseteq A} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C\subseteq B} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C\subseteq A\cup B \text{ and } C\cap A, \ C\cap B\neq\emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)}$$
$$= E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right] + \phi$$
$$\geq E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right]$$

This theorem suggests an immediate corollary.

**Corollary 3.** Adding skills to an individual's skill set will never decrease her average degree in a cost minimizing collaboration network.

*Proof.* From Theorem 2,  $E[d(A \cup a)] + E[d(A \cap a)] = E[d(A \cup a)] \ge E[d(A)] + E[d(a)]$ , and so  $E[d(A \cup a)] - E[d(A)] \ge E[d(a)] \ge 0$ . □

# 4 The Importance of Skill Combinations and Complex Heterogeneity

The result presented in Theorem 2 has some implications for who we think is important in a collaborative community. In particular, it indicates that it is important to consider an individual's *combination* of skills, rather than looking at each of her skills in isolation. When actors are able to consider a potential collaborator's skills in combination, the result is much different than one would see in a model where skills are evaluated individually. As an illustration, suppose there are two different problem solving populations, faced with the same problem,  $\omega = \{a, b, c\}$ . In population A, every individual has exactly one skill, which we can think of as her speciality or "type":  $\Psi(a) = \frac{1}{3}$ ,  $\Psi(b) = \frac{1}{3}$ ,  $\Psi(c) = \frac{1}{3}$ . In population B, the skills are distributed independently. This means that and the probability of having skill set A is  $\Psi(A) = \prod_{i \in A} p_i$ . Let  $p_i = Prob(have skill i) = \frac{1}{3}$  for i = a, b, c.

These two populations have much in common: in both, the three skills occur in equal proportion, and both populations average one skill per person. However, as illustrated in Figure 4, the collaboration network is much different in the two different populations.

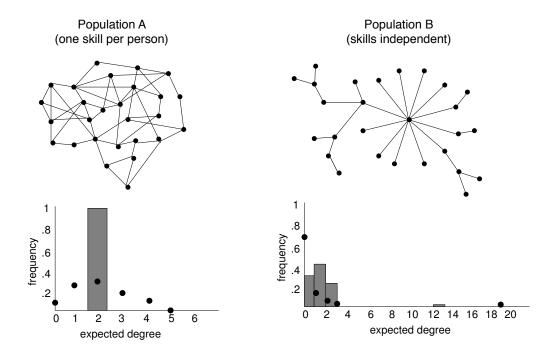


Figure 4: A population with types and a population with independent skills.

This pair of examples highlights the value of a more detailed treatment of skills in modeling problem solving. Although it is possible to model the heterogeneity of problem solvers more simply–for example, via a one-dimensional ability level (Gautier, 2002, Shi, 2002), by giving each individual a type or speciality (Hamilton *et al.*, 2000), or some combination of the two (Jovanovic, 1994), these modeling choices are not necessarily benign.

Allowing for correlations between skills can generate linking behavior that non-monotonic in the number of skills an individual has–a case that is both relevant empirically and not present in other models of collaboration networks. For example, consider the population summarized in the first panel of Figure 5. The problem requires five skills,  $S = \{a, b, c, d, e\}$ ,

Population 1						
	а	b	с	d	е	In-degree
Individual 1	х	х	х			1 1/2
Individual 2	х	х	х			1 1/2
Individual 3				х	х	3
Individual 4				х		1/2
Individual 5					х	1/2

Demulation 1

Population 2

	а	b	с	d	е	In-degree
Individual 1	х	х	Х			1 1/2
Individual 2	х	Х	Х			1 1/2
Individual 3				х	х	3
Individual 4	х	х		х		1/2
Individual 5	х	х			х	1/2

Figure 5: Two populations of problem solvers with five skills  $S = \{a, b, c, d, e\}$ -n copies of each individual yields a population of size N = 5n.

which are distributed across a population of N = 5n as indicated. Each skill is held by exactly 2n individuals, and thus no skill is rarer than the others. Traditionally, we might condense the information contained in this table into a single-dimensional measure of ability– individuals 1 and 2 have the most skills, and therefore, we might expect them to have the most links. However, despite having fewer skills, individual 3 will, on average have more links. This is because while neither of individual 3's skills are rare by themselves, in combination, they are both rare, and useful to a large fraction of the population. This means that agent 3 will have more links than a tally of her individual skills might predict—-in other words, the value of a combination of skills may be greater than the sum of its parts. Figure 6 shows another example. In this case, skills d and e are more common than skill a, but the individual with skills d and e gets more links, showing that an individual with common skills might even have more links than an individual with rarer skills.

Examples of this phenomenon are not difficult to find. For example, consider a tech entrepreneur who has experience in both computer programming and marketing. Neither of those skills are rare individually, but together they are quire rare. Individuals with this combination of skills will be in high demand, and will have more collaborative connections than those without that combination of skills. This further suggests that in problem solving communities such as those in knowledge-based industries, entrepreneurial firms, and academic research, models in which individuals are scored on a one-dimensional ability scale

	_	-	_			
	а	b	с	d	е	In-degree
Individual 1	х	Х	Х			2 1/2
Individual 2	х	Х	Х			2 1/2
Individual 3				x	х	3 1/3
Individual 4			х	x		2/3
Individual 5				x		2/3
Individual 6		х			х	2/3
Individual 7					х	2/3

Population 3

Figure 6: A third populations of problem solvers with five skills  $S = \{a, b, c, d, e\}-n$  copies of each individual yields a population of size N = 7n.

will fail to capture the full effect of variation between individuals.

Pushing this point a bit further suggests another implication of Theorem 2–because degree is a supermodular function of a problem solver's complete set of skills, it is not generically possible to assign prices to individual skills in a way that captures her degree on the social network. This means that examining the supply and demand of single skills in isolation does not necessarily capture an individual's value to a community of problem solvers.

# **Corollary 4.** There need exist no vector of prices, $\mu$ , such that $\sum_{a \in A} \mu_a = d(A)$ .

To further emphasize this point, consider the skill distribution shown in the second panel of in Figure 5. This distribution is identical to that in the first panel, except that individuals 4 and 5 have been given an extra skill. However, gaining these skills does not influence the degree of either. In fact, endowing them with those skills does not change any part of the degree distribution. Skills a and b have value to individuals 1 and 2, but not to individuals 3 or 4–clearly, no linear weighting of the individual skills could produce such a pattern. Two characteristics of this model contribute to this result. First, skills are bundled within a person. Thus, in evaluating a collaborator, that person's combination of skills must be considered as a unit. Second, the individuals in the model have an incentive to minimize the number of collaborators they work with. Together, these two factors mean that an individual's value to the collaborative community may be more than the sum of her individual skills.

These results have implications for empirical models of labor market outcomes. If we assume that skills contribute to outcomes independently, then we will underestimate the value of particular combinations of skills and overestimate the value of other combinations. Moreover, this non-linear relationship between skills and degree means that if we observe skills imperfectly, the amount of variation in outcomes that is explained by an observed set of skills will decline dramatically in the set of skills we can observe. Additionally, the fact that an individual's degree depends on both the distribution of skills in the population and the set of skills she already has suggests that in collaborative fields, optimal training decisions are highly individualized.<sup>11</sup>

# 5 The Distribution of Skills and the Structure of the Collaboration Network: The Bernoulli Skills Model

In this section, I look at how the distribution of skills in the population affects the degree distribution of the collaboration network.

#### 5.1 The Bernoulli Skills Model

Thus far, I have considered results for a general skill population. However, when we turn to the effects of skill populations on network structure, it is difficult to obtain clear predictions using such a general construction. Therefore, in this section I will consider a special case, where skills are distributed independently with equal probability-that is,  $Prob(a_i \in A | a_j \in A) = Prob(a_i \in A) = p \forall i \neq j \in S$ . I call this special case the Bernoulli Skills Model because each individual's skill set can be thought of as the result of a set of M Bernoulli trials, each with probability p of success. This means that the distribution of skill set sizes in the population is binomial, implying that the fraction problem solvers who have a particular set of k skills is  $\Psi(A) = p^k (1-p)^{M-k}$ , and the fraction having any k

<sup>&</sup>lt;sup>11</sup>To see this, consider the Shapley value of an additional skill. When an each individual requires only one collaborator to solve a problem, the individual's degree is , (the results are similar for the case where multiple collaborators may be required). Using degree,  $d(A_i) = \sum_{C \subseteq A_i} \frac{\Psi(S \setminus C)}{\sum_{D \subseteq S \setminus C} \Psi(C \cup D)}$ , as a value function (note that d(.) satisfies both  $d(\emptyset) = 0$  and superadditivity), the Shapely value of a skill, a, to individual i:  $\phi_{a,i}(d) = \sum_{B \subseteq A_i \setminus \{a\}} \frac{1}{\binom{|A_i|}{|B|}} \left( \sum_{C \subseteq B} \frac{\Psi(S \setminus (C \cup a))}{\sum_{D \subseteq S \setminus (C \cup a)} \Psi((C \cup a) \cup D)} \right)$ , which depends on both the existing skill population and the individual's current set of skills.

skills is  $\binom{M}{k} p^k (1-p)^{M-k}$ .<sup>12</sup> This special case has several characteristics which make it a useful starting point. First, because skills are uncorrelated and occur with equal frequency, individuals with the same number of skills will, in expectation, have the same degree. This means that the only variable of interest is skill set size, k. Second, because this model has only 2 parameters–M and p–it is easy to see how changes in the distribution of skills in the population affect the structure of the collaboration network.

#### 5.2 Degree Distribution of the Bernoulli Skills Model

Let  $\Delta$  denote the distribution of expected degree. That is,  $\Delta(d)$  is the fraction of the population who have expected degree d, where the expectation is taken over all  $g \in \Gamma(\Psi)$ . In this particular case, I will use a convenient shorthand:  $\Delta_{M,p}$  represents the distribution of expected degree when M skills are independently distributed with probability p.

Theorem 5 states the closed form expression for the degree of an individual in the Bernoulli Skills Model.

**Theorem 5.** Suppose each individual in a population faces a problem,  $\omega(S)$ , requiring M skills. If the skills are distributed independently with  $Prob(a) = p \forall a \in S$ , then the expected degree of an individual with k skills is

$$E\left[d(k)\right] = p^M\left[\left(\frac{1-p+p^2}{p^2}\right)^k - 1\right]$$

Proof. Since  $\Sigma(\Psi) = 2^S$ , every individual needs to make only one link. Thus, we can write  $E[d(A)] = \sum_{C \subseteq A} \frac{\delta(C)}{\sigma(C)}$ , where  $\delta(C)$  is the fraction of the population who need skill set C and  $\sigma(C)$  is the fraction of the population who can provide skill set C. Since the skills are independent, we can separate this sum according to the size of the skill set required. If we start with the individuals lacking exactly one skill in  $A_i$  and end with the individuals

<sup>&</sup>lt;sup>12</sup>Note that as before, I will be using the *frequency* distribution. This means that implicitly I will be assuming that the population is large enough for every skill combination in the distribution to be represented at least once:  $N = p^{-M}$ . For small M, this assumption poses no serious problem. However, as M increases, population sizes become quite large. Thus, the Bernoulli Skills model will provide more insight in cases where there are fewer relevant skills.

needing all of the skills, we obtain the following sum:

$$E[d(A)] = \sum_{i \in A} \frac{p^{M-1} (1-p)}{p} + \sum_{i,j \in A} \frac{p^{M-2} (1-p)}{p^2} + \dots + \frac{p^{M-k} (1-p)^k}{p^k}$$
$$= \sum_{i=1}^k \binom{k}{i} \frac{p^{M-i} (1-p)^i}{p^i}$$
$$= p^M \left[ \left( \frac{1-p+p^2}{p^2} \right)^k - 1 \right]$$

Note that when skills are independent, an individual's degree on the network depends only on the size of her skill set, k. Therefore, in the Bernoulli Skills Model, it is appropriate to interpret the size of an individual's skill set as her "ability"–something that we cannot do in the more general case (recall Figure 5 in the previous section). Moreover, in the Bernoulli Skills Model, degree is a superadditive function of the number of skills a person has: E[d(k+1)] > E[d(k)] + E[d(1)]. This suggests a corollary to Theorem 5.

**Corollary 6.** Suppose each individual in a population faces a problem,  $\omega(S)$ , requiring M skills. If the skills are distributed independently with  $Prob(a) = p \forall a \in S$ , then degree is a superadditive function of the size of her skill set,  $k = |A_i|$ . Moreover, her degree is strictly increasing in k.

However, we still cannot price the skills individually in such a way that we capture degree, despite the fact that skills are independently distributed.

**Theorem 7.** Suppose each individual in a population faces a problem,  $\omega(S)$ , requiring M skills. If the skills are distributed independently with  $Prob(a) = p \forall a \in S$ , then there exists no vector of prices,  $\mu$ , such that  $\sum_{a \in A} \mu_a = E[d(A)]$  for all  $A \subseteq S$ .

*Proof.* Any such vector would be required to set  $\mu_a = d(a) = p^{M-2}(1-p)$  for all  $a \in S$ . But that would imply that  $d(A) = kp^{M-2}(1-p)$  for |A| = k. This is clearly not true for k > 1.

This superadditive relationship between skills and degree has a dramatic affect on the

distribution of links in the population. The distribution of degree in the network is

$$\Delta_{M,p}\left(d\right) = \frac{1}{\left(1-p\right)^{M}} \binom{M}{k\left(M,p,d\right)} \left(\frac{p}{1-p}\right)^{k\left(M,p,d\right)}$$

where  $k(M, p, d) = \frac{\ln\left(\frac{d}{p^M+1}\right)}{\ln\left(\frac{1-p+p^2}{p^2}\right)}$ . Note that although the distribution of "ability" (skill set size) in the Bernoulli Skills Model is binomial, and thus symmetric, the distribution of degree is highly skewed. Individuals with a few extra skills will have many additional combinations of skills, so individuals with marginally larger skill sets will have dramatically more links. As a result, the distribution of degree has much higher variance than the distribution of ability. Figure 7 shows an example of ability and degree distributions with M = 10 and  $p = \frac{1}{2}$ .

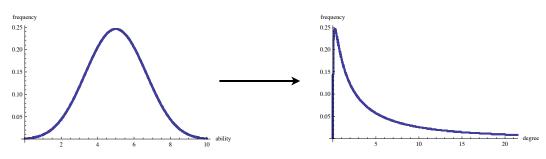


Figure 7: Distributions of ability and degree in a Bernoulli Skills Network with M = 10 and  $p = \frac{1}{2}$ .

The resulting network has the kind of long-tailed degree distribution we expect to see in a collaboration network. Interestingly, this long-tailed distribution emerges, despite the lack of a preferential attachment dynamic. If such a dynamic were added on top of this model, the degree distribution would become even more skewed.

This transformation of the ability distribution has an empirical implication-individuals who have a disproportionate number of links in a collaboration network may not have a disproportionate number of skills. In particular, individuals who are only slightly more skillful may be dramatically more important in the collaborative community. This means that collaborative communities may have "superstars"-individuals with a disproportionate influence in the community-even when skills are distributed relatively evenly in the population-while great innovators like Thomas Edison are undoubtedly more skilled than the average entrepreneur, they are likely not 1000 times as skilled, as their levels of productive output might suggest. Moreover, modeling individuals as having skill sets may account for some unexplained variation in collaborative outcomes.

## 5.3 Problem Difficulty and the Distribution of Degree: Comparative Statics on the Bernoulli Skills Model

While most collaboration networks share the skewed degree distribution, there is some variation between them. In this section, I use comparative statics of the Bernoulli skills model to examine how changes in the distribution of skills change the amount of inequality in the distribution of links.

In the following, I will use the Gini coefficient as my measure of distributional equality (See Appendix C for a discussion of the gini coefficient in this case). Using the fact that  $\Delta_{M,p}(d) = \binom{M}{k(M,p,d)} p^{k(M,p,d)} (1-p)^{M-k(M,p,d)}$  for  $k(M,p,d) = \frac{\ln\left(\frac{d}{pM}-1\right)}{\ln\left(\frac{1-p+p^2}{p^2}\right)}$  the gini coefficient of the degree distribution  $\Delta_{M,p}$  is

$$G(M,p) = 1 - \frac{(1-p)^{2M}}{(1-p^{M})} \left[ \sum_{k=0}^{M} \left( \binom{M}{k} \left( \frac{p}{1-p} \right)^{k} * \sum_{j=0}^{k} \left( \binom{M}{j} \left( \frac{p}{1-p} \right)^{j} * d(M,p,k) \right) \right) \right]$$

As is conventional, values of G(p, M) closer to 1 indicate a more unequal distribution of degree, while values closer to 0 indicate a more equitable distribution of degree.

Recall that the Bernoulli Skills model has only two parameters: the number of skills required to solve the problem (M) and the fraction of those skills possessed by the average individual (p). Figure 9 shows the Gini coefficient for various values of M and Figure 8 shows it for various values of p. The Gini coefficient is increasing in M, meaning that as problems require more skills, the distribution of degree becomes increasingly skewed towards a few high-degree nodes. The Gini coefficient is decreasing in p, meaning that as the probability of having a given skill goes down, the network becomes more skewed (see Figure 8). These results are summarized in Theorem 8.

**Theorem 8.** Suppose each individual in a population faces a problem,  $\omega(S)$ , requiring M skills and further suppose the skills are distributed independently with  $Prob(a) = p \forall a \in S$ . Let G(p, M) be the Gini Coefficient for the resulting degree distribution,  $\Delta_{M,p}$ . Then

1. G(p, M) is strictly increasing in M. That is, the distribution of links in the collaboration network is more uneven when the problem being solved requires more skills. 2. As G(p, M) is strictly decreasing in p. That is, the distribution of links in the collaboration network is more uneven as the probability of having each skill gets lower.

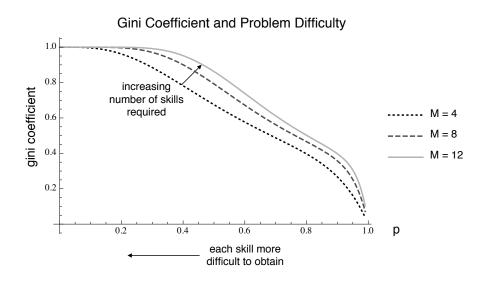


Figure 8: The gini coefficient, G(p, M), for different values of M and p.

Together, these results can be interpreted in terms of the difficulty of the problem being faced. A problem is more difficult if it requires many skills  $(M \uparrow)$ , or if the average individual has only a few of them  $(p \downarrow)$ . This means that the above can be recast as a comparative static on problem difficulty. As problems become increasingly difficult  $(M \uparrow \text{ and } p \downarrow)$ , the degree distribution of the network becomes more skewed, and a small number of "superstars" will dominate the collaborative community. Figures 9 and 10 illustrate the changes in the network as these parameters shift.

## 6 Skill Ladders: Skill Hierarchy and Degree

In the previous section, I considered a special case in which skills are entirely uncorrelated. In this section, I consider the effect of correlations between skills. I divide the skills into multiple skill ladders, where the skills within a ladder build on one another. I show that when skills are correlated in this way, the degree distribution of the collaboration network becomes even more unequal than when skills are uncorrelated. This suggests that when

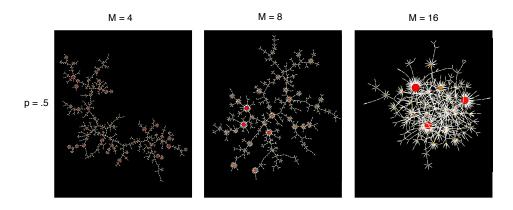


Figure 9: Examples of Bernoulli Skills Networks with N = 1000

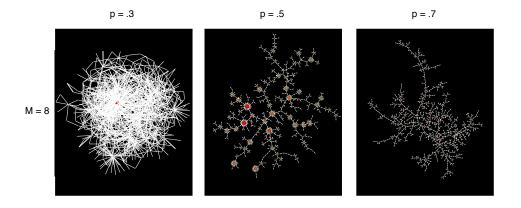


Figure 10: Examples of Bernoulli Skills Networks with N = 1000

skills in a community are structured in this way, a very small number of individuals will tend to dominate the collaboration network.

#### 6.1 The Ladder Model

In order to consider the effect of correlations between skills, in this section, I will need a second special case, which I will call the Ladder Model. I will define a *ladder* to be an ordered set of skills,  $L = \{a_1, a_2, a_3...a_l\} \subseteq S$ , such that Prob (have  $a_i|$  have  $a_{i+1}) = 1$  (that is, such that any individual who has the  $i^{th}$  skill in the set must have all of the skills that precede it in the set).<sup>13</sup> Here, I consider a special case where the skills in S are partitioned

<sup>&</sup>lt;sup>13</sup>Page, 2007 introduces this concept of skill ladders.

into *m* ladders of equal length.<sup>14</sup> The set of ladders is denoted  $\hat{S} = \{L_1...L_m\}$ . Figure 11 shows an example with 12 skills arranged into four ladders.

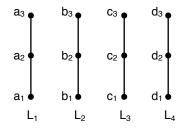


Figure 11: 12 skills arranged into four ladders of equal length.

I will call an individual who has all of the skills in a single ladder an "expert" in that ladder, and I will call the set of ladders that individual i is an expert in  $\hat{A}_i \subseteq \hat{S}$ .

One additional assumption will allow me to compare the results in this section to the results of the Bernoulli skills model. I will assume that the conditional probability of having the next skill in a ladder is the same for all skills-that is, Prob (have  $a_i|$  have  $a_{i-1}) = p$  for all  $a_i$ .<sup>15</sup> The probability of being an expert in a ladder of l skills is then  $p^l$ . Thanks to this assumption, the case where m = M corresponds to the Bernoulli skills model. On the other extreme, m = 1, and all of the skills are arranged in a single ladder. Before considering ladders of arbitrary length, I will first look at this case, where m = 1.

#### 6.2 Example: a single ladder of skills

Suppose the skills in the set S comprise a single ladder of length M. Because Prob (have  $a_i$ |have  $a_{i+1}$ ) =  $1 \forall i \in S$ , an individual's skill set can be represented by the number of skills she has  $(|A_i| = k$  implies  $A_i = \{a_1, a_2...a_k\}$ ). Because individuals can be ranked in order of the number of skills they have, this case corresponds to a model in which each individual has a one-dimensional ability measure.

The linking behavior in this case is very simple. The only individuals who have skill  $a_M$  are those who also have skills  $a_1...a_{M-1}$ . Anyone who doesn't have all M skills links

<sup>&</sup>lt;sup>14</sup>Obviously, since the length of a ladder is an integer, there will only be equal-length ladders if m divides M evenly. To simplify the exposition, I have written the following as if this is true. However, all of the the following results hold if the ladders are equal length up to integer constraints, which allows for cases where m does not divide M evenly.

<sup>&</sup>lt;sup>15</sup>In other words, I assume that putting a skill at the end of a ladder doesn't change the essential difficulty of obtaining that skill. If one instead makes a skill at the top easier to obtain (eg: because it builds on previous experience) or harder to obtain (eg: because they are more demanding than the skills that came before), then the results remain qualitatively unchanged.

to someone who does. The resulting collaboration network is a set of isolated stars, each with  $\frac{1-p^M}{p^M}$  links, on average. Figure 12 compares the network structure in the case with one skill ladder (m = 1) to the network structure in the Bernoulli skills model, where skills are independent (m = M). The populations that make up these two networks are similar–they have the same number of problem solvers, the same number of skills, and the same probability of having an additional skill. This means that the probability of an individual having all of the skills required to solve the problem is the same in both networks. Moreover, in both, exactly one individual has all of the skills required. However, the two networks have a much different structure. When all of the skills build on each other, the network is centered around a single, high-degree node. When skills are independent, the network structure is much more distributed.<sup>16</sup>

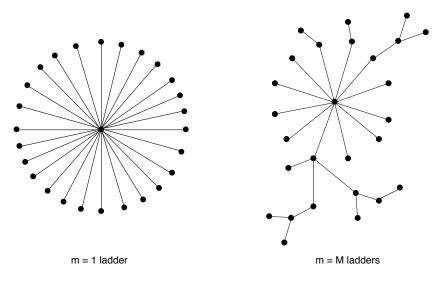


Figure 12: Ladder Skills Networks with N = 27 and M = 3.

<sup>&</sup>lt;sup>16</sup>Note that this could be interpreted as a prediction about organizational structures in different industries. In industries where value is created through the exploitation of existing knowledge (eg: manufacturing), skills tend to build on one another, and it is viable to rank workers by ability. However, as demand for workers shifts towards industries that create value by generating new knowledge, we expect skills to be arranged in ladders less often. This model predicts a corresponding shift from hierarchical organizational structures towards more distributed organizational structures. Empirical evidence seems to support that prediction. Traditional organizational structures were hierarchical (the theoretical underpinnings are explored in Rosen, 1982). However, evidence indicates that organizational structure within firms is changing–hierarchical structures are flattening, and workplaces are becoming more decentralized (see, for example, Bresnahan *et al.*, 2002 and Rajan & Wulf, 2006).

#### 6.3 Results for *m* ladders

Now consider the more general result. Suppose the skills are arranged in m equal-length ladders. As in the previous example, only experts obtain links. Thus, a model with mladders reduces to a Bernoulli skills model with m independent skills. Theorem 9 presents a closed-form expression for a individual problem solver's degree in the case with m skill ladders.

**Theorem 9.** If  $\Psi$  is a distribution of skills such that  $\hat{S} = \{L_1...L_m\}$  is a partition of S into m equal-length ladders with  $Prob\left(have a_i^j| have a_{(i-1)}^j\right) = p$ , then an individual with the skill set A will have expected degree  $E\left[d(A)\right] = p^{\frac{M}{m}} \left[\left(\frac{1-p^{\frac{M}{m}}+p^{2\frac{M}{m}}}{p^{2\frac{M}{m}}}\right)^k - 1\right]$ , where k is the number of disciplines the individual is an expert in.

Proof. The ladders are of equal length, so the length of a single ladder is  $\frac{M}{m}$ , and the probability that an individual is an expert in any one ladder is  $p^{\frac{M}{m}}$ . An individual receives a link only if she is an expert in a field. Define a new set of skills that correspond to the set of ladders:  $\hat{S} = \{L_1...L_m\}$ . The individual's new skill set is  $\hat{A}_i$ , where  $L_k \in \hat{A}_i$  if she is an expert in ladder  $L_k$ . Each of these new skills has a probability equal to the probability of being an expert in that field, so define  $\hat{p} = p^{\frac{M}{m}}$ . The probability of being an expert in a particular ladder is independent of the probability of being an expert in any other ladder, so this problem reduces to one with m independent skills, with probability  $p^{\frac{M}{m}}$ . The result then is a simple extension of Theorem 5.

We can now do a comparative static on the number of skill ladders, to see how the amount of hierarchy in the skill pool affects the structure of the collaboration network. Theorem 10 indicates that in communities where skills are more hierarchical, the degree distribution of the collaboration network is more skewed, and a few experts become extremely high-degree hubs in the network. Figure 13 shows how the gini coefficient depends on the number of ladders for the case where M = 6 (note that the bottom curve (m = 6) is equivalent to the Bernoulli Skills Model with M = 6 in Figure 8).

**Theorem 10.** Suppose S skills are arranged in m ladders of equal length, with constant conditional probability  $\operatorname{Prob}\left(\operatorname{have} a_i^j | \operatorname{have} a_{i-1}^j\right) = p$  and  $\operatorname{Prob}\left(\operatorname{have} a_1^j\right) = p \;\forall j = 1...m$ . The gini coefficient of the resulting network is decreasing in the number of ladders, m. That is, when there are fewer skill ladders, the degree distribution becomes increasingly uneven.

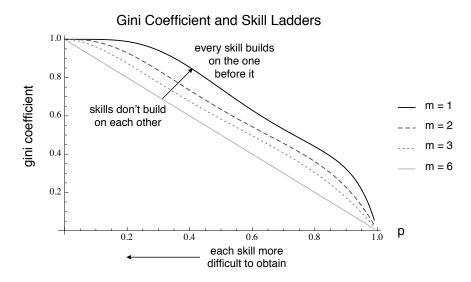


Figure 13: The gini coefficient, G(p, M, m), for different values of m and p and M = 6.

*Proof.* A model with m skill ladders is equivalent to a Bernoulli Skills model with m skills and  $p = p^{M/m}$ . The gini coefficient is thus  $G\left(p, \frac{M}{m}\right)$ . Since this function is increasing in the second argument (see Theorem 8), it is decreasing in m.

## 7 Discussion: Labor Market Implications

The more general treatment of individual heterogeneity that I present in this paper has some interesting implications for labor markets knowledge-based fields, where workers are richly heterogeneous and collaborate to solve problems.<sup>17</sup> In particular, given that there is a relationship between the number of collaborators an individual has and her output,<sup>18</sup> the model presented in this paper can provide insight into individual heterogeneity and output. The results of Section 4 strongly indicate that the decisions we make when modeling heterogeneity matter. A full exploration of the effects of complex heterogeneity on labor markets is outside the scope of this work, but I will provide a brief discussion here.

Empirically, output in creative and knowledge-based fields is highly concentrated among

 $<sup>^{17}</sup>$ If we were to recast the relationship between collaborators as a relationship between employees and firms, then an extension of this model would also provide insights into the affect of heterogeneity on labor in a non-collaborative context.

<sup>&</sup>lt;sup>18</sup>In the Bernoulli Skills model, where each individual has a complement, the relationship is one-to-one.

a small number of people.<sup>19</sup> This long-tailed distribution of productive output has implications for the distribution of wages and welfare, and thus there has been considerable interest in understanding why such a concentration in labor demand occurs. Rosen, 1981's model of production can induce a long-tailed distribution when there is a high premium on quality, and production technology decouples effort from output quantity (eg: in creative industries, where a single performance or album can be enjoyed by many consumers). However, such technologies are not relevant in knowledge-based industries, where effort is not decoupled from output volume. The long-tailed distribution of collaborative interactions in this paper suggests an alternative explanation for the observed variation in output. Moreover, the prediction that output will become more skewed as problems become more difficult suggests an empirical test in the context of aggregate labor demand.

The non-linear relationship between skill sets and output also has implications for the explanation of variation in labor outcomes when individual heterogeneity is imperfectly observed. If we assume that skills make a linear contribution to output, then we would expect a linear decline in the amount of variation explained. However, if the relationship between individual skills and output is non-linear, as is suggested by the model in this paper, then the amount of variation in output explained by observables will drop nonlinearly as well. This suggests that decreases in the amount of variation in labor outcomes explained by observed agent heterogeneity could be the result of a shift from manufacturing–where skill could reasonably be considered to be a one-dimensional variable–to knowledge-based industries, where individual heterogeneity is more complex. A more thorough exploration of the implications of skill-based labor heterogeneity on labor markets would make an interesting extension.

### 8 Conclusion

In this paper, I have presented a model of collaboration network formation in which individuals have heterogeneous skill sets and collaborate to solve problems that none of them could solve as individuals. The result is a collaboration network, the structure of which depends on the distribution of skills in the underlying community. This framework is extremely

<sup>&</sup>lt;sup>19</sup>Newman, 2001, Moody, 2004, Acedo *et al.*, 2006, and Goyal *et al.*, 2006 show that a small number of academics produce the majority of papers written, Rosen, 1981 observes a similar pattern in in music, film, and textbook writing, and Uzzi & Spiro, 2005 notes a similarly skewed distribution of output among directors, producers, and other creative artists on Broadway.

flexible and has the potential to provide insights into a wide variety of questions that could not previously be answered. Here, I have used an independent distribution of skills to look at how the difficulty of problem-solving tasks might affect the structure of the collaboration network. But the same general framework can be used to explore a variety of questions about the collaborative process. For example, one could use correlations between related skills to explore how skill specialization affects the structure of a collaborative community. A more elaborate set of correlations between skills might provide insight into why some skills are under-provided.

One of the strengths of the framework introduced here is that it lends itself to extension and modification. One could use changes to the payoff structure to examine how community structure depends on team dynamics. For example, how does the structure of the community change if teams benefit from skill overlap as well as skill complementarity? And how does that change the role of individuals with bridging skills? Time is another interesting dimension for future work. A dynamic model would allow for a more complete model of search. Individuals might find new collaborators by searching the networks of their current collaborators. Such a model might also incorporate a geographic component, making it more difficult for geographically dispersed individuals to collaborate. In a longer-run model there is room to explore the role of learning. This model assumes a fixed skill population and endowments. However, individuals clearly develop new skills over time. A model of the skill acquisition decision would pave the way for a more dynamic model, in which the network and population both evolve over time. In that case, there may be characteristics of the distribution of problems that influence the long-run distribution of skills and network structure. In the even longer term, it would be interesting to see how shocks to the distribution of problems (eg: a dramatic shift in the outlook of a field) would affect the skill population and collaboration network. In particular, what factors affect the robustness of a problem-solving population to changes in the distribution of problems faced? Finally, the results of this paper are evidence that more complex models of individual heterogeneity are both tractable and crucial to understanding the structure and function of collaborative communities.

# Appendices: The Formation of Collaboration Networks among Individuals with Heterogeneous Skills

#### Appendix A: Pairwise Stability and Efficiency

Briefly, a network is pairwise stable if no individual would prefer to terminate an existing link, and if no pair of individuals would prefer to add a link. Although this definition is usually used in undirected networks, it works equally well in the current context. Formally, a directed collaboration network, g, is *pairwise stable* if

- 1. for all  $ij \in g$ ,  $u_i(g) \ge u_i(g-ij)$  and  $u_j(g) \ge u_j(g-ij)$
- 2. for all  $ij \notin g$ , if  $u_j (g + ij) > u_j (g)$  then  $u_i (g + ij) < u_i (g)$

Together, these two conditions ensure that links are mutual. That is, if a network is pairwise stable, then both ends agree to maintain the link.

**Theorem.** Any complementarity network,  $g \in \Gamma(\Psi)$ , is pairwise stable. In other words,  $\forall ij \in g \ u_i(g) \ge u_i(g-ij) \text{ and } u_j(g) \ge u_j(g-ij) \text{ and for all } ij \notin g, \text{ if } u_j(g+ij) > u_j(g)$ then  $u_i(g+ij) < u_i(g)$ .

Proof. First, consider whether any individual wishes to unilaterally remove a link,  $ij \in g$ . Removing this incoming link costs individual j her share of the payoff from solving i's problem  $\left(\frac{1}{|C_i|+1} \geq 0\right)$ , and thus she will never choose to do so. Individual i chose a minimal set of collaborators that allowed her to solve her problem, so removing this outgoing link means that she can no longer solve the problem. This would cost her the payoff  $\left(\frac{1}{|C_i|+1} \geq 0\right)$  from solving the problem, and thus she will also never choose to do so. Finally, note that no individual would ever want to add an outgoing link to a cost-minimizing collaboration network, because having chosen a minimal set of collaborators, any additional link would require her to further split her prize.

**Theorem.** Any cost minimizing collaboration network,  $g \in \Gamma(\Psi)$ , is strongly efficient. In other words,  $\sum_{i} u_i(g) \ge \sum_{i} u_i(g') \forall g' \in G$ .

*Proof.* Because all value is generated from solving problems, the maximum possible value in the network is N. Since solving problems is incentive compatible and there is no loss, the problem solvers always extract the maximum value from the network.

#### Appendix B: General Proof of Theorem 2

An individual with the set  $A \cup B$  will be able to help anyone needing any subset of those skills. Let  $\delta(C)$  be the demand for a particular set of skills, C. In the general case,  $\delta(C) = \Psi(S \setminus C) + \sum_{D: \Psi(C \cup D)=0} \Psi(S \setminus (C \cup D))$ . The fraction who can supply the set Cis  $\sigma(C) = \sum_{D \subseteq S \setminus C} \Psi(C \cup D)$ . Note that  $\delta(C)$  and  $\sigma(C)$  depend only on the particulars of the problem (S), the distribution of skills  $(\Psi)$ , and the subset of skills (C). Thus, any individual with the skill set  $A \cup B$  has expected degree

$$E\left[d\left(A\cup B\right)\right] = \sum_{C\subseteq A\cup B} \frac{\delta\left(C\right)}{\sigma\left(C\right)}$$

We can divide the problems that an individual with  $A \cup B$  can solve into three groups:

- 1. Requires only skills from set  $A: C \subseteq A$
- 2. Requires only skills from set B, including at least one found only in  $B : \{C \mid C \subseteq B \text{ and } \exists b \in C \text{ st } b \in B \setminus A\}$
- 3. Requires at least one skill from each set that can only be found in that set:  $\{C \mid C \subseteq A \cup B, where \exists a, b \in C \text{ s.t. } a \in A \setminus B \text{ and } b \in B \setminus A\}$

Using this partition, we can write

$$\begin{split} E\left[d\left(A\cup B\right)\right] &= \sum_{C\subseteq A} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C\subseteq B \text{ and } C\cap B\neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C\subseteq A\cup B \text{ and } C\cap A, \ C\cap B\neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} \\ &= E\left[d\left(A\right)\right] + \sum_{C\subseteq B \text{ and } C\cap B\neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \phi \end{split}$$

which implies that

$$\begin{split} E\left[d\left(A\cup B\right)\right] + E\left[d\left(A\cap B\right)\right] &= E\left[d\left(A\right)\right] + \sum_{C\subseteq B \text{ and } C\cap B\neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \phi + E\left[d\left(A\cap B\right)\right)\right] \\ &= E\left[d\left(A\right)\right] + \left(\sum_{C\subseteq B \text{ and } C\cap B\neq \emptyset} \frac{\delta\left(C\right)}{\sigma\left(C\right)} + \sum_{C\subseteq A\cap B} \frac{\delta\left(C\right)}{\sigma\left(C\right)}\right) + \phi \\ &= E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right] + \phi \\ &\geq E\left[d\left(A\right)\right] + E\left[d\left(B\right)\right] \end{split}$$

#### Appendix C: Discussion of the Gini Coefficient in the discrete case

The Gini coefficient measures the area between the Lorenz curve of a distribution (in this case, the distribution of expected degree), and the line of equality. In the case of a discrete distribution with values  $y_0...y_N$  where  $y_i < y_{i+1}$ , the Lorenz curve is a piecewise function connecting points  $(F_i, D_i)$  where  $F_i = \sum_{k=0}^i \Delta(y_k)$  is the fraction of individuals with strictly less than  $y_i$  links, and  $D_i = \frac{\sum_{k=0}^i \Delta(y_k)y_k}{\sum_{k=0}^k \Delta(y_k)y_k}$  is the fraction of the total number of links they hold. See Figure 14 for an example. The gini coefficient for a discrete distribution is given by  $G = 1 - \sum_{i=1}^N D_i (F_i - F_{i-1})$ . Lower values of the gini coefficient indicate a more equal distribution of links across individuals in the population, and higher values indicate a more skewed distribution of links. The coefficient is which is 0 when the distribution is perfectly equal (ie: the bottom x% of the population holds exactly x% of the links) and 1 when all of the links are held by a single individual.

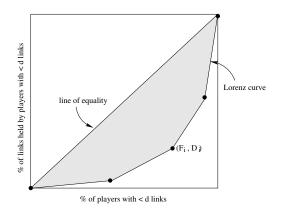


Figure 14: An example of the Gini coefficient for a discrete distribution,  $\Delta(y)$ . In this case, the random variable y takes on one of five values,  $y_0...y_4$ . The Gini coefficient is the area of the shaded region between the line of equality and the Lorenz curve.

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