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A NEW TWO-URN MODEL

MAY-RU CHEN,* *National Sun Yat-sen University*
SHOOU-REN HSIAU ** *** AND
TING-HSIN YANG,** *National Changhua University of Education*

Abstract

We propose a two-urn model of Pólya type as follows. There are two urns, urn A and urn B. At the beginning, urn A contains r_A red and w_A white balls and urn B contains r_B red and w_B white balls. We first draw m balls from urn A and note their colors, say i red and $m-i$ white balls. The balls are returned to urn A and bi red and $b(m-i)$ white balls are added to urn B. Next, we draw ℓ balls from urn B and note their colors, say j red and $\ell - j$ white balls. The balls are returned to urn B and aj red and $a(\ell - j)$ white balls are added to urn A. Repeat the above action n times and let X_n be the fraction of red balls in urn A and Y_n the fraction of red balls in urn B. We first show that the expectations of X_n and Y_n have the same limit, and then use martingale theory to show that X_n and Y_n converge almost surely to the same limit.

Keywords: Two-urn model; martingale

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1. Introduction

The study of urn models has a long history. James Bernoulli (1713) may be the first person to mention problems in the language of urns (cf. [10]). Up to 1977, the results on urn models were summarized in the book 'Urn Models and Their Applications' [9]. This book stimulated many probabilists and statisticians to investigate different kinds of urn models. After two decades, Kotz and Balakrishnan [10] published a survey paper: 'Advances in Urn Models During the Past Two Decades'. It covered almost all of the different kinds of urn models and their properties. One of the most famous urn models was introduced by Eggenberger and Pólya [5], and is usually called the Pólya urn; it is described as follows. An urn initially contains r red and w white balls. A ball is drawn at random and then replaced together with c balls of the same color. The procedure is repeated *ad infinitum*. It is well-known that the fraction of red balls converges almost surely to a beta distributed random variable with parameters r/c and w/c.

Various generalizations of the Pólya urn can be found in the literature. Hill *et al.* [8] and Higueras *et al.* [7] extended the result of Eggenberger and Pólya to an urn model in which the probability of adding balls is a function of the fraction of red balls. Bagchi and Pal [2] considered a generalization of the Pólya urn model in which the rule for adding balls follows a matrix. Subsequently, Gouet [6] showed that if the matrix satisfies some suitable conditions,

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* Postal address: Department of Applied Mathematics, National Sun Yat-sen University, 70 Lien-hai Rd., Kaohsiung 804, Taiwan, R. O. C. Email address: mayru@faculty.nsysu.edu.tw
** Postal address: Department of Mathematics, National Changhua University of Education, No. 1 Jin-De Road, Changhua 500, Taiwan, R. O. C.
*** Email address: srhsiau@cc.ncue.edu.tw

the limit of the fraction of red balls degenerates to a constant. The generalization of the Pólya urn given by Pemantle [13] is a time-dependent urn model. Chen and Wei [4] used martingales to study a generalization of the Pólya urn in which at least two balls are drawn at each time step. Later, Renlund [14] studied a generalized Pólya urn (similar to Chen and Wei's) via stochastic approximation. Recently, Mahmoud [11] wrote a book named 'Pólya urn model', which includes several classical urn models and presents different techniques to investigate various structural properties of Eggenberger–Pólya urn schemes.

One famous generalization of the Pólya urn is the Ehrenfest urn model, which is a two-urn model and may be described as follows. There are two urns, say A and B, and N balls, numbered consecutively from 1 to N, distributed in the two urns. At each step, an integer between 1 and N is chosen at random, uniformly, and the ball whose number has been chosen is moved to the opposite urn. The procedure is repeated n times and the probability distribution of the number of balls in each urn is considered. Much of the literature on the Ehrenfest urn model is surveyed in [10]. Some of the literature (e.g. [3] and [12]) studied other two-urn models.

In this paper we introduce a new two-urn model, which is described as follows. Assume that there are two urns, urn A and urn B. Suppose that, at the beginning, there are s_0x_0 red and $s_0(1-x_0)$ white balls in urn A, and t_0y_0 red and $t_0(1-y_0)$ white balls in urn B. Note that s_0 and t_0 are the respective number of balls in urns A and B, and x_0 and y_0 are the respective fraction of red balls in urns A and B. First, draw $m \le s_0$ balls (without replacement) from urn A and note their colors, say i red and $m-i$ white balls. Then return the balls to urn A and add bi red and $b(m-i)$ white balls to urn B. Next, draw $\ell \le t_0 + bm$ balls (without replacement) from urn B and note their colors, say j red balls and $\ell - j$ white balls. Then return the balls to urn B and add aj red and $a(\ell - j)$ white balls to urn A. Repeat the above action *ad infinitum*.

After the nth action, let $R_A(n)$ and $R_B(n)$ be the respective numbers of red balls in the urns A and B. At this stage, there are $s_0 + a\ell n$ balls in urn A and $t_0 + bmn$ balls in urn B. Denote $s_0 + a\ell n = s_n$ and $t_0 + bmn = t_n$ for $n \ge 1$. Let $X_n = R_A(n)/s_n$ and $Y_n = R_B(n)/t_n$, that is, X_n and Y_n are the respective fractions of red balls in the urns A and B after the nth action. Let $\mathcal{F}_n$ denote the σ-field generated by $X_1, X_2, \dots, X_n$ and $Y_1, Y_2, \dots, Y_n$. Then $(R_A(n), R_B(n))$ is clearly a Markov chain. For clarity, let

$$R_A(n+1) = R_A(n) + a\ell U_{n+1}, \qquad R_B(n+1) = R_B(n) + bmV_{n+1},$$

where mV_n and ℓU_n are the respective numbers of red balls drawn from the urns A and B in the nth action. Then

$$\mathbb{P}\biggl(V_{n+1} = \frac{i}{m} \biggm| \mathcal{F}_n\biggr) = \frac{\dbinom{s_nX_n}{i}\dbinom{s_n - s_nX_n}{m-i}}{\dbinom{s_n}{m}},$$

$$\mathbb{P}\biggl(U_{n+1} = \frac{j}{\ell} \biggm| \mathcal{F}_n, V_{n+1}\biggr) = \frac{\dbinom{t_{n+1}Y_{n+1}}{j}\dbinom{t_{n+1} - t_{n+1}Y_{n+1}}{\ell-j}}{\dbinom{t_{n+1}}{\ell}},$$

giving

$$\mathbb{P}\biggl(U_{n+1} = \frac{j}{\ell} \biggm| \mathcal{F}_n\biggr) = \sum_{i=0}^{m} \frac{\dbinom{t_nY_n + bi}{j}\dbinom{t_{n+1} - t_nY_n - bi}{\ell-j}}{\dbinom{t_{n+1}}{\ell}} \frac{\dbinom{s_nX_n}{i}\dbinom{s_n - s_nX_n}{m-i}}{\dbinom{s_n}{m}}.$$

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Moreover, we have, for $n \geq 0$,

$$X_{n+1} = \frac{R_A(n+1)}{s_{n+1}} = \frac{s_n X_n + a\ell U_{n+1}}{s_{n+1}} = \alpha_n X_n + \bar{\alpha}_n U_{n+1} = X_n + \bar{\alpha}_n (U_{n+1} - X_n), \tag{1.1}$$

$$Y_{n+1} = \frac{R_B(n+1)}{t_{n+1}} = \frac{t_n Y_n + bm V_{n+1}}{t_{n+1}} = \beta_n Y_n + \bar{\beta}_n V_{n+1} = Y_n + \bar{\beta}_n (V_{n+1} - Y_n), \tag{1.2}$$

where $\alpha_n = s_n/s_{n+1} = 1 - \bar{\alpha}_n$ and $\beta_n = t_n/t_{n+1} = 1 - \bar{\beta}_n$.

From the above conditional distributions of V_{n+1}, it follows that mV_{n+1} is a (conditional) hypergeometric random variable with parameters s_n, $s_n X_n$, and m. Hence,

$$\mathbb{E}(V_{n+1} \mid \mathcal{F}_n) = \frac{1}{m}\mathbb{E}(mV_{n+1} \mid \mathcal{F}_n) = \frac{1}{m}\frac{ms_n X_n}{s_n} = X_n. \tag{1.3}$$

Similarly, since ℓU_{n+1} is a conditional hypergeometric random variable with parameters t_{n+1}, $t_{n+1}Y_{n+1}$, and ℓ, given $\mathcal{F}_n$ and V_{n+1}, we have

$$\mathbb{E}(U_{n+1} \mid \mathcal{F}_n, V_{n+1}) = \frac{1}{\ell}\mathbb{E}(\ell U_{n+1} \mid \mathcal{F}_n, V_{n+1}) = \frac{1}{\ell}\frac{\ell t_{n+1} Y_{n+1}}{t_{n+1}} = \beta_n Y_n + \bar{\beta}_n V_{n+1}.$$

Thus,

$$\begin{aligned} \mathbb{E}(U_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}(\mathbb{E}(U_{n+1} \mid \mathcal{F}_n, V_{n+1}) \mid \mathcal{F}_n) \\ &= \mathbb{E}(\beta_n Y_n + \bar{\beta}_n V_{n+1} \mid \mathcal{F}_n) \\ &= \beta_n Y_n + \bar{\beta}_n X_n. \end{aligned} \tag{1.4}$$

Combining identities (1.1)–(1.4) yields

$$\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = X_n - \bar{\alpha}_n \beta_n (X_n - Y_n), \tag{1.5}$$

$$\mathbb{E}(Y_{n+1} \mid \mathcal{F}_n) = Y_n + \bar{\beta}_n (X_n - Y_n). \tag{1.6}$$

From (1.5) and (1.6), it is easy to derive

$$\mathbb{E}(X_{n+1} - Y_{n+1}) = \alpha_n \beta_n \mathbb{E}(X_n - Y_n) = \cdots = (x_0 - y_0)\prod_{i=0}^{n} \alpha_i \beta_i = \frac{s_0 t_0 (x_0 - y_0)}{s_{n+1} t_{n+1}}, \tag{1.7}$$

and, hence, we have $\lim_{n\to\infty} \mathbb{E}(X_n - Y_n) = 0$ since $\lim_{n\to\infty} 1/(s_{n+1}t_{n+1}) = 0$.

Motivated by the above result, it is natural to consider the limit of $\mathbb{E}X_n$. In Section 2 we first derive the limit of $\mathbb{E}X_n$ and this implies that $\mathbb{E}X_n$ and $\mathbb{E}Y_n$ have the same limit since $\lim_{n\to\infty} \mathbb{E}(X_n - Y_n) = 0$. In Section 3 we strengthen the result to the strong convergence; we prove that X_n and Y_n converge almost surely to the same limit. This is a more involved work. In fact we can use martingale theory to prove that $X_n + Y_n$ converges almost surely and $X_n - Y_n$ converges to 0 almost surely; these two results then imply the claim.

2. Limits of $\mathbb{E}X_n$ and $\mathbb{E}Y_n$

In this section we study the limits of $\mathbb{E}X_n$ and $\mathbb{E}Y_n$. Taking the expectation of both sides of (1.5) and (1.6), and using (1.7) repeatedly, we obtain

$$\mathbb{E}X_n = x_0 - \sum_{j=0}^{n-1} \frac{a\ell s_0 t_0 (x_0 - y_0)}{s_j s_{j+1} t_{j+1}} \quad \text{and} \quad \mathbb{E}Y_n = y_0 + \sum_{j=0}^{n-1} \frac{bms_0 t_0 (x_0 - y_0)}{s_j t_j t_{j+1}}. \tag{2.1}$$

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Note that

$$\sum_{j=0}^{\infty} \frac{1}{s_j s_{j+1} t_{j+1}} \le \sum_{j=1}^{\infty} \frac{1}{j^3} < \infty \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{1}{s_j t_j t_{j+1}} \le \sum_{j=1}^{\infty} \frac{1}{j^3} < \infty.$$

Thus, by (2.1), we obtain

$$\lim_{n\to\infty} \mathbb{E}X_n = x_0 - \sum_{j=0}^{\infty} \frac{a\ell s_0 t_0 (x_0 - y_0)}{s_j s_{j+1} t_{j+1}} \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}Y_n = y_0 + \sum_{j=0}^{\infty} \frac{bm s_0 t_0 (x_0 - y_0)}{s_j t_j t_{j+1}}.$$

Moreover, since $\lim_{n\to\infty} \mathbb{E}(X_n - Y_n) = 0$, we have

$$\lim_{n\to\infty} \mathbb{E}X_n = \lim_{n\to\infty} \mathbb{E}Y_n.$$

It is interesting to note that if $x_0 = y_0$ then, in view of (2.1), $\mathbb{E}X_n = \mathbb{E}Y_n = x_0$ for each n, and so $\lim_{n\to\infty} \mathbb{E}X_n = \lim_{n\to\infty} \mathbb{E}Y_n = x_0$. In the case $x_0 \ne y_0$ we find a simpler form of $\lim_{n\to\infty} \mathbb{E}X_n$ under the assumptions $a\ell = bm$ and $s_0 = t_0 + ka\ell$, $k \in \mathbb{Z}$.

Theorem 2.1. *In the two-urn model, assume that $x_0 \ne y_0$, $a\ell = bm$, and $s_0 = t_0 + ka\ell$ for $k \in \mathbb{Z}$. Then*

$$\lim_{n\to\infty} \mathbb{E}X_n = \begin{cases} y_0 - (x_0 - y_0)\left(\dfrac{s_0}{a\ell k} - \dfrac{s_0 t_0}{a\ell k(k+1)} \displaystyle\sum_{j=0}^{k} \frac{1}{s_0 + a\ell j}\right) & \text{if } k \le -2, \\ \frac{1}{2}(x_0 + y_0) & \text{if } k = -1, \\ y_0 + (x_0 - y_0)\left(\displaystyle\sum_{j=0}^{\infty} \frac{s_0^2}{(s_0 + a\ell j)^2} - \frac{s_0}{a\ell}\right) & \text{if } k = 0, \\ x_0 + (x_0 - y_0)\left(\dfrac{t_0}{a\ell} - \displaystyle\sum_{j=1}^{\infty} \frac{s_0 t_0}{(t_0 + a\ell j)^2}\right) & \text{if } k = 1, \\ x_0 + (x_0 - y_0)\left(\dfrac{t_0}{a\ell k} - \dfrac{s_0 t_0}{a\ell k(k-1)} \displaystyle\sum_{j=1}^{k-1} \frac{1}{t_0 + a\ell j}\right) & \text{if } k \ge 2. \end{cases}$$

In particular, if $x_0 = 1$, $y_0 = 0$, $s_0 = t_0 = 1$, and $a = b = \ell = m = 1$ then we have $\lim_{n\to\infty} \mathbb{E}X_n = \pi^2/6 - 1$.

Proof. Since $\lim_{n\to\infty} \mathbb{E}X_n = \lim_{n\to\infty} \mathbb{E}Y_n$ we have

$$\lim_{n\to\infty} \mathbb{E}X_n = \tfrac{1}{2} \lim_{n\to\infty} \mathbb{E}(X_n + Y_n).$$

By the assumption $a\ell = bm$ and (2.1), it follows that

$$\mathbb{E}(X_n + Y_n) = x_0 + y_0 + s_0 t_0 (x_0 - y_0) \sum_{j=0}^{n-1} \frac{a\ell}{s_j t_{j+1}} \left(\frac{1}{t_j} - \frac{1}{s_{j+1}}\right). \tag{2.2}$$

If $s_0 = t_0 - a\ell$ then $s_{j+1} = t_j$ for all j and 2.2 gives $\mathbb{E}(X_n + Y_n) = x_0 + y_0$. This implies that

$$\lim_{n\to\infty} \mathbb{E}X_n = \tfrac{1}{2}(x_0 + y_0).$$

If $s_0 = t_0$ then $s_j = t_j$ for all j and so, from (2.2),

$$\begin{aligned}
\mathbb{E}(X_n + Y_n) &= x_0 + y_0 + s_0^2(x_0 - y_0)\sum_{j=0}^{n-1}\left(\frac{1}{s_j} - \frac{1}{s_{j+1}}\right)^2 \\
&= x_0 + y_0 + s_0^2(x_0 - y_0)\sum_{j=0}^{n-1}\left[\frac{2}{a\ell}\left(\frac{1}{s_{j+1}} - \frac{1}{s_j}\right) + \frac{1}{s_j^2} + \frac{1}{s_{j+1}^2}\right] \\
&= x_0 + y_0 + s_0^2(x_0 - y_0)\left[\frac{2}{a\ell}\left(\frac{1}{s_n} - \frac{1}{s_0}\right) - \frac{1}{s_0^2} - \frac{1}{s_n^2} + \sum_{j=0}^{n}\frac{2}{s_j^2}\right] \\
&= x_0 + y_0 + (x_0 - y_0)\left[\frac{2s_0^2}{a\ell}\left(\frac{1}{s_n} - \frac{1}{s_0}\right) - 1 - \frac{s_0^2}{s_n^2} + \sum_{j=0}^{n}\frac{2s_0^2}{s_j^2}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}X_n &= \frac{1}{2}\left\{x_0 + y_0 + (x_0 - y_0)\left[-\frac{2s_0}{a\ell} - 1 + \sum_{j=0}^{\infty}\frac{2s_0^2}{s_j^2}\right]\right\} \\
&= y_0 + (x_0 - y_0)\left[\sum_{j=0}^{\infty}\frac{s_0^2}{(s_0 + a\ell j)^2} - \frac{s_0}{a\ell}\right].
\end{aligned}$$

If $s_0 = t_0 + ka\ell,\ k \geq 2$, then $s_j = t_{j+k}$ for all j and so, from (2.2),

$$\begin{aligned}
&\mathbb{E}(X_n + Y_n) \\
&= x_0 + y_0 + s_0 t_0(x_0 - y_0)\sum_{j=0}^{n-1}\frac{(k+1)a^2\ell^2}{t_j t_{j+1} t_{j+k} t_{j+k+1}} \\
&= x_0 + y_0 + s_0 t_0(x_0 - y_0)\sum_{j=0}^{n-1}\left[\frac{1}{a\ell k}\left(\frac{1}{t_j} - \frac{1}{t_{j+k+1}}\right) - \frac{k+1}{a\ell k(k-1)}\left(\frac{1}{t_{j+1}} - \frac{1}{t_{j+k}}\right)\right] \\
&= x_0 + y_0 + (x_0 - y_0)\left[\frac{s_0 t_0}{a\ell k}\left(\frac{1}{t_0} + \frac{1}{s_0} - \frac{1}{t_n} - \frac{1}{t_{n+k}}\right)\right. \\
&\qquad\left. - \frac{2s_0 t_0}{a\ell k(k-1)}\left(\sum_{j=1}^{k-1}\frac{1}{t_j} - \sum_{j=n+1}^{n+k-1}\frac{1}{t_j}\right)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{E}X_n &= \frac{1}{2}\left\{x_0 + y_0 + (x_0 - y_0)\left[\frac{s_0 + t_0}{a\ell k} - \frac{2s_0 t_0}{a\ell k(k-1)}\sum_{j=1}^{k-1}\frac{1}{t_j}\right]\right\} \\
&= x_0 + (x_0 - y_0)\left(\frac{t_0}{a\ell k} - \frac{s_0 t_0}{a\ell k(k-1)}\sum_{j=1}^{k-1}\frac{1}{t_j}\right) \\
&= x_0 + (x_0 - y_0)\left(\frac{t_0}{a\ell k} - \frac{s_0 t_0}{a\ell k(k-1)}\sum_{j=1}^{k-1}\frac{1}{t_0 + a\ell j}\right),
\end{aligned}$$

where the second equality follows from $s_0 = t_0 + ka\ell$. Because the proofs for $k = 1$ and $k \leq -2$ are similar to the proofs for $k = 0$ and $k \geq 2$, respectively, we omit them here.

3. Strong convergence of X_n

In this section we study the long term behavior of the processes $\{X_n\}_{n\geq 1}$ and $\{Y_n\}_{n\geq 1}$. Because $\mathbb{E}X_n$ and $\mathbb{E}Y_n$ converge to the same limit, we are motivated to prove that X_n and Y_n converge almost surely to the same limit. To this end, we construct a martingale and a supermartingale in terms of $X_n + Y_n$ and $X_n - Y_n$.

Theorem 3.1. *In the two-urn model,*

$$\left\{X_n + Y_n + \sum_{k=0}^{n-1}(\bar{\alpha}_k\beta_k - \bar{\beta}_k)(X_k - Y_k)\right\}_{n\geq 1} \tag{3.1}$$

is a bounded martingale and X_n+Y_n converges almost surely; moreover, $\{(X_n-Y_n)^2+4/n\}_{n\geq 1}$ is a nonnegative supermartingale.

Proof. From (1.5) and (1.6), we have, for any $\ell, m \geq 1$,

$$\mathbb{E}[X_{n+1} + Y_{n+1} \mid \mathcal{F}_n] = X_n + Y_n - (\bar{\alpha}_n\beta_n - \bar{\beta}_n)(X_n - Y_n),$$

so $\{X_n + Y_n + \sum_{k=0}^{n-1}(\bar{\alpha}_k\beta_k - \bar{\beta}_k)(X_k - Y_k)\}_{n\geq 1}$ is a martingale. Furthermore, since $\bar{\alpha}_n\beta_n - \bar{\beta}_n = O(n^{-2})$, the compensator

$$\sum_{k=0}^{n-1}(\bar{\alpha}_k\beta_k - \bar{\beta}_k)(X_k - Y_k)$$

converges almost surely. This implies that the martingale (3.1) is bounded and, therefore, convergent almost surely. Hence, $X_n + Y_n$ converges almost surely.

Next, from (1.1) and (1.2), it follows that

$$\begin{aligned}
&(X_{n+1} - Y_{n+1})^2 \\
&\quad = (X_n - Y_n)^2 + [\bar{\alpha}_n(U_{n+1} - X_n) - \bar{\beta}_n(V_{n+1} - Y_n)]^2 \\
&\qquad + 2(X_n - Y_n)[\bar{\alpha}_n(U_{n+1} - X_n) - \bar{\beta}_n(V_{n+1} - Y_n)] \\
&\quad \leq (X_n - Y_n)^2 + \frac{4}{(n+1)^2} + 2(X_n - Y_n)[\bar{\alpha}_n(U_{n+1} - X_n) - \bar{\beta}_n(V_{n+1} - Y_n)],
\end{aligned}$$

where the last inequality holds since $0 < \bar{\alpha}_n, \bar{\beta}_n \leq 1/(n+1)$ and $0 \leq X_n, Y_n, U_{n+1}, V_{n+1} \leq 1$. This together with (1.3) and (1.4) implies that

$$\begin{aligned}
&\mathbb{E}[(X_{n+1} - Y_{n+1})^2 \mid \mathcal{F}_n] \\
&\quad \leq (X_n - Y_n)^2 + \frac{4}{(n+1)^2} + 2(X_n - Y_n)\mathbb{E}[\bar{\alpha}_n(U_{n+1} - X_n) - \bar{\beta}_n(V_{n+1} - Y_n) \mid \mathcal{F}_n] \\
&\quad = (1 - 2\bar{\alpha}_n\beta_n - 2\bar{\beta}_n)(X_n - Y_n)^2 + \frac{4}{(n+1)^2}.
\end{aligned} \tag{3.2}$$

Since $1 - 2\bar{\alpha}_n\beta_n - 2\bar{\beta}_n < 1$, we have, from (3.2),

$$\mathbb{E}\left[(X_{n+1} - Y_{n+1})^2 + \frac{4}{n+1} \,\middle|\, \mathcal{F}_n\right] \leq (X_n - Y_n)^2 + \frac{4}{(n+1)^2} + \frac{4}{n+1} \leq (X_n - Y_n)^2 + \frac{4}{n}.$$

Hence, $\{(X_n - Y_n)^2 + 4/n\}_{n\geq 1}$ is a nonnegative supermartingale.

We are now ready to prove our main result.

Theorem 3.2. *In the two-urn model, the fractions of red balls in urns A and B converge almost surely to the same limit.*

Proof. In view of Theorem 3.1, $X_n + Y_n$ converges almost surely. If we can further prove that $X_n - Y_n$ converges to 0 almost surely, then the proof is complete. Again Theorem 3.1 and the supermartingale convergence theorem (see [1, p. 419]) imply that $(X_n - Y_n)^2 + 4/n$ converges almost surely and so does $(X_n - Y_n)^2$ since $4/n$ converges to 0. Let $Z_n = (X_n - Y_n)^2$ and $Z_n \to Z$ almost surely. We claim that $\mathbb{E}Z_n \to 0$. If so, then, by the dominated convergence theorem, $\mathbb{E}Z = 0$ and, therefore, $Z = 0$ almost surely. This implies that $X_n - Y_n$ converges to 0 almost surely. In the following, we prove that $\mathbb{E}Z_n \to 0$.

From (3.2), we have

$$\mathbb{E}Z_{n+1} \leq (1 - 2\bar{\alpha}_n\beta_n - 2\bar{\beta}_n)\mathbb{E}Z_n + \frac{4}{(n+1)^2}. \tag{3.3}$$

Since $0 < \alpha_n, \beta_n < 1$, it follows that

$$1 - 2\bar{\alpha}_n\beta_n - 2\bar{\beta}_n = 1 - 2\bar{\alpha}_n\beta_n - 2(1 - \beta_n) = 2\beta_n(1 - \bar{\alpha}_n) - 1 = 2\alpha_n\beta_n - 1 \leq \alpha_n\beta_n,$$

which, combined with (3.3), implies that

$$\mathbb{E}Z_{n+1} \leq \alpha_n\beta_n\mathbb{E}Z_n + \frac{4}{(n+1)^2}. \tag{3.4}$$

Note that $0 < \alpha_n\beta_n < 1$, $\prod_{i=1}^n \alpha_i\beta_i = s_1t_1/s_{n+1}t_{n+1} \to 0$ and $\sum_{n=1}^\infty 4/(n+1)^2 < \infty$. Hence, applying Lemma 3.1 below to (3.4) gives the desired conclusion, $\mathbb{E}Z_n \to 0$.

Lemma 3.1. *Suppose that $\{x_n\}_{n\geq 1}$, $\{a_n\}_{n\geq 1}$, and $\{b_n\}_{n\geq 1}$ are nonnegative real sequences satisfying $x_{n+1} \leq a_nx_n + b_n$, where $0 < a_n < 1$ for $n \geq 1$. If $\prod_{i=1}^n a_i \to 0$ and $\sum_{n=1}^\infty b_n < \infty$, then $x_n \to 0$.*

Proof. First, note that $x_{n+1} \leq x_n + b_n$ since $0 < a_n < 1$. Thus,

$$x_{n+1} \leq x_1 + \sum_{i=1}^n b_i \leq x_1 + \sum_{i=1}^\infty b_i,$$

which implies that $\{x_n\}_{n\geq 1}$ is uniformly bounded by a positive constant M.

Given $\varepsilon > 0$, choose n_0 such that $\sum_{n=n_0}^\infty b_i < \varepsilon/(1 + M)$, and then choose $n_1 > n_0$ such that $\prod_{i=n_0}^n a_i < \varepsilon/(1 + M)$ whenever $n > n_1$. Now, for each $n > n_1$, we have

$$\begin{aligned} x_{n+1} &\leq a_nx_n + b_n \\ &\leq a_na_{n-1}x_{n-1} + b_{n-1} + b_n \\ &\vdots \\ &\leq \biggl(\prod_{i=n_0}^n a_i\biggr)x_{n_0} + \sum_{i=n_0}^n b_i \\ &\leq \frac{\varepsilon M}{1+M} + \frac{\varepsilon}{1+M} \\ &= \varepsilon. \end{aligned}$$

Hence, $x_n \to 0$, as required.

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