# STATE-DEPENDENT FRACTIONAL POINT PROCESSES

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ABSTRACT. The aim of this paper is the analysis of the fractional Poisson process where the state probabilities  $p_k^{\nu_k}(t), t \ge 0$ , are governed by time-fractional equations of order  $0 < \nu_k \le 1$  depending on the number k of events occurred up to time t. We are able to obtain explicitly the Laplace transform of  $p_k^{\nu_k}(t)$  and various representations of state probabilities. We show that the Poisson process with intermediate waiting times depending on  $\nu_k$  differs from that constructed from the fractional state equations (in the case  $\nu_k = \nu$ , for all k, they coincide with the time-fractional Poisson process). We also introduce a different form of fractional state-dependent Poisson process as a weighted sum of homogeneous Poisson processes. Finally we consider the fractional birth process governed by equations with state-dependent fractionality.

#### 1. INTRODUCTION

We first consider a state-dependent time-fractional Poisson process  $N(t), t \ge 0$ , whose state probabilities  $p_k^{\nu_k}(t) = \Pr\{N(t) = k\}$  are governed by the following equations

(1.1) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda p_k^{\nu_k}(t) + \lambda p_{k-1}^{\nu_{k-1}}(t), \quad k \ge 0, \ t > 0, \ \nu_k \in (0, 1], \ \lambda > 0, \\ p_k^{\nu_k}(0) = \begin{cases} 1, \quad k = 0, \\ 0, \quad k \ge 1, \end{cases} \end{cases}$$

where  $p_k^{\nu_k}(t) = 0$ , if  $k \in \mathbb{Z}^- \setminus \{0\}$ . These equations are obtained by replacing, in the governing equations of the homogeneous Poisson process, the ordinary derivative with the Dzhrbashyan–Caputo fractional derivative that is [16]

(1.2) 
$$\frac{\mathrm{d}^{\nu}}{\mathrm{d}t^{\nu}}f(t) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_{0}^{t} (t-s)^{m-\nu-1} f^{(m)}(s) \,\mathrm{d}s, & m-1 < \nu < m, \\ \frac{\mathrm{d}^{m}f}{\mathrm{d}t^{m}}, & \nu = m. \end{cases}$$

We remark that in (1.1), the order of the fractional derivatives depend on the number of events occurred up to time t. By definition we have that

(1.3) 
$$\frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = \frac{1}{\Gamma(1-\nu_k)} \int_0^t (t-s)^{-\nu_k} \frac{\mathrm{d}}{\mathrm{d}s} p_k^{\nu_k}(s) \,\mathrm{d}s, \qquad 0 < \nu_k < 1.$$

Hence the dependence of  $p_k^{\nu_k}(t)$  on the past is twofold. On one side, the fractional derivative depends on the whole time span [0, t] through the weight function. On the other side the number of events occurred up to the time t modifies the power of the weight function. This means that the memory effect can play an increasing or decreasing role, in the case of a monotonical structure of the sequence of fractional orders  $\nu_k$ .

For example, if  $\nu_k$  decreases with k, the memory function tends to be constant and to give the same weight to the whole time span [0, t]. We notice that state-depending fractionality was considered in different contexts by Fedotov et al. [5].

For  $\nu_k = \nu$ , for all k, the system (1.1) coincides with the one governing the classical fractional Poisson process considered for example by Beghin and Orsingher [3], where the fractional derivative is meant in the Dzhrbashyan–Caputo sense as in this case. Of course, if  $\nu_k = 1$ , for all k, we retrieve the governing equation for the homogeneous Poisson process. Some papers devoted to various forms of fractional Poisson processes have appeared in the last decades. In Hilfer and Anton [7] the authors introduced for the first time the Mittag-Leffler waiting-time density in the theory of continuous-time random walks. The time-fractional Poisson process was then explicitly considered by Repin and Saichev [18]. Starting from this paper, different approaches to fractional Poisson processes were considered. In Mainardi et al. [10], for example, the authors considered renewal processes with Mittag-Leffler distributed intertimes. A slightly different approach to

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the fractional Poisson process was developed in Laskin [8], where the fractional derivative appearing in the equations governing the state probabilities coincides with the Riemann-Liouville derivative. More recently Beghin and Orsingher [3] and Meerschaert et al. [12] studied the subordination of the Poisson process to the inverse stable subordinator, discussing the relation with fractional Poisson processes. Another type of fractional Poisson process was developed in Orsingher and Polito [15] where a space-fractionality is considered. Physical applications of the fractional Poisson processes are discussed, for example, in Laskin [9], where a new family of quantum coherent states has been studied.

By solving equation (1.1), we obtain that

(1.4) 
$$\int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) \mathrm{d}t = \frac{\lambda^{k} s^{\nu_{0}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}} + \lambda)}, \qquad s > 0.$$

The inversion of (1.4) is by no means a simple matter and we have been able to obtain an explicit result for  $p_0^{\nu_0}$  and  $p_1^{\nu_1}$  in terms of generalized Mittag–Leffler functions defined as (see for example Saxena et al. [19])

(1.5) 
$$E_{\nu,\beta}^{m}(x) = \sum_{k=0}^{\infty} \frac{x^{k} \Gamma(m+k)}{k! \Gamma(\nu k+\beta) \Gamma(m)}, \qquad \nu, \beta, m \in \mathbb{R}^{+}, \ x \in \mathbb{R}$$

We give also the distribution  $p_k^{\nu_k}(t)$  of the Poisson process with fractionality  $\nu_k$  depending on the number of events k, in terms of subordinators and their inverses (see formula (2.26) below).

A part of our paper is devoted to the construction of a point process  $\mathcal{N}(t)$ ,  $t \ge 0$ , with intertime  $U_k$  between the kth and (k+1)th event distributed as

(1.6) 
$$\Pr\{U_k > t\} = E_{\nu_k, 1}(-\lambda t^{\nu_k}).$$

The Laplace transform of the univariate distributions of  $\mathcal{N}(t), t \ge 0$ , is

(1.7) 
$$\int_0^\infty e^{-st} \Pr\{\mathcal{N}(t) = k\} \mathrm{d}t = \lambda^k \frac{s^{\nu_k - 1}}{\prod_{j=0}^k (s^{\nu_j} + \lambda)},$$

which slightly differs from (1.4). From this point of view the state-dependent fractional Poisson process differs from the time-fractional Poisson process because the approach based on the construction by means of independent inter-event times  $U_k$  and the one based on fractional equations (1.1), do not lead to the same one-dimensional distribution. We show that the probabilities  $p_k(t) = \Pr{\mathcal{N}(t) = k}$  are solutions to the fractional integral equations

(1.8) 
$$p_k(t) - p_k(0) = -\lambda I^{\nu_k} p_k(t) + \lambda I^{\nu_{k-1}} p_{k-1}(t),$$

where  $I^{\nu_k}$  is the Riemann–Liouville fractional integral

(1.9) 
$$(I^{\nu_k} f)(t) = \frac{1}{\Gamma(\nu_k)} \int_0^t (t-s)^{\nu_k - 1} f(s) \mathrm{d}s, \qquad \nu_k > 0$$

A third definition of the state-dependent fractional Poisson process, say  $\hat{N}(t)$ , with distribution

(1.10) 
$$\Pr\{\hat{N}(t) = j\} = \frac{\frac{(\lambda t)^{j}}{\Gamma(\nu_{j}j+1)} \frac{1}{E_{\nu_{j},1}(\lambda t)}}{\sum_{j=0}^{+\infty} \frac{(\lambda t)^{j}}{\Gamma(\nu_{j}j+1)} \frac{1}{E_{\nu_{j},1}(\lambda t)}}, \quad j \ge 0,$$

is introduced and analyzed in Section 3. The distribution

(1.11) 
$$\Pr\{\hat{N}_{\nu}(t) = j\} = \frac{(\lambda t)^{k}}{\Gamma(\nu j + 1)} \frac{1}{E_{\nu_{j},1}(\lambda t)}$$

investigated in Beghin and Orsingher [3], has been proved to be a weighted sum of Poisson distributions in Balakrishnan and Kozubowski [1] and Beghin and Macci [2].

Finally, we analyze the state-dependent nonlinear pure birth process with one initial progenitor, where the state probabilities  $p_k^{\nu_k}(t)$  satisfy the fractional equations

(1.12) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda_k p_k^{\nu_k}(t) + \lambda_{k-1} p_{k-1}^{\nu_{k-1}}(t), \quad k \ge 1, \ t > 0, \ \nu_k \in (0,1], \\ p_k^{\nu_k}(0) = \begin{cases} 1, \quad k = 1, \\ 0, \quad k \ge 2. \end{cases} \end{cases}$$

The Laplace transform of the solution to (1.12) reads

(1.13) 
$$\int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) dt = \left(\prod_{j=1}^{k-1} \lambda_{j}\right) \frac{s^{\nu_{1}-1}}{\prod_{j=1}^{k} (s^{\nu_{j}} + \lambda_{j})}.$$

A similar and more general state-dependent fractional birth-death process was recently tackled by Fedotov et al. [5], where possible applications to chemotaxis are sketched.

The case where  $\nu_k = \nu$ , for all k in (1.12), has been dealt with in Orsingher and Polito [14]. An attempt to apply this fractional birth process was discussed in Garra and Polito [6] in relation to tumoral growth models and ETAS (*Epidemic Type Aftershock Sequences*) model in statistical seismology.

The dependence of the state probabilities of the point processes considered here from the structure of  $\nu_k$ , requires a further investigation which certainly implies a numerical approach.

1.1. Notation. For the sake of clarity we briefly summarize the notation used for the different point processes analyzed in the following sections.

First we indicate with N(t),  $t \ge 0$ , the counting process associated with the variable-order differencedifferential equations (1.1). In particular, the state probabilities  $p_k^{\nu_k}(t) = \Pr\{N(t) = k\}$ ,  $k \ge 0$ , represent the probability of being in state k at a fixed time  $t \ge 0$ . The point process constructed and studied in Section 3 by means of independent but non i.d. inter-arrival times is instead indicated by a calligraphic  $\mathcal{N}(t)$ . Both processes, when  $\nu_k = \nu$  for all  $k \ge 0$ , reduce to the time-fractional Poisson process  $N_{\nu}(t)$  treated for example in Beghin and Orsingher [3]. In the same article it is also considered the alternative definition for a fractional Poisson process characterized by the distribution (1.11) and denoted here by  $\hat{N}_{\nu}(t)$ ,  $t \ge 0$ . We refer to its direct generalization in a state-dependent sense as  $\hat{N}(t)$  for which the distribution becomes that in (1.10). Lastly, the linear fractional pure birth process with state-dependent order of fractionality presented in the last section is simply indicated as  $N_{lin}(t)$ ,  $t \ge 0$ .

## 2. The state-dependent fractional Poisson process

We first consider a state-dependent time-fractional Poisson process N(t),  $t \ge 0$ , whose state probabilities  $p_k^{\nu_k}(t) = \Pr\{N(t) = k\}$  are governed by equations (1.1). We have the following result

Theorem 2.1. The Laplace transform of the solution to the state-dependent time-fractional equations

(2.1) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda p_k^{\nu_k}(t) + \lambda p_{k-1}^{\nu_{k-1}}(t), \quad k \ge 0, \ t > 0, \ \nu_k \in (0, 1], \\ p_k^{\nu_k}(0) = \begin{cases} 1, \quad k = 0, \\ 0, \quad k \ge 1, \end{cases} \end{cases}$$

reads

(2.2) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = \int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) dt = \frac{\lambda^{k} s^{\nu_{0}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}} + \lambda)},$$

where the fractional derivative appearing in (2.1) is in the sense of Dzhrbashyan-Caputo.

*Proof.* We can solve equation (2.1) by means of an iterative procedure, as follows. The equation related to k = 0

(2.3) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_0}}{\mathrm{d}t^{\nu_0}} p_0^{\nu_0}(t) = -\lambda p_0^{\nu_0}(t), \quad t > 0, \ \nu_0 \in (0, 1], \\ p_0^{\nu_0}(0) = 1, \end{cases}$$

has solution  $p_0^{\nu_0}(t) = E_{\nu_0,1}(-\lambda t^{\nu_0})$ , with Laplace transform

(2.4) 
$$\tilde{p}_0^{\nu_0}(s) = \int_0^{+\infty} e^{-st} p_0^{\nu_0}(t) \mathrm{d}t = \frac{s^{\nu_0 - 1}}{\lambda + s^{\nu_0}},$$

where

(2.5) 
$$E_{\nu_0,1}(-\lambda t^{\nu_0}) = \sum_{k=0}^{\infty} \frac{(-\lambda t^{\nu_0})^k}{\Gamma(\nu_0 k+1)}$$

is the Mittag–Leffler function.

For k = 1, the equation

(2.6) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_1}}{\mathrm{d}t^{\nu_1}} p_1^{\nu_1}(t) = -\lambda p_1^{\nu_1}(t) + \lambda p_0^{\nu_0}(t), \quad t > 0, \ \nu_1 \in (0, 1], \\ p_1^{\nu_1}(0) = 0, \end{cases}$$

has solution with Laplace transform

(2.7) 
$$\tilde{p}_1^{\nu_1}(s) = \int_0^{+\infty} e^{-st} p_1^{\nu_1}(t) \mathrm{d}t = \frac{\lambda s^{\nu_0 - 1}}{\lambda + s^{\nu_0}} \frac{1}{\lambda + s^{\nu_1}}.$$

By iterating this procedure, we arrive at

(2.8) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = \int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) dt = \frac{\lambda^{k} s^{\nu_{0}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}} + \lambda)}.$$

**Remark 2.2.** A direct approach based on the inversion of the Laplace transform of (2.8) is clumsy and cumbersome. We give the explicit evaluation of  $p_1^{\nu_1}(t)$ . In this case, from (2.7), we have that

(2.9) 
$$\tilde{p}_{1}^{\nu_{1}}(s) = \int_{0}^{+\infty} e^{-st} p_{1}^{\nu_{1}}(t) dt$$
$$= \frac{\lambda s^{\nu_{0}-1}}{\lambda^{2} + \lambda(s^{\nu_{0}} + s^{\nu_{1}}) + s^{\nu_{0}+\nu_{1}}}$$
$$= \frac{\lambda s^{\nu_{0}-1}}{\lambda^{2} + s^{\nu_{0}+\nu_{1}}} \frac{1}{1 + \frac{\lambda(s^{\nu_{0}} + s^{\nu_{1}})}{\lambda^{2} + s^{\nu_{0}+\nu_{1}}}}$$
$$= \lambda s^{\nu_{0}-1} \sum_{m=0}^{\infty} \frac{(-\lambda(s^{\nu_{0}} + s^{\nu_{1}}))^{m}}{(\lambda^{2} + s^{\nu_{0}+\nu_{1}})^{m+1}}$$
$$= \lambda s^{\nu_{0}-1} \sum_{m=0}^{\infty} \frac{(-\lambda)^{m}}{(\lambda^{2} + s^{\nu_{0}+\nu_{1}})^{m+1}} \sum_{r=0}^{m} \binom{m}{r} s^{\nu_{0}r+\nu_{1}(m-r)}.$$

The inversion of (2.9) involves the generalized Mittag-Leffler function, defined as (see, for example, Saxena et al. [19])

(2.10) 
$$E_{\nu,\beta}^{m}(-\lambda t^{\nu}) = \sum_{k=0}^{\infty} \frac{(-\lambda t^{\nu})^{k} \Gamma(m+k)}{k! \Gamma(\nu k+\beta) \Gamma(m)},$$

where  $\nu, \beta, m \in \mathbb{R}^+$ .

Indeed, we recall the following relation.

(2.11) 
$$\int_{0}^{+\infty} e^{-st} t^{\beta-1} E^{m}_{\nu,\beta}(-\lambda t^{\nu}) \mathrm{d}t = \frac{s^{\nu m-\beta}}{(\lambda+s^{\nu})^{m}}.$$

In view of (2.9) and (2.11), we arrive at

(2.12) 
$$p_1^{\nu_1}(t) = \sum_{m=0}^{\infty} (-1)^m \lambda^{m+1} \sum_{r=0}^m \binom{m}{r} t^{\nu_0(m-r)+\nu_1r+\nu_1} E_{\nu_0+\nu_1,\nu_0(m-r)+\nu_1r+\nu_1+1}^{m+1} (-\lambda^2 t^{\nu_0+\nu_1}).$$

For the case  $\nu_1 = \nu_0 = \nu$ , formula (2.12) becomes

$$(2.13) \qquad p_{1}^{\nu}(t) = \sum_{m=0}^{\infty} (-1)^{m} \lambda^{m+1} \sum_{r=0}^{m} \binom{m}{r} t^{\nu(m+1)} E_{2\nu,\nu(m+1)+1}^{m+1} (-\lambda^{2} t^{2\nu}) 
= \sum_{m=0}^{\infty} (-1)^{m} (\lambda t^{\nu})^{m+1} 2^{m} E_{2\nu,\nu(m+1)+1}^{m+1} (-\lambda^{2} t^{2\nu})^{r} 
= \sum_{m=0}^{\infty} (-1)^{m} (\lambda t^{\nu})^{m+1} 2^{m} \sum_{r=0}^{\infty} \binom{m+r}{r} \frac{(-1)^{r} (\lambda^{2} t^{2\nu})^{r}}{\Gamma(2\nu r + \nu(m+1) + 1)} 
= \sum_{m=0}^{\infty} (-1)^{m} (\lambda t^{\nu})^{m+1} 2^{m} \sum_{r=0}^{\infty} \binom{-(m+1)}{r} (\lambda^{2} t^{2\nu})^{r} \frac{1}{2\pi i} \int_{Ha} e^{w} w^{-2\nu r - \nu(m+1) - 1} dw 
= \sum_{m=0}^{\infty} (-1)^{m} (\lambda t^{\nu})^{m+1} 2^{m} \frac{1}{2\pi i} \int_{Ha} e^{w} w^{-\nu(m+1) - 1} \left[ \sum_{r=0}^{\infty} \binom{-(m+1)}{r} (\lambda^{2} t^{2\nu} w^{-2\nu})^{r} \right] dw 
= \sum_{m=0}^{\infty} (-1)^{m} \lambda t^{\nu})^{m+1} 2^{m} \frac{1}{2\pi i} \int_{Ha} e^{w} \frac{w^{-\nu(m+1) - 1}}{(\lambda^{2} t^{2\nu} w^{-2\nu} + 1)^{m+1}} dw$$

$$= \frac{1}{2\pi i} \int_{Ha} e^w \frac{\lambda t^\nu w^{-\nu-1}}{\lambda^2 t^{2\nu} w^{-2\nu} + 1} \left[ \sum_{m=0}^\infty (-1)^m \left( \frac{2w^{-\nu} \lambda t^\nu}{\lambda^2 t^{2\nu} w^{-2\nu} + 1} \right)^m \right] \mathrm{d}w$$
$$= \frac{1}{2\pi i} \int_{Ha} \frac{\lambda t^\nu w^{\nu-1} e^w}{(w^\nu + \lambda t^\nu)^2} \mathrm{d}w$$
$$= \frac{\lambda t^\nu}{\nu} E_{\nu,\nu} (-\lambda t^\nu),$$

where we have used in the last equality the fact that

(2.14) 
$$E_{\nu,\nu}(x) = \nu \frac{\mathrm{d}}{\mathrm{d}x} E_{\nu,1}(x) = \frac{\nu}{2\pi i} \frac{\mathrm{d}}{\mathrm{d}x} \int_{Ha} \frac{e^w w^{\nu-1}}{w^\nu - x} \mathrm{d}w = \frac{\nu}{2\pi i} \int_{Ha} \frac{e^w w^{\nu-1}}{(w^\nu - x)^2} \mathrm{d}w,$$

and we have applied the contour-integral representation of the reciprocal of the Gamma function

(2.15) 
$$\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_{Ha} e^u u^{-x} \mathrm{d}u$$

where Ha stands for the Hankel contour (see formula 5.9.2, pg. 139 in Olver et al. [13]).

We notice that equation (2.13) gives the result obtained for the time-fractional Poisson process in Beghin and Orsingher [3] as expected. Moreover by considering that

(2.16) 
$$\int_0^{+\infty} e^{-st} \frac{\lambda t^{\nu}}{\nu} E_{\nu,\nu}(-\lambda t^{\nu}) \mathrm{d}t = \frac{\lambda s^{\nu-1}}{(\lambda+s^{\nu})^2},$$

we retrieve, for the case  $\nu = \nu_0 = \nu_1$  that

(2.17) 
$$p_1^{\nu}(t) = \frac{\lambda t^{\nu}}{\nu} E_{\nu,\nu}(-\lambda t^{\nu}),$$

 $a \pm \infty$ 

that is the result obtained for the time-fractional Poisson process (see formula (2.11) of Beghin and Orsingher [3]).

By applying formula (34) of Saxena et al. [19] it is possible to give an explicit expression for  $p_k^{\nu_k}(t)$ , for any  $k \ge 2$ , in terms of cumbersome sums of generalized Mittag-Leffler functions.

**Remark 2.3.** A different way to give a representation of the state probability in the state-dependent Poisson process is given by the following integral approach; starting from (2.8), we have

 $u_{0} = 1$ 

(2.18) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = \int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) dt = \lambda^{k} \frac{s^{\nu_{0}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}} + \lambda)}$$
$$= \left( \int_{0}^{\infty} e^{-\lambda w_{0}} s^{\nu_{0}-1} e^{-w_{0}s^{\nu_{0}}} dw_{0} \right) \left( \prod_{j=1}^{k} \int_{0}^{\infty} e^{-\lambda w_{j}} \lambda e^{-w_{j}s^{\nu_{j}}} dw_{j} \right).$$

For the following developments, it is useful to recall that the inverse process of a  $\nu$ -stable subordinator  $\mathcal{H}^{\nu}(t), t \ge 0$ , namely  $\mathcal{L}^{\nu}(t), t \ge 0$ , is such that

(2.19) 
$$\Pr{\mathcal{L}^{\nu}(t) < x} = \Pr{\mathcal{H}^{\nu}(x) > t}, \qquad x, t \ge 0$$

Hence the relation between the law  $l_{\nu}(x,t)$  of the process  $\mathcal{L}^{\nu}(t)$  and the law  $h_{\nu}(x,t)$  of the process  $\mathcal{H}^{\nu}(t)$  is given by (see for example D'Ovidio et al. [4])

(2.20) 
$$l_{\nu}(x,t) = \frac{\Pr\{\mathcal{L}^{\nu}(t) \in \mathrm{d}x\}}{\mathrm{d}x} = \frac{\partial}{\partial x} \Pr\{\mathcal{H}^{\nu}(x) > t\} = \frac{\partial}{\partial x} \int_{t}^{\infty} h_{\nu}(s,x) \mathrm{d}s,$$

 $or \ otherwise$ 

(2.21) 
$$\int_{t}^{\infty} \Pr\{\mathcal{H}^{\nu}(x) \in \mathrm{d}w\} = \int_{0}^{x} \Pr\{\mathcal{L}^{\nu}(t) \in \mathrm{d}z\}.$$

Hence the density of the inverse process  $\mathcal{L}^{\nu}(t)$  reads

(2.22) 
$$\Pr\{\mathcal{L}^{\nu}(t) \in \mathrm{d}x\} = \frac{\partial}{\partial x} \int_{t}^{\infty} \Pr\{\mathcal{H}^{\nu}(x) \in \mathrm{d}w\}$$

Therefore the Laplace transform of  $l_{\nu}(x,t)$  is given by

(2.23) 
$$\tilde{l}_{\nu}(x,s) = \int_{0}^{\infty} e^{-st} l_{\nu}(x,t) dt = \int_{0}^{\infty} e^{-st} \frac{d}{dx} \left[ \int_{t}^{+\infty} \Pr\{\mathcal{H}^{\nu}(x) \in dw\} \right] dt$$
$$= \frac{d}{dx} \int_{0}^{\infty} \Pr\{\mathcal{H}^{\nu}(x) \in dw\} \int_{0}^{w} e^{-st} dt$$

$$= \frac{1}{s} \frac{\mathrm{d}}{\mathrm{d}x} \left[ \int_0^\infty (1 - e^{-sw}) \Pr\{\mathcal{H}^\nu(x) \in \mathrm{d}w\} \right] = s^{\nu-1} e^{-xs^\nu}$$

where we used the fact that

(2.24) 
$$\tilde{h}_{\nu}(x,s) = \int_{0}^{+\infty} e^{-st} h_{\nu}(x,t) dt = e^{-xs^{\nu}}$$

We also notice that the explicit form of the law of the inverse of the stable subordinator is known in terms of Wright functions [4]. Going back to equation (2.18) and in view of (2.23), we can write

$$(2.25) \qquad \tilde{p}_{k}^{\nu_{k}}(s) = \left(\int_{0}^{\infty} e^{-\lambda w_{0}} \mathrm{d}w_{0} \int_{0}^{\infty} e^{-st} l_{\nu_{0}}(w_{0}, t) \mathrm{d}t\right) \left(\prod_{j=1}^{k} \lambda \int_{0}^{\infty} e^{-\lambda w_{j}} \mathrm{d}w_{j} \int_{0}^{\infty} e^{-sx} h_{\nu_{j}}(x, w_{j}) \mathrm{d}x\right) \\ = \int_{0}^{\infty} \mathrm{d}w_{0} e^{-\lambda w_{0}} \cdots \int_{0}^{\infty} \mathrm{d}w_{k} e^{-\lambda w_{k}} \left[\int_{0}^{\infty} e^{-st} l_{\nu_{0}}(w_{0}, t) \mathrm{d}t \prod_{j=1}^{k} \lambda \int_{0}^{\infty} e^{-sx} h_{\nu_{j}}(x, w_{j}) \mathrm{d}x\right].$$

Hence by inverting the Laplace transform we obtain

(2.26) 
$$p_k^{\nu_k}(t) = \lambda^k \int_0^\infty \mathrm{d}w_0 e^{-\lambda w_0} \int_0^\infty \mathrm{d}w_1 e^{-\lambda w_1} \cdots \int_0^\infty \mathrm{d}w_k e^{-\lambda w_k} \left[ l_{\nu_0}(w,t) * h_{\nu_1,\cdots,\nu_k}(w_1,\cdots,w_k,t) \right],$$

where the symbol \* stands for the convolution of the law of the inverse stable subordinator  $l_{\nu_0}$  and the distribution of the sum of k independent stable subordinators  $h_{\nu_1,\dots,\nu_k}(w_1,\dots,w_k,t)$ . In other words  $l_{\nu_0}(w,t) * h_{\nu_1,\dots,\nu_k}(w_1,\dots,w_k,t)$  is the distribution of the r.v.

(2.27) 
$$\mathcal{L}^{\nu_0}(t) + \sum_{j=1}^k \mathcal{H}^{\nu_j}(t).$$

**Remark 2.4.** Another interesting characterization of the state-probabilities of the above process is given by the following observation. First of all, since for m = 1,  $E_{\nu,\beta}^1(\cdot) = E_{\nu,\beta}(\cdot)$ , from (2.11) we have that

(2.28) 
$$\int_{0}^{+\infty} e^{-st} t^{\nu-1} E_{\nu,\nu}(-\lambda t^{\nu}) dt = \frac{1}{\lambda + s^{\nu}},$$
(2.29) 
$$\int_{0}^{+\infty} e^{-st} E_{\nu,1}(-\lambda t^{\nu}) dt = \frac{s^{\nu-1}}{\lambda + s^{\nu}}.$$

Hence, from (2.8), we find that

(2.30) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = \frac{\lambda^{k} s^{\nu_{0}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}} + \lambda)}$$
$$= \left[ \int_{0}^{+\infty} e^{-st} E_{\nu_{0},1}(-\lambda t^{\nu}) dt \right] \prod_{j=1}^{k} \left[ \int_{0}^{+\infty} e^{-st} \lambda t^{\nu_{j}-1} E_{\nu_{j},\nu_{j}}(-\lambda t^{\nu_{j}}) dt \right].$$

On the other hand, from (2.25), we have

$$(2.31) \qquad \tilde{p}_{k}^{\nu_{k}}(s) = \left(\int_{0}^{\infty} e^{-\lambda w_{0}} \mathrm{d}w_{0} \int_{0}^{\infty} e^{-st} l_{\nu_{0}}(w_{0}, t) \mathrm{d}t\right) \left(\prod_{j=1}^{k} \lambda \int_{0}^{\infty} e^{-\lambda w_{j}} \mathrm{d}w_{j} \int_{0}^{\infty} e^{-sx} h_{\nu_{j}}(x, w_{j}) \mathrm{d}x\right) \\ = \left(\int_{0}^{\infty} e^{-st} \tilde{l}_{\nu_{0}}(\lambda, t) \mathrm{d}t\right) \left(\prod_{j=1}^{k} \lambda \int_{0}^{\infty} e^{-sx} \tilde{h}_{\nu_{j}}(x, \lambda) \mathrm{d}x\right),$$

which clearly coincides with (2.30).

By inverting the Laplace transform, we obtain the following result

(2.32) 
$$p_{k}^{\nu_{k}}(t) = E_{\nu_{0},1}(-\lambda t^{\nu_{0}}) \underset{j=1}{\overset{k}{\underset{j=1}{\times}}} \lambda t^{\nu_{j}-1} E_{\nu_{j},\nu_{j}}(-\lambda t^{\nu_{j}})$$
$$= \int_{0}^{\infty} E_{\nu_{0},1}\left(-\lambda(t-s)^{\nu_{0}}\right) g(s) ds,$$

where g(s) is the k-th time iterated convolution of the functions

$$h_j(t) = \lambda t^{\nu_j - 1} E_{\nu_j, \nu_j} \left( -\lambda t^{\nu_j} \right)$$

We notice that, the last equation can be written in terms of the Prabhakar operator, that is an integral operator involving a Mittag-Leffler function as kernel [17]. From equation (2.32) we have an integral representation, in explicit form given by

$$(2.33) \quad p_1^{\nu_1}(t) = \int_0^t E_{\nu_0,1}(-\lambda(t-s)^{\nu_0})E_{\nu_1,\nu_1}(-\lambda s^{\nu_1})s^{\nu_1-1}ds$$

$$p_2^{\nu_2}(t) = \int_0^t ds_1 E_{\nu_0,1}(-\lambda(t-s_1)^{\nu_0})\int_0^{s_1} ds_2 s_2^{\nu_1-1}E_{\nu_1,\nu_1}(-\lambda s_2^{\nu_1})(s_1-s_2)^{\nu_2-1}E_{\nu_2,\nu_2}(-\lambda(s_1-s_2)^{\nu_2})$$

$$\vdots$$

$$p_k^{\nu_k}(t) = \int_0^t ds_1 E_{\nu_0,1}(-\lambda(t-s_1)^{\nu_0})\int_0^{s_1} ds_2\cdots\times$$

$$\times \int_0^{s_k} ds_k s_k^{\nu_{k-1}-1}E_{\nu_{k-1},\nu_{k-1}}(-\lambda s_k^{\nu_{k-1}})(s_{k-1}-s_k)^{\nu_k-1}E_{\nu_k,\nu_k}(-\lambda(s_{k-1}-s_k)^{\nu_k}).$$

In order to find the mean value of the distribution  $p_k^{\nu_k}(t)$ , we multiply all the terms of (2.1) for k and sum over all the states so that

(2.34) 
$$\sum_{k=0}^{\infty} k \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda \sum_{k=0}^{\infty} k p_k^{\nu_k}(t) + \lambda \sum_{k=0}^{\infty} k p_{k-1}^{\nu_{k-1}}(t)$$
$$= -\lambda \sum_{k=0}^{\infty} k p_k^{\nu_k}(t) + \lambda \sum_{k=0}^{\infty} (k+1) p_k^{\nu_k}(t) = \lambda.$$

In the case  $\nu_k = \nu$ , for all k, we have

(2.35) 
$$\frac{\mathrm{d}^{\nu}}{\mathrm{d}t^{\nu}}\sum_{k=0}^{\infty}kp_{k}^{\nu}(t) = \frac{\mathrm{d}^{\nu}}{\mathrm{d}t^{\nu}}\mathbb{E}(N_{\nu}(t)) = \lambda,$$

whose solution is given by  $\mathbb{E}(N_{\nu}(t)) = \frac{\lambda t^{\nu}}{\Gamma(\nu+1)}$  (see formula (2.7) in Beghin and Orsingher [3]). We notice that it is possible to find an interesting summation formula by using the Laplace transform in equation (2.34). Indeed we have

(2.36) 
$$\sum_{k=0}^{\infty} k s^{\nu_k} \tilde{p}_k^{\nu_k}(s) = \lambda s^{-1},$$

and recalling that

(2.37) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = \frac{\lambda^{\kappa} s^{\nu_{0}-1}}{\prod_{i=0}^{k} (s^{\nu_{j}} + \lambda)}$$

we find that

(2.38) 
$$\sum_{k=0}^{\infty} \frac{k\lambda^k s^{\nu_0+\nu_k}}{\prod_{j=0}^k (s^{\nu_j}+\lambda)} = \lambda.$$

This summation formula is not trivial and we can check that it works for example in the special case  $\nu = \nu_k$  for all k.

(2.39) 
$$\sum_{k=1}^{\infty} \frac{k\lambda^k s^{2\nu}}{(s^{\nu}+\lambda)^{k+1}} = \frac{s^{2\nu}}{s^{\nu}+\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^k}{(s^{\nu}+\lambda)^k}$$
$$= \frac{\lambda s^{2\nu}}{s^{\nu}+\lambda} \left[ \frac{d}{dw} \sum_{k=1}^{\infty} \frac{w^k}{(s^{\nu}+\lambda)^k} \right]_{w=\lambda}$$
$$= \frac{\lambda s^{2\nu}}{s^{\nu}+\lambda} \left[ \frac{d}{dw} \frac{w}{s^{\nu}+\lambda-w} \right]_{w=\lambda}$$
$$= \frac{\lambda s^{2\nu}}{s^{\nu}+\lambda} \left[ \frac{s^{\nu}+\lambda}{(s^{\nu}+\lambda-w)^2} \right]_{w=\lambda} = \lambda$$

**Remark 2.5.** We notice that for the probability generating function G(u,t) of the process N(t),  $t \ge 0$ , the following representation holds for  $u \in [0,1]$ 

(2.40) 
$$\int_0^\infty e^{-st} G(u,t) dt = \sum_{k=0}^\infty u^k \int_0^\infty e^{-st} \Pr\{N(t) = k\} dt$$

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$$= \int_0^\infty e^{-st} \Pr\{\min_{\substack{0 \le k \le N(t)}} X_k > 1 - u\} dt$$
$$= \sum_{k=0}^\infty \frac{\lambda^k u^k s^{\nu_0 - 1}}{\prod_{j=0}^k (\lambda + s^{\nu_j})},$$

where  $X_k$ ,  $k \ge 1$ , are *i.i.d.* random variables uniform in [0, 1].

The representation of the probability generating function as

(2.41) 
$$G(u,t) = \Pr\{\min_{0 \le k \le N(t)} X_k > 1 - u\},\$$

follows the same lines of the time and space fractional Poisson processes described in Orsingher and Polito [15]. In (2.40), the driving process is the state-dependent Poisson process.

## 3. Alternative forms of the state-dependent Poisson process

We construct now a point process with independent but not i.d. inter-arrival times. In particular, the waiting time  $U_k$  between the kth and (k + 1)th arrival is distributed with p.d.f.

(3.1) 
$$f_{U_k}(t) = \lambda t^{\nu_k - 1} E_{\nu_k, \nu_k}(-\lambda t^{\nu_k}), \ t > 0$$

Let us now call  $\mathcal{N}(t), t \ge 0$ , such process and we have the following theorem.

**Theorem 3.1.** The state probabilities  $p_k(t)$  of the process  $\mathcal{N}(t), t \ge 0$ , are governed by the integral equation

(3.2) 
$$p_k(t) - p_k(0) = -\lambda I^{\nu_k} p_k(t) + \lambda I^{\nu_{k-1}} p_{k-1}(t), \qquad t \ge 0, \ \nu_k \in (0,1],$$

where  $I^{\nu}$  is the fractional integral in the sense of Riemann-Liouville (see (1.9)). Moreover, their Laplace transforms are given by

(3.3) 
$$\int_0^\infty e^{-st} \Pr\{\mathcal{N}(t) = k\} \mathrm{d}t = \lambda^k \frac{s^{\nu_k - 1}}{\prod_{j=0}^k (s^{\nu_j} + \lambda)}$$

Proof. First, we observe that the Laplace transform of the state probabilities, can be directly calculated by using the definition of the process  $\mathcal{N}(t)$ 

$$4) \qquad \int_{0}^{\infty} e^{-st} \Pr\{\mathcal{N}(t) = k\} dt \\ = \int_{0}^{\infty} e^{-st} dt \left[ \int_{0}^{t} \Pr(U_{0} + \dots + U_{k-1} \in dy) - \int_{0}^{t} \Pr(U_{0} + \dots + U_{k} \in dy) \right] \\ = \frac{1}{s} \int_{0}^{\infty} e^{-sy} \left[ \Pr(U_{0} + \dots + U_{k-1} \in dy) - \Pr(U_{0} + \dots + U_{k} \in dy) \right] \\ = \frac{1}{s} \left[ \frac{\lambda^{k}}{\prod_{j=0}^{k-1} (\lambda + s^{\nu_{j}})} - \frac{\lambda^{k+1}}{\prod_{j=0}^{k} (\lambda + s^{\nu_{j}})} \right] \\ = \frac{1}{s} \frac{\lambda^{k} (\lambda + s^{\nu_{j}}) - \lambda^{k+1}}{\prod_{j=0}^{k} (\lambda + s^{\nu_{k}})} \\ = \lambda^{k} \frac{s^{\nu_{k}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}} + \lambda)}.$$

We notice that, unfortunately, it does not coincide with (2.8).

Hence we have two distinct processes that can be matched only by assuming that  $\nu_k = \nu$  for each k = 0, 1... (in other words in the time-fractional Poisson case).

We can also find in explicit way the integral equation governing the probabilities  $p_k(t) = \Pr\{\mathcal{N}(t) = k\}$ . We start from the ordinary difference-differential equation, governing the Poisson process

(3.5) 
$$\frac{\mathrm{d}p_k}{\mathrm{d}t}(t) = -\lambda p_k(t) + \lambda p_{k-1}(t),$$

.

with initial conditions

(3.6) 
$$p_k(0) = \begin{cases} 1 & k = 0, \\ 0 & k \ge 1. \end{cases}$$

By integration with respect to t, we have the equivalent integral equation

(3.7) 
$$p_k(t) - p_k(0) = -\lambda \int_0^t p_k(s) ds + \lambda \int_0^t p_{k-1}(s) ds$$

In order to obtain a fractional generalization of the last equation, we replace the first-order integral in the right hand side of (3.7), with state-dependent fractional integrals, i.e.

(3.8) 
$$p_k(t) - p_k(0) = -\lambda I^{\nu_k} p_k(t) + \lambda I^{\nu_{k-1}} p_{k-1}(t), \quad t \ge 0, \ \nu_k \in (0,1], \ k \ge 0,$$

where  $I^{\nu_k}$  is the fractional integral in the sense of Riemann–Liouville. For k = 0, we have

(3.9) 
$$p_0(t) - 1 = -\lambda I^{\nu_0} p_0(t),$$

whose solution is simply given by  $p_0(t) = E_{\nu_0,1}(-\lambda t^{\nu_0})$ . With k = 1, we obtain

(3.10) 
$$p_1(t) = -\lambda I^{\nu_1} p_1(t) + \lambda I^{\nu_0} p_0(t),$$

whose Laplace transform, after some simple calculation, is given by

(3.11) 
$$\tilde{p}_1(t) = \frac{\lambda s^{\nu_1 - 1}}{(s^{\nu_0} + \lambda)(s^{\nu_1} + \lambda)},$$

and coincides with (3.4) in the case k = 1. Then, it is immediate to prove that, for any order  $k \ge 1$ , the Laplace transform of  $p_k(t)$ , is given by (3.4). This proves that (3.8) is the governing equation for  $\mathcal{N}(t)$ ,  $t \ge 0$ , as claimed.

In order to highlight the relation between the two processes N(t),  $t \ge 0$ , and  $\mathcal{N}(t)$ ,  $t \ge 0$ , by rearranging (2.37), we can write the following

(3.12) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = s^{\nu_{0}-\nu_{k}} \frac{\lambda^{k} s^{\nu_{k}-1}}{\prod_{j=0}^{k} (s^{\nu_{j}}+\lambda)}.$$

Therefore if  $(\nu_k - \nu_0) > 0$  for a fixed k we have that

(3.13) 
$$p_{k}^{\nu_{k}}(t) = \frac{1}{\Gamma(\nu_{k} - \nu_{0})} \int_{0}^{t} (t - y)^{(\nu_{k} - \nu_{0}) - 1} \Pr\{\mathcal{N}(y) = k\} dy$$
$$= I^{\nu_{k} - \nu_{0}} \Pr\{\mathcal{N}(t) = k\}, \qquad t \ge 0,$$

where  $I^{\nu_k - \nu_0}$  is the Riemann–Liouville fractional integral. Note that since the Riemann–Liouville fractional derivative (that we indicate here with  $D^{\alpha}$ ) is the left-inverse operator to the Riemann–Liouville fractional integral we also obtain the related relation

(3.14) 
$$D^{\nu_k - \nu_0} p_k^{\nu_k}(t) = \Pr\{\mathcal{N}(t) = k\}, \qquad t \ge 0, \ (\nu_k - \nu_0) > 0.$$

Conversely, in view of (3.4), we can write

(3.15) 
$$\int_0^\infty e^{-st} \Pr\{\mathcal{N}(t) = k\} \mathrm{d}t = s^{\nu_k - \nu_0} \frac{\lambda^k s^{\nu_0 - 1}}{\prod_{j=0}^k (s^{\nu_j} + \lambda)},$$

and thus if  $(\nu_0 - \nu_k) > 0$ , for a fixed k, we obtain that

(3.16) 
$$\Pr\{\mathcal{N}(t) = k\} = \frac{1}{\Gamma(\nu_0 - \nu_k)} \int_0^t (t - y)^{(\nu_0 - \nu_k) - 1} p_k^{\nu_k}(y) \, \mathrm{d}y$$
$$= I^{\nu_0 - \nu_k} p_k^{\nu_k}(t), \qquad t \ge 0,$$

and that (3.17)

$$D^{\nu_0 - \nu_k} \Pr\{\mathcal{N}(t) = k\} = p_k^{\nu_k}(t), \qquad t \ge 0, \ (\nu_0 - \nu_k) > 0.$$

Finally, we have the following relation between the state probabilities of the two processes

(3.18) 
$$p_k^{\nu_k}(t) = \begin{cases} I^{\nu_k - \nu_0} \Pr\{\mathcal{N}(t) = k\}, & \nu_k > \nu_0, \\ D^{\nu_0 - \nu_k} \Pr\{\mathcal{N}(t) = k\}, & \nu_k < \nu_0. \end{cases}$$

In order to deepen the meaning of this relation, we consider as an example the relation between  $p_1^{\nu_1}(t)$  and  $\Pr\{\mathcal{N}(t)=1\}$ .

By inverting the Laplace transform (3.4), we obtain that

(3.19) 
$$\Pr\{\mathcal{N}(t)=1\} = \sum_{m=0}^{\infty} (-1)^m \lambda^{m+1} \sum_{r=0}^m \binom{m}{r} t^{\nu_0(m-r)+\nu_1r+\nu_0} E^{m+1}_{\nu_0+\nu_1,\nu_0(m-r)+\nu_1r+\nu_0+1}(-\lambda^2 t^{\nu_0+\nu_1}),$$

by calculation similar to those given above for  $p_1^{\nu_1}(t)$ . Recalling that (Mathai and Haubold [11], page 123)

(3.20) 
$$I^{\alpha}[t^{\gamma-1}E^{m}_{\beta,\gamma}(at^{\beta})] = t^{\alpha+\gamma-1}E^{m}_{\beta,\alpha+\gamma}(at^{\beta})$$

and assuming, for example  $\nu_1 > \nu_0$ , we find that

$$(3.21) \qquad I^{\nu_1 - \nu_0} \Pr\{\mathcal{N}(t) = 1\} \\ = \sum_{m=0}^{\infty} (-1)^m \lambda^{m+1} \sum_{r=0}^m \binom{m}{r} I^{\nu_1 - \nu_0} \left( t^{\nu_0(m-r) + \nu_1 r + \nu_0} E^{m+1}_{\nu_0 + \nu_1, \nu_0(m-r) + \nu_1 r + \nu_0 + 1} (-\lambda^2 t^{\nu_0 + \nu_1}) \right) \\ = \sum_{m=0}^{\infty} (-1)^m \lambda^{m+1} \sum_{r=0}^m \binom{m}{r} t^{\nu_0(m-r) + \nu_1 r + \nu_1} E^{m+1}_{\nu_0 + \nu_1, \nu_0(m-r) + \nu_1 r + \nu_1 + 1} (-\lambda^2 t^{\nu_0 + \nu_1}) = p_1^{\nu_1}(t),$$

as expected.

Moreover, we observe that, since  $p_0^{\nu_0}(t) = \Pr{\{\mathcal{N}(t) = 0\}} = E_{\nu_0,1}(-\lambda t^{\nu_0})$ , we have

(3.22) 
$$\sum_{k=1}^{\infty} p_k^{\nu_k}(t) = \sum_{k=1}^{\infty} \Pr\{\mathcal{N}(t) = k\} = 1 - E_{\nu_0,1}(-\lambda t^{\nu_0})$$

In view of (3.18), this implies that

(3.23) 
$$\sum_{k=1}^{\infty} \Pr\{\mathcal{N}(t) = k\} = \sum_{k=1}^{\infty} p_k^{\nu_k}(t) = \sum_{k: \ \nu_k > \nu_0} I^{\nu_k - \nu_0} \Pr\{\mathcal{N}(t) = k\} + \sum_{k: \ \nu_k < \nu_0} D^{\nu_0 - \nu_k} \Pr\{\mathcal{N}(t) = k\}.$$

The second process we construct here, denoted by  $\hat{N}(t), t \ge 0$ , is given by the following generalization of the Poisson process, whose univariate probabilities are given by

(3.24) 
$$\Pr\{\widehat{N}(t) = j\} = \frac{\frac{(\lambda t)^j}{\Gamma(\nu_j j + 1)} \frac{1}{E_{\nu_j,1}(\lambda t)}}{\sum_{j=0}^{+\infty} \frac{(\lambda t)^j}{\Gamma(\nu_j j + 1)} \frac{1}{E_{\nu_j,1}(\lambda t)}}, \qquad j \ge 0,$$

where  $\lambda > 0$ ,  $0 < \nu_j \leq 1$ . We can treat it as a generalized Poisson process with state-dependent probabilities. Indeed, we notice that, if  $\nu_j = 1$ , for all j, we have

(3.25) 
$$\Pr\{\hat{N}(t) = j\} = \frac{\frac{(\lambda t)^j}{\Gamma(j+1)} \frac{1}{e^{\lambda t}}}{\sum_{j=0}^{+\infty} \frac{(\lambda t)^j}{\Gamma(j+1)} \frac{1}{e^{\lambda t}}} = \frac{(\lambda t)^j}{j!} e^{-\lambda t} = \Pr\{N(t) = j\},$$

that is the state probability of the homogeneous Poisson process.

A similar construction was adopted in Beghin and Orsingher [3]. We notice that an analogous generalization was used by Sixdeniers et al. [20] in quantum mechanics, in relation to Mittag-Leffler type coherent states. We now recall from Balakrishnan and Kozubowski [1] that the distribution (3.24) can be regarded as a weighted Poisson sum. Indeed we notice that

(3.26) 
$$\frac{(\lambda t)^{j}}{\Gamma(\nu_{j}j+1)} \frac{1}{E_{\nu_{j},1}(\lambda t)} = \frac{\frac{j!}{\Gamma(\nu_{j}j+1)} \operatorname{Pr}\{N(t)=j\}}{\sum_{k=0}^{+\infty} \frac{k!}{\Gamma(\nu_{j}k+1)} \operatorname{Pr}\{N(t)=k\}}$$

Hence we have

(3.27) 
$$\Pr\{\hat{N}(t) = j\} = \frac{\frac{\sum_{k=0}^{j!} \Pr\{N(t) = j\}}{\sum_{k=0}^{k=0} \frac{k!}{\Gamma(\nu_j j + 1)} \Pr\{N(t) = k\}}}{\sum_{j=0}^{+\infty} \frac{\frac{j!}{\Gamma(\nu_j j + 1)} \Pr\{N(t) = j\}}{\sum_{k=0}^{+\infty} \frac{j!}{\Gamma(\nu_j k + 1)} \Pr\{N(t) = k\}}}$$

The probability generating function of (3.24) is given by

(3.28) 
$$G(u,t) = \sum_{k=0}^{\infty} u^k \operatorname{Pr}\{\hat{N}(t) = k\}$$
$$= \frac{\sum_{k=0}^{+\infty} \frac{(\lambda u t)^k}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}}{\sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}}$$

In the case  $\nu_j = \nu$ , for all  $j \ge 0$ , we have

(3.29) 
$$G(u,t) = \frac{\sum_{k=0}^{+\infty} \frac{(\lambda ut)^k}{\Gamma(\nu k+1)} \frac{1}{E_{\nu,1}(\lambda t)}}{\sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{\Gamma(\nu k+1)} \frac{1}{E_{\nu,1}(\lambda t)}} = \frac{E_{\nu,1}(u\lambda t)}{E_{\nu,1}(\lambda t)}$$

that coincides with the equation (4.4) of Beghin and Orsingher [3]. By means of the generating function we can also find the explicit form of the mean value of the distribution (3.24), i.e.

$$(3.30) \qquad \qquad \mathbb{E}\hat{N}(t) = \frac{\lambda t \sum_{k=0}^{+\infty} \frac{k(\lambda t)^{k-1}}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}}{\sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}} = \frac{\lambda t \sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{\nu_{k+1} \Gamma(\nu_{k+1} k+\nu_{k+1})} \frac{1}{E_{\nu_k,1}(\lambda t)}}{\sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}},$$

such that, when  $\nu_k = \nu$  for all k, we recover the case considered in Beghin and Orsingher [3] and in Beghin and Macci [2], i.e.

(3.31) 
$$\mathbb{E}\widehat{N}_{\nu}(t) = \frac{\lambda t E_{\nu,\nu}(\lambda t)}{\nu E_{\nu,1}(\lambda t)}.$$

We now consider a sequence of a random number of non-negative i.i.d. random variables with distribution  $F(\beta) = \Pr(X_i \leq \beta), i \geq 1$  and represented by  $\hat{N}(t)$ . The distribution of the maximum and minimum of this sequence is given by

(3.32) 
$$\Pr\{\max\left(X_1,\cdots,X_{\widehat{N}(t)}\right) < \beta\} = \frac{\sum_{k=0}^{+\infty} \frac{(\lambda F(\beta)t)^k}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}}{\sum_{k=0}^{+\infty} \frac{(\lambda t)^k}{\Gamma(\nu_k k+1)} \frac{1}{E_{\nu_k,1}(\lambda t)}}$$

(3.33) 
$$\Pr\{\min\left(X_{1}, \cdots, X_{\widehat{N}(t)}\right) > \beta\} = \frac{\sum_{k=0}^{+\infty} \frac{(\lambda[1-F(\beta)]t)^{k}}{\Gamma(\nu_{k}k+1)} \frac{1}{E_{\nu_{k},1}(\lambda t)}}{\sum_{k=0}^{+\infty} \frac{(\lambda t)^{k}}{\Gamma(\nu_{k}k+1)} \frac{1}{E_{\nu_{k},1}(\lambda t)}}$$

In the case  $\nu = \nu_k = 1$ , for all k, we recover the distribution of the maximum and minimum of the homogeneous Poisson process.

### 4. STATE DEPENDENT FRACTIONAL PURE BIRTH PROCESSES

In this section we consider a different point process which can be generalized in a state-dependent sense as we have done for the fractional Poisson process. We thus analyze a state-dependent fractional pure birth process (see Orsingher and Polito [14] for the fractional case with constant order), where the probabilities are governed by the following equations

(4.1) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda_k p_k^{\nu_k}(t) + \lambda_{k-1} p_{k-1}^{\nu_{k-1}}(t), \quad k \ge 1, \ t > 0, \ \nu_k \in (0, 1], \\ p_k^{\nu_k}(0) = \begin{cases} 1, \quad k = 1, \\ 0, \quad k \ge 2. \end{cases} \end{cases}$$

As in the Section 2 the Laplace transform of the solution to (4.1) can be found rather easily. This is done in the following proposition.

**Proposition 4.1.** The Laplace transform of the solution to the state-dependent fractional pure-birth process (4.1) reads

(4.2) 
$$\tilde{p}_{k}^{\nu_{k}}(s) = \int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) dt = \left(\prod_{j=1}^{k-1} \lambda_{j}\right) \frac{s^{\nu_{1}-1}}{\prod_{j=1}^{k} (s^{\nu_{j}} + \lambda_{j})},$$

where the fractional derivative appearing in (4.1) is in the sense of Dzhrbashyan-Caputo.

*Proof.* We can solve equation (4.1) by means of an iterative procedure, as follows. The equation related to k = 1

(4.3) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_1}}{\mathrm{d}t^{\nu_1}} p_1^{\nu_1}(t) = -\lambda_1 p_1^{\nu_1}(t), \quad t > 0, \ \nu_1 \in (0, 1], \\ p_1^{\nu_1}(0) = 1, \end{cases}$$

has solution  $p_1^{\nu_1}(t) = E_{\nu_1,1}(-\lambda t^{\nu_1})$ . For k = 2, the equation

(4.4) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_2}}{\mathrm{d}t^{\nu_2}} p_2^{\nu_2}(t) = -\lambda_2 p_2^{\nu_2}(t) + \lambda_1 p_1^{\nu_1}(t), \quad t > 0, \ \nu_2 \in (0, 1], \\ p_2^{\nu_2}(0) = 0, \end{cases}$$

has solution with Laplace transform

(4.5) 
$$\tilde{p}_{2}^{\nu_{2}}(s) = \int_{0}^{+\infty} e^{-st} p_{2}^{\nu_{2}}(t) dt = \frac{\lambda_{1} s^{\nu_{1}-1}}{\lambda_{1} + s^{\nu_{1}}} \frac{1}{\lambda_{2} + s^{\nu_{2}}}$$

whose inverse is given by (see (2.9))

$$(4.6) p_2^{\nu_2}(t) = \sum_{m=0}^{\infty} (-1)^m \sum_{r=0}^m \binom{m}{r} \lambda_1^{r+1} \lambda_2^{m-r} t^{\nu_2(m-r)+\nu_1r+\nu_2} E_{\nu_1+\nu_2,\nu_2(m-r)+\nu_1r+\nu_2+1}^{m+1} (-\lambda_1\lambda_2 t^{\nu_1+\nu_2}).$$

By iterating this procedure, we arrive immediately at

(4.7) 
$$\hat{p}_{k}^{\nu_{k}}(s) = \int_{0}^{+\infty} e^{-st} p_{k}^{\nu_{k}}(t) dt = \left(\prod_{j=1}^{k-1} \lambda_{j}\right) \frac{s^{\nu_{1}-1}}{\prod_{j=1}^{k} (s^{\nu_{j}} + \lambda_{j})},$$

as claimed.

By recalling (2.28), we obtain the explicit expression of the state probabilities  $p_k^{\nu_k}(t), k \ge 1, t \ge 0$ , as

(4.8) 
$$p_k^{\nu_k}(t) = E_{\nu_1,1}(-\lambda_1 t^{\nu_1}) \underset{j=1}{\overset{k}{\ast}} \lambda_j t^{\nu_j - 1} E_{\nu_j,\nu_j}(-\lambda_j t^{\nu_j}),$$

where the convolution is in the sense of equation (2.32).

We now consider the state dependent linear birth process, denoted by  $N_{lin}(t)$ ,  $t \ge 0$ . This means that we take  $\lambda_k = \lambda k$  in (4.1). We have the following

**Theorem 4.2.** Let us consider the state dependent linear birth process  $N_{lin}(t)$ ,  $t \ge 0$ , governed by

(4.9) 
$$\begin{cases} \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda k p_k^{\nu_k}(t) + \lambda (k-1) p_{k-1}^{\nu_{k-1}}(t), \ k \ge 1, \ t > 0, \ \nu_k \in (0,1], \\ p_k^{\nu_k}(0) = \begin{cases} 1, & k = 1, \\ 0, & k \ge 2, \end{cases} \end{cases}$$

then the following relation holds

(4.10) 
$$\sum_{k=1}^{\infty} k^m \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = \lambda \sum_{j=1}^{m-1} \binom{m}{j} \mathbb{E} N_{lin}^{m-j+1}.$$

*Proof.* In order to find explicit relations for the moments of the distribution  $N_{lin}(t)$ , we multiply both sides of equation (4.9) by  $k^m$  and sum over all the states, obtaining

$$(4.11) \qquad \sum_{k=1}^{\infty} k^m \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda \sum_{k=1}^{\infty} k^{m+1} p_k^{\nu_k}(t) + \lambda \sum_{k=1}^{\infty} k^m (k-1) p_{k-1}^{\nu_{k-1}}(t) = -\lambda \sum_{k=1}^{\infty} k^{m+1} p_k^{\nu_k}(t) + \lambda \sum_{k=1}^{\infty} k(k+1)^m p_k^{\nu_k}(t) = -\lambda \sum_{k=1}^{\infty} k^{m+1} p_k^{\nu_k}(t) + \lambda \sum_{k=1}^{\infty} \sum_{j=0}^m \binom{m}{j} k^{m-j+1} p_k^{\nu_k}(t) = \lambda \sum_{j=1}^m \binom{m}{j} \sum_{k=1}^{\infty} k^{m-j+1} p_k^{\nu_k}(t) = \lambda \sum_{j=1}^m \binom{m}{j} \mathbb{E} N_{lin}^{m-j+1},$$

**Remark 4.3.** We can consider in a explicit way the relations involving first and second moments. For example, if we multiply (4.9) for k and sum over all the states, we obtain that

(4.12) 
$$\sum_{k=1}^{\infty} k \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda \sum_{k=1}^{\infty} k^2 p_k^{\nu_k}(t) + \lambda \sum_{k=1}^{\infty} k(k-1) p_{k-1}^{\nu_{k-1}}(t)$$
$$= \lambda \sum_{k=1}^{\infty} k p_k^{\nu_k}(t) = \lambda \mathbb{E} N_{lin}(t).$$

In the same way, for the second moment, we multiply (4.9) for  $k^2$ , obtaining

(4.13) 
$$\sum_{k=1}^{\infty} k^2 \frac{\mathrm{d}^{\nu_k}}{\mathrm{d}t^{\nu_k}} p_k^{\nu_k}(t) = -\lambda \sum_{k=1}^{\infty} k^3 p_k^{\nu_k}(t) + \lambda \sum_{k=1}^{\infty} k^2 (k-1) p_{k-1}^{\nu_{k-1}}(t)$$
$$= \lambda \sum_{k=1}^{\infty} k p_k^{\nu_k}(t) + 2\lambda \sum_{k=1}^{\infty} k^2 p_k^{\nu_k}(t)$$
$$= \lambda \mathbb{E} N_{lin}(t) + 2\lambda \mathbb{E} (N_{lin})^2(t).$$

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