# On the Acceleration of the Multi-Level Monte Carlo Method 

Kristian Debrabant ${ }^{\text {a }}$ and Andreas Rößler ${ }^{\text {b }}$<br>${ }^{a}$ University of Southern Denmark, Department of Mathematics and Computer Science (IMADA), Campusvej 55, 5230 Odense M, Denmark<br>${ }^{\text {b }}$ Universität zu Lübeck, Institut für Mathematik, Ratzeburger Allee 160, 23562 Lübeck, Germany


#### Abstract

The multi-level Monte Carlo method proposed by M. Giles (2008) approximates the expectation of some functionals applied to a stochastic process with optimal order of convergence for the mean-square error. In this paper, a modified multi-level Monte Carlo estimator is proposed with significantly reduced computational costs. As the main result, it is proved that the modified estimator reduces the computational costs asymptotically by a factor $(p / \alpha)^{2}$ if weak approximation methods of orders $\alpha$ and $p$ are applied in case of computational costs growing with same order as variances decay.


Key words: Multi-level Monte Carlo, Monte Carlo, variance reduction, weak approximation, stochastic differential equation
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## 1 Introduction

The multi-level Monte Carlo method proposed in [7] approximates the expectation of some functional applied to some stochastic processes like e. g. solutions of stochastic differential equations (SDEs) at a lower computational complexity than classical Monte Carlo simulation, see also [5] 8 ,9]. Multi-level Monte Carlo approximation is applied in many fields like mathematical finance [1,6], for SDEs driven by a Lévy process [3], by fractional Brownian motion [11] or for

[^0]stochastic PDEs [13]. The main idea of this article is to reduce the computational costs additionally by applying the multi-level Monte Carlo method as a variance reduction technique for some higher order weak approximation method. As a result, the computational effort can be significantly reduced while the optimal order of convergence for the root mean-square error is preserved.

The outline of this paper is as follows. We give a brief introduction to the main ideas and results of the multi-level Monte Carlo method in Section 2, Based on these results, in Section 3 we present as the main result a modified multilevel Monte Carlo algorithm that allows to reduce the computational costs significantly. Depending on the relationship between the orders of variance reduction and of the growth of the costs, there exists a reduction of the computational costs by a factor depending on the weak order of the underlying numerical method. As an example, the modified multi-level Monte Carlo algorithm is applied to the problem of weak approximation for stochastic differential equations driven by Brownian motion in Section 4.

## 2 Multi-level Monte Carlo simulation

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with some filtration $\left(\mathcal{F}_{t}\right)_{t \geq 0}$ and let $X=$ $\left(X_{t}\right)_{t \in I}$ denote an adapted stochastic process on the interval $I=\left[t_{0}, T\right]$ that belongs to a space $\mathbb{X}$ that may be infinite dimensional. In the following, we are interested in the approximation of $\mathrm{E}_{\mathrm{P}}(f(X))$ for some functional $f \in \mathbb{F}$ where $\mathbb{F}$ denotes a suitable class of functionals that are of interest. Further, let an equidistant discretization $I_{h}=\left\{t_{0}, t_{1}, \ldots, t_{N}\right\}$ with $0 \leq t_{0}<t_{1}<\ldots<t_{N}=T$ of the time interval $I$ with step size $h$ be given. Then, we consider a probability $\operatorname{space}(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathrm{P}})$ with some filtration $\left(\tilde{\mathcal{F}}_{t}\right)_{t \in I_{h}}$ and we denote by $Y_{\tilde{\mathcal{F}}}=\left(Y_{t}\right)_{t \in I_{h}}$ a discrete time approximation of $X$ on the grid $I_{h}$, adapted to $\left(\tilde{\mathcal{F}}_{t}\right)_{t \in I_{h}}$. Thus, we consider the approximation $Y \in \mathbb{X}_{h}$ of $X \in \mathbb{X}$ on a finite dimensional space $\mathbb{X}_{h}$. Here, the probability spaces $(\Omega, \mathcal{F}, \mathrm{P})$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathrm{P}})$ may be but do not have to be equal and we assume that $Y$ approximates $X$ in the weak sense with some order $p>0$, i.e.

$$
\begin{equation*}
\left|\mathrm{E}_{\tilde{\mathrm{P}}}(f(Y))-\mathrm{E}_{\mathrm{P}}(f(X))\right|=O\left(h^{p}\right) \tag{1}
\end{equation*}
$$

for all $f \in \mathbb{F}$.
In order to approximate the expectation of $f(X)$ we apply the multi-level Monte Carlo estimator introduced in [7]. For some fixed $M \in \mathbb{N}$ with $M \geq 2$ and some $L \in \mathbb{N}$ we define the step sizes $h_{l}=\frac{T}{M^{l}}$ and let $Y^{l}=\left(Y_{t}\right)_{t \in I_{h_{l}}}$ denote the discrete time approximation process on the grid $I_{h_{l}}$ based on step size $h_{l}$ for $l=0,1, \ldots, L$. Here, we consider the approximations $Y^{l} \in \mathbb{X}_{h_{l}}$ for $l=0,1, \ldots, L$ of $X \in \mathbb{X}$ on a sequence $\mathbb{X}_{h_{0}} \subset \mathbb{X}_{h_{1}} \subset \ldots \subset \mathbb{X}_{h_{L}}$ of finite
dimensional subspaces. Then, the multi-level Monte Carlo estimator is defined by

$$
\begin{equation*}
\hat{Y}_{M L}=\sum_{l=0}^{L} \hat{Y}^{l} \tag{2}
\end{equation*}
$$

for some $L \in \mathbb{N}$ using the estimators $\hat{Y}^{0}=\frac{1}{N_{0}} \sum_{i=1}^{N_{0}} f\left(Y^{0(i)}\right)$ and

$$
\begin{equation*}
\hat{Y}^{l}=\frac{1}{N_{l}} \sum_{i=1}^{N_{l}}\left(f\left(Y^{l(i)}\right)-f\left(Y^{l-1(i)}\right)\right) \tag{3}
\end{equation*}
$$

for $l=1, \ldots, L$. Then, we get

$$
\begin{equation*}
\mathrm{E}_{\tilde{\mathrm{P}}}\left(\hat{Y}_{M L}\right)=\mathrm{E}_{\tilde{\mathrm{P}}}\left(f\left(Y^{0}\right)\right)+\sum_{l=1}^{L} \mathrm{E}_{\tilde{\mathrm{P}}}\left(f\left(Y^{l}\right)-f\left(Y^{l-1}\right)\right) \tag{4}
\end{equation*}
$$

Here, we have to point out, that both approximations $Y^{l^{(i)}}$ and $Y^{l-1^{(i)}}$ are simulated simultaneously based on the same realisation of the underlying driving random process whereas $\left(Y^{l^{(i)}}, Y^{l-1(i)}\right)$ and $\left(Y^{l^{(j)}}, Y^{l-1}(j)\right.$ are independent realisations for $i \neq j$.

Now, there are two sources of errors for the approximation. On the one hand, we have a systematical error that depends on the dimension of $\mathbb{X}_{h_{l}}$ due to the discrete time approximation $Y^{l} \in \mathbb{X}_{h_{l}}$ based on step size $h_{l}$ which is given by the bias of the method. On the other hand, there is a statistical error from the estimator for the expectation of $f\left(Y^{l}\right)$ by the Monte Carlo simulation. Therefore, we consider the root mean-square error

$$
\begin{equation*}
e\left(\hat{Y}_{M L}\right)=\left(\mathrm{E}_{\tilde{\mathrm{P}}}\left(\left|\hat{Y}_{M L}-\mathrm{E}_{\mathrm{P}}(f(X))\right|^{2}\right)\right)^{1 / 2} \tag{5}
\end{equation*}
$$

of the multi-level Monte Carlo method in the following. In order to rate the performance of an approximation method, we will analyse the root mean-square error of the method compared to the computational costs. Therefore, we denote by $C(Y)$ the computational costs of the approximation method $Y$. In order to determine $C(Y)$, one may use a cost model where e.g. each operation or evaluation of some function is charged with the price of one unit, i.e. one counts the number of needed mathematical operations or function evaluations. Further, each random number that has to be generated to compute $Y$ may also be charged with the price of one unit.

It is well known that the optimal order of convergence for the classical Monte Carlo estimator $\hat{Y}_{M C}=\frac{1}{N} \sum_{i=1}^{N} f\left(Y^{(i)}\right)$ is given by

$$
e\left(\hat{Y}_{M C}\right)=O\left(\left(1 / C\left(\hat{Y}_{M C}\right)\right)^{\frac{p}{2 p+1}}\right)
$$

where $p$ is the weak order of convergence of the approximations $Y$, see Duffie and Glynn [4]. Thus, higher order weak approximation methods result in a
higher order of convergence with respect to the root mean-square error. Clearly, the best root mean-square order of convergence that can be achieved is at most $1 / 2$. However, the order bound $1 / 2$ can not be reached by any weak order $p$ approximation method in the case of the classical Monte Carlo simulation. Therefore, in order to attain the optimal order of convergence for the root mean-square error we apply the multi-level Monte Carlo estimator (2). The following theorem due to Giles [7] is presented in a slightly generalized version suitable for our considerations.

Theorem 2.1. For some $L \in \mathbb{N}$, let $Y^{l}$ denote the approximation process on the grid $I_{h_{l}}$ with respect to step size $h_{l}=\frac{T}{M^{l}}$ for each $l=0,1, \ldots, L$, respectively. Suppose that there exist some constants $\alpha>0$ and $c_{1, \alpha}, c_{2,0}, c_{2}, c_{2, L}>0$ and $\beta, \beta_{L}>0$ such that for the bias

$$
\text { 1) }\left|\mathrm{E}_{\mathrm{P}}(f(X))-\mathrm{E}_{\tilde{\mathrm{P}}}\left(f\left(Y^{L}\right)\right)\right| \leq c_{1, \alpha} h_{L}^{\alpha}
$$

and for the variances
2) $\operatorname{Var}_{\tilde{\mathrm{P}}}\left(f\left(Y^{0}\right)\right) \leq c_{2,0} h_{0}^{\beta}$,
3) $\operatorname{Var}_{\tilde{\mathrm{P}}}\left(f\left(Y^{l}\right)-f\left(Y^{l-1}\right)\right) \leq c_{2} h_{l}^{\beta} \quad$ for $l=1, \ldots, L-1$,
4) $\operatorname{Var}_{\tilde{\mathrm{P}}}\left(f\left(Y^{L}\right)-f\left(Y^{L-1}\right)\right) \leq c_{2, L} h_{L}^{\beta_{L}}$.

Further, assume that there exist constants $c_{3,0}, c_{3}, c_{3, L}>0$ and $\gamma, \gamma_{L} \geq 1$ such that for the computational costs
5) $C\left(Y^{0}\right) \leq c_{3,0} T h_{0}^{-\gamma}$,
6) $C\left(Y^{l}, Y^{l-1}\right) \leq c_{3} T h_{l}^{-\gamma} \quad$ for $l=1, \ldots, L-1$,
7) $C\left(Y^{L}, Y^{L-1}\right) \leq c_{3, L} T h_{L}^{-\gamma_{L}}$.

Then, for some arbitrarily prescribed error bound $\varepsilon>0$ there exist values $L$ and $N_{l}$ for $l=0,1, \ldots, L$, such that the root mean-square error of the multi-level Monte Carlo estimator $\hat{Y}_{M L}$ has the bound

$$
\begin{equation*}
e\left(\hat{Y}_{M L}\right)<\varepsilon \tag{6}
\end{equation*}
$$

with computational costs bounded by

$$
C\left(\hat{Y}_{M L}\right) \leq \begin{cases}c_{4} \varepsilon^{-2} & \text { if } \beta>\gamma, \beta_{L} \geq \gamma_{L}, \alpha \geq \frac{1}{2} \max \left\{\gamma, \gamma_{L}\right\}  \tag{7}\\ c_{4} \varepsilon^{-2}(\log (\varepsilon))^{2} & \text { if } \beta=\gamma, \beta_{L} \geq \gamma_{L}, \alpha \geq \frac{1}{2} \max \left\{\gamma, \gamma_{L}\right\} \\ c_{4} \varepsilon^{-2-\frac{\max \left\{\gamma-\beta, \gamma_{L}-\beta_{L}\right\}}{\alpha}} & \text { if } \beta<\gamma, \alpha \geq \frac{\max \left\{\gamma, \gamma_{L}\right\}-\max \left\{\gamma-\beta, \gamma_{L}-\beta_{L}\right\}}{2}\end{cases}
$$

for some positive constant $c_{4}$.
In order to apply Theorem 2.1 and the multi-level Monte Carlo method, one has to determine the values $\alpha, \beta, \beta_{L}>0$ as well as $\gamma, \gamma_{L} \geq 1$. Firstly, $\alpha$ denotes the weak order of convergence for the bias of the finite dimensional approximation
$Y^{L} \in \mathbb{X}_{h_{L}}$ as the dimension of the approximation subspace increases. This value is well known for commonly applied approximations $Y^{L}$. Because the approximations $\left(Y^{l}\right)_{l \geq 0}$ converge to $X$ in the weak sense, the differences of two successive approximations $\left(f\left(Y^{l}\right)-f\left(Y^{l-1}\right)\right)_{l \geq 1}$ converge to zero as the dimensions of the subspaces increase. Then, usually their variances will also tend to zero with some order $\beta$ and $\beta_{L}$ for the approximations applied on levels $0,1, \ldots, L-1$ and on level $L$, respectively. Here, we want to point out that estimates of type 1)-4) in Theorem 2.1 are rather natural and turn out to be no considerable restriction for typical applications. Finally, the computational costs to evaluate two correlated approximations $Y^{l}$ and $Y^{l-1}$ on the finite dimensional subspaces $\mathbb{X}_{h_{l}}$ and $\mathbb{X}_{h_{l-1}}$ depend on the dimensions of the subspaces that are proportional to $h_{l}^{-1}$. For commonly used discrete time approximations, one typically has $\gamma=\gamma_{L}=1$.

The calculations for the proof follow the lines of the original proof due to Giles [7]. Considering the mean square-error

$$
\begin{equation*}
e\left(\hat{Y}_{M L}\right)=\left(\left|\mathrm{E}_{\mathrm{P}}(f(X))-\mathrm{E}_{\tilde{\mathrm{P}}}\left(f\left(Y^{L}\right)\right)\right|^{2}+\operatorname{Var}_{\tilde{\mathrm{P}}}\left(\hat{Y}_{M L}\right)\right)^{1 / 2}<\varepsilon \tag{8}
\end{equation*}
$$

we make use of the weight $q \in] 0,1[$ and claim that

$$
\begin{equation*}
\left|\mathrm{E}_{\mathrm{P}}(f(X))-\mathrm{E}_{\tilde{\mathrm{P}}}\left(f\left(Y^{L}\right)\right)\right|^{2}<q \varepsilon^{2} \quad \text { and } \quad \operatorname{Var}_{\tilde{\mathrm{P}}}\left(\hat{Y}_{M L}\right)<(1-q) \varepsilon^{2} \tag{9}
\end{equation*}
$$

Then, we can calculate $L$ from the bias and we have to solve the minimization problem

$$
\begin{equation*}
\min _{N_{l}: 0 \leq l \leq L} C\left(\hat{Y}_{M L}\right) \tag{10}
\end{equation*}
$$

under the constraint that $\operatorname{Var}_{\tilde{\mathrm{P}}}\left(\hat{Y}_{M L}\right)<(1-q) \varepsilon^{2}$. As a result of this, we obtain the following values for $L$ and $N_{l}$ :

$$
\begin{equation*}
L=\left\lceil\frac{\log \left(q^{-\frac{1}{2}} c_{1, \alpha} \varepsilon^{-1} T^{\alpha}\right)}{\alpha \log (M)}\right\rceil \tag{11}
\end{equation*}
$$

and $N_{0}=\left\lceil\frac{1}{1-q} \varepsilon^{-2} h_{0}^{\frac{\beta+\gamma}{2}}\left(\frac{c_{2,0}}{c_{3,0}}\right)^{\frac{1}{2}} \kappa\right\rceil$,

$$
\begin{equation*}
N_{l}=\left\lceil\frac{1}{1-q} \varepsilon^{-2} h_{l}^{\frac{\beta+\gamma}{2}}\left(\frac{c_{2}}{c_{3}}\right)^{\frac{1}{2}} \kappa\right\rceil \tag{12}
\end{equation*}
$$

for $l=1, \ldots, L-1$ and $N_{L}=\left\lceil\frac{1}{1-q} \varepsilon^{-2} h_{L}^{\frac{\beta_{L}+\gamma_{L}}{2}}\left(\frac{c_{2, L}}{c_{3, L}}\right)^{\frac{1}{2}} \kappa\right\rceil$ for some $\left.q \in\right] 0,1[$ where

- In case of $\beta>\gamma$ and $\beta_{L} \geq \gamma_{L}$ or in case of $\beta<\gamma$ and $\gamma_{L}-\beta_{L} \leq \gamma-\beta$ :

$$
\begin{equation*}
\kappa=\left(c_{2,0} c_{3,0}\right)^{\frac{1}{2}} T^{\frac{\beta-\gamma}{2}}+\left(c_{2} c_{3}\right)^{\frac{1}{2}} \frac{\left(M^{-1} T\right)^{\frac{\beta-\gamma}{2}}-h_{L}^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}}+\left(c_{2, L} c_{3, L}\right)^{\frac{1}{2}} h_{L}^{\frac{\beta_{L}-\gamma_{L}}{2}} . \tag{13}
\end{equation*}
$$

- In case of $\beta=\gamma$ and $\beta_{L} \geq \gamma_{L}$ :

$$
\begin{equation*}
\kappa=\left(c_{2,0} c_{3,0}\right)^{\frac{1}{2}}+(L-1)\left(c_{2} c_{3}\right)^{\frac{1}{2}}+\left(c_{2, L} c_{3, L}\right)^{\frac{1}{2}} h_{L}^{\frac{\beta_{L}-\gamma_{L}}{2}} . \tag{14}
\end{equation*}
$$

## 3 The improved multi-level Monte Carlo estimator

The order of convergence of the multi-level Monte Carlo estimator $\hat{Y}_{M L}$ given in (2) is optimal in the given framework. However, the computational costs can be reduced if a modified estimator is applied. As yet, the estimator $\hat{Y}_{M L}$ is based on some weak order $\alpha$ approximations $Y^{l}$ for $l=0,1, \ldots, L$ on each level. Now, let us apply some cheap low order weak approximation $Y^{l}$ on levels $l=0,1, \ldots, L-1$ combined with some probably expansive high order weak approximation $\check{Y}^{L}$ on the finest level $L$. The idea is, that the approximations $Y^{l}$ contribute a variance reduction while the approximation $\check{Y}^{L}$ results in a small bias of the multi-level Monte Carlo estimator, thus reducing the number of levels needed to attain a prescribed accuracy.

Let $Y$ be an order $\alpha$ weak approximation method and let $\check{Y}$ be an order $p$ weak approximation method applied on the finest level. Further, let $L=L_{p}$ with

$$
\begin{equation*}
L_{p}=\left\lceil\frac{\log \left(q^{-\frac{1}{2}} c_{1, p} \varepsilon^{-1} T^{p}\right)}{p \log (M)}\right\rceil \tag{15}
\end{equation*}
$$

denote the number of levels in order to indicate the dependence on the weak order $p$. Then, we define the modified multi-level Monte Carlo estimator by

$$
\begin{equation*}
\hat{Y}_{M L(\alpha, p)}=\sum_{l=0}^{L_{p}} \hat{Y}^{l} \tag{16}
\end{equation*}
$$

with the estimators $\hat{Y}^{l}$ for $l=0,1, \ldots, L_{p}-1$ based on the order $\alpha$ weak approximations $Y^{l}$ as defined in Section 2, however now applying the modified estimator

$$
\begin{equation*}
\hat{Y}^{L_{p}}=\frac{1}{N_{L_{p}}} \sum_{i=1}^{N_{L_{p}}}\left(f\left(\check{Y}^{L_{p}}\right)^{(i)}-f\left(Y^{L_{p}-1}\right)^{(i)}\right) \tag{17}
\end{equation*}
$$

which combines the weak order $\alpha$ approximations $Y^{L_{p}-1}$ with the weak order $p$ approximations $\check{Y}^{L_{p}}$. Clearly, all conditions of Theorem 2.1 have to be fulfilled for $Y^{L}$ replaced by $\check{Y}^{L}$. Then, in the case of $p>\alpha$, the improved multi-level Monte Carlo estimator $\hat{Y}_{M L(\alpha, p)}$ features significantly reduced computational costs compared to the originally proposed estimator $\hat{Y}_{M L}=\hat{Y}_{M L(\alpha, \alpha)}$.

Definition 3.1. Let conditions 1)-7) of Theorem 2.1 be fulfilled and suppose that there exist constants $\hat{c}_{3,0}, \hat{c}_{3}, \hat{c}_{3, L_{p}}, \delta_{i}>0$ and $\hat{c}_{3,0}^{(i)}, \hat{c}_{3}^{(i)}, \hat{c}_{3, L_{p}}^{(i)} \geq 0$ such that for the computational costs

5') $C\left(Y^{0}\right)=\hat{c}_{3,0} T h_{0}^{-\gamma}+\sum_{i=1}^{k} \hat{c}_{3,0}^{(i)} T h_{0}^{-\gamma+\delta_{i}}$,
6') $C\left(Y^{l}, Y^{l-1}\right)=\hat{c}_{3} T h_{l}^{-\gamma}+\sum_{i=1}^{k} \hat{c}_{3}^{(i)} T h_{l}^{-\gamma+\delta_{i}} \quad$ for $l=1, \ldots, L_{p}-1$,
$\left.7^{\prime}\right) C\left(\check{Y}^{L_{p}}, Y^{L_{p}-1}\right)=\hat{c}_{3, L_{p}} T h_{L_{p}}^{-\gamma_{L_{p}}}+\sum_{i=1}^{k} \hat{c}_{3, L_{p}}^{(i)} T h_{L_{p}}^{-\gamma_{L_{p}}+\delta_{i}}$
with some $\gamma, \gamma_{L_{p}} \geq 1$ such that $\gamma-\delta_{i} \geq 1$ and $\gamma_{L_{p}}-\delta_{i} \geq 1$. Then, the multi-level Monte Carlo estimator $\hat{Y}_{M L(\alpha, p)}$ based on a weak order $\alpha>0$ approximation scheme on levels $0,1, \ldots, L_{p}-1$ and some weak order $p>\alpha$ approximation scheme on level $L_{p}$ has reduced computational costs:
i) In case of $\beta>\gamma$ and $\beta-\gamma<\beta_{L_{p}}-\gamma_{L_{p}}$, there exists some $\varepsilon_{0}>0$ such that for all $\left.\varepsilon \in] 0, \varepsilon_{0}\right]$ it holds

$$
\begin{equation*}
\frac{C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)}>1 \tag{18}
\end{equation*}
$$

provided that $\alpha \geq \frac{\gamma}{2}, p \geq \frac{1}{2} \max \left\{\gamma, \gamma_{L_{p}}\right\}$ and $p>\frac{1}{4} \max \left\{\beta+\gamma, \beta-\gamma+2 \gamma_{L_{p}}\right\}$. In case of $\beta>\gamma$ and $\beta-\gamma=\beta_{L_{p}}-\gamma_{L_{p}}$ then (18) holds if in addition $c_{2} c_{3}>\left(1-M^{\frac{\gamma-\beta}{2}}\right)^{2} c_{2, L_{p}} c_{3, L_{p}}$ and $\hat{c}_{3}^{2} \frac{c_{2}}{c_{3}}>\left(1-M^{\frac{\gamma-\beta}{2}}\right)^{2} \hat{c}_{3, L_{p}}^{2} \frac{c_{2}, L_{p}}{c_{3, L_{p}}}$. Further, for $0<\beta-\gamma \leq \beta_{L_{p}}-\gamma_{L_{p}}$ it holds $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=O\left(\varepsilon^{-2}\right)$ if $\alpha>0$ and $p \geq \frac{1}{2} \max \left\{\gamma, \gamma_{L_{p}}\right\}$.
ii) In case of $\beta=\gamma$ and $\beta_{L_{p}} \geq \gamma_{L_{p}}$ and if $p \geq \frac{1}{2} \max \left\{\gamma, \gamma_{L_{p}}\right\}, \alpha \geq \frac{\gamma}{2}$, it holds

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)} \geq\left(\frac{p}{\alpha}\right)^{2} \tag{19}
\end{equation*}
$$

and $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=O\left(\varepsilon^{-2}(\log (\varepsilon))^{2}\right)$ if $\alpha>0$ and $p \geq \frac{1}{2} \max \left\{\gamma, \gamma_{L_{p}}\right\}$.
iii) In case of $\beta<\gamma$ and $\gamma-\beta=\gamma_{L_{p}}-\beta_{L_{p}}$ it holds

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \frac{C\left(\hat{Y}_{M L(p, p)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)} \geq & M^{2(\gamma-\beta)}\left(\frac{\hat{c}_{3} c_{2}}{\hat{c}_{3, L_{p}} c_{2, L_{p}}}+\frac{\hat{c}_{3}\left(c_{2} c_{3, L_{p}}\right)^{1 / 2}}{\hat{c}_{3, L_{p}}\left(c_{2, L_{p}} c_{3}\right)^{1 / 2}}\left(M^{\frac{\gamma-\beta}{2}}-1\right)\right. \\
& \left.+\left(\frac{c_{2} c_{3}}{c_{2, L_{p}} c_{3, L_{p}}}\right)^{1 / 2}\left(M^{\frac{\gamma-\beta}{2}}-1\right)+\left(M^{\frac{\gamma-\beta}{2}}-1\right)^{2}\right)^{-1} \tag{20}
\end{align*}
$$

if $p>\frac{1}{2}\left(\max \left\{\gamma, \gamma_{L_{p}}\right\}-\gamma+\beta\right)$. If the parameter $\left.q \in\right] 0,1[$ is chosen as

$$
\begin{equation*}
q=\frac{\gamma-\beta}{\gamma-\beta+2 p} \tag{21}
\end{equation*}
$$

then the computational costs $C\left(\hat{Y}_{M L(\alpha, p)}\right)$ are asymptotically minimal. In general, if $\beta<\gamma$ or if $\beta_{L_{p}}<\gamma_{L_{p}}$ then it holds $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=$ $O\left(\varepsilon^{-2-\frac{\max \left\{\gamma-\beta, \gamma_{L_{p}}-\beta_{L_{p}}\right\}}{p}}\right)$ for $p \geq \frac{1}{2}\left(\max \left\{\gamma, \gamma_{L_{p}}\right\}-\min \left\{\gamma-\beta, \gamma_{L_{p}}-\beta_{L_{p}}\right\}\right)$.

We note, that in relations $\left.5^{\prime}\right)-7^{\prime}$ ) of Proposition 3.1 a more detailed polynomial dependence of the computational costs from the dimension of the approximation subspaces has to be taken into account. E.g., standard discrete time approximation methods possess polynomial computational costs and the constants are known explicitly.

Proof. In the following, we will first state some basic formulas and conditions used in the remaining part of the proof. Then we will calculate lower and upper bounds for the computational costs in the case $\beta \neq \gamma$. Those will then be used to prove first i) and then iii). Finally, case ii) with $\beta=\gamma$ is considered.

Basic formulas. Assume that $\varepsilon<1$. Let $\delta_{0}=0, \hat{c}_{3,0}^{(0)}=\hat{c}_{3,0}, \hat{c}_{3}^{(0)}=\hat{c}_{3}$ and $\hat{c}_{3, L_{p}}^{(0)}=\hat{c}_{3, L_{p}}$. Then, the computational costs for $\hat{Y}_{M L(\alpha, p)}$ are

$$
\begin{align*}
C\left(\hat{Y}_{M L(\alpha, p)}\right)= & \sum_{i=0}^{k} \hat{c}_{3,0}^{(i)} T h_{0}^{-\gamma+\delta_{i}} N_{0}+\sum_{i=0}^{k} \sum_{l=1}^{L_{p}-1} \hat{c}_{3}^{(i)} T h_{l}^{-\gamma+\delta_{i}} N_{l}  \tag{22}\\
& +\sum_{i=0}^{k} \hat{c}_{3, L_{p}}^{(i)} T h_{L_{p}}^{-\gamma_{L_{p}}+\delta_{i}} N_{L_{p}}
\end{align*}
$$

with $L=L_{p}=\left\lceil\frac{\log \left(q^{-\frac{1}{2}} c_{1, p} \varepsilon^{-1} T^{p}\right)}{p \log (M)}\right\rceil$ and $N_{l}$ for $l=0,1, \ldots, L_{p}$ given in (12). Without loss of generality, suppose that $\delta_{i} \neq \delta_{j}$ for $i \neq j$ and that $\delta_{k}=\frac{\gamma-\beta}{2}$ with $\hat{c}_{3,0}^{(k)}=\hat{c}_{3}^{(k)}=\hat{c}_{3, L_{p}}^{(k)}=0$ in the case of $\beta \geq \gamma$. In the following, we make use of the two estimates

$$
\begin{align*}
L_{\alpha} & \geq \frac{\log \left(\varepsilon^{-1}\right)}{\alpha \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, \alpha} T^{\alpha}\right)}{\alpha \log (M)},  \tag{23}\\
L_{p}-1 & \leq \frac{\log \left(\varepsilon^{-1}\right)}{p \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, p} T^{p}\right)}{p \log (M)} . \tag{24}
\end{align*}
$$

Lower bound for $\beta \neq \gamma$. Let $\beta \neq \gamma$. Then, we obtain the lower bound

$$
\begin{aligned}
C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon) \geq & \frac{T \kappa \varepsilon^{-2}}{1-q} \sum_{i=0}^{k}\left(h_{0}^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0}}{c_{3,0}}\right)^{1 / 2}+\sum_{l=1}^{L_{\alpha}} h_{l}^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3}^{(i)}\left(\frac{c_{2}}{c_{3}}\right)^{1 / 2}\right) \\
\geq & \frac{T}{1-q} \varepsilon^{-2}\left[\sum_{i=0}^{k} T^{\beta-\gamma+\delta_{i}} \hat{c}_{3,0}^{(i)} c_{2,0}\right. \\
& +\sum_{i=0}^{k} T^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2} c_{3}}{c_{3,0}}\right)^{1 / 2} \frac{T^{\frac{\beta-\gamma}{2}}-h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}}}{M^{\frac{\beta-\gamma}{2}}-1} \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}} \cdot \frac{T^{\frac{\beta-\gamma}{2}+\delta_{i}}-h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}+\delta_{i}}}{M^{\frac{\beta-\gamma}{2}+\delta_{i}}-1}
\end{aligned}
$$

$$
\begin{align*}
& +\hat{c}_{3}^{(k)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}}\left(\frac{\log \left(\varepsilon^{-1}\right)}{\alpha \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, \alpha} T^{\alpha}\right)}{\alpha \log (M)}\right) \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)} c_{2} \frac{T^{\frac{\beta-\gamma}{2}+\delta_{i}}-h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}+\delta_{i}}}{M^{\frac{\beta-\gamma}{2}+\delta_{i}}-1} \cdot \frac{T^{\frac{\beta-\gamma}{2}}-h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}}}{M^{\frac{\beta-\gamma}{2}}-1} \\
& \left.+\hat{c}_{3}^{(k)} c_{2} \frac{T^{\frac{\beta-\gamma}{2}}-h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}}}{M^{\frac{\beta-\gamma}{2}}-1}\left(\frac{\log \left(\varepsilon^{-1}\right)}{\alpha \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, \alpha} T^{\alpha}\right)}{\alpha \log (M)}\right)\right] \tag{25}
\end{align*}
$$

where $\hat{c}_{3, L_{\alpha}}^{(i)}=\hat{c}_{3}^{(i)}, c_{2, L_{\alpha}}=c_{2}, c_{3, L_{\alpha}}=c_{3}, \beta_{L_{\alpha}}=\beta$ and $\gamma_{L_{\alpha}}=\gamma$ for $\hat{Y}_{M L(\alpha, \alpha)}$.
Upper bound for $\beta \neq \gamma$. Next, we calculate for the case of $\beta \neq \gamma$ the upper bound
$C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)$

$$
\begin{aligned}
\leq & \frac{T \kappa \varepsilon^{-2}}{1-q} \sum_{i=0}^{k}\left(h_{0}^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0}}{c_{3,0}}\right)^{1 / 2}+\sum_{l=1}^{L_{p}-1} h_{l}^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3}^{(i)}\left(\frac{c_{2}}{c_{3}}\right)^{1 / 2}\right. \\
& \left.+h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}}+\delta_{i} \hat{c}_{3, L_{p}}^{(i)}\left(\frac{c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2}\right) \\
& +T \sum_{i=0}^{k}\left(\hat{c}_{3,0}^{(i)} h_{0}^{-\gamma+\delta_{i}}+\hat{c}_{3}^{(i)} \sum_{l=1}^{L_{p}-1} h_{l}^{-\gamma+\delta_{i}}+\hat{c}_{3, L_{p}}^{(i)} h_{L_{p}}^{-\gamma_{L_{p}}+\delta_{i}}\right) \\
\leq & \frac{T}{1-q} \varepsilon^{-2}\left[\sum _ { i = 0 } ^ { k } \hat { c } _ { 3 , 0 } ^ { ( i ) } \left(c_{2,0} T^{\beta-\gamma+\delta_{i}}+\left(\frac{c_{2,0} c_{2} c_{3}}{c_{3,0}}\right)^{1 / 2} \Lambda_{0} T^{\frac{\beta-\gamma}{2}+\delta_{i}}\right.\right. \\
& \left.+\left(\frac{c_{2,0} c_{2, L_{p} p} c_{3, L_{p}}}{c_{3,0}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}+\delta_{i}} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}}\right) \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)}\left(\frac{c_{2,0} c_{3,0} c_{2}}{c_{3}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}} \Lambda_{i}+\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)} c_{2} \Lambda_{i} \Lambda_{0} \\
& +\left(\hat{c}_{3}^{(k)} c_{2} \Lambda_{0}+\hat{c}_{3}^{(k)}\left(\frac{c_{2} c_{2, L_{p}} c_{3, L_{p}}}{c_{3}}\right)^{1 / 2} h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}}\right. \\
& \left.+\hat{c}_{3}^{(k)}\left(\frac{c_{2,0} c_{3,0} c_{2}}{c_{3}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}}\right)\left(\frac{\log \left(\varepsilon^{-1}\right)}{p \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, p} T^{p}\right)}{p \log (M)}\right) \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2, L_{p}} c_{3, L_{p}}}{c_{3}}\right)^{1 / 2} \Lambda_{i} h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}} \\
& +\sum_{i=0}^{k} \hat{c}_{3, L_{p}}^{(i)}\left(\frac{c_{2,0} c_{3,0} c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}} h_{L_{L_{p}}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}+\delta_{i}} \\
& +\sum_{i=0}^{k} \hat{c}_{3, L_{p}}^{(i)}\left(\left(\frac{c_{2} c_{3} c_{2, L_{p}}}{c_{3, L_{p}}^{1 / 2}}\right)^{1 / 2} \Lambda_{0} h_{L_{p}}^{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}+\delta_{i}\right.
\end{aligned}+c_{\left.\left.2, L_{p} h_{L_{p}}^{\beta_{L_{p}-\gamma_{L_{p}}+\delta_{i}}}\right)\right]}
$$

$$
\begin{equation*}
+T \sum_{i=0}^{k}\left(\hat{c}_{3,0}^{(i)} T^{\delta_{i}-\gamma}+\hat{c}_{3}^{(i)} \frac{\left(M^{-1} T\right)^{\delta_{i}-\gamma}-h_{L_{p}}^{\delta_{i}-\gamma}}{1-M^{\gamma-\delta_{i}}}+\hat{c}_{3, L_{p}}^{(i)} h_{L_{p}}^{\delta_{i}-\gamma_{L_{p}}}\right) \tag{26}
\end{equation*}
$$

with $\Lambda_{i}=\frac{\left(M^{-1} T\right)^{\frac{\beta-\gamma}{2}+\delta_{i}}-h_{L_{p}}^{\frac{\beta-\gamma}{2}+\delta_{i}}}{1-M^{\frac{\gamma-\beta}{2}-\delta_{i}}}$ for $i=0, \ldots, k-1$.
Proof of $i$ ). In case of $\beta>\gamma$ and $\beta_{L_{p}}>\gamma_{L_{p}}$, we prove that there exists some $\varepsilon_{0}>0$ such that for all $\left.\left.\varepsilon \in\right] 0, \varepsilon_{0}\right]$ it follows $C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)>C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)$. From the lower bound 25) for $C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)$ and the upper bound 26) for $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)$ we get the estimate

$$
\begin{align*}
& C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)-C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon) \\
& \geq \frac{T}{1-q} \varepsilon^{-2}\left(\sum_{i=0}^{k-1} T^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2} c_{3}}{c_{3,0}}\right)^{1 / 2} \frac{h_{L_{p}}^{\frac{\beta-\gamma}{2}}-M^{\frac{\gamma-\beta}{2}} h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}}\right. \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}} \cdot \frac{h_{L_{p}}^{\frac{\beta-\gamma}{2}+\delta_{i}}-M^{\frac{\gamma-\beta}{2}-\delta_{i}} h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}+\delta_{i}}}{1-M^{\frac{\gamma-\beta}{2}-\delta_{i}}} \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)} c_{2}\left(\frac{\left(M^{-1} T\right)^{\frac{\beta-\gamma}{2}+\delta_{i}}\left(h_{L_{p}}^{\frac{\beta-\gamma}{2}}-M^{\frac{\gamma-\beta}{2}} h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}}\right)}{\left(1-M^{\frac{\gamma-\beta}{2}-\delta_{i}}\right)\left(1-M^{\frac{\gamma-\beta}{2}}\right)}\right. \\
& \left.+\frac{\left(M^{-1} T\right)^{\frac{\beta-\gamma}{2}}\left(h_{L_{p}}^{\frac{\beta-\gamma}{2}+\delta_{i}}-M^{\frac{\gamma-\beta}{2}-\delta_{i}} h_{L_{\alpha}}^{\frac{\beta-\gamma}{2}+\delta_{i}}\right)-h_{L_{p}}^{\beta-\gamma+\delta_{i}}+M^{\gamma-\beta-\delta_{i}} h_{L_{\alpha}}^{\beta-\gamma+\delta_{i}}}{\left(1-M^{\frac{\gamma-\beta}{2}-\delta_{i}}\right)\left(1-M^{\frac{\gamma-\beta}{2}}\right)}\right) \\
& -\sum_{i=0}^{k-1} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2, L_{p}} c_{3, L_{p}}}{c_{3,0}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}+\delta_{i}} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}} \\
& -\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2, L_{p}} c_{3, L_{p}}}{c_{3}}\right)^{1 / 2} \Lambda_{i} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}} \\
& -\sum_{i=0}^{k-1} \hat{c}_{3, L_{p}}^{(i)}\left(\frac{c_{2,0} c_{3,0} c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2} T^{\frac{\beta-\gamma}{2}} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}+\delta_{i}} \\
& \left.-\sum_{i=0}^{k-1} \hat{c}_{3, L_{p}}^{(i)}\left(\left(\frac{c_{2} c_{3} c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2} \Lambda_{0} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}}+\delta_{i} c_{2, L_{p}} h_{L_{p}}^{\beta_{L_{p}}-\gamma_{L_{p}}+\delta_{i}}\right)\right) \\
& -T \sum_{i=0}^{k-1}\left(\hat{c}_{3,0}^{(i)} T^{\delta_{i}-\gamma}+\hat{c}_{3}^{(i)} \frac{\left(M^{-1} T\right)^{\delta_{i}-\gamma}-h_{L_{p}}^{\delta_{i}-\gamma}}{1-M^{\gamma-\delta_{i}}}+\hat{c}_{3, L_{p}}^{(i)} h_{L_{p}}^{\delta_{i}-\gamma_{L_{p}}}\right) . \tag{27}
\end{align*}
$$

In the following, we make use of the estimates $M^{-1} c_{1, \alpha}^{-\frac{1}{\alpha}} q^{\frac{1}{2 \alpha}} \varepsilon^{\frac{1}{\alpha}} \leq h_{L_{\alpha}} \leq$ $c_{1, \alpha}^{-\frac{1}{\alpha}} q^{\frac{1}{2 \alpha}} \varepsilon^{\frac{1}{\alpha}}$ and $M^{-1} c_{1, p}^{-\frac{1}{p}} q^{\frac{1}{2 p}} \varepsilon^{\frac{1}{p}} \leq h_{L_{p}} \leq c_{1, p}^{-\frac{1}{p}} q^{\frac{1}{2 p}} \varepsilon^{\frac{1}{p}}$, i.e. we have $h_{L_{p}} \rightarrow 0$ and $h_{L_{\alpha}} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Multiplying both sides of 27 with $\frac{1-q}{T} \varepsilon^{2} h_{L_{p}}^{-\frac{\min \left\{\beta-\gamma, \beta_{L_{p}}-\gamma_{L_{p}}\right\}}{2}}$ and taking into account the assumptions $4 p>\beta+\gamma$ and $4 p>\beta-\gamma+2 \gamma_{L_{p}}$ results in

$$
\begin{align*}
& \frac{1-q}{T} \varepsilon^{2} h_{L_{p}}^{-\frac{\min \left\{\beta-\gamma, \beta_{L_{p}}-\gamma_{L_{p}}\right\}}{2}}\left(C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)-C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)\right) \\
& \geq\left[\sum_{i=0}^{k-1} T^{\frac{\beta-\gamma}{2}+\delta_{i}} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0}}{c_{3,0}}\right)^{1 / 2}\left(\left(c_{2} c_{3}\right)^{1 / 2} \frac{h_{L_{p}}^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}}-\left(c_{2, L_{p}} c_{3, L_{p}}\right)^{1 / 2} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}}\right)\right. \\
& +T^{\frac{\beta-\gamma}{2}}\left(c_{2,0} c_{3,0}\right)^{1 / 2}\left(\hat{c}_{3}^{(0)}\left(\frac{c_{2}}{c_{3}}\right)^{1 / 2} \frac{h_{L_{p}}^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}}-\hat{c}_{3, L_{p}}^{(0)}\left(\frac{c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2} h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}}\right) \\
& +\sum_{i=0}^{k-1} \hat{c}_{3}^{(i)} c_{2}{ }^{1 / 2} \frac{\left(M^{-1} T\right)^{\frac{\beta-\gamma}{2}+\delta_{i}}}{1-M^{\frac{\gamma-\beta}{2}-\delta_{i}}}\left(\left(c_{2}\right)^{1 / 2} \frac{h_{L_{p}}^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}}-\left(\frac{c_{2, L_{p}} c_{3, L_{p}}}{c_{3}}\right)^{1 / 2} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}}\right) \\
& +\frac{\left(M^{-1} T\right)^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}} c_{2}^{1 / 2}\left(\hat{c}_{3}^{(0)} c_{2}^{1 / 2} \frac{h_{L_{p}}^{\frac{\beta-\gamma}{2}}}{1-M^{\frac{\gamma-\beta}{2}}}-\hat{c}_{3, L_{p}}^{(0)}\left(\frac{c_{3} c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2} h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}}\right) \\
& \left.+o\left(h^{\frac{\min \left\{\beta_{L_{p}}-\gamma_{L_{p}}, \beta-\gamma\right\}}{2}}\right)\right] h_{L_{p}}^{-\frac{\min \left\{\beta-\gamma, \beta_{L_{p}}-\gamma_{L_{p}}\right\}}{2}} . \tag{28}
\end{align*}
$$

As a result of (28) it follows that in the case of $\beta-\gamma<\beta_{L_{p}}-\gamma_{L_{p}}$ there exists some $\varepsilon_{0}>0$ such that

$$
\begin{equation*}
\frac{C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)}>1 \tag{29}
\end{equation*}
$$

for all $\left.\varepsilon \in] 0, \varepsilon_{0}\right]$. In the case of $\beta-\gamma=\beta_{L_{p}}-\gamma_{L_{p}}$ there exists some $\varepsilon_{0}>0$ such that 29 holds for all $\left.\varepsilon \in] 0, \varepsilon_{0}\right]$ if $c_{2} c_{3}>\left(1-M^{\frac{\gamma-\beta}{2}}\right)^{2} c_{2, L_{p}} c_{3, L_{p}}$ and $\left(\hat{c}_{3}^{(0)}\right)^{2} \frac{c_{2}}{c_{3}}>\left(1-M^{\frac{\gamma-\beta}{2}}\right)^{2}\left(\hat{c}_{3, L_{p}}^{(0)}\right)^{2} \frac{c_{2, L_{p}}}{c_{3, L_{p}}}$. Finally, $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=O\left(\varepsilon^{-2}\right)$ follows from (26).

Proof of iii). In case of $\beta<\gamma$ and $\beta<2 p$, we have to compare the dominating terms as $\varepsilon \rightarrow 0$. Therefore, we get from the lower bound that

$$
\begin{align*}
C\left(\hat{Y}_{M L(p, p)}\right)(\varepsilon) \geq & \frac{q^{\frac{\beta-\gamma}{2 p}}}{1-q} \varepsilon^{-2-\frac{\gamma-\beta}{p}} T \hat{c}_{3, L_{p}}^{(0)} c_{2, L_{p}} c_{1, p}^{\frac{\gamma-\beta}{p}} M^{\gamma-\beta}\left(M^{\frac{\beta-\gamma}{2}}-1\right)^{-2} \\
& +o\left(\varepsilon^{-2-\frac{\gamma-\beta}{p}}\right) \tag{30}
\end{align*}
$$

and from the upper bound

$$
C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon) \leq \frac{q^{\frac{\beta-\gamma}{2 p}}}{1-q} \varepsilon^{-2-\frac{\gamma-\beta}{p}} T c_{1, p}^{\frac{\gamma-\beta}{p}}\left(\frac{\hat{c}_{3}^{(0)} c_{2}}{\left(1-M^{\frac{\gamma-\beta}{2}}\right)^{2}}-\frac{\hat{c}_{3}^{(0)}\left(c_{2} c_{2, L_{p}} c_{3, L_{p}}\right)^{1 / 2}}{c_{3}^{1 / 2}\left(1-M^{\frac{\gamma-\beta}{2}}\right)}\right.
$$

$$
\begin{equation*}
\left.-\frac{\hat{c}_{3, L_{p}}^{(0)}\left(c_{2} c_{3} c_{2, L_{p}}\right)^{1 / 2}}{c_{3, L_{p}}^{1 / 2}\left(1-M^{\frac{\gamma-\beta}{2}}\right)}+\hat{c}_{3, L_{p}}^{(0)} c_{2, L_{p}}\right)+o\left(\varepsilon^{-2-\frac{\gamma-\beta}{p}}\right) \tag{31}
\end{equation*}
$$

Making use of these two estimates (30) and (31), this results in the estimate (20) where $\beta_{L_{p}}<\gamma_{L_{p}}$ because we require that $\beta_{L_{p}}-\gamma_{L_{p}}=\beta-\gamma<0$.

In general, it follows that $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=O\left(\varepsilon^{-2-\frac{\max \left\{\gamma-\beta, \gamma_{L_{p}}-\beta_{L_{p}}\right\}}{p}}\right)$ due to the upper bound (26) for $\beta<\gamma$ and any $\beta_{L_{p}}>0, \gamma_{L_{p}} \geq 1$. Further, there is an asymptotically optimal choice for the parameter $q \in] 0,1[$ such that the computational costs are asymptotically minimal. Calculating a lower bound for $C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)$ and taking into account the upper bound (31), we get

$$
\begin{equation*}
C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=\frac{1}{1-q} \varepsilon^{-2-\frac{\gamma-\beta}{p}} q^{\frac{\beta-\gamma}{2 p}} C+o\left(\varepsilon^{-2-\frac{\gamma-\beta}{p}}\right) \tag{32}
\end{equation*}
$$

with some constant $C>0$ independent of $q$ and $\varepsilon$. Now, we have to find some $\hat{q} \in] 0,1[$ such that

$$
\begin{equation*}
C \varepsilon^{-2-\frac{\gamma-\beta}{p}} \frac{\hat{q}^{\frac{\beta-\gamma}{2 p}}}{1-\hat{q}}=\min _{q \in] 0,1[ } C \varepsilon^{-2-\frac{\gamma-\beta}{p}} \frac{q^{\frac{\beta-\gamma}{2 p}}}{1-q} \tag{33}
\end{equation*}
$$

for all $0<\varepsilon<1$. Solving this minimization problem leads to

$$
\begin{equation*}
\hat{q}=\frac{\gamma-\beta}{\gamma-\beta+2 p} \tag{34}
\end{equation*}
$$

which is asymptotically the optimal choice for $q \in] 0,1[$ in case of $\beta<\gamma$.
Lower bound for $\beta=\gamma$. In case of $\beta=\gamma$, we get the following lower bound

$$
\begin{aligned}
C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon) \geq & \frac{T}{1-q} \varepsilon^{-2}\left(\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)} c_{2,0} h_{0}^{\delta_{i}}+\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2} c_{3}}{c_{3,0}}\right)^{1 / 2} L_{\alpha} h_{0}^{\delta_{i}}\right. \\
& +\hat{c}_{3}^{(0)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2} L_{\alpha}+\sum_{i=1}^{k} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2} \frac{T^{\delta_{i}}-h_{L_{\alpha}}^{\delta_{i}}}{M^{\delta_{i}}-1} \\
& \left.+\hat{c}_{3}^{(0)} c_{2} L_{\alpha}^{2}+\sum_{i=1}^{k} \hat{c}_{3}^{(i)} c_{2} L_{\alpha} \frac{T^{\delta_{i}}-h_{L_{\alpha}}^{\delta_{i}}}{M^{\delta_{i}}-1}\right) \\
\geq & \frac{T}{1-q} \varepsilon^{-2}\left(\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)} c_{2,0} T^{\delta_{i}}+\left(\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2} c_{3}}{c_{3,0}}\right)^{1 / 2} T^{\delta_{i}}\right.\right. \\
& \left.+\hat{c}_{3}^{(0)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2}+\sum_{i=1}^{k} \hat{c}_{3}^{(i)} c_{2} \frac{T^{\delta_{i}}-h_{L_{\alpha}}^{\delta_{i}}}{M^{\delta_{i}}-1}\right) \\
& \times\left(\frac{\log \left(\varepsilon^{-1}\right)}{\alpha \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, \alpha} T^{\alpha}\right)}{\alpha \log (M)}\right)+\hat{c}_{3}^{(0)} c_{2}\left(\frac{\log \left(\varepsilon^{-1}\right)}{\alpha \log (M)}\right)^{2}
\end{aligned}
$$

$$
\begin{align*}
& +2 \hat{c}_{3}^{(0)} c_{2} \frac{\log \left(\varepsilon^{-1}\right) \log \left(q^{-\frac{1}{2}} c_{1, \alpha} T^{\alpha}\right)}{\alpha^{2}(\log (M))^{2}}+\hat{c}_{3}^{(0)} c_{2} \frac{\left(\log \left(q^{-\frac{1}{2}} c_{1, \alpha} T^{\alpha}\right)\right)^{2}}{(\alpha \log (M))^{2}} \\
& +\sum_{i=1}^{k} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2,0} c_{3,0}}{c_{3}}\right)^{1 / 2} \frac{T^{\delta_{i}}-h_{L_{\alpha}}^{\delta_{i}}}{M^{\delta_{i}}-1} \tag{35}
\end{align*}
$$

where $\hat{c}_{3, L_{\alpha}}^{(i)}=\hat{c}_{3}^{(i)}, c_{2, L_{\alpha}}=c_{2}, c_{3, L_{\alpha}}=c_{3}, \beta_{L_{\alpha}}=\beta$ and $\gamma_{L_{\alpha}}=\gamma$ for $\hat{Y}_{M L(\alpha, \alpha)}$.
Upper bound for $\beta=\gamma$. Next, we calculate for $\beta=\gamma$ the upper bound

$$
\begin{align*}
& C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon) \\
& \leq \frac{T}{1-q} \varepsilon^{-2}\left[\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)} c_{2,0} T^{\delta_{i}}+\sum_{i=1}^{k} \hat{c}_{3}^{(i)}\left(\frac{c_{2,0} c_{3,0} c_{2}}{c_{3}}\right)^{1 / 2} \Lambda_{i}\right. \\
&+\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2, L_{p}} c_{3, L_{p}}}{c_{3,0}}\right)^{1 / 2} T^{\delta_{i}} h_{L_{p}}^{\frac{\beta_{L_{p}}-\gamma_{L_{p}}}{2}} \\
&+\left(\hat{c}_{3}^{(0)}\left(\frac{c_{2,0} c_{3,0} c_{2}}{c_{3}}\right)^{1 / 2}+\sum_{i=0}^{k} \hat{c}_{3,0}^{(i)}\left(\frac{c_{2,0} c_{2} c_{3}}{c_{3,0}}\right)^{1 / 2} T^{\delta_{i}}\right. \\
&+\hat{c}_{3}^{(0)}\left(\frac{c_{2} c_{2, L_{p}} c_{3, L_{p}}}{c_{3}}\right)^{1 / 2} h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}} \\
&\left.+\sum_{i=0}^{k} \hat{c}_{3, L_{p}}^{(i)}\left(\frac{c_{2} c_{3} c_{2, L_{p}}}{c_{3, L_{p}}}\right)^{1 / 2} h_{L_{L_{p}}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}+\delta_{i}}+\sum_{i=1}^{k} \hat{c}_{3}^{(i)} c_{2} \Lambda_{i}\right) \\
&\left.\times\left(\frac{\log \left(\varepsilon^{-1}\right)}{p \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, p} T^{p}\right)}{p \log (M)}\right)^{2}\right) \\
&+\hat{c}_{3}^{(0)} c_{2}\left(\frac{\log \left(\varepsilon^{-1}\right)}{p \log (M)}+\frac{\log \left(q^{-\frac{1}{2}} c_{1, p} T^{p}\right)}{p \log (M)}\right)^{2} \\
&+\sum_{i=1}^{k} \hat{c}_{3}^{(i)}\left(\frac{c_{2} c_{2, L_{p}} c_{3, L_{p}}}{c_{3}}\right)^{1 / 2} \Lambda_{i} h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}} \\
&\left.+\sum_{i=0}^{k} \hat{c}_{3, L_{p}}^{(i)}\left(\left(\frac{c_{2,0} c_{3,0} c_{2, L_{p}}}{c_{3, L_{p}}^{1 / 2}}\right)^{1 / h_{L_{p}}^{\frac{\beta_{L_{p}-\gamma_{L_{p}}}^{2}}{2}}+\delta_{i}}+c_{2, L_{p}} h_{L_{p}}^{\beta_{L_{p}-\gamma_{L_{p}}+\delta_{i}}}\right)\right] \\
&+T \sum_{i=0}^{k}\left(\hat{c}_{3,0}^{(i)} T^{\delta_{i}-\gamma}+\hat{c}_{3}^{(i)} \frac{\left.(M)^{-1} T\right)^{\delta_{i}-\gamma}-h_{L_{p}}^{\delta_{i}-\gamma}}{1-M^{\gamma-\delta_{i}}}+\hat{c}_{3, L_{p}}^{(i)} h_{L_{p}}^{\delta_{i}-\gamma_{L_{p}}}\right) \tag{36}
\end{align*}
$$

where we applied the relation (24).
Proof of ii). Suppose that $\beta_{L_{p}} \geq \gamma_{L_{p}}$ and $\gamma, \gamma_{L_{p}} \leq 2 p$. Then, we get from the upper bound (36) that

$$
C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)=O\left(\varepsilon^{-2}(\log (\varepsilon))^{2}\right)
$$

Further, comparing the lower and the upper bounds (35) and (36), we asymp-
totically obtain that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{C\left(\hat{Y}_{M L(\alpha, \alpha)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)} \geq \lim _{\varepsilon \rightarrow 0} \frac{\frac{T}{1-q} \varepsilon^{-2} \hat{c}_{3}^{(0)} c_{2}\left(\frac{\log \left(\varepsilon^{-1}\right)}{\alpha \log (M)}\right)^{2}+o\left(\varepsilon^{-2}(\log (\varepsilon))^{2}\right)}{\frac{T}{1-q} \varepsilon^{-2} \hat{c}_{3}^{(0)} c_{2}\left(\frac{\log \left(\varepsilon^{-1}\right)}{p \log (M)}\right)^{2}+o\left(\varepsilon^{-2}(\log (\varepsilon))^{2}\right)}=\frac{p^{2}}{\alpha^{2}} \tag{37}
\end{equation*}
$$

which proves statement (19). This completes the proof.
Remark 3.2. Especially, if $c_{3}=\hat{c}_{3}$ and $c_{3, L_{p}}=\hat{c}_{3, L_{p}}$, then it follows in case of $\beta<\gamma$ and $\beta<2 p$ that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{C\left(\hat{Y}_{M L(p, p)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)} \geq M^{\gamma-\beta}\left(1-M^{\frac{\beta-\gamma}{2}}\left(1-\left(\frac{c_{2} c_{3}}{c_{2, L_{p}} c_{3, L_{p}}}\right)^{1 / 2}\right)\right)^{-2} \tag{38}
\end{equation*}
$$

Thus, if $c_{2} c_{3}<c_{2, L_{p}} c_{3, L_{p}}$ it follows directly that

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} \frac{C\left(\hat{Y}_{M L(p, p)}\right)(\varepsilon)}{C\left(\hat{Y}_{M L(\alpha, p)}\right)(\varepsilon)}>1 \tag{39}
\end{equation*}
$$

## 4 Numerical examples in case of SDEs

For illustration of the improvement that can be realized with the proposed modified multi-level Monte Carlo estimator, we consider the problem of weak approximation for stochastic differential equations (SDEs)

$$
\begin{equation*}
\mathrm{d} X_{t}=a\left(X_{t}\right) \mathrm{d} t+\sum_{j=1}^{m} b^{j}\left(X_{t}\right) \mathrm{d} B_{t}^{j} \tag{40}
\end{equation*}
$$

with initial value $X_{t_{0}}=x_{0} \in \mathbb{R}^{d}$ driven by $m$-dimensional Brownian motion.
In the following, we compare for several numerical examples the root meansquare errors (5) versus the computational costs for the multi-level Monte Carlo estimator $\hat{Y}_{M L}$ proposed in [5],677] and described in Section 2 with the proposed modified multi-level Monte Carlo estimator $\hat{Y}_{M L(\alpha, p)}$ described in Section 3. As a measure for the computational costs, we count the number of evaluations of the drift and diffusion functions taking into account the dimension $d$ of the solution process as well as the dimension $m$ of the driving Brownian motion.

In the following, we consider on each level $l=0,1, \ldots, L$ an equidistant discretization $I_{h_{l}}=\left\{t_{0}, \ldots, t_{\frac{T}{2^{l}}}\right\}$ of $\left[t_{0}, T\right]$ with step size $h_{l}=\frac{T}{2^{2}}$. Further, we denote by $Y_{n}=Y_{t_{n}}$ the approximation at time $t_{n}$. In case of the multilevel Monte Carlo estimator $\hat{Y}_{M L}$ we apply on each level $l=0,1, \ldots, L$ the

Euler-Maruyama scheme on the grid $I_{h_{l}}$ given by $Y_{0}=x_{0}$ and

$$
\begin{equation*}
Y_{n+1}=Y_{n}+a\left(Y_{n}\right) h_{n}+\sum_{j=1}^{m} b^{j}\left(Y_{n}\right) I_{(j), n} \tag{41}
\end{equation*}
$$

where $h_{n}=h_{l}$ and $I_{(j), n}=B_{t_{n+1}}^{j}-B_{t_{n}}^{j}$ for $n=0,1, \ldots, \frac{T}{2^{l}}-1$. The EulerMaruyama scheme converges with order $1 / 2$ in the mean-square sense and with order $\alpha=1$ in the weak sense to the solution of the considered SDE 40) at time $T$ [10].

On the other hand, for the modified multi-level Monte Carlo estimator $\hat{Y}_{M L(\alpha, p)}$ the Euler-Maruyama scheme is applied on levels $0,1, \ldots, L_{p}-1$ whereas on level $L_{p}$ a second order weak stochastic Runge-Kutta (SRK) scheme RI6 proposed in [12] is applied. The SRK scheme RI6 on level $L_{p}$ is defined on the grid $I_{h_{L_{p}}}$ by $\check{Y}_{0}=x_{0}$,

$$
\begin{align*}
\check{Y}_{n+1} & =\check{Y}_{n}+\frac{1}{2}\left(a\left(\check{Y}_{n}\right)+a(\Upsilon)\right) h_{n}+\frac{1}{2} \sum_{k=1}^{m}\left(b^{k}\left(\Upsilon_{+}^{(k)}\right)-b^{k}\left(\Upsilon_{-}^{(k)}\right)\right) \frac{\hat{I}_{(k, k), n}}{\sqrt{h_{n}}} \\
& +\sum_{k=1}^{m}\left(\frac{1}{2} b^{k}\left(\check{Y}_{n}\right)+\frac{1}{4} b^{k}\left(\Upsilon_{+}^{(k)}\right)+\frac{1}{4} b^{k}\left(\Upsilon_{-}^{(k)}\right)\right) I_{(k), n} \\
& +\frac{1}{2} \sum_{k=1}^{m}\left(b^{k}\left(\hat{\Upsilon}_{+}^{(k)}\right)-b^{k}\left(\hat{\Upsilon}_{-}^{(k)}\right)\right) \sqrt{h_{n}}  \tag{42}\\
& -\sum_{k=1}^{m}\left(\frac{1}{2} b^{k}\left(\check{Y}_{n}\right)-\frac{1}{4} b^{k}\left(\hat{\Upsilon}_{+}^{(k)}\right)-\frac{1}{4} b^{k}\left(\hat{\Upsilon}_{-}^{(k)}\right)\right) I_{(k), n}
\end{align*}
$$

where $h_{n}=h_{L_{p}}$ and $I_{(k), n}=B_{t_{n+1}}^{k}-B_{t_{n}}^{k}$ for $n=0,1, \ldots, \frac{T}{2^{L_{p}}}-1$ with stages

$$
\begin{align*}
\Upsilon & =\check{Y}_{n}+a\left(\check{Y}_{n}\right) h_{n}+\sum_{j=1}^{m} b^{j}\left(\check{Y}_{n}\right) I_{(j), n}, \\
\Upsilon_{ \pm}^{(k)} & =\check{Y}_{n}+a\left(\check{Y}_{n}\right) h_{n} \pm b^{k}\left(\check{Y}_{n}\right) \sqrt{h_{n}}, \quad \hat{\Upsilon}_{ \pm}^{(k)}=\check{Y}_{n} \pm \sum_{\substack{j=1 \\
j \neq k}}^{m} b^{j}\left(\check{Y}_{n}\right) \frac{\hat{I}_{(k, j), n}}{\sqrt{h_{n}}} \tag{43}
\end{align*}
$$

where $\hat{I}_{(k, k), n}=\frac{1}{2}\left(I_{(k), n}^{2}-h_{n}\right)$ and

$$
\hat{I}_{(k, j)_{n}}= \begin{cases}\frac{1}{2}\left(I_{(k), n} I_{(j), n}-\sqrt{h_{n}} \tilde{I}_{(k), n}\right) & \text { if } k<j  \tag{44}\\ \frac{1}{2}\left(I_{(k), n} I_{(j), n}+\sqrt{h_{n}} \tilde{I}_{(j), n}\right) & \text { if } j<k\end{cases}
$$

based on independent random variables $\tilde{I}_{(k), n}$ with $\mathrm{P}\left(\tilde{I}_{(k), n}= \pm \sqrt{h_{n}}\right)=\frac{1}{2}$. Thus, we have $\alpha=1$ and $p=2$ for the modified multi-level Monte Carlo estimator $\hat{Y}_{M L(\alpha, p)}$ in the following. Further, for both schemes the variance decays with the same order as the computational costs increase, i. e. $\beta=\beta_{L_{p}}=\gamma=\gamma_{L_{p}}=1$. Then, the optimal order of convergence attained by the multi-level Monte Carlo method is $O\left(\varepsilon^{-2}(\log (\varepsilon))^{2}\right)$ due to Theorem 2.1. For the presented simulations,


Fig. 1. Error vs. computational effort for SDE 45) using $f(x)=x$ (left) and $f(x)=x^{2}$ (right).
we denote by MLMC EM the numerical results for $\hat{Y}_{M L}$ based on the EulerMaruyama scheme only and by MLMC SRK the results for $\hat{Y}_{M L(\alpha, p)}$ based on the combination of the Euler-Maruyama scheme and the SRK scheme RI6.

As a first example, we consider the scalar linear SDE with $d=m=1$ given by

$$
\begin{equation*}
\mathrm{d} X_{t}=r X_{t} \mathrm{~d} t+\sigma X_{t} \mathrm{~d} B_{t}, \quad X_{0}=0.1, \tag{45}
\end{equation*}
$$

using the parameters $r=1.5$ and $\sigma=0.1$. We choose $T=1$ and apply the functionals $f(x)=x$ and $f(x)=x^{2}$, see Figure 1. The presented simulations are calculated using the prescribed error bounds $\varepsilon=4^{-j}$ for $j=0,1, \ldots, 5$. In Figure 1 we can see the significantly reduced computational effort for the estimator $Y_{M L(1,2)}$ (MLMC SRK) compared to the estimator $\hat{Y}_{M L}$ (MLMC EM) in case of a linear and a nonlinear functional.

The second example is a nonlinear scalar SDE with $d=m=1$ given by

$$
\begin{equation*}
\mathrm{d} X_{t}=\frac{1}{2} X_{t}+\sqrt{X_{t}^{2}+1} \mathrm{~d} t+\sqrt{X_{t}^{2}+1} \mathrm{~d} B_{t}, \quad X_{0}=0 . \tag{46}
\end{equation*}
$$

We apply the functional

$$
f(x)=\left(\log \left(x+\sqrt{x^{2}+1}\right)\right)^{3}-6\left(\log \left(x+\sqrt{x^{2}+1}\right)\right)^{2}+8 \log \left(x+\sqrt{x^{2}+1}\right) .
$$

Then, the approximated expectation is given by

$$
\mathrm{E}\left(f\left(X_{t}\right)\right)=t^{3}-3 t^{2}+2 t .
$$

Here, the results presented in Figure 2 (left) are calculated for $T=2$ applying the prescribed error bounds $\varepsilon=4^{-j}$ for $j=0,1, \ldots, 6$. Here, the improved estimator $\hat{Y}_{M L(1,2)}$ performs much better than $\hat{Y}_{M L}$ also for nonlinear functionals and a nonlinear SDE. Finally, we consider a nonlinear multi-dimensional SDE with a $d=4$ dimensional solution process driven by an $m=6$ dimensional Brownian motion with non-commutative noise:


Fig. 2. Error vs. computational effort for the nonlinear SDE (46) (left) and SDE (47) (right) with non-commutative noise.

$$
\begin{align*}
& \mathrm{d}\left(\begin{array}{l}
X_{t}^{1} \\
X_{t}^{2} \\
X_{t}^{3} \\
X_{t}^{4}
\end{array}\right)=\left(\begin{array}{l}
\frac{243}{154} X_{t}^{1}-\frac{27}{77} X_{t}^{2}+\frac{23}{154} X_{t}^{3}-\frac{65}{154} X_{t}^{4} \\
\frac{27}{77} X_{t}^{1}-\frac{243}{154} X_{t}^{2}+\frac{65}{154} X_{t}^{3}-\frac{23}{154} X_{t}^{4} \\
\frac{5}{154} X_{t}^{1}-\frac{61}{154} X_{t}^{2}+\frac{162}{77} X_{t}^{3}-\frac{36}{77} X_{t}^{4} \\
\frac{61}{154} X_{t}^{1}-\frac{5}{154} X_{t}^{2}+\frac{36}{77} X_{t}^{3}-\frac{162}{77} X_{t}^{4}
\end{array}\right) \mathrm{d} t \\
& +\left(\begin{array}{c}
\frac{1}{13} \\
\frac{1}{14} \\
\frac{1}{13} \\
\frac{1}{15}
\end{array}\right) \mathrm{d} B_{t}^{1}+\frac{1}{8} \sqrt{\left(X_{t}^{4}\right)^{2}+\left(X_{t}^{1}\right)^{2}+\frac{1}{11}}\left(\begin{array}{c}
\frac{1}{14} \\
\frac{1}{16} \\
\frac{1}{16} \\
\frac{1}{12}
\end{array}\right) \mathrm{d} B_{t}^{2} \\
& +\frac{1}{12} \sqrt{\left.\left(X_{t}^{2}\right)^{2}+\left(X_{t}^{3}\right)^{2}+\frac{2}{6}\right)^{2}+\left(X_{t}^{2}\right)^{2}+\frac{1}{9}}\left(\begin{array}{c}
\frac{1}{5} \\
\frac{1}{5} \\
\frac{1}{5} \\
\frac{1}{6}
\end{array}\right) \mathrm{d} B_{t}^{3}+\frac{1}{14} \sqrt{\left(X_{t}^{3}\right)^{2}+\left(X_{t}^{4}\right)^{2}+\frac{3}{29}}\left(\begin{array}{c}
\frac{1}{9} \\
\frac{1}{8} \\
\frac{1}{9}
\end{array}\right) \mathrm{d} B_{t}^{4} \\
& +\frac{1}{11}  \tag{47}\\
& \frac{1}{10} \sqrt{\left(X_{t}^{1}\right)^{2}+\left(X_{t}^{3}\right)^{2}+\frac{1}{13}}\left(\begin{array}{c}
\frac{1}{12} \\
\frac{1}{15} \\
\frac{1}{13} \\
\frac{1}{11}
\end{array}\right) \mathrm{d} B_{t}^{5}+\frac{1}{11} \sqrt{\left(X_{t}^{2}\right)^{2}+\left(X_{t}^{4}\right)^{2}+\frac{2}{25}}\left(\begin{array}{c}
\frac{1}{13} \\
\frac{1}{16} \\
\frac{1}{13}
\end{array}\right) \mathrm{d} B_{t}^{6}
\end{align*}
$$

with initial condition $X_{0}=\left(\frac{1}{8}, \frac{1}{8}, 1, \frac{1}{8}\right)^{T}$. Then, the approximated first moment of the solution is given by $E\left(X_{T}^{i}\right)=X_{0}^{i} \exp (2 T)$ for $i=1,2,3,4$. The simulation results calculated at $T=1$ for the error bounds $\varepsilon=4^{-j}$ for $j=0,1, \ldots, 6$ are presented in Figure 2 (right). Again, in the multi-dimensional non-commutative noise case the proposed estimator $\hat{Y}_{M L(1,2)}$ needs significantly less computational effort compared to the estimator $\hat{Y}_{M L}$ which reveals the theoretical results (19) in Proposition 3.1.

## 5 Conclusions

In this paper we proposed a modification of the multi-level Monte Carlo method introduced by M. Giles which combines approximation methods of different orders of weak convergence. This modified multi-level Monte Carlo method attains the same mean square order of convergence like the originally proposed method that is in some sense optimal. However, the newly proposed multi-level Monte Carlo estimator can attain significantly reduced computational costs. As an example, there is a reduction of costs by a factor $(p / \alpha)^{2}$ for the problem of weak approximation for SDEs driven by Brownian motion in case of $\beta=\gamma$. This has been approved by some numerical examples for the case of $p=2$ and $\alpha=1$ where four times less calculations are needed compared to the standard multi-level Monte Carlo estimator. Here, we want to point out that there also exist higher order weak approximation schemes, e. g. $p=3$ in case of SDEs with additive noise [2], that may further improve the benefit of the modified multi-level Monte Carlo estimator. Future research will consider the application of this approach to, e.g., more general SDEs like SDEs driven by Lévy processes [3] or fractional Brownian motion [11] and to the numerical solution of SPDEs [13]. Further, the focus will be on numerical schemes that feature not only high orders of convergence but also minimized constants for the variance estimates.

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[^0]:    Email addresses: debrabant@imada.sdu.dk (Kristian Debrabant), roessler@math.uni-luebeck.de (Andreas Rößler).

