

On A New Class of Tempered Stable Distributions: Moments and Regular Variation

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Abstract

We extend the class of tempered stable distributions first introduced in Rosiński 2007 [19]. Our new class allows for more structure and more variety of tail behaviors. We discuss various subclasses and the relation between them. To characterize the possible tails we give detailed results about finiteness of various moments. We also give necessary and sufficient conditions for the tails to be regularly varying. This last part allows us to characterize the domain of attraction to which a particular tempered stable distribution belongs.

1 Introduction

Tempered stable distributions were defined in Rosiński 2007 [19] as a class of models obtained by modifying the Lévy measures of stable distributions by multiplying their densities by completely monotone functions. This allows for models that are similar to stable distributions in some central region, but possess lighter (i.e. tempered) tails. It has been observed that these models provide a good fit to data in a variety of applications. These include mathematical finance [9] [14], biostatistics [2] [18], computer science [25], and physics [8] [17]. An explanation for why such models might appear in applications is given in [11].

The purpose of this paper is two-fold. First, we provide necessary and sufficient conditions for tempered stable distributions to have regularly varying tails. This is an important question both from a theoretical perspective, since it will allow us to classify which domain of attraction a tempered stable distribution belongs to, and from an applied point of view, since such models are often used in practice. Our second purpose is to introduce the class of p -tempered α -stable distributions, where $p > 0$ and $\alpha < 2$. The parameter p controls the amount of tempering, while α is the index of stability of the corresponding stable distribution. Clearly the case where $\alpha \leq 0$ no longer has any meaning in terms of tempering stable distributions, however it allows the class to be more flexible. In fact, within certain subclasses, the case where $\alpha \leq 0$ has been shown to provide a good fit to data, see e.g. [2] or [9].

This class combines a number of important subclasses that have been studied separately in the literature. In particular, when $p = 1$ and $\alpha \in (0, 2)$ it coincides with

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Rosiński's [19] tempered stable distributions. When $p = 2$ and $\alpha \in [0, 2)$ it coincides with the class of tempered infinitely divisible distributions defined in [6]. If we allow the distributions to have a Gaussian part, then we would have the class $J_{\alpha,p}$ defined in [15]. This, in turn, contains important subclasses including the Thorin class (when $p = 1$ and $\alpha = 0$), the Goldie-Steutel-Bondesson class (when $p = 1$ and $\alpha = -1$), the class of type M distributions (when $p = 2$ and $\alpha = 0$), and the class of type G distributions (when $p = 2$ and $\alpha = -1$). For more information on these classes see [4], [3], and the references therein.

This paper is structured as follows. In Section 2 we define p -tempered α -stable distributions and state some basic results. We show that, as with tempered stable distributions, for a fixed α and p , all elements of this class are uniquely determined by a Rosiński measure R and a shift b . The remaining two sections are concerned with relating the tails of the Rosiński measure to the tails of the distribution. In Section 3 we give necessary and sufficient conditions for the existence of moments and exponential moments. We also give explicit formulas for the cumulants. Finally, in Section 4 we give necessary and sufficient conditions for the tails to be regularly varying. Specifically, we show that the tails of a p -tempered α -stable distribution are regularly varying if and only if the tails of the corresponding Rosiński measure are regularly varying.

Before proceeding, recall that the characteristic function of an infinitely divisible distribution μ on \mathbb{R}^d can be written as $\hat{\mu}(z) = \exp\{C_\mu(z)\}$ where

$$C_\mu(z) = -\frac{1}{2}\langle z, Az \rangle + i\langle b, z \rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z, x \rangle} - 1 - i\frac{\langle z, x \rangle}{1 + |x|^2} \right) M(dx), \quad (1)$$

A is a symmetric nonnegative-definite $d \times d$ matrix, $b \in \mathbb{R}^d$, and M satisfies

$$M(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} (|x|^2 \wedge 1) M(dx) < \infty. \quad (2)$$

The measure μ is uniquely identified by the Lévy triplet (A, M, b) and we write $\mu = ID(A, M, b)$.

2 p -Tempered α -Stable Distributions

Recall that for $\alpha \in (0, 2)$ the Lévy measure of an α -stable distribution with spectral measure σ is given by

$$L(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(ru) r^{-\alpha-1} dr \sigma(du), \quad B \in \mathfrak{B}(\mathbb{R}^d). \quad (3)$$

By analogy, we define the following.

Definition 1. Fix $\alpha < 2$ and $p > 0$. An infinitely divisible probability measure μ is called a **p -tempered α -stable distribution** if it has no Gaussian part and its Lévy measure is given by

$$M(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(ru) q(r^p, u) r^{-\alpha-1} dr \sigma(du), \quad B \in \mathfrak{B}(\mathbb{R}^d), \quad (4)$$

where σ is a finite Borel measure on \mathbb{S}^{d-1} and $q : (0, \infty) \times \mathbb{S}^{d-1} \mapsto (0, \infty)$ is a Borel function such that for all $u \in \mathbb{S}^{d-1}$ $q(\cdot, u)$ is completely monotone and

$$\lim_{r \rightarrow \infty} q(r, u) = 0. \quad (5)$$

We denote the class of p -tempered α -stable distributions by TS_α^p . If, in addition, $\lim_{r \downarrow 0} q(r, u) = 1$ for every $u \in \mathbb{S}^{d-1}$ then μ is called a **proper p -tempered α -stable distribution**.

Remark 2. The complete monotonicity of $q(\cdot, u)$ implies that for each $u \in \mathbb{S}^{d-1}$ the function $q(r, u)$ is differentiable and monotonely decreasing in r . Moreover, by Bernstein's Theorem (see e.g. Theorem 1a in Section XIII.4 of [10]),

$$q(r^p, u) = \int_{(0, \infty)} e^{-r^p s} Q_u(ds) \quad (6)$$

for some measurable family $\{Q_u\}_{u \in \mathbb{S}^{d-1}}$ of Borel measures on $(0, \infty)$. For a guarantee that we can take the family to be measurable see Remark 3.2 in [4]. Note that the condition $\lim_{r \downarrow 0} q(r, u) = 1$ for every $u \in \mathbb{S}^{d-1}$ is equivalent to the condition that $\{Q_u\}_{u \in \mathbb{S}^{d-1}}$ is a family of probability measures.

Remark 3. From (6) it follows that as p increases the tails of M (as given in (4)) go to zero quicker. In this sense p controls the extent to which the tails of the Lévy measure are tempered.

Remark 4. For $\alpha \in (0, 2)$ and $p > 0$, all proper p -tempered α -stable distributions belong to the class of generalized tempered stable distributions defined in [20]. Many important results about their Lévy processes are given there. These include short time behavior, conditions for absolute continuity with respect to the underlying stable process, and a series representation, see Theorems 3.1, 4.1, and 5.5 in [20] for details.

Remark 5. From Theorem 15.10 in [24] it follows that p -tempered α -stable distributions are selfdecomposable if and only if $q(r^p, u)r^{-\alpha}$ is a decreasing function of r for every $u \in \mathbb{S}^{d-1}$. By Remark 2 this always holds when $\alpha \in [0, 2)$. Thus, when $\alpha \in [0, 2)$, p -tempered α -stable distributions inherit properties of selfdecomposable distributions. In particular, if they are nondegenerate then they are absolutely continuous with respect to Lebesgue measure in d -dimensions and when $d = 1$ they are unimodal.

Following [19], we will reparametrize the Lévy measure M into a form that is often easier to work with. Let Q be a Borel measure on \mathbb{R}^d given by

$$Q(A) = \int_{\mathbb{S}^{d-1}} \int_{(0, \infty)} 1_A(ru) Q_u(dr) \sigma(du), \quad A \in \mathfrak{B}(\mathbb{R}^d). \quad (7)$$

Note that $Q(\{0\}) = 0$. Define a Borel measure R on \mathbb{R}^d by

$$R(A) = \int_{\mathbb{R}^d} 1_A \left(\frac{x}{|x|^{1+1/p}} \right) |x|^{\alpha/p} Q(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d), \quad (8)$$

and again note that $R(\{0\}) = 0$. To get the inverse transformation we have

$$Q(A) = \int_{\mathbb{R}^d} 1_A \left(\frac{x}{|x|^{p+1}} \right) |x|^\alpha R(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d). \quad (9)$$

The following result extends Theorem 2.3 of [19].

Theorem 6. Fix $p > 0$. Let M be given by (4) and let R be given by (8).

1. We can write

$$M(A) = \int_{\mathbb{R}^d} \int_0^\infty 1_A(tx) t^{-1-\alpha} e^{-t^p} dt R(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d), \quad (10)$$

or equivalently,

$$M(A) = p^{-1} \int_{\mathbb{R}^d} \int_0^\infty 1_A(t^{1/p}x) t^{-1-\alpha/p} e^{-t} dt R(dx), \quad A \in \mathfrak{B}(\mathbb{R}^d). \quad (11)$$

2. (10) defines a Lévy measure if and only if either $R = 0$ or the following hold:

$$\alpha < 2, \quad R(\{0\}) = 0, \quad (12)$$

and

$$\begin{aligned} \int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\alpha) R(dx) &< \infty \quad \text{if } \alpha \in (0, 2), \\ \int_{\mathbb{R}^d} (|x|^2 \wedge [1 + \log^+ |x|]) R(dx) &< \infty \quad \text{if } \alpha = 0, \\ \int_{\mathbb{R}^d} (|x|^2 \wedge 1) R(dx) &< \infty \quad \text{if } \alpha < 0. \end{aligned} \quad (13)$$

Moreover, when R satisfies these conditions, M is the Lévy measure of a p -tempered α -stable distribution and it uniquely determines R .

3. A p -tempered α -stable distribution is proper if and only if in addition to (12) and (13) R satisfies

$$\int_{\mathbb{R}^d} |x|^\alpha R(dx) < \infty. \quad (14)$$

4. If R satisfies (14) then in (4) the measure σ is given by

$$\sigma(B) = \int_{\mathbb{R}^d} 1_B\left(\frac{x}{|x|}\right) |x|^\alpha R(dx), \quad B \in \mathfrak{B}(\mathbb{S}^{d-1}). \quad (15)$$

Note that for all $\alpha < 2$ the conditions in (13) imply the necessity of

$$\int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\alpha) R(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1) R(dx) < \infty. \quad (16)$$

Before proving Theorem 6, we will translate the integrability conditions on R into integrability conditions on $\{Q_u\}_{u \in \mathbb{S}^{d-1}}$ and σ .

Corollary 7. Fix $p > 0$, let M be given by (4), and let $\{Q_u\}$ be as in (6). M is a Lévy measure if and only if either

$$Q_u(\mathbb{R}_+) = 0 \quad \sigma\text{-a.e.}$$

or $\alpha < 2$ and

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \int_0^\infty \left(t^{-(2-\alpha)/p} \wedge 1 \right) Q_u(dt) \sigma(du) &< \infty \quad \alpha \in (0, 2) \\ \int_{\mathbb{S}^{d-1}} \int_0^\infty \left(t^{-2/p} \wedge [1 + \log^+(t^{-1/p})] \right) Q_u(dt) \sigma(du) &< \infty \quad \alpha = 0 \\ \int_{\mathbb{S}^{d-1}} \int_0^\infty \left(t^{-(2-\alpha)/p} \wedge t^{\alpha/p} \right) Q_u(dt) \sigma(du) &< \infty \quad \alpha < 0. \end{aligned}$$

Note that these conditions guarantee that for any $p > 0$ and for σ -a.e. u (5) holds and $\int_{\mathbb{R}^d} e^{-r^p s} Q_u(ds) < \infty$.

Proof of Theorem 6. We omit most parts of the proof because they are similar to the case when $p = 1$ and $\alpha \in (0, 2)$, which is given in [19]. We only show that when M is given by (10) it is a Lévy measure if and only if (12) and (13) hold. Assume $R \neq 0$, since the other case is trivial. We have

$$M(\{0\}) = \int_{\mathbb{R}^d} \int_0^\infty 1_{\{0\}}(tx) t^{-\alpha-1} e^{-t^p} dt R(dx) = \int_{\{0\}} \int_0^\infty t^{-1-\alpha} e^{-t^p} dt R(dx),$$

which equals zero if and only if $R(\{0\}) = 0$.

Now assume $\int (|x|^2 \wedge 1) M(dx) < \infty$. For any $\epsilon > 0$

$$\begin{aligned} \infty &> \int_{|x| \leq 1} |x|^2 M(dx) = \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} e^{-t^p} dt R(dx) \\ &\geq \int_{|x| \leq 1/\epsilon} |x|^2 \int_0^\epsilon t^{1-\alpha} e^{-t^p} dt R(dx) \geq e^{-\epsilon^p} \int_{|x| \leq 1/\epsilon} |x|^2 \int_0^\epsilon t^{1-\alpha} dt R(dx). \end{aligned}$$

Since $R \neq 0$, for this to be finite for all $\epsilon > 0$ it is necessary that $\alpha < 2$. Taking $\epsilon = 1$ gives the necessity of $\int_{|x| \leq 1} |x|^2 R(dx) < \infty$. Observe that

$$\begin{aligned} \infty &> \int_{|x| \geq 1} M(dx) = \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\geq \int_{|x| \geq 1} \int_{|x|^{-1}}^\infty t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\geq \int_1^\infty t^{-1-\alpha} e^{-t^p} dt \int_{|x| \geq 1} R(dx) + e^{-1} \int_{|x| \geq 1} \int_{|x|^{-1}}^1 t^{-1-\alpha} dt R(dx). \end{aligned}$$

This implies the necessity of $\int_{|x| \geq 1} R(dx) < \infty$ and $\int_{|x| \geq 1} \int_{|x|^{-1}}^1 t^{-1-\alpha} dt R(dx) < \infty$. When $\alpha < 0$ we are done. When $\alpha = 0$ this implies the finiteness of $\int_{|x| \geq 1} \log |x| R(dx)$, and when $\alpha \in (0, 2)$ it implies the finiteness of $\int_{|x| \geq 1} |x|^\alpha R(dx)$. Thus (12) and (13) hold.

Now assume that (12) and (13) hold. We have

$$\begin{aligned} \int_{|x| \leq 1} |x|^2 M(dx) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} e^{-t^p} dt R(dx) \\ &\leq \int_{|x| \leq 1} |x|^2 R(dx) \int_0^\infty t^{1-\alpha} e^{-t^p} dt + \int_{|x| > 1} |x|^2 \int_0^{|x|^{-1}} t^{1-\alpha} dt R(dx) \\ &= p^{-1} \Gamma\left(\frac{2-\alpha}{p}\right) \int_{|x| \leq 1} |x|^2 R(dx) + (2-\alpha)^{-1} \int_{|x| > 1} |x|^\alpha R(dx) < \infty. \end{aligned}$$

Let $D = \sup_{t \geq 1} t^{2-\alpha} e^{-t^p}$. We have

$$\begin{aligned} \int_{|x| \geq 1} M(dx) &= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\leq D \int_{|x| \leq 1} \int_{|x|^{-1}}^\infty t^{-3} dt R(dx) + \int_{|x| > 1} \int_{|x|^{-1}}^\infty t^{-1-\alpha} e^{-t^p} dt R(dx). \end{aligned}$$

The first integral in the above equals $.5D \int_{|x| \leq 1} |x|^2 R(dx)$, which is assumed finite. The second integral can be written as

$$\int_{|x| > 1} \int_{|x|^{-1}}^1 t^{-1-\alpha} e^{-t^p} dt R(dx) + \int_1^\infty t^{-1-\alpha} e^{-t^p} dt \int_{|x| > 1} R(dx)$$

Of these, the second integral is finite since $\int_{|x| > 1} R(dx) < \infty$. The first is bounded by $\int_{|x| > 1} \frac{|x|^{\alpha-1}}{\alpha} R(dx)$ when $\alpha \neq 0$ and by $\int_{|x| > 1} \log |x| R(dx)$, when $\alpha = 0$. The fact that both of these are assumed to be finite gives the result. \square

Definition 8. *The unique measure in (8) is called the **Rosiński measure** of the corresponding p -tempered α -stable distribution.*

Remark 9. *For $\alpha \in (0, 2)$ and $p = 1$ the Rosiński measure was called the spectral measure in [19]. For $\alpha \in [0, 2)$ and $p = 2$ the Rosiński measure was introduced in a slightly different parametrization in [6].*

Remark 10. *Fix $\alpha < 2$, $p > 0$, and let $\mu \in TS_\alpha^p$ with Rosiński measure R . Then $\mu = ID(0, M, b)$ for some $b \in \mathbb{R}^d$ and M uniquely determined by R . We write $TS_\alpha^p(R, b)$ to denote this distribution.*

Theorem 6 shows that for a fixed $p > 0$ and $\alpha < 2$, the Rosiński measure is uniquely determined by the Lévy measure. This leaves the question of whether all of the parameters are jointly identifiable. Unfortunately this is not the case. As we will show below, even for a fixed $p > 0$ the parameters α and R are not jointly identifiable. However, using ideas similar to those in [19], we will show that for a fixed $p > 0$, in the subclass of proper tempered stable distribution, they are jointly identifiable. On the other hand, for a fixed $\alpha < 2$, even in the subclass of proper tempered stable distributions, the parameters p and R are not jointly identifiable. We begin with the following lemma.

Lemma 11. *Fix $\alpha < 2$, $p > 0$, and let M be the Lévy measure of a p -tempered α -stable distribution with Rosiński measure $R \neq 0$.*

1. *The map $s \mapsto s^\alpha M(|x| > s)$ is decreasing and $\lim_{s \rightarrow \infty} s^\alpha M(|x| > s) = 0$.*
2. *If $\alpha \in (0, 2)$ then*

$$\lim_{s \downarrow 0} s^\alpha M(|x| > s) = \frac{1}{\alpha} \int_{\mathbb{R}^d} |x|^\alpha R(dx)$$

and if $\alpha \leq 0$ then

$$\lim_{s \downarrow 0} s^\alpha M(|x| > s) = \infty.$$

3. *If $\alpha < 0$ then*

$$\lim_{s \downarrow 0} s^\alpha M(|x| < s) = \frac{1}{|\alpha|} \int_{\mathbb{R}^d} |x|^\alpha R(dx)$$

and if $\alpha \in [0, 2)$ then for all $s > 0$

$$M(|x| < s) = \infty.$$

This lemma extends Corollary 2.5 in [19]. Note that it implies that in the subclass of proper tempered stable distributions both $\lim_{s \downarrow 0} s^\alpha M(|x| > s) = \infty$ and $M(|x| < s) = \infty$ if and only if $\alpha = 0$.

Proof. We begin with the first part. Since

$$\begin{aligned} s^\alpha M(|x| > s) &= s^\alpha \int_{\mathbb{R}^d} \int_{s|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-(st)^p} dt R(dx), \end{aligned} \quad (17)$$

the map $s \mapsto s^\alpha M(|x| > s)$ is decreasing. For large enough s , the integrand in (17) is bounded by $t^{-1-\alpha} e^{-t^p}$, which is integrable. Thus by dominated convergence $\lim_{s \rightarrow \infty} s^\alpha M(|x| > s) = 0$.

For the second part, by (17) and the Monotone Convergence Theorem

$$\lim_{s \downarrow 0} s^\alpha M(|x| > s) = \int_{\mathbb{R}^d} \int_{|x|^{-1}}^{\infty} t^{-1-\alpha} dt R(dx).$$

Thus if $\alpha \in (0, 2)$ then

$$\lim_{s \downarrow 0} s^\alpha M(|x| > s) = \frac{1}{\alpha} \int_{\mathbb{R}^d} |x|^\alpha R(dx),$$

and if $\alpha \leq 0$ then

$$\lim_{s \downarrow 0} s^\alpha M(|x| > s) = \infty.$$

Now for the third part. If $\alpha \in [0, 2)$ then for all $s > 0$

$$\begin{aligned} M(|x| < s) &= \int_{\mathbb{R}^d} \int_0^{s|x|^{-1}} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &\geq \int_{\mathbb{R}^d} e^{-(s/|x|)^p} \int_0^{s|x|^{-1}} t^{-1-\alpha} dt R(dx) = \infty. \end{aligned}$$

If $\alpha < 0$ then

$$\begin{aligned} \lim_{s \downarrow 0} s^\alpha M(|x| < s) &= \lim_{s \downarrow 0} s^\alpha \int_{\mathbb{R}^d} \int_0^{s|x|^{-1}} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &= \lim_{s \downarrow 0} \int_{\mathbb{R}^d} \int_0^{|x|^{-1}} t^{-1-\alpha} e^{-(st)^p} dt R(dx) \\ &= \int_{\mathbb{R}^d} \int_0^{|x|^{-1}} t^{-1-\alpha} dt R(dx) = \frac{1}{|\alpha|} \int_{\mathbb{R}^d} |x|^\alpha R(dx), \end{aligned}$$

where the third line follows by the Monotone Convergence Theorem. \square

Combining Lemma 11 with (14) gives the following.

Proposition 1. *In the subclass of proper tempered stable distributions with parameter $p > 0$ fixed, the parameters R and α are jointly identifiable.*

However, in general, the parameters α and p are not identifiable. This will become apparent from the following results.

Proposition 2. Fix $\alpha < 2$, $\beta \in (\alpha, 2)$, and let $K = \int_0^\infty s^{\beta-\alpha-1} e^{-s^p} ds$. If $\mu = TS_\beta^p(R, b)$ and

$$R'(A) = K^{-1} \int_{\mathbb{R}^d} \int_0^1 1_A(ux) u^{-\beta-1} (1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \quad (18)$$

then R' is the Rosiński measure of a p -tempered α -stable distribution and $\mu = TS_\alpha^p(R', b)$.

Proof. First we will show that R' is the Rosiński measure of some p -tempered α -stable distribution. Let $C = \max_{u \in [0, .5]} (1-u^p)^{(\beta-\alpha)/p-1}$. We have

$$\begin{aligned} K \int_{|x| \leq 1} |x|^2 R'(dx) &= \int_{\mathbb{R}^d} |x|^2 \int_0^{1 \wedge |x|^{-1}} u^{1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \\ &\leq \int_{|x| \leq 2} |x|^2 R(dx) \int_0^1 u^{1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du \\ &\quad + C \int_{|x| > 2} |x|^2 \int_0^{|x|^{-1}} u^{1-\beta} du R(dx) \\ &= \int_{|x| \leq 2} |x|^2 R(dx) \int_0^1 u^{1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du \\ &\quad + \frac{C}{2-\beta} \int_{|x| \geq 2} |x|^\beta R(dx) < \infty. \end{aligned}$$

If $\alpha \in (0, 2)$ then

$$\begin{aligned} K \int_{|x| > 2} |x|^\alpha R'(dx) &= \int_{|x| \geq 2} |x|^\alpha \int_{|x|^{-1}}^{1/2} u^{\alpha-1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \\ &\quad + \int_{|x| \geq 2} |x|^\alpha \int_{1/2}^1 u^{\alpha-1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \\ &\leq C \int_{|x| \geq 2} |x|^\alpha \int_{|x|^{-1}}^\infty u^{\alpha-1-\beta} du R(dx) \\ &\quad + \int_{|x| > 2} |x|^\beta R(dx) \int_{1/2}^1 u^{\alpha-1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du. \end{aligned}$$

Here the first integral equals $\frac{C}{\beta-\alpha} \int_{|x| \geq 2} |x|^\beta R(dx) < \infty$ and the second is also finite. Now assume $\alpha = 0$ and fix $\epsilon \in (0, \beta)$. By 4.1.37 in [1] there exists a $C_\epsilon > 0$ such that for all $u > 0$, $\log u \leq C_\epsilon u^\epsilon$. Thus

$$K \int_{|x| > 2} \log |x| R'(dx) \leq KC_\epsilon \int_{|x| > 2} |x|^\epsilon R'(dx),$$

which is finite by arguments similar to the previous case. When $\alpha < 0$

$$\begin{aligned} K \int_{|x|>2} R'(dx) &= \int_{|x|>2} \int_{|x|^{-1}}^1 u^{-1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du R(dx) \\ &\leq C \int_{|x|>2} \int_{|x|^{-1}}^1 u^{-1-\beta} du R(dx) \\ &\quad + \int_{|x|>2} R(dx) \int_{1/2}^1 u^{-1-\beta} (1-u^p)^{(\beta-\alpha)/p-1} du. \end{aligned}$$

Here the second integral is finite. For $\beta \neq 0$ the first equals $\frac{C}{\beta} \int_{|x|>2} (|x|^\beta - 1) R(dx)$ which is finite, and for $\beta = 0$ it equals $\int_{|x|>2} \log |x| R(dx) < \infty$.

Now, let M' be the Lévy measure of $TS_\alpha^p(R', b)$. By (10)

$$\begin{aligned} M'(A) &= K^{-1} \int_{\mathbb{R}^d} \int_0^\infty \int_0^1 1_A(utx) t^{-1-\alpha} e^{-t^p} u^{-\beta-1} (1-u^p)^{\frac{\beta-\alpha}{p}-1} du dt R(dx) \\ &= K^{-1} \int_{\mathbb{R}^d} \int_0^\infty \int_0^t 1_A(vx) t^{\beta-\alpha-1} e^{-t^p} v^{-\beta-1} (1-v^p/t^p)^{\frac{\beta-\alpha}{p}-1} dv dt R(dx) \\ &= K^{-1} \int_{\mathbb{R}^d} \int_0^\infty \int_v^\infty 1_A(vx) t^{p-1} e^{-t^p} v^{-\beta-1} (t^p - v^p)^{\frac{\beta-\alpha}{p}-1} dt dv R(dx) \\ &= K^{-1} \int_{\mathbb{R}^d} \int_0^\infty 1_A(vx) e^{-v^p} v^{-\beta-1} dv R(dx) \int_0^\infty e^{-s^p} s^{\beta-\alpha-1} ds \\ &= \int_{\mathbb{R}^d} \int_0^\infty 1_A(vx) e^{-v^p} v^{-\beta-1} dv R(dx), \end{aligned}$$

where the second line follows by the substitution $v = ut$ and the fourth by the substitution $s^p = t^p - v^p$. \square

To show a similar result for the parameter p we need some additional notation. For $r \in (0, 1)$, let f_r be the density of the r -stable distribution with

$$\int_0^\infty e^{-tx} f_r(x) dx = e^{-t^r}. \quad (19)$$

Such a density exists by Proposition 1.2.12 in [22]. However, the only case where an explicit formula is known is

$$f_{.5}(s) = (2\sqrt{\pi})^{-1} e^{-1/(4s)} s^{-3/2} 1_{[s>0]} \quad (20)$$

(see Examples 2.13 and 8.11 in [24]). From Theorem 5.4.1 in [27] it follows that if $X \sim f_r$ and $\beta \geq 0$ then

$$E|X|^{-\beta} < \infty. \quad (21)$$

Proposition 3. Fix $\alpha < 2$, $0 < p < q$. If $\mu = TS_\alpha^p(R, b)$ and

$$R'(A) = \int_{\mathbb{R}^d} \int_0^\infty 1_A(s^{-1/q}x) s^{\alpha/q} f_{p/q}(s) ds R(dx) \quad (22)$$

then R' is the Rosiński measure of a q -tempered α -stable distribution and $\mu = TS_\alpha^q(R', b)$. Moreover, μ is a proper p -tempered α -stable distribution if and only if it is a proper q -tempered α -stable distribution.

This implies that for a fixed α the parameters p and R are not jointly identifiable even within the subclass of proper tempered stable distributions.

Proof. First we show that R' is, in fact, the Rosiński measure of a q -tempered α -stable distribution. We have

$$\begin{aligned} \int_{|x| \leq 1} |x|^2 R'(dx) &= \int_{\mathbb{R}^d} |x|^2 \int_{|x|^q}^{\infty} s^{-(2-\alpha)/q} f_{p/q}(s) ds R(dx) \\ &\leq \int_{|x| \leq 1} |x|^2 \int_0^{\infty} s^{-(2-\alpha)/q} f_{p/q}(s) ds R(dx) \\ &\quad + \int_{|x| > 1} |x|^\alpha R(dx) \int_0^{\infty} f_{p/q}(s) ds < \infty. \end{aligned}$$

If $\alpha \neq 0$ and $\beta = \alpha \vee 0$ then

$$\begin{aligned} \int_{|x| > 1} |x|^\beta R'(dx) &= \int_{\mathbb{R}^d} |x|^\beta \int_0^{|x|^q} s^{-(\beta-\alpha)/q} f_{p/q}(s) ds R(dx) \\ &\leq \int_{|x| \leq 1} |x|^2 \int_0^{\infty} s^{-(2-\alpha)/q} f_{p/q}(s) ds R(dx) \\ &\quad + \int_{|x| > 1} |x|^\beta \int_0^{\infty} s^{-(\beta-\alpha)/q} f_{p/q}(s) ds R(dx) < \infty. \end{aligned}$$

If $\alpha = 0$ then

$$\begin{aligned} \int_{|x| > 1} \log |x| R'(dx) &= \int_{\mathbb{R}^d} \int_0^{|x|^q} \log |x| s^{-1/q} f_{p/q}(s) ds R(dx) \\ &\leq \int_{|x| \leq 1} |x|^2 R(dx) \int_0^{\infty} s^{-2/q} f_{p/q}(s) ds \\ &\quad + \int_{|x| > 1} \log |x| R(dx) \int_0^{\infty} f_{p/q}(s) ds \\ &\quad + \int_{|x| > 1} R(dx) \int_0^{\infty} s^{-1/q} f_{p/q}(s) ds < \infty, \end{aligned}$$

where the inequality uses the fact that $\log |x| \leq |x|$ (see 4.1.36 in [1]).

If M' is the Lévy measure of $TS_\alpha^q(R', b)$ then by (10)

$$\begin{aligned} M'(A) &= \int_{\mathbb{R}^d} \int_0^{\infty} \int_0^{\infty} 1_A(s^{-1/q} tx) t^{-1-\alpha} e^{-t^q} dt s^{\alpha/q} f_{p/q}(s) ds R(dx) \\ &= \int_{\mathbb{R}^d} \int_0^{\infty} 1_A(vx) v^{-1-\alpha} \int_0^{\infty} e^{-v^q s} f_{p/q}(s) ds dv R(dx) \\ &= \int_{\mathbb{R}^d} \int_0^{\infty} 1_A(vx) v^{-1-\alpha} e^{-v^p} dv R(dx). \end{aligned}$$

The last part follows from (14) and the fact that

$$\int_{\mathbb{R}^d} |x|^\alpha R'(dx) = \int_{\mathbb{R}^d} |x|^\alpha R(dx) \int_0^{\infty} s^{-\alpha/q} s^{\alpha/q} f_{p/q}(s) ds = \int_{\mathbb{R}^d} |x|^\alpha R(dx).$$

This completes the proof. \square

Propositions 2 and 3 give a constructive proof of the following result, a version of which was shown in [15].

Corollary 12. Fix $\alpha < 2$, $p > 0$, and let $\mu \in TS_\alpha^p$.

1. For any $q \geq p$, $\mu \in TS_\alpha^q$.
2. For any $\beta \leq \alpha$, $\mu \in TS_\beta^p$.

We end this section by characterizing when a p -tempered α -stable distribution is β -stable for some $\beta \in (0, 2)$.

Proposition 4. Fix $\alpha < 2$, $p > 0$, and $\beta \in (0, 2)$. Let μ be a β -stable distribution with spectral measure $\sigma \neq 0$. If $\beta \leq \alpha$ then $\mu \notin TS_\alpha^p$. If $\beta \in (0 \vee \alpha, 2)$ then $\mu = TS_\alpha^p(R, b)$ and

$$R(A) = K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(r\xi) r^{-1-\beta} dr \sigma(d\xi), \quad A \in \mathfrak{B}(\mathbb{R}^d), \quad (23)$$

where $K = \int_0^\infty t^{\beta-\alpha-1} e^{-t^p} dt$.

Note that

$$\int_{\mathbb{R}^d} |x|^\alpha R(dx) = K^{-1} \sigma(\mathbb{S}^{d-1}) \int_0^\infty r^{-(\beta-\alpha)-1} dr = \infty.$$

Thus, by Part 3 of Theorem 6, no stable distributions are proper p -tempered α -stable.

Proof. If $\mu \in TS_\alpha^p$ then its Lévy measure can be written as (4). By uniqueness of the polar decomposition of Lévy measures (see Lemma 2.1 in [4]) the function $q(r, u) = r^{(\alpha-\beta)/p}$. This is not completely monotone when $\beta < \alpha$, and it does not satisfy (5) when $\beta = \alpha$.

Now assume that $\beta > \alpha$ and let R be as in (23). In this case $R(\{0\}) = 0$ and for any $\gamma \in [0, \beta)$

$$\int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\gamma) R(dx) = K^{-1} \sigma(\mathbb{S}^{d-1}) \int_0^\infty (r^{1-\beta} \wedge r^{\gamma-\beta-1}) dr < \infty.$$

Thus, by Theorem 6, R is the Rosiński measure of a p -tempered α -stable distribution. If M is the Lévy measure of $TS_\alpha^p(R, b)$ then

$$\begin{aligned} M(A) &= K^{-1} \int_{\mathbb{S}^{d-1}} \int_0^\infty \int_0^\infty 1_A(rt\xi) t^{-1-\alpha} e^{-t^p} dt r^{-1-\beta} dr \sigma(d\xi) \\ &= K^{-1} \int_0^\infty t^{\beta-\alpha-1} e^{-t^p} dt \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(r\xi) r^{-1-\beta} dr \sigma(d\xi) \\ &= \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_A(r\xi) r^{-1-\beta} dr \sigma(d\xi), \end{aligned}$$

which is the Lévy measure of μ . □

3 Moments

In this section we give necessary and sufficient conditions for the finiteness of moments and exponential moments. We also give explicit formulas for the cumulants when they exist. This is useful, for instance, in parameter estimation by the method of moments. First we introduce some notation. For any $x \in \mathbb{R}^d$ let x_i be the i th component. For simplicity, throughout this section, we will use M to denote the Lévy measure of a p -tempered α -stable distribution.

Let k be a d -dimensional vector of nonnegative integers. Let C_μ be as in (1). Recall that we define the cumulant

$$c_k = (-i)^{\sum k_i} \frac{\partial^{\sum k_i}}{\partial z_d^{k_d} \dots \partial z_1^{k_1}} C_\mu(z) \Big|_{z=0}, \quad (24)$$

when the derivative exists and is continuous in a neighborhood of zero. Cumulants can be uniquely expressed in terms of moments. Let $X \sim \mu$. When $k_i = 1$ and $k_j = 0$ for all $j \neq i$ then $c_k = EX_i$, when $k_i = 2$ and $k_j = 0$ for all $j \neq i$ then $c_k = \text{var}(X_i)$, and when for some $i \neq j$ $k_i = k_j = 1$ and $k_\ell = 0$ for all $\ell \neq i, j$ then $c_k = \text{cov}(X_i, X_j)$. In the statement of the following theorem, we adopt the convention that $0^0 = 1$.

Theorem 13. Fix $\alpha < 2$, $p > 0$ and let $\mu = TS_\alpha^p(R, b)$.

1. If $\alpha \in (0, 2)$ and $q_1, \dots, q_d \geq 0$ with $q := \sum_{j=1}^d q_j < \alpha$ then

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \mu(dx) \leq \int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty. \quad (25)$$

2. If $\alpha \in (0, 2)$ then

$$\int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \iff \int_{|x|>1} |x|^\alpha \log |x| R(dx) < \infty. \quad (26)$$

Additionally, if $q_1, \dots, q_d \geq 0$ with $\sum_{j=1}^d q_j = \alpha$ then

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \mu(dx) < \infty \quad (27)$$

if and only if

$$\int_{|x|>1} \left(\prod_{j=1}^d |x_j|^{q_j} \right) \log |x| R(dx) < \infty. \quad (28)$$

3. If $q > (\alpha \vee 0)$ then

$$\int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty \iff \int_{|x|>1} |x|^q R(dx) < \infty. \quad (29)$$

Additionally, if $q_1, \dots, q_d \geq 0$ with $\sum_{j=1}^d q_j > (\alpha \vee 0)$ then

$$\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{r_j} \right) \mu(dx) < \infty \text{ for all } r_k \in [0, q_k], \ k = 1, \dots, d \quad (30)$$

if and only if

$$\int_{|x|>1} \left(\prod_{j=1}^d |x_j|^{r_j} \right) R(dx) < \infty \text{ for all } r_k \in [0, q_k], \ k = 1, \dots, d. \quad (31)$$

4. Let q_1, \dots, q_d be nonnegative integers and let $q = \sum_{i=1}^d q_i$. Further if $q = \alpha$ assume that (28) holds and if $q > \alpha$ that (31) holds. If $q_i = q = 1$ for some i then

$$c_{(q_1, \dots, q_d)} = b_i + \int_{\mathbb{R}^d} \int_0^\infty x_i \frac{|x|^2}{1 + |x|^2 t^2} t^{2-\alpha} e^{-t} dt R(dx). \quad (32)$$

If $q \geq 2$ then

$$c_{(q_1, \dots, q_d)} = p^{-1} \Gamma\left(\frac{n-\alpha}{p}\right) \int_{\mathbb{R}^d} \left(\prod_{j=1}^d x_j^{q_j} \right) R(dx). \quad (33)$$

For proper 1-tempered α -stable distributions with $\alpha \in (0, 2)$ a somewhat weaker version of Part (iv) above was given in [26].

Proof. By Corollary 25.8 in [24], the condition $\int_{\mathbb{R}^d} |x|^q \mu(dx) < \infty$ is equivalent to the condition $\int_{|x|>1} |x|^q M(dx) < \infty$. Similarly, by Theorem 1 in [23] the condition $\int_{\mathbb{R}^d} \left(\prod_{j=1}^d |x_j|^{r_j} \right) \mu(dx) < \infty$ for all $r_k \in [0, q_k]$, $k = 1, \dots, d$ is equivalent to the condition $\int_{|x|>1} \left(\prod_{j=1}^d |x_j|^{r_j} \right) M(dx) < \infty$ for all $r_k \in [0, q_k]$, $k = 1, \dots, d$.

We will now transfer the integrability conditions from M to R . Let $f_q(x)$ be either $|x|^q$ or $\prod_{j=1}^d |x_j|^{r_j}$ where $\sum_{j=1}^d r_j = q$. By (10)

$$\int_{|x|>1} f_q(x) M(dx) = \int_{\mathbb{R}^d} \int_{|x|^{-1}}^\infty f_q(x) t^{q-1-\alpha} e^{-t^p} dt R(dx).$$

From here (26) and Parts 1 and 2 follow by arguments similar to those in Proposition 2.7 of [19]. The second half of Part 2 essentially follows from arguments similar to those in Proposition 2.7 of [19] as well, but to guarantee that the integral remains finite for all $r_k \in [0, q_k]$ we use (25).

For general infinitely divisible distributions, the form of the cumulants in terms of the Lévy measure is given in Theorems 5.1 and 5.2 of [12]. From this, Part 4 follows by using (10) and simplifying. \square

In the rest of this section we will give conditions for the finiteness of certain exponential moments.

Theorem 14. Fix $\alpha < 2$, $p \in (0, 1]$, and $\theta > 0$. Let $\mu = TS_\alpha^p(R, b)$.

1. If $\alpha \in (0, 2)$ then

$$\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty \iff R(\{|x| > \theta^{-1/p}\}) = 0.$$

2. If $\alpha < 0$ then $\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty$ if and only if

$$R(\{|x| \geq \theta^{-1/p}\}) = 0 \text{ and } \int_{0 < |x|^{-p} - \theta < 1} (|x|^{-p} - \theta)^{\alpha/p} R(dx) < \infty.$$

3. If $\alpha = 0$ then $\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty$ if and only if

$$R(\{|x| \geq \theta^{-1/p}\}) = 0 \text{ and } \int_{0 < |x|^{-p} - \theta < 1} |\log(|x|^{-p} - \theta)| R(dx) < \infty.$$

This implies that unless $R = 0$ it is impossible to have $\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx) < \infty$ for all $\theta > 0$. Note that in Parts 2 and 3 we have the condition, $R(\{|x| \geq \theta^{-1/p}\}) = 0$, whereas in Part 1 we have a similar condition, but with strict inequality. Note also that the set $\{0 < |x|^{-p} - \theta < 1\} = \{(1 + \theta)^{-1/p} < |x| < \theta^{-1/p}\}$. The latter form may be somewhat more appealing, but it loses emphasis on why the integrals may diverge.

Proof. The proof of Part 1 is similar to the proof of Proposition 2.7 in [19]. Now fix $\alpha \leq 0$. By Corollary 25.8 in [24] the finiteness of $\int_{\mathbb{R}^d} e^{\theta|x|^p} \mu(dx)$ is equivalent to the finiteness of $\int_{|x| > 1} e^{\theta|x|^p} M(dx)$. We have

$$\begin{aligned} \int_{|x| > 1} e^{\theta|x|^p} M(dx) &= \int_{\mathbb{R}^d} \int_{|x|^{-1}}^{\infty} e^{(\theta|x|^p - 1)t^p} t^{-\alpha-1} dt R(dx) \\ &\geq \int_{|x|^p \geq \theta^{-1}} \int_{\theta^{1/p}}^{\infty} t^{-\alpha-1} dt R(dx). \end{aligned}$$

This shows the necessity of $R(\{|x| \geq \theta^{-1/p}\}) = 0$ in both Parts 2 and 3. We will henceforth assume that this property holds both when showing necessity and sufficiency. We have

$$\begin{aligned} \int_{|x| > 1} e^{\theta|x|^p} M(dx) &= \int_{|x| < \theta^{-1/p}} \int_{|x|^{-1}}^{\infty} e^{(\theta|x|^p - 1)t^p} t^{-1-\alpha} dt R(dx) \\ &= p^{-1} \int_{0 < |x|^{-p} - \theta} (1 - \theta|x|^p)^{\alpha/p} \int_{|x|^{-p} - \theta}^{\infty} e^{-u} u^{-1-\alpha/p} du R(dx). \end{aligned}$$

This can be divided into two parts

$$\begin{aligned} &p^{-1} \int_{1 \leq |x|^{-p} - \theta} (|x|^{-p} - \theta)^{-|\alpha|/p} |x|^{-|\alpha|} \int_{|x|^{-p} - \theta}^{\infty} e^{-u} u^{-1+|\alpha|/p} du R(dx) \\ &+ p^{-1} \int_{0 < |x|^{-p} - \theta < 1} (|x|^{-p} - \theta)^{-|\alpha|/p} |x|^{-|\alpha|} \int_{|x|^{-p} - \theta}^{\infty} e^{-u} u^{-1+|\alpha|/p} du R(dx) \\ &=: p^{-1}(I_1 + I_2). \end{aligned}$$

Let $C_\theta := \sup_{u>1} e^{-u} u^{-1+|\alpha|/p} (u+\theta)^{(2+|\alpha|)/p+1}$. We have,

$$\begin{aligned} I_1 &\leq \int_{1 \leq |x|^{-p-\theta}} |x|^{-|\alpha|} \int_{|x|^{-p-\theta}}^{\infty} e^{-u} u^{-1+|\alpha|/p} du R(dx) \\ &\leq C_\theta \int_{1 \leq |x|^{-p-\theta}} |x|^{-|\alpha|} \int_{|x|^{-p-\theta}}^{\infty} (u+\theta)^{-(2+|\alpha|)/p-1} du R(dx) \\ &= C_\theta \frac{p}{2-\alpha} \int_{|x| \leq (1+\theta)^{-1/p}} |x|^2 R(dx) < \infty. \end{aligned}$$

Thus finiteness is determined by I_2 .

If $\alpha < 0$ and $0 < |x|^{-p} - \theta < 1$, we have

$$\int_1^{\infty} e^{-u} u^{-1+|\alpha|/p} du \leq \int_{|x|^{-p-\theta}}^{\infty} e^{-u} u^{-1+|\alpha|/p} du \leq \Gamma(|\alpha|/p)$$

and

$$\theta^{|\alpha|/p} \leq |x|^{-|\alpha|} \leq (1+\theta)^{|\alpha|/p}.$$

Thus, when $\alpha < 0$, I_2 is finite if and only if

$$\int_{0 < |x|^{-p-\theta} < 1} (|x|^{-p} - \theta)^{-|\alpha|/p} R(dx) < \infty$$

If $\alpha = 0$ then for $0 < |x|^{-p} - \theta < 1$, we have

$$\int_{|x|^{-p-\theta}}^{\infty} e^{-u} u^{-1} du = \int_1^{\infty} e^{-u} u^{-1} du + \int_{|x|^{-p-\theta}}^1 e^{-u} u^{-1} du,$$

where the first integral is finite. For the second, we have

$$\int_{|x|^{-p-\theta}}^1 e^{-u} u^{-1} du \leq \int_{|x|^{-p-\theta}}^1 u^{-1} du = -\log(|x|^{-p} - \theta),$$

and

$$\int_{|x|^{-p-\theta}}^1 e^{-u} u^{-1} du \geq e^{-1} \int_{|x|^{-p-\theta}}^1 u^{-1} du = -e^{-1} \log(|x|^{-p} - \theta).$$

Thus, when $\alpha = 0$, the finiteness of I_2 is equivalent to the finiteness of

$$-\int_{0 < |x|^{-p-\theta} < 1} \log(|x|^{-p} - \theta) R(dx) = \int_{0 < |x|^{-p-\theta} < 1} |\log(|x|^{-p} - \theta)| R(dx).$$

This completes the proof. \square

Theorem 15. Fix $\alpha < 2$, $p > 0$, and let $\mu = TS_\alpha^p(R, b)$.

1. If $q \in (0, 1]$ with $q < p$ then for any $\theta > 0$

$$\int_{\mathbb{R}^d} e^{\theta|x|^q} \mu(dx) < \infty \quad (34)$$

whenever

$$\int_{|x|>1} e^{A_{p,q}(\theta|x|^q)^{p/(p-q)}} |x|^{-\alpha q/(p-q)} R(dx) < \infty, \quad (35)$$

where $A_{p,q} = (q/p)^{q/(p-q)} (1 - q/p)$.

2. If $R \neq 0$ then $\int_{\mathbb{R}^d} e^{\theta|x|\log|x|} \mu(dx) = \infty$ for every $\theta > 0$.

For the case where $\alpha \in (0, 2)$, $p = 2$, and $q = 1$, a necessary and sufficient condition for (34) is given in [6]. Their method of proof is easily extended to the case when $\alpha < 2$ and $p = 2q$. In this case, the necessary and sufficient condition for (34) is

$$\int_{|x|>1} e^{\theta^2|x|^{2q}/4} |x|^{-q-\alpha} R(dx) < \infty. \quad (36)$$

Proof. We begin with Part 1. Fix $c = \left(\frac{2p}{\theta q}\right)^{1/p}$. By Corollary 25.8 in [24], the problem is equivalent to the finiteness of

$$\begin{aligned} \int_{|x|>1} e^{\theta|x|^q} M(dx) &= \int_{|x|\leq c} \int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) \\ &\quad + \int_{|x|>c} \int_{|x|^{-1}}^{(\theta|x|^q)^{1/(p-q)}} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) \\ &\quad + \int_{|x|>c} \int_{(\theta|x|^q)^{1/(p-q)}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{-1-\alpha} dt R(dx) \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

For the first integral, we have

$$\begin{aligned} I_1 &= \int_{|x|\leq c} \int_{|x|^{-1}}^{\infty} e^{\theta|x|^q t^q - t^p} t^{1-\alpha} t^{-2} dt R(dx) \\ &\leq \int_{|x|\leq c} |x|^2 R(dx) \int_{c^{-1}}^{\infty} e^{\theta c^q t^q - t^p} t^{1-\alpha} dt < \infty. \end{aligned}$$

For the third integral, by the substitution $u = t^{p-q}/(\theta|x|^q)$, we have

$$\begin{aligned} I_3 &= \frac{1}{(p-q)} \int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} \times \\ &\quad \times \int_1^{\infty} e^{-(1-1/u)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du R(dx) \\ &\leq \frac{1}{(p-q)} \int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} R(dx) \times \\ &\quad \times \int_1^{\infty} e^{-(1-1/u)(u\theta c^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du. \end{aligned}$$

Clearly this is finite for $\alpha \in [0, 2)$. We will show that it is, in fact, always finite when

$I_2 < \infty$. To see this, observe that, after the substitution $u = t^{p-q}/(\theta|x|^q)$, we have

$$\begin{aligned} I_2 &= \frac{1}{(p-q)} \int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} \times \\ &\quad \times \int_{|x|^{-p}/\theta}^1 e^{(1/u-1)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du R(dx) \\ &\geq \frac{1}{(p-q)} \int_{|x|>c} (\theta|x|^q)^{-\alpha/(p-q)} R(dx) \int_{c^{-p}/\theta}^1 u^{-1-\alpha/(p-q)} du. \end{aligned} \quad (37)$$

Thus everything is determined by I_2 .

Note that, as a function of u , $(1/u-1)(u\theta|x|^q)^{p/(p-q)}$ is strictly increasing until $u = q/p$, where it attains a maximum and is then decreasing. Thus

$$\begin{aligned} &\int_{|x|^{-p}/\theta}^{q/(2p)} e^{(1/u-1)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du \\ &\leq e^{(2p/q-1)(\theta|x|^q q/(2p))^{p/(p-q)}} (|x|^{p\theta})^{0 \vee [1+\alpha/(p-q)]} \end{aligned} \quad (38)$$

and for some constant $C > 0$

$$\int_{q/(2p)}^1 e^{(1/u-1)(u\theta|x|^q)^{p/(p-q)}} u^{-1-\alpha/(p-q)} du \leq C e^{(p/q-1)[qp^{-1}\theta|x|^q]^{p/(p-q)}}. \quad (39)$$

Note that $(p/q-1)(q/p)^{p/(p-q)} = A_{p,q}$. Now observing that the right side in (39) goes to infinity faster than the right side in (38), and combining this with (37) gives Part 1.

Now to show Part 2. For any $h > 0$, let $T_h = \{|x| > h\}$. Assume that $R \neq 0$. Since $R(\{0\}) = 0$ there exists an $\epsilon > 0$ such that $R(T_\epsilon) > 0$. Thus for any $h > 0$

$$\begin{aligned} M(T_h) &= \int_{\mathbb{R}^d} \int_{h|x|^{-1}}^\infty e^{-t^p} t^{-1-\alpha} dt R(dx) \\ &\geq \int_{|x|>\epsilon} \int_{h\epsilon^{-1}}^\infty e^{-t^p} t^{-1-\alpha} dt R(dx) = R(T_\epsilon) \int_{h\epsilon^{-1}}^\infty e^{-t^p} t^{-1-\alpha} dt > 0. \end{aligned}$$

From here the result follows by Theorem 26.1 in [24]. \square

4 Regular Variation

In this section we give necessary and sufficient conditions for tempered stable distributions to have regularly varying tails. To simplify the notation, we adopt the following convention. For $c \in \mathbb{R}$ and real-valued functions f, g with g strictly positive in some neighborhood of infinity we write $f(t) \sim cg(t)$ as $t \rightarrow \infty$ to mean

$$\lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = c.$$

We now recall what it means for a measure to have regularly varying tails.

Definition 16. Fix $\varrho \geq 0$. Let R be a Borel measure on \mathbb{R}^d such that for some $T > 0$

$$R(\{x \in \mathbb{R}^d : |x| > T\}) < \infty \quad (40)$$

and for all $s > 0$

$$R(\{x \in \mathbb{R}^d : |x| > s\}) > 0. \quad (41)$$

We say that R has **regularly varying tails with index ϱ** if there exists a finite Borel measure $\sigma \neq 0$ on \mathbb{S}^{d-1} such that for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$\lim_{r \downarrow 0} \frac{R(|x| > rt : \frac{x}{|x|} \in D)}{R(|x| > r)} = t^{-\varrho} \frac{\sigma(D)}{\sigma(\mathbb{S}^{d-1})}. \quad (42)$$

When this holds we write $R \in RV_{-\varrho}(\sigma)$.

Clearly a measure $R \in RV_{-\varrho}(\sigma)$ if and only if there exists a slowly varying function ℓ such that for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$

$$R(|x| > t, x/|x| \in D) \sim \sigma(D)t^{-\varrho}\ell(t) \text{ as } t \rightarrow \infty. \quad (43)$$

It is well-known (see e.g. [5]) that if $R \in RV_{-\varrho}(\sigma)$ then

$$\int_{|x| \geq T} |x|^\gamma R(dx) \begin{cases} < \infty & \text{if } \gamma < \varrho \\ = \infty & \text{if } \gamma > \varrho \end{cases}. \quad (44)$$

Let $\mu = TS_\alpha^p(R, b)$. If $\alpha \in (0, 2)$ then Theorem 13 implies that $\int_{\mathbb{R}^d} |x|^\varrho \mu(dx) < \infty$ for all $\varrho \in [0, \alpha)$, and hence, by (44) μ cannot have regularly varying tails with index $\varrho < \alpha$. However, other tail indices are possible. We will now categorize when μ has regularly varying tails.

Theorem 17. Fix $\alpha < 2$, $p > 0$. Let $\mu = TS_\alpha^p(R, b)$ and let M be the Lévy measure of μ . If $\varrho > \alpha \vee 0$ then

$$\mu \in RV_{-\varrho}^\infty(\sigma) \iff M \in RV_{-\varrho}^\infty(\sigma) \iff R \in RV_{-\varrho}^\infty(\sigma). \quad (45)$$

Moreover, if $M \in RV_{-\varrho}^\infty(\sigma)$ then for all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ and $\sigma(D) > 0$

$$\lim_{r \rightarrow \infty} \frac{R(|x| > r, x/|x| \in D)}{M(|x| > r, x/|x| \in D)} = \frac{p}{\Gamma\left(\frac{\varrho - \alpha}{p}\right)}.$$

Before proving the theorem let us state a useful corollary. Recall that for $\gamma \in (0, 2)$ a probability measure μ is in the domain of attraction of a γ -stable distribution with spectral measure $\sigma \neq 0$ if and only if $\mu \in RV_{-\gamma}(\sigma)$. See e.g. [21] or [16] although they make the additional assumption that the limiting stable distribution is full.

Corollary 18. Fix $\alpha < 2$, $p > 0$, and let $\mu = TS_\alpha^p(R, b)$. If $\sigma \neq 0$ is a finite Borel measure on \mathbb{S}^{d-1} and $\gamma \in (0 \vee \alpha, 2)$ then μ is in the domain of attraction of a γ -stable with spectral measure σ if and only if $R \in RV_{-\gamma}(\sigma)$.

In Theorem 17, the relationship between the regular variation of μ and M is well known, see for example [13]. A proof of the fact that $R \in RV_{-\varrho}^\infty(\sigma)$ implies that $M \in RV_{-\varrho}^\infty(\sigma)$ can be accomplished using standard tools. However the other direction requires heavier machinery. For brevity, we use the same approach for both directions.

Let $k : (0, \infty) \mapsto \mathbb{R}$ be a Borel function. The **Mellin transform** of k is defined by

$$\hat{k}(z) = \int_0^\infty u^{z-1} k(1/u) du \quad (46)$$

for all $z \in \mathbb{C}$ for which the integral converges. We will need the following result, which combines Theorems 4.4.2 and 4.9.1 in [7].

Lemma 19. *Let $-\infty < \gamma < \rho < \tau < \infty$, $c \in \mathbb{R}$, and let ℓ be a slowly varying function. Assume that k is a continuous and non-negative function on $(0, \infty)$ such that*

$$\sum_{-\infty < n < \infty} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \leq x \leq e^{n+1}} k(x) < \infty \quad (47)$$

and

$$\hat{k}(z) \neq 0 \quad \text{when } \Re z = \rho. \quad (48)$$

Let U be a monotone, right continuous function on $(0, \infty)$ with

$$\limsup_{r \downarrow 0} \frac{|U(r)|}{r^\gamma} < \infty. \quad (49)$$

Then

$$\int_{(0, \infty)} k(x/t) dU(t) \sim c \rho \hat{k}(\rho) x^\rho \ell(x) \quad \text{as } x \rightarrow \infty \quad (50)$$

if and only if

$$U(x) \sim c x^\rho \ell(x) \quad \text{as } x \rightarrow \infty. \quad (51)$$

Let $\mu = TS_\alpha^p(R, b)$, let M be the Lévy measure of μ , and assume that $\sigma \neq 0$ is a finite Borel measure on \mathbb{S}^{d-1} . For all $D \in \mathfrak{B}(\mathbb{S}^{d-1})$ with $\sigma(\partial D) = 0$ define for $r > 0$

$$M_D(r) = M(|x| > r, x/|x| \in D)$$

and

$$R_D(r) = R(|x| > r, x/|x| \in D).$$

Note that for any integrable function $f : \mathbb{R} \rightarrow \mathbb{R}$

$$\int_{x/|x| \in D} f(|x|) R(dx) = - \int_{(0, \infty)} f(x) dR_D(x). \quad (52)$$

Lemma 20. *If $\varrho > \alpha \vee 0$ and $\ell \in RV_0^\infty$ then*

$$M_D(r) \sim \sigma(D) p^{-1} \Gamma\left(\frac{\varrho - \alpha}{p}\right) r^{-\varrho} \ell(r) \quad \text{as } r \rightarrow \infty$$

if and only if

$$R_D(r) \sim \sigma(D) r^{-\varrho} \ell(r) \quad \text{as } r \rightarrow \infty.$$

Proof. For simplicity, let $\beta = \alpha \vee 0$. Note that by (10) and (52)

$$\begin{aligned} M_D(r) &= \int_{x/|x| \in D} \int_{r|x|^{-1}}^{\infty} t^{-1-\alpha} e^{-t^p} dt R(dx) \\ &= - \int_{(0, \infty)} \int_{r/x}^{\infty} t^{-1-\alpha} e^{-t^p} dt dR_D(x) \\ &= - \int_{(0, \infty)} k(r/x) R_D(dx), \end{aligned}$$

where

$$k(s) = \int_s^{\infty} t^{-\alpha-1} e^{-t^p} dt = p^{-1} \int_{s^p}^{\infty} t^{-\alpha/p-1} e^{-t} dt.$$

For $\Re z < -\beta$

$$\begin{aligned} \hat{k}(z) &= \int_0^{\infty} u^{z-1} k(1/u) du = \int_0^{\infty} u^{z-1} \int_{1/u}^{\infty} t^{-1-\alpha} e^{-t^p} dt du \\ &= \int_0^{\infty} u^{z+\alpha-1} \int_1^{\infty} t^{-1-\alpha} e^{-(t/u)^p} dt du \\ &= \int_0^{\infty} u^{-z-\alpha-1} e^{-u^p} du \int_1^{\infty} t^{z-1} dt = -\frac{1}{pz} \Gamma\left(\frac{-z-\alpha}{p}\right). \end{aligned}$$

Thus, since $-\varrho < -\beta$

$$k(-\varrho) = \frac{1}{p\varrho} \Gamma\left(\frac{\varrho-\alpha}{p}\right).$$

From here, the result will follow by ma 19. We just need to verify that the assumptions hold.

It is easy to see that k is a continuous, non-negative function on $(0, \infty)$ and that $\hat{k}(z)$ has no zeros. Fix $\tau \in (-\varrho, -\beta)$, $\gamma < -(\varrho \vee 2)$, and let $C = \sup_{t \geq 1} t^{-\alpha/p-1} e^{-t/2}$. Note that $\gamma < \tau < 0$. We have

$$\begin{aligned} p \sum_{n=0}^{\infty} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \leq x \leq e^{n+1}} k(x) \\ &= \sum_{n=0}^{\infty} e^{|\gamma|n} \int_{e^{np}}^{\infty} t^{-\alpha/p-1} e^{-t/2} e^{-t/2} dt \\ &\leq C \sum_{n=0}^{\infty} e^{|\gamma|n} \int_{e^{np}}^{\infty} e^{-t/2} dt = 2C \sum_{n=0}^{\infty} e^{|\gamma|n} e^{-e^{np}/2} < \infty \end{aligned}$$

and

$$\begin{aligned} p \sum_{-\infty < n \leq -1} \max\{e^{-\gamma n}, e^{-\tau n}\} \sup_{e^n \leq x \leq e^{n+1}} k(x) \\ &= \sum_{n=1}^{\infty} e^{-|\tau|n} \int_{e^{-np}}^{\infty} t^{-\alpha/p-1} e^{-t} dt. \end{aligned}$$

When $\alpha < 0$ this is bounded by

$$\int_0^\infty t^{-\alpha/p-1} e^{-t} dt \sum_{n=1}^\infty e^{-|\tau|n} < \infty.$$

When $\alpha = 0$ it is bounded by

$$\sum_{n=1}^\infty e^{-|\tau|n} \left(\int_{e^{-np}}^1 t^{-1} dt + \int_1^\infty e^{-t} dt \right) = \sum_{n=1}^\infty e^{-|\tau|n} (np + e^{-1}) < \infty.$$

When $\alpha \in (0, 2)$ it is bounded by

$$\sum_{n=1}^\infty e^{-|\tau|n} \int_{e^{-np}}^\infty t^{-\alpha/p-1} dt = \frac{p}{\alpha} \sum_{n=1}^\infty e^{-|\tau|n+\alpha n} = \frac{p}{\alpha} \sum_{n=1}^\infty e^{-(|\tau|-\alpha)n} < \infty.$$

Recall that $\gamma < -2$. Note that $-R_D(r)$ is a right continuous, monotonely increasing function on $(0, \infty)$ with

$$\begin{aligned} \limsup_{r \downarrow 0} \frac{|-R_D(r)|}{r^\gamma} &\leq \limsup_{r \downarrow 0} r^2 \int_{|x|>r} R(dx) \\ &\leq \limsup_{r \downarrow 0} \int_{1>|x|>r} |x|^2 R(dr) + \limsup_{r \downarrow 0} r^2 R(|x| \geq 1) \\ &\leq \int_{|x|<1} |x|^2 R(dr) < \infty. \end{aligned}$$

This completes the proof of Lemma 20. \square

The proof of Theorem 17 follows immediately from Lemma 20 and (43).

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