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INJECTIVES IN FINITELY GENERATED UNIVERSAL HORN CLASSES

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Abstract. Let **K** be a finite set of finite structures. We give a syntactic characterization of the property: every element of **K** is injective in **ISP(K)**. We use this result to establish that $\mathscr{A}$ is injective in **ISP**($\mathscr{A}$) for every two-element algebra $\mathscr{A}$.

§0. Introduction. Let **K** be a finite set of finite structures for a first-order language. In this paper we give a syntactic characterization (Theorem 4) of the property that each member of **K** is injective in **ISP(K)**, the universal Horn class generated by **K**. We then show that $\mathbf{K} = \{\mathscr{A}\}$ has this property for every two-element algebra $\mathscr{A}$.

This paper was motivated by the following question: for which two-element algebras $\mathscr{A}$ does **ISP**($\mathscr{A}$) have the amalgamation property? The property stated above is stronger than the amalgamation property, so the answer is: for every two-element algebra.

Model companions appear in Lemma 1. For their definition and elementary properties see [8], but note that we replace a theory by its class of models. Theorem 9 rests on E. Post's classification [9] of all two-valued clones of operations, a summary of which appears in [7].

The results of §1 are joint work of the authors, the individual contributions being inextricably combined but of equal magnitude. §2 is due to R. Willard, and the details of §3 were also worked out by R. Willard based on an explanation of the results in [1] given by M. H. Albert.

§1. The syntactic characterization. Let **M** be a class of structures for some first-order language. The classes **I(M)**, **S(M)**, **P(M)** and $\mathbf{P}_{\mathrm{fin}}(\mathbf{M})$ are the closures of **M** under isomorphism, substructures, products, and finite products respectively. $\mathbf{M}_{\mathrm{fin}}$ is the class of all finite members of **M**. Arrows and hooked arrows between members of **M** denote homomorphisms and embeddings respectively. A member $\mathscr{D}$ of **M** is *injective in* **M** if every diagram 1(a) in **M** can be completed to a commuting diagram 1(b).

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$$\begin{tikzcd} \mathscr{A} \arrow[r, hook] \arrow[d] & \mathscr{B} \\ \mathscr{D} & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, hook] \arrow[d] & \mathscr{B} \arrow[dl] \\ \mathscr{D} & \end{tikzcd}$$

1(a) 1(b)

$\mathbf{M}^{\mathrm{inj}}$ denotes the class of all members of $\mathbf{M}$ which are injective in $\mathbf{M}$. We say that $\mathbf{M}$ has *enough injectives* (EI) if $\mathbf{M} \subseteq \mathbf{IS}(\mathbf{M}^{\mathrm{inj}})$, and that $\mathbf{M}$ has the *transferability property* (TP) if every diagram 2(a) in $\mathbf{M}$ can be completed in $\mathbf{M}$ to a commuting diagram 2(b).

$$\begin{tikzcd} \mathscr{A} \arrow[r, hook] \arrow[d] & \mathscr{B} \\ \mathscr{C} & \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, hook] \arrow[d] & \mathscr{B} \arrow[d] \\ \mathscr{C} \arrow[r, hook] & \mathscr{D} \end{tikzcd}$$

2(a) 2(b)

REMARK. EI implies TP, and $\mathbf{P}(\mathbf{M}^{\mathrm{inj}}) \cap \mathbf{M} = \mathbf{M}^{\mathrm{inj}}$.

A first-order formula is *existential-positive* ($\exists^+$) if it is both existential and positive; likewise *open-positive* (O^+). A &at formula is a conjunction of atomic formulas; an $\exists$&at formula is one of the form $\exists \vec{y}\, \phi(\vec{y})$ where ϕ is &at. We say that $\mathbf{M}$ has the property $\exists^+ \equiv O^+$ if every $\exists^+$ formula (in the language of $\mathbf{M}$) is equivalent modulo $\mathbf{M}$ to an O^+ formula; likewise $\exists$&at $\equiv$ &at.

REMARK. $\exists$&at $\equiv$ &at implies $\exists^+ \equiv O^+$; if $\mathbf{P}_{\mathrm{fin}}(\mathbf{M}) \subseteq \mathbf{IS}(\mathbf{M})$, then $\mathbf{M}$ has $\exists^+ \equiv O^+$ iff it has $\exists$&at $\equiv$ &at.

A member $\mathscr{A}$ of $\mathbf{M}$ is *algebraically closed in* $\mathbf{M}$ if for every $\mathscr{B} \in \mathbf{M}$, every embedding $\alpha: \mathscr{A} \hookrightarrow \mathscr{B}$, every $\exists^+$ formula $\phi(\vec{x})$[3] and all $\vec{a} \in \mathscr{A}$, $\mathscr{B} \models \phi(\alpha\vec{a})$ implies $\mathscr{A} \models \phi(\vec{a})$. The class of all members of $\mathbf{M}$ which are algebraically closed in $\mathbf{M}$ is denoted $\mathbf{M}^{\mathrm{ac}}$.

We first establish some connections between $\mathbf{M}^{\mathrm{inj}}$, $\mathbf{M}^{\mathrm{ac}}$, $\exists^+ \equiv O^+$ and TP.

LEMMA 1. *Let* $\mathbf{M}$ *be a universal class.*

a) *If there is a class* $\mathbf{N}$ *such that* $\mathbf{M} = \mathbf{IS}(\mathbf{N})$ *and* $\mathbf{N}$ *has* $\exists^+ \equiv O^+$, *then* $\mathbf{M}$ *has TP.*

b) *The converse is true if* $\mathbf{M}$ *has a model companion.*

PROOF. a) This is a consequence of Theorem 2.1(c) of [2]; however, we prove it directly. Suppose that we have the following diagram in $\mathbf{M}$:

$$\begin{tikzcd} \mathscr{A} \arrow[r, hook, "\alpha"] \arrow[d, "h"'] & \mathscr{B} \\ \mathscr{C} & \end{tikzcd}$$

Choose $\beta: \mathscr{B} \hookrightarrow \mathscr{B}' \in \mathbf{N}$ and $\gamma: \mathscr{C} \hookrightarrow \mathscr{C}' \in \mathbf{N}$. Suppose that $\phi(\vec{x}, \vec{y})$ is an O^+ formula, $\vec{a} \in \mathscr{A}$, $\vec{b} \in \mathscr{B}$ and $\mathscr{B} \models \phi(\alpha\vec{a}, \vec{b})$. Pick an O^+ formula $\psi(\vec{x})$ such that $\mathbf{N} \models \exists \vec{y}\, \phi \leftrightarrow \psi$.

[3] In order to avoid annoying exceptions, we reserve the symbol $\vec{x}$ (but not $\vec{y}$) for *nonempty* tuples of variables, i.e. n-tuples for $n \geq 1$. Otherwise, our notation is standard.

Then $\mathscr{B}' \models \exists\vec{y}\,\phi(\beta\alpha\vec{a}, \vec{y})$, so $\mathscr{B}' \models \psi(\beta\alpha\vec{a})$, $\mathscr{A} \models \psi(\vec{a})$ and $\mathscr{C}' \models \psi(\gamma h\vec{a})$; hence $\mathscr{C}' \models \exists\vec{y}\,\phi(\gamma h\vec{a}, \vec{y})$. It now follows by compactness that there is an elementary extension $\mathscr{D}$ of $\mathscr{C}'$ and a homomorphism $g: \mathscr{B} \to \mathscr{D}$ such that $g \circ \alpha = (\mathrm{incl} \circ \gamma) \circ h$, as desired.

b) Let $\mathbf{N}$ be the model companion of $\mathbf{M}$. All we need to know is that $\mathbf{N}$ is elementary, $\mathbf{N} \subseteq \mathbf{M}^{\mathrm{ac}}$, and $\mathbf{M} = \mathbf{IS}(\mathbf{N})$. Let $\phi(\vec{x})$ be an $\exists^+$ formula. If $\mathbf{N} \models \neg\phi(\vec{x})$ then clearly $\mathbf{N} \models \phi(\vec{x}) \leftrightarrow \psi(\vec{x}, \vec{y})$, where $\psi(\vec{x}, \vec{y})$ is the matrix of $\phi(\vec{x})$. Otherwise we use the following variation of Lemma 3.2.1 in [6]: if $\mathbf{N}$ is elementary and $\phi(\vec{x})$ is a formula such that $\mathbf{N} \not\models \neg\phi(\vec{x})$, then ϕ is equivalent modulo $\mathbf{N}$ to an O^+ formula iff for all $\mathscr{A}$, $\mathscr{B} \in \mathbf{N}$, all $\vec{a} \in \mathscr{A}$, and every homomorphism h from $\langle\vec{a}\rangle$ (= the substructure of $\mathscr{A}$ generated by $\vec{a}$) to $\mathscr{B}$, $\mathscr{A} \models \phi(\vec{a})$ implies $\mathscr{B} \models \phi(h\vec{a})$.

Suppose that in our case we have the diagram below, with $\mathscr{A}$, $\mathscr{B} \in \mathbf{N}$:

$$\begin{array}{ccc} \langle\vec{a}\rangle & \xhookrightarrow{\mathrm{incl}} & \mathscr{A} \models \phi(\vec{a}) \\ {\scriptstyle h}\downarrow & & \\ \mathscr{B} & & \end{array}$$

Since $\mathbf{M}$ has TP there are $\mathscr{C} \in \mathbf{M}$, $\alpha: \mathscr{B} \hookrightarrow \mathscr{C}$, and $g: \mathscr{A} \to \mathscr{C}$ such that $g \circ \mathrm{incl} = \alpha \circ h$. Then $\mathscr{C} \models \phi(g\vec{a})$, i.e., $\mathscr{C} \models \phi(\alpha h\vec{a})$ and hence $\mathscr{B} \models \phi(h\vec{a})$ as desired, since ϕ is $\exists^+$ and $\mathscr{B} \in \mathbf{M}^{\mathrm{ac}}$.

LEMMA 2. *Let* $\mathbf{M}$ *be a universal class and let* $\mathbf{N}$ *be a class such that* $\mathbf{M} = \mathbf{IS}(\mathbf{N})$ *and* $\mathbf{N}$ *has* $\exists^+ \equiv O^+$. *Then*

a) $\mathbf{N} \subseteq \mathbf{M}^{\mathrm{ac}}$, *and*

b) $\mathbf{M}^{\mathrm{ac}}$ *has* $\exists^+ \equiv O^+$.

PROOF. a) Let $\mathscr{A} \in \mathbf{N}$. Given $\alpha: \mathscr{A} \hookrightarrow \mathscr{B} \in \mathbf{M}$, find $\beta: \mathscr{B} \hookrightarrow \mathscr{C} \in \mathbf{N}$. If $\phi(\vec{x})$ is $\exists^+$, $\vec{a} \in \mathscr{A}$ and $\mathscr{B} \models \phi(\alpha\vec{a})$, then $\mathscr{C} \models \phi(\beta\alpha\vec{a})$. Clearly $\mathbf{N}^{\mathrm{ac}} = \mathbf{N}$ since $\mathbf{N}$ has $\exists^+ \equiv O^+$; hence $\mathscr{A} \models \phi(\vec{a})$.

b) Let φ, ψ be $\exists^+$, O^+ formulas respectively such that $\mathbf{N} \models \phi \leftrightarrow \psi$. An argument like the previous one yields $\mathbf{M}^{\mathrm{ac}} \models \phi \leftrightarrow \psi$.

LEMMA 3. *Let* $\mathbf{M}$ *be a class of structures.*

a) $\mathbf{M}^{\mathrm{inj}} \subseteq \mathbf{M}^{\mathrm{ac}}$.

b) $(\mathbf{M}^{\mathrm{ac}})_{\mathrm{fin}} \subseteq \mathbf{M}^{\mathrm{inj}}$ *if* $\mathbf{M}$ *has TP.*

PROOF. a) Let $\mathscr{A} \in \mathbf{M}^{\mathrm{inj}}$. Given $\alpha: \mathscr{A} \hookrightarrow \mathscr{B} \in \mathbf{M}$, use the injectivity of $\mathscr{A}$ in the diagram

$$\begin{array}{ccc} \mathscr{A} & \xhookrightarrow{\alpha} & \mathscr{B} \\ {\scriptstyle 1}\downarrow & & \\ \mathscr{A} & & \end{array}$$

b) Suppose $\mathscr{C} \in (\mathbf{M}^{\mathrm{ac}})_{\mathrm{fin}}$ and we have the following diagram in $\mathbf{M}$:

$$\begin{array}{ccc} \mathscr{A} & \xhookrightarrow{\alpha} & \mathscr{B} \\ {\scriptstyle h}\downarrow & & \\ \mathscr{C} & & \end{array}$$

As in the proof of Lemma 1(b), for every $\exists^+$ formula $\phi(\vec{x})$ and all $\vec{a} \in \mathscr{A}$, $\mathscr{B} \models \phi(\alpha\vec{a})$ implies $\mathscr{C} \models \phi(h\vec{a})$. As in the proof of Lemma 1(a), it follows that there is an elementary extension $\mathscr{D}$ of $\mathscr{C}$ and a homomorphism $g: \mathscr{B} \to \mathscr{D}$ such that $g \circ \alpha = \text{incl} \circ h$. But $\mathscr{C}$ is finite, so $\mathscr{D} = \mathscr{C}$ as desired.

We now establish the main result of this section.

THEOREM 4. *Let* $\mathbf{K}$ *be a finite set of finite structures. Then* $\mathbf{K} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$ *iff* $\mathbf{K}$ *has* $\exists\&\text{at} \equiv \&\text{at}$.

PROOF. ($\Rightarrow$). Suppose that $\mathbf{K} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$. Then $\mathbf{P}(\mathbf{K}) \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$ by the first remark of this section, so $\mathbf{ISP}(\mathbf{K})$ has EI and hence TP. $\mathbf{ISP}(\mathbf{K})$ has a model companion [5]; thus by Lemmas 1(b) and 2(b), $\mathbf{ISP}(\mathbf{K})^{\text{ac}}$ has $\exists^+ \equiv O^+$ and hence $\exists\&\text{at} \equiv \&\text{at}$ (see the second remark). It remains to note that $\mathbf{K} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{ac}}$ by Lemma 3(a) and the hypothesis.

($\Leftarrow$). Suppose that $\mathbf{K}$ has $\exists\&\text{at} \equiv \&\text{at}$. Then $\mathbf{P}(\mathbf{K})$ does likewise (since, if ϕ and ψ are $\exists\&\text{at}$, then $\forall\vec{x}(\phi(\vec{x}) \leftrightarrow \psi(\vec{x}))$ is equivalent to a Horn sentence and so is preserved under products). Thus $\mathbf{ISP}(\mathbf{K})$ has TP by Lemma 1(a), $\mathbf{K} \subseteq (\mathbf{ISP}(\mathbf{K})^{\text{ac}})_{\text{fin}}$ by Lemma 2(a), and $(\mathbf{ISP}(\mathbf{K})^{\text{ac}})_{\text{fin}} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$ by Lemma 3(b).

§2. A transfer theorem. In this section we reformulate Theorem 4 in the style of clones and use this reformulation to prove a transfer theorem (Corollary 5).

Let $\mathbf{K}$ be a finite set of finite structures for a first order language. K denotes the set of universes of the members of $\mathbf{K}$. We say that $\mathbf{K}$ is *regular* if no two members share a common universe. We wish to compare regular sets $\mathbf{K}$ and $\mathbf{K}'$, not necessarily for the same language, such that $K = K'$. The insistence on distinct universes is simply to ensure that there is, in this case, a natural 1-1 correspondence between the structures of $\mathbf{K}$ and those of $\mathbf{K}'$.

Let K be any finite set of nonempty finite sets. An *n-ary operation* (*relation*) on K is a map with domain K which assigns to each $A \in K$ an n-ary operation (relation) on A. We use $\mathscr{O}_K$ ($\mathscr{R}_K$) to denote the set of all n-ary operations on K, $n \geq 0$ (all n-ary relations on K, $n \geq 1$). We distinguish an operation f from its graph (which is a relation). $\mathscr{L}_K$ is the (disjoint) union of $\mathscr{O}_K$ and $\mathscr{R}_K$.

Let $\mathbf{K}$ be a regular set of structures for some language. $L_{\mathbf{K}}$ is the set of operations and relations on K defined by the symbols in the language; thus $L_{\mathbf{K}} \subseteq \mathscr{L}_K$. Conversely, any $L \subseteq \mathscr{L}_K$ can be considered as a language (in the obvious way) which has a natural interpretation in K. The regular set of L-structures so obtained is denoted K_L.

Fix K. For $f \in \mathscr{O}_K$, graph(f) denotes the graph of f. If $\mathbf{c}$ is a constant (nullary operation) on K, then c denotes the unary operation on K defined by $c(x) = \mathbf{c}$. Define Su$(\mathscr{L}_K)$ to be the set of subsets of $\mathscr{L}_K$. We define the maps C, $C_{\&\text{at}}$, and $C_{\exists\&\text{at}}$ from Su$(\mathscr{L}_K)$ to Su$(\mathscr{L}_K)$ as follows. For $L \subseteq \mathscr{L}_K$, $f \in \mathscr{O}_K$, $p \in \mathscr{R}_K$, and constant $\mathbf{c} \in \mathscr{O}_K$:

$p \in C_{\&\text{at}}(L)$ iff p is definable in K_L by a &at formula;

$f \in C_{\&\text{at}}(L)$ iff graph$(f) \in C_{\&\text{at}}(L)$;

$C_{\exists\&\text{at}}$ is defined similarly;

$p \in C(L)$ iff $p \in C_{\&\text{at}}(L)$;

$f \in C(L)$ iff f is definable in K_L by an L-term, provided that the arity of f is not zero;

$\mathbf{c} \in C(L)$ iff $c(x) \in C(L)$.

It is straightforward to show that these maps are monotone increasing, $L \subseteq C(L) \subseteq C_{\&\text{at}}(L) \subseteq C_{\exists\&\text{at}}(L)$ for all $L \subseteq \mathscr{L}_K$, $C_{\exists\&\text{at}}$ is idempotent and $C_{\&\text{at}}C = C_{\&\text{at}}$.

Theorem 4 now has the following reformulation: for any regular $\mathbf{K}$, $\mathbf{K} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$ iff $C_{\exists\&\text{at}}(L_{\mathbf{K}}) = C_{\&\text{at}}(L_{\mathbf{K}})$. From this we deduce

COROLLARY 5. *Let* $\mathbf{K}$ *and* $\mathbf{K}'$ *be regular sets of structures (not necessarily for the same language) such that* $K = K'$. *Suppose that* $L_{\mathbf{K}} \subseteq C(L_{\mathbf{K}'})$ *and* $L_{\mathbf{K}'} \subseteq C_{\exists\&\text{at}}(L_{\mathbf{K}})$. *Then* $\mathbf{K} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$ *implies* $\mathbf{K}' \subseteq \mathbf{ISP}(\mathbf{K}')^{\text{inj}}$.

PROOF. Let $L = L_{\mathbf{K}}$ and $L' = L_{\mathbf{K}'}$. By Theorem 4 we have $C_{\exists\&\text{at}}(L) = C_{\&\text{at}}(L)$, and we must show that $C_{\exists\&\text{at}}(L') \subseteq C_{\&\text{at}}(L')$. First, monotonicity yields $C_{\exists\&\text{at}}(L') \subseteq C_{\exists\&\text{at}}(C_{\exists\&\text{at}}(L))$. By idempotence and the sentence above, this last equals $C_{\&\text{at}}(L)$. Finally $C_{\&\text{at}}(L) \subseteq C_{\&\text{at}}(C(L')) = C_{\&\text{at}}(L')$.

REMARK. It is clear from the proof that the condition $L_{\mathbf{K}} \subseteq C(L_{\mathbf{K}'})$ can be replaced by the more general (but less wieldy) $C_{\&\text{at}}(L_{\mathbf{K}}) \subseteq C_{\&\text{at}}(L_{\mathbf{K}'})$.

We close this section by stating a trivial transfer result.

LEMMA 6. *Suppose that* $\mathbf{K}$ *is a regular set of structures,* $\mathbf{c}$ *is a constant on* K, *and* $\mathbf{K_c}$ *is the set of expansions of the members of* $\mathbf{K}$ *to include* $\mathbf{c}$.

a) *If* $\mathbf{K} \subseteq \mathbf{ISP}(\mathbf{K})^{\text{inj}}$ *then* $\mathbf{K_c} \subseteq \mathbf{ISP}(\mathbf{K_c})^{\text{inj}}$.

b) *The converse is true if the automorphism group of each member of* $\mathbf{K}$ *is transitive.*

§3. Application to two-element algebras. In this section we prove the claim that $\mathscr{A} \in \mathbf{ISP}(\mathscr{A})^{\text{inj}}$ for every two-element algebra $\mathscr{A}$. Our method is to use Corollary 5 and Lemma 6 to reduce the case of an arbitrary two-element algebra to cases for which the claim is already known to be true. We start with Post's classification of the clones of operations on the set $\mathbf{2} = \{0, 1\}$ as presented in [7].

We write $\mathscr{O}_2$ for $\mathscr{O}_{\{2\}}$, etc.; $\vee$, $\wedge$, $\neg$, $\mathbf{0}$, $\mathbf{1}$, are the Boolean operations on $\mathbf{2}$. Define $x \to y := (\neg x) \vee y$, $(x, y, z) := x \wedge (y \vee z)$, $d_3(x, y, z) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, and $\text{RC}(x, y, z) := d_3(x, y, \neg z)$. The group operation on $\mathbf{2}$ with identity 0 is $+$. For L, $L' \subseteq \mathscr{O}_2$ we write $L \equiv L'$ if L and L' generate the same clone; that is, $C(L) \cap \mathscr{O}_2 = C(L') \cap \mathscr{O}_2$.

Lemma 6(a) has a number of obvious applications including, for example, $L_1 = \{+, \neg\} \equiv \{+, \mathbf{1}\}$ reducing to $\{+\} = L_3$ and $C_3 = \{+, \wedge\} \equiv \{\vee, \wedge, \text{RC}, \mathbf{0}\}$ reducing to $\{\vee, \wedge, \text{RC}\} \equiv C_4$. Lemma 6(b) can be applied to subsets of $\{d_3, x + y + z, \neg\}$; for example [1], $D_2 = \{d_3\}$ reduces to $\{d_3, \mathbf{0}\} \equiv \{(x, y, z), d_3, \mathbf{0}\} = F_7^3$ and $D_1 = \{d_3, x + y + z\}$ reduces to $\{d_3, x + y + z, \mathbf{0}\} \equiv \{+, \wedge\} = C_3$. Finally, [1] shows that $\{\wedge, \vee\} \subseteq C_{\&\text{at}}(\{\to\}) \cap C_{\&\text{at}}(\{(x, y, z)\})$ and $\text{RC} \in C_{\&\text{at}}(\{\wedge, \vee\})$, from which we deduce

LEMMA 7. *For any* $L \subseteq C(\{\vee, \wedge, \text{RC}\})$, $L \cup \{\to\}$ *reduces to* $\{\to\} = F_4$ *and* $L \cup \{(x, y, z)\}$ *reduces to* $\{(x, y, z)\} = F_6$.

PROOF. Consider $L \cup \{\to\}$. We establish the hypothesis of Corollary 5; that is, $\{\to\} \subseteq C(L \cup \{\to\})$ and $L \cup \{\to\} \subseteq C_{\exists\&\text{at}}(\{\to\})$. It suffices to show that $L \subseteq C_{\exists\&\text{at}}(\{\to\})$, and indeed

$$\begin{aligned} L &\subseteq C(\{\vee, \wedge, \text{RC}\}) \subseteq C(C_{\&\text{at}}(\{\vee, \wedge\})) \subseteq C(C_{\&\text{at}}(C_{\&\text{at}}(\{\to\}))) \\ &\subseteq (C_{\exists\&\text{at}})^3(\{\to\}) = C_{\exists\&\text{at}}(\{\to\}). \end{aligned}$$

The preceding reductions allow us to reduce an arbitrary clone of $\mathscr{O}_2$ to one of the following (or a dual): $\varnothing$, $\{\neg\}$, $\{+\}$, $\{\vee\}$, $\{(x, y, z)\}$, $\{\to\}$.

LEMMA 8. *$\{(x,y,z)\}$ and $\{\rightarrow\}$ reduce to $\{\vee, \wedge\}$.*

PROOF. It suffices to reduce $\{(x,y,z)\}$ to $\{\vee, \wedge\}$ and $\{\rightarrow\}$ to $\{\vee, \wedge, \mathbf{1}\}$. We use the remark following Corollary 5. Thus in the first case we must show that $C_{\&\mathrm{at}}(\{\vee, \wedge\}) \subseteq C_{\&\mathrm{at}}(\{(x,y,z)\})$ and $\{(x,y,z)\} \subseteq C_{\exists\&\mathrm{at}}(\{\vee, \wedge\})$. The latter inclusion is immediate. Concerning the former, note that every atomic $\{\vee, \wedge\}$-formula is equivalent (in $\mathbf{2}$) to a conjunction of formulas of the form

$$\bigwedge_i x_i \leq \bigvee_j \bigwedge_k y_{jk}.$$

Since $\wedge$ is an $\{(x,y,z)\}$-term, it suffices to show that each $x \leq y_1 \vee \cdots \vee y_n$ is definable (in $\mathbf{2}$) by a &at $\{(x,y,z)\}$-formula. Indeed for $n \geq 2$

$$x \leq y_1 \vee \cdots \vee y_n \quad \text{iff} \quad (x, y_1, (x, y_2, (\cdots (x, y_{n-1}, y_n) \cdots))) = x.$$

The proof for $\{\rightarrow\}$ is similar: $\vee$ and $\mathbf{1}$ are $\{\rightarrow\}$-terms, and

$$x_1 \wedge \cdots \wedge x_n \leq y \quad \text{iff} \quad (x_1 \rightarrow y) \vee \cdots \vee (x_n \rightarrow y) = \mathbf{1}.$$

We can now prove

THEOREM 9. *$\mathscr{A} \in \mathbf{ISP}(\mathscr{A})^{\mathrm{inj}}$ for every two-element algebra $\mathscr{A}$.*

PROOF. $\mathbf{ISP}(\langle \mathbf{2}, \{\vee, \wedge\}\rangle)$ is the class of distributive lattices; it is known that the two-element lattice is injective there [3, p. 113]. The situation is the same for $\mathbf{ISP}(\langle \mathbf{2}, \{\vee\}\rangle)$, the class of semilattices [4]. $\mathbf{ISP}(\langle \mathbf{2}, \{+\}\rangle)$ is the class of vector spaces over the two-element field, and hence *every* member is injective by elementary linear algebra. The claim can be checked directly for $\mathbf{ISP}(\langle \mathbf{2}, \{\neg\}\rangle)$ and for $\mathbf{ISP}(\langle \mathbf{2}, \varnothing\rangle)$. All other cases reduce to these by the previous results of this section.

We conclude this paper with an application to varieties generated by two-element algebras.

COROLLARY 10. *Every variety generated by a two-element algebra has EI.*

PROOF. Let $\mathscr{A}$ be a two-element algebra and $\mathbf{V}(\mathscr{A})$ the variety it generates. Since $\mathbf{V}(\mathscr{A})$ has, up to isomorphism, only finitely many subdirectly irreducible members [10], it suffices, by Birkhoff's theorem and the first remark of §1, to show that every maximal subdirectly irreducible member of $\mathbf{V}(\mathscr{A})$ is injective in $\mathbf{V}(\mathscr{A})$.

If $\mathbf{V}(\mathscr{A})$ is simply $\mathbf{ISP}(\mathscr{A})$ together with trivial algebras then the claim follows by Theorem 9. It is known (e.g. [10]) that there are only a handful of exceptions to this situation: those $\mathscr{A}$ whose clones are generated by one of the following sets: $\{\mathbf{0}, \mathbf{1}\}$, $\{\neg\}$, $\{\neg, \mathbf{0}\}$, $\{+, \mathbf{1}\}$, and $\{x + y + z, \neg\}$. The claim can be verified directly in each of these exceptional cases.

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