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Efficient, incremental coverage of space with a continuous curve

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SUMMARY

This paper is concerned with algorithmic techniques for the incremental generation of continuous curves that can efficiently cover an abstract surface. We introduce the notion of low-discrepancy curves as an extension of the notion of low-discrepancy sequences—such that sufficiently smooth curves with low-discrepancy properties can be defined and generated. We then devise a procedure for lifting these curves, that efficiently cover the unit cube, to abstract surfaces, such as nonlinear manifolds. We present algorithms that yield suitable *fair* mappings between the unit cube and the abstract surface. We demonstrate the application of these ideas using some concrete examples of interest in robotics.

KEYWORDS: space coverage, low-discrepancy curve, nonlinear manifold

1. Introduction

This paper is concerned with algorithmic techniques for the incremental generation of continuous curves that can efficiently cover an abstract surface. This addresses the canonical problem of searching a given abstract surface using a robot that is constrained to move along a single continuous path. This canonical setting encompasses several applications that arise in practice. For concreteness, consider the following scenario—we are tasked with devising a motion strategy for a robot that must explore a bounded area in order to locate a feature (e.g., a wreck or rescue victim) using very local sensory modalities (e.g., bump sensors on a mobile robot, or perhaps vision in a low-visibility foggy environment). Any physical robot can only move along continuous paths and the goal is to create a suitable path that enables the robot to efficiently search the entire area in question. Moreover, it is desirable to have the robot perform this search incrementally, so that it may continue its operation over an arbitrary time horizon, with the assurance that it has always achieved the best coverage possible up to that point. This paper will provide an algorithmic procedure that yields such paths in space.

In a more general setting, uniform coverage via a search process is a natural requirement in many applications involving adaptation and learning. Many such processes can be posed as global search problems in abstract spaces, such as the configuration space of a robot or the parameter space of a complex dynamical system—soluble by the algorithms presented in this paper.

Our approach to solving this problem involves two steps. First, we generate curves that uniformly and incrementally cover a model space, e.g., the unit square. We generalize the well established theory of low-discrepancy sequences in such a way that sufficiently smooth curves with low-discrepancy properties can be defined and generated. In addition to the types of curves that we present in this paper, one may also tap into a sizeable literature on ergodic theory[1] to construct alternate curves with different coverage properties. These curves may then be lifted to induce suitable low-discrepancy curves in abstract spaces, such as nonlinear manifolds. This is achieved through the definition and determination of an area and fairness-preserving map. Given a suitable parametrization of the space to be covered, this procedure yields a curve that can cover it uniformly, optimally in a low-discrepancy sense, and incrementally. This latter step ensures that our algorithm is applicable in a wide variety of applications, where one is interested in covering an abstract space given in terms of a parametrization and metric.

Low-discrepancy point sets and sequences[2] have a successful history within robotics, and within computer science, in general.[3] They have been used in sampling-based motion planning and area coverage applications.[4] In recent work,[5,6] techniques have been proposed for generating sequences in an incremental fashion. How does this relate to the problem posed above? We are interested in incrementally generating continuous paths. Low-discrepancy sequences can, in fact, be generated incrementally, but the process is necessarily discontinuous and a robot could not easily utilize these points to synthesize an appropriate continuous path. Many other constructions that are possible using such point sets, e.g., exploring trees, are also not suitable for use by a robot that must *incrementally* trace out a *single continuous* path in order to cover space.

The notion of solving a path planning problem in a model space and mapping the solution to an abstract space has been used in some prior work, e.g., in ref. [7], navigation functions are generated for model sphere worlds and mapped to more complex environments that are the images of a

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suitable map from the original space. However, we are not aware of prior attempts to use such a notion in the context of efficient and incremental space coverage.

2. On the Notion of Low-Discrepancy Point Sets and Sequences

The definition of discrepancy of a finite set X was introduced to quantify the homogeneity of finite-dimensional point sets[8]

$$D(X) = \sup_R |m(R) - p(R)|. \tag{1}$$

In Eq. (1), referring to the discrepancy of a point set in a unit cube, R runs over all d-dimensional rectangles $[0, r]^d$ with $0 \le r \le 1$, $m(R)$ stands for the Lebesgue measure of R, and $p(R)$ is the ratio of the number of points of X in R and the number of all points of X. The lower the discrepancy, the better or more homogeneous is the distribution of the point set. The discrepancy of an infinite sequence $X = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$ is a new sequence of positive real numbers $D(X_n)$, where X_n stands for the first n elements of X.

There exists a point set of given length that realizes the lowest discrepancy. It is known (Roth bound[9]) that the following inequality holds true for all finite sequences X_n of length n in the d-dimensional unit cube.

$$D(X_n) \ge B_d \frac{(\log n)^{\frac{d-1}{2}}}{n}. \tag{2}$$

B_d depends only on d. Except for the trivial case $d = 1$, it is unknown whether the theoretical lower bound is attainable. Many schemes to build finite sequences X_n of length n do exist that deliver a slightly worse limit,

$$D(X_n) \ge B_d \frac{(\log n)^{d}}{n}. \tag{3}$$

There are also infinite sequences X with the above lower bound, Eq. (3), for all subsequences consisting of the first n elements. The latter result leads to the definition of low-discrepancy infinite sequences X—where the inequality (3) must be valid for all subsequences of the first n elements, with B_d as an appropriate constant.

Many low-discrepancy sequences in d-dimensional unit cubes can be constructed as combinations of one-dimensional low-discrepancy sequences. Popular low-discrepancy sequences are based on schemes introduced by van der Corput,[10] Halton,[11] Sobol,[12] and Niederreiter.[8]

One of the primary motivations for investigations into these sequences arises from high-dimensional function approximation and Monte-Carlo integration. In this setting, there is a well-known[8] relationship between integrals I, approximations I_n, and an infinite sequence $X = \{x_1, x_2, \ldots, x_n, \ldots\}$ in d-dimensions, known as the Koksma–Hlawka inequality

$$|I(f) - I_n(f)| \le V(f) D(X_n), \tag{4}$$

$$I(f) = \int_0^1 f(x)\,dx, \tag{5}$$

$$I_n(f) = \frac{1}{n}\sum_{i=1}^{n} f(x_i), \tag{6}$$

where $V(f)$ is the variation of the function in the sense of Hardy and Krause.

3. Low-Discrepancy Curves in the Unit Square

One of the earliest known quasi-random sequences is the Richtmyer sequence,[13,14] which illustrates a simple but general result in ergodic dynamics[15,16] Let $x_n = \{n\alpha\}$ (i.e., $[n\alpha]$ mod 1) and $X = \{x_1, x_2, \ldots, x_n, \ldots\}$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is irrational and $\alpha_1, \ldots, \alpha_d$ are linearly independent over the rational numbers. Then for almost all α in $\mathbb{R}^d$ and for positive ϵ, with the exception of a set of points that has zero Lebesgue measure,

$$D(X_n) = O\left(\frac{\log^{d+1+\epsilon} n}{n}\right). \tag{7}$$

The Richtmeyer sequence is probably the only quasi-random sequence based on a linear congruential algorithm.[17] This is useful because it suggests a natural extension to the generation of curves.

Let C be a given piecewise smooth and finite curve in the unit square S. Furthermore, let R be an arbitrary aligned rectangle in S with lower left corner $(0, 0)$. Let L be the length of the given curve in S and l be the length of the subcurve of C that lies in R. In case of well-distributed curves, the ratio l/L should represent the area $A(R)$ of R reasonably well. This gives rise to the following definition of discrepancy of a given finite piecewise smooth curve in S:

$$D(C) = \sup_R \left| m(R) - \frac{l}{L} \right|. \tag{8}$$

It would be desirable to construct curves C with the property that the discrepancy is always small. More precisely, we will call an infinite and piecewise sufficiently smooth curve $C : \mathbb{R}^+ \mapsto S$, in natural parametrization, a low-discrepancy curve if for all positive arc lengths L, the curves $C_L = C/[0, L]$ satisfy the inequalities (the function F must be defined appropriately)

$$D(C_L) \le F(L). \tag{9}$$

In fact, a piecewise smooth curve in natural parametrization generates sequences $\{x_1, x_2, \ldots, x_n, \ldots\}$ by setting $x_n = C_n(n\Delta)$, where Δ is a fixed positive number. Then, from Eq. (7),

$$F(L) = O\left(\frac{\log^{3+\epsilon} L}{L}\right), d = 2 \tag{10}$$

We now present a constructive procedure for generating the curve. For $\alpha = (\alpha_1, \alpha_2)$, let $C_A(\alpha)$ be the piecewise linear

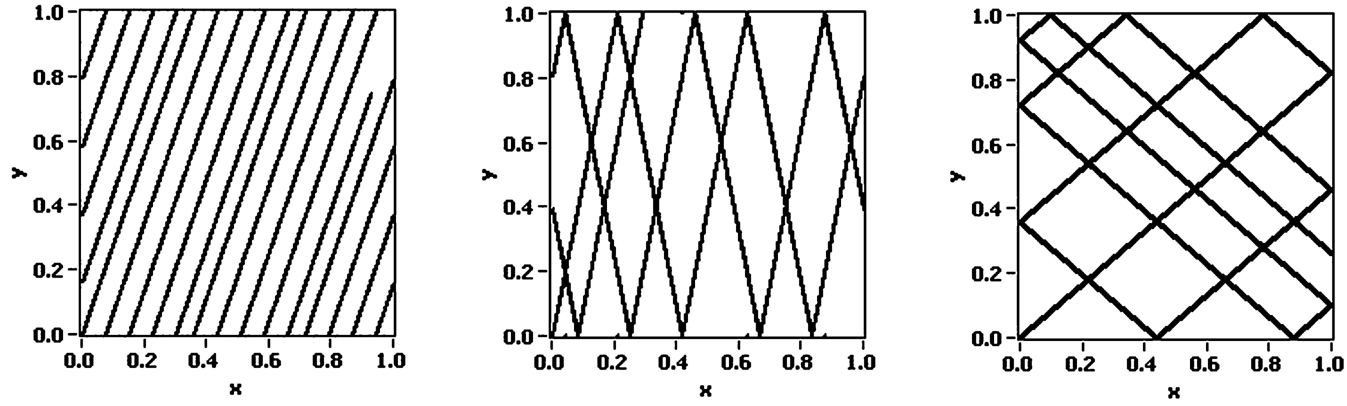

Fig. 1. Various low-discrepancy curves for the unit square: $C_{\mathcal{A}}$, $C_{\mathcal{B}}$, $C_{\mathcal{C}}$ from left to right.

curve $(t\alpha_1 \bmod 1, t\alpha_2 \bmod 1) = (\{t\alpha_1\}, \{t\alpha_2\})$, where t is in $\mathbb{R}^+$. In fact, we can define three classes of curves in the unit square, as shown in Fig. 1. For the simplest type of curves, let us call this class $C_{\mathcal{A}}$, the right-left and top-bottom edges are identified, so that the curve jumps from one edge to the other upon hitting it. In this scheme, all curves are parallel and continue indefinitely when the square tiles the plane. In the second type of curves, class $C_{\mathcal{B}}$, we introduce reflections at the top and bottom edges, but preserve the same identification between right and left edges. The third type of curves, class $C_{\mathcal{C}}$, involves reflections on all edges. The latter curve is continuous.

At this point, we will prove that these curves are, in fact, low discrepancy. For one of the steps in the proof, we will use the following classic result by Weyl (see ref. [3] for details) which states that, for a suitably ergodic curve, the "time-course" of the continuous curve is equivalent to a spatial integral.

Theorem 1 (Ergodic Principle). *If γ is irrational, then the sequence $\{n\gamma\} = n\gamma \pmod 1$ obeys the ergodic principle, which states that for a function f, continuously differentiable from $\mathbb{R}$ to $\mathbb{C}$ and of period 1, $\frac{1}{n}\sum_{k=0}^{n-1} f(\{k\gamma\})$ converges to $\int_0^1 f(x)\,dx$.*

Theorem 2. *For almost all numbers α in $\mathbb{R}^2$, $C_{\mathcal{A}}(\alpha)$ is a low-discrepancy curve in the sense of Eqs.* 9 *and* 10.

Proof.
The proof of this theorem is based on establishing the relation between the area of an axis-aligned rectangle and the length of the curve segment contained within. In essence, we want to compute the quantity in Eq. (8) and show that it is bounded appropriately. This requires us to say that $\frac{l}{L}$ as $L \to \infty$ converges to the measure $m(R)$.

Without loss of generality, we assume $\alpha_1, \alpha_2 > 0$. $C_{\mathcal{A}}(\alpha)$ intersects the axes at $(x = 0, y_n = \{\frac{n\alpha_2}{\alpha_1}\})$ and $(x_n = \{\frac{n\alpha_1}{\alpha_2}\}, y = 0)$, where n is an arbitrary natural number. For almost all α_1, α_2 all three of the quantities (α_1, α_2), $\frac{\alpha_1}{\alpha_2}$, and $\frac{\alpha_2}{\alpha_1}$ generate low-discrepancy sequences in the sense of Eq. (7), in $\mathbb{R}^2$, $\mathbb{R}$, and $\mathbb{R}$, respectively. In other words, the aforementioned sequences x_n, y_n form low-discrepancy sequences in [0, 1].

Now, for the class of curves $C_{\mathcal{A}}$, all curve segments between points of intersection with the edges of the square are parallel to each other. So, by reasoning about the distribution of these points of intersection, we may arrive at conclusions about the distribution of the curves themselves. Moreover, Weyl's result above assures us that an integral over these points of intersection is identical to the corresponding "time-averaged" quantity $\frac{l}{L}$ in Eq. (8).

Let $\tan\phi = \frac{\alpha_2}{\alpha_1}$ and $[0, a] \times [0, b]$ be a rectangle with $0 < a, b < 1$. Depending on the relative values of $(\alpha_1, \alpha_2), a, b$ the integrals take on specific forms. We will explain the case when $\frac{b}{a} \le \tan\phi$, $b < 1 - \tan\phi$ (see Fig. 2) in some detail, the other cases being similar.

We divide the unit square into three parts, as shown in Fig. 2. Then, I_1, I_2, and I_3 are real numbers that represent the average length that the (α_1, α_2) lines corresponding to these regions have in common with the rectangle $[0, a] \times [0, b]$. Asymptotically, with curve length $L \to \infty$, these quantities may be represented as follows (based on simple geometric

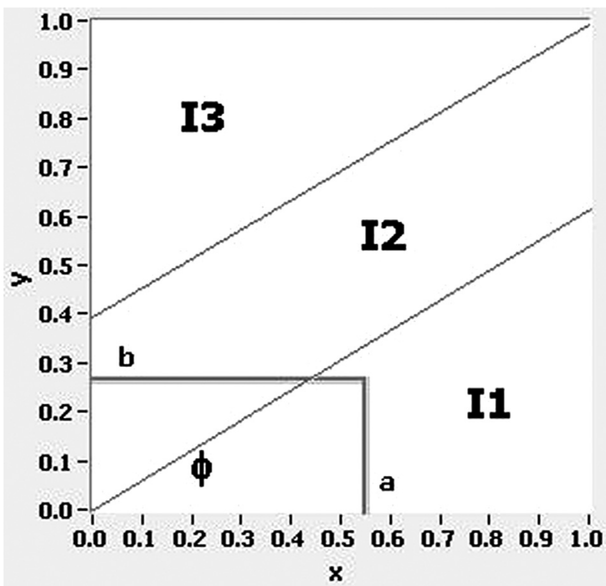

Fig. 2. Definition of the integrals I_1, I_2, and I_3.

considerations):

$$I_1 = \int_0^{a-\frac{b}{\tan\phi}} \frac{b}{\sin\phi}dx + \int_{a-\frac{b}{\tan\phi}}^{a} \frac{a-x}{\cos\phi}dx$$

$$= \frac{ab}{\sin\phi} - \frac{b^2\cos\phi}{2\sin^2\phi}, \tag{11}$$

$$I_2 = \int_0^b \frac{b-y}{\sin\phi}dy = \frac{b^2}{2\sin\phi}, \tag{12}$$

$$I_3 = 0, \tag{13}$$

$$I_1\sin\phi + (I_2 + I_3)\cos\phi = ab. \tag{14}$$

The final term stands for the average length that (α_1, α_2) lines in $[0,1]\times[0,1]$ with slope $\tan\phi$ have in common with the rectangle $[0,a]\times[0,b]$. As expected, it is exactly the area of the rectangle.

In practice, with a finite length curve, what is the discrepancy? We can estimate this using the Koksma-Hlawka inequality (4). For instance, I_1 is approximated using finite length curve segments of the form,

$$l_1(x_i) = \begin{cases} \dfrac{b}{\sin\phi} & : \quad 0 \le x_i \le a - \dfrac{b}{\tan\phi} \\ \dfrac{a-x_i}{\cos\phi} & : \quad a - \dfrac{b}{\tan\phi} \le x_i \le a \\ 0 & : \quad a \le x_i \le 1 \end{cases}.$$

Using a finite number, n, of such segments, we have the discrepancy,

$$\left| I_1 - \sum_{i=1}^{n} l_1(x_i) \right| = O\left(\frac{\log^{2+\epsilon} n}{n}\right), \tag{15}$$

where $x_i = \{\frac{i\alpha_1}{\alpha_2}\}$.

The sum stands for the length of that part of the given curve that lies in $[0,a]\times[0,b]$. The same argument applies for the integrals I_2 and I_3. This means that a finitely generated curve segment of type C_A also has low discrepancy. Moreover, note that we are reasoning about a one-dimensional sequence of points of intersection. So, the constant 3 (for $d=2$) in Eq. (10) is now replaced with 2 ($d=1$).

We can now develop two variations of Theorem 2.

Theorem 3. *For almost all numbers α in $\mathbb{R}^2$, $C_{\mathcal{B}}(\alpha)$ and $C_{\mathcal{C}}(\alpha)$ are low-discrepancy curves in the sense of Eqs.* (9) *and* (10).

Proof.
We begin with a remark. One can show that Theorem 2 is still valid when in the definition of low-discrepancy curves, a broader class of rectangles R is considered, i.e., rectangles where we replace $[0,a]\times[0,b]$ with the more generic $[c,a]\times[d,b]$. The literature on geometric discrepancy[2] includes several related proofs and expanded discussion on such ideas.

Curves of type $\mathcal{B}$: Such a curve can be translated into an equivalent version acting in $[0,1]\times[0,2]$, by simply mirroring the square. To this end, reflections at the upper edge (see Fig. 1) are ignored. What results is an equivalent scheme as type $\mathcal{A}$ in $[0,1]\times[0,2]$. For almost all choices of α, the resulting curve in $[0,1]\times[0,2]$ is low discrepancy. The relation between the original space and the new one is straightforward. The original curve goes through a rectangle $R=[0,a]\times[0,b]$, if and only if the derived curve in $[0,1]\times[0,2]$ goes through $[0,a]\times[0,b]$ or through $[0,a]\times[2-b,2]$ (see the remark at the beginning of this proof). The latter implies that $C_{\mathcal{B}}(\alpha)$ satisfies Eqs. (9) and (10).

Curves of type $\mathcal{C}$: We essentially repeat the arguments from type $\mathcal{B}$. For almost all α, curves of type $\mathcal{B}$ in $[0,2]\times[0,1]$ are low discrepancy. Such curves can be generated when reflections at the right edge are ignored. The mirrored version of this curve goes through a rectangle $R=[0,a]\times[0,b]$, if and only if the original curve in $[0,2]\times[0,1]$ goes through $[0,a]\times[0,b]$ or $[2-a,2]\times[0,b]$ (see the remark at the beginning of this proof). The latter implies that $C_{\mathcal{C}}(\alpha)$ satisfies Eqs. (9) and (10). □

Curves $C_{\mathcal{C}}(\alpha)$ can be regarded as first examples of continuous trajectories in a unit square that offer low-discrepancy behavior. In real area coverage scenarios, they are highly efficient compared to alternate techniques, as we will demonstrate in Section 5.

Finally, these results can be generalized to higher dimensions, using the same style of argument. We state this theorem without proof.

Theorem 4. *For almost all numbers α in $\mathbb{R}^d$, $C_{\mathcal{A}}(\alpha)$, $C_{\mathcal{B}}(\alpha)$, and $C_{\mathcal{C}}(\alpha)$ are low-discrepancy curves in the generalized sense of Eqs.* (9) *and* (10) *in d-dimensional unit cubes.*

In practice, the question arises as to how these curves can be realized on digital computers. For this purpose, it is reasonable to assume that we have the ability to generate rational numbers of user-specified arbitrary precision. In this case, the error due to the rational approximation of α in Theorem 2, 3, and 4 is correspondingly small, so that finite lengths of the resulting curves generate discrepancies that are low.

Theorem 5. *If $\{n\alpha\}$ generates a low-discrepancy sequence for some irrational number α, then $\exists N \gg 1$, such that $\{n\frac{p}{q}\}$, $n=1,\ldots,N$, also generates a low-discrepancy sequence, where $p,q \in \mathbb{N}$.*

Proof.
The Hurwitz theorem in number theory[18] states that there are infinitely many $\frac{p}{q}$ with the property $|\alpha - \frac{p}{q}| < \frac{1}{\sqrt{5}q^2}$.

Given any irrational number, it is possible to find a sufficiently large $q \in \mathbb{N}$, such that the error of the rational approximation is small. Let q_1 be such a number, with

$q_1 > N \gg 1$. Then,

$$\left| n \left(\alpha - \frac{p_1}{q_1} \right) \right| < q_1 . \frac{1}{q_1^2} = \frac{1}{q_1} < \frac{1}{N}. \qquad (16)$$

Now, assume that $\{(n\alpha) \bmod 1\}$ generates a low-discrepancy point set, of discrepancy $D(\alpha)$ for $n = 1, \ldots, N$. When α is irrational, $\{(n\alpha) \bmod 1\}$ does not intersect the vertices of the unit square. Then, there always exists a neighborhood where elements of $\{(nx) \bmod 1\}$, $n = 1, \ldots, N$, are continuous functions of x. This implies that $D(x)$ is a continuous function of x in this neighborhood.

By selecting suitable p_1, q_1 that approximate α well, to within $\epsilon = \frac{1}{q_1}$, we arrive at the bound, $|D(x) - D(\alpha)| < \delta$, where δ is a suitably small constant. This implies that a curve constructed using these sequences, according to the procedure shown in Theorem 2, and using rational approximations to α, is still low-discrepancy. ■

4. Low-Discrepancy Curves over Abstract Surfaces

In many common applications, we deal with surfaces that can be described as (covering) maps of the unit square. We would like to extend our constructions to these surfaces. However, the standard parametrization of these maps rarely respects our stated requirements of fairness and low-discrepancy. In such situations, we may suitably modify the map via reparametrization. When the surface in question is a nonlinear manifold (see ref. [19, 20] for an introduction to the language of manifold theory), i.e., a space whose metric properties vary from point to point, the metric can be further modified via a *fair* reparametrization. We will now discuss how to achieve this, such that the low-discrepancy properties are preserved.

Given an abstract surface S with a metric defined for (u, v) in $[0, 1]^2$,

$$ds^2 = E(u, v)\, du^2 + F(u, v)\, du\, dv + G(u, v)\, dv^2, \quad (17)$$

where $E(u, v)$, $F(u, v)$, $G(u, v)$ are differentiable functions in u and v and $EG - F^2$ is positive. The area element dA is defined by,

$$dA = \sqrt{E(u, v)G(u, v) - F^2(u, v)}\, du \wedge dv. \qquad (18)$$

The function,

$$\Psi(u, v) = \sqrt{E(u, v)G(u, v) - F^2(u, v)} \qquad (19)$$

is nonnegative in $[0, 1]^2$ and $\Psi^2(u, v)$ is differentiable.

Let $\alpha = (\alpha_1, \alpha_2)$ be a given irrational vector (direction) in $\mathbb{R}^2$. According to the definitions (17) and (18), line and area elements of S for this specific direction $((du, dv) = (\alpha_1 dw, \alpha_2 dw))$ satisfy,

$$\frac{ds}{dw} = \sqrt{E(u, v)\alpha_1^2 + F(u, v)\alpha_1\alpha_2 + G(u, v)\alpha_2^2}, \quad (20)$$

$$\frac{dA}{dw^2} = \sqrt{E(u, v)G(u, v) - F^2(u, v)}\alpha_1\alpha_2. \qquad (21)$$

From this, we define the quantity Q which describes our notion of fairness,

$$Q = \frac{\frac{ds}{dw}}{\frac{dA}{dw^2}} = \frac{\sqrt{E(u, v)\alpha_1^2 + F(u, v)\alpha_1\alpha_2 + G(u, v)\alpha_2^2}}{\sqrt{E(u, v)G(u, v) - F^2(u, v)}\alpha_1\alpha_2}. \qquad (22)$$

Definition 1.
A piecewise smooth curve $C : \mathbb{R}^+ \mapsto S$ lying on an abstract surface S is called a low-discrepancy curve based on a vector $\alpha = (\alpha_1, \alpha_2)$, if and only if:

1. C is S-filling, i.e., C comes arbitrarily close to any point of S.
2. There is a parametrization of S, where Q in Eq. (22) is constant for all (u, v).
3. In any regular point of C, the tangent vector is parallel to $\alpha = (\alpha_1, \alpha_2)$.

The following algorithm is based on Definition 1.

Algorithm 1.

1. Find a parametrization of S that satisfies conditions 2 and 3 in Definition 1. See also Remark 1 below.
2. Generate a curve in S based on the image of a low-discrepancy curve in the unit square according to Theorem 3.

Remark 1. The parametrization in step 2 is not unique. In all examples, an originally given natural parametrization is modified using replacements $u \mapsto h(u)$ or $v \mapsto h(v)$, where h is smooth.

An abstract d-dimensional surface is defined by,

$$ds^2 = \sum_{i,j=1}^{d} g_{ij}(u_1, u_2, \ldots, u_d)\, du_i\, du_j, \qquad (23)$$

where the matrix consisting of $g_{ij} : [0, d]^d \mapsto \mathbb{R}$ is always symmetric, differentiable, and positive semi-definite. An embedding of an abstract space as in Eq. (23) in an m-dimensional Euclidean space is a map f of the hypercube $[0, 1]^d$ with $f_1(u_1, u_2, \ldots, u_d)$, $f_2(u_1, u_2, \ldots, u_d), \ldots$, $f_m(u_1, u_2, \ldots, u_d)$: $\mathbb{R}^d \mapsto \mathbb{R}^m$, where the Riemannian metric of this embedding is described by Eq. (23). Usually, this definition is too restrictive. Instead, local maps, i.e., coordinate patches, should be used where these patches cover the whole space under consideration.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ be a given vector (direction) in $\mathbb{R}^d$. According to Eq. (23), the line and volume/content elements for a specific direction $(du_1, du_2, \ldots, du_d) = (\alpha_1 du, \alpha_2 du, \ldots, \alpha_d du)$ are:

$$\frac{ds}{du} = \sqrt{\sum_{i,j=1}^{d} g_{ij}(u_1, u_2, \ \ldots, u_d)\alpha_i\alpha_j}, \qquad (24)$$

$$\frac{dV}{du^n} = \sqrt{\det(g_{ij}(u_1, u_2, \ \ldots, u_d))\alpha_1\alpha_2 \ldots \alpha_d}. \quad (25)$$

From this,

$$Q = \frac{\frac{ds}{du}}{\frac{dV}{du^n}} = \frac{\sqrt{\sum_{i,j=1}^{d} g_{ij}(u_1, u_2, \ldots, u_d)\alpha_i\alpha_j}}{\sqrt{\det(g_{ij}(u_1, u_2, \ldots, u_d))}\alpha_1\alpha_2\ldots\alpha_d}. \quad (26)$$

Definition 2.
A piecewise smooth curve $C : \mathbb{R}^+ \mapsto S$ in the given Riemannian space S is called a low-discrepancy curve based on a vector direction $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ if and only if:

1. C is S-filling, i.e., C comes arbitrarily close to any point of S.
2. There is a parametrization of S, where Q in eq. (26) is constant for all $(u_1, u_2, \ldots, u_d)$.
3. In any regular point of C, the tangent vector is parallel to $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$.

Remark 2. In higher dimensions, i.e., $d \geq 3$, the above definition may yield a very large class of solutions, which may need to be further specialized through the use of additional requirements. We will return to this issue in Section 6.1.

5. Examples of the Coverage of Various Spaces

In this section, we demonstrate the use of the proposed technique to cover various surfaces. These examples are chosen to correspond to geometrical shapes and surfaces that are commonly encountered in robotics.

5.1. Covering the annulus

Consider an annulus, i.e., a ring, that has a standard parametrization given by $x(u, v) = (u\cos v, u\sin v)$, where $0 \leq u_0 \leq u \leq u_1$ and $0 \leq v \leq 2\pi$.

Let us consider a map, such that $u \mapsto g(u)$ and $v \mapsto v$. Then the ring can be reparameterized by $x(u, v) = (g(u)\cos v, g(u)\sin v)$. Now, we can apply the method developed earlier. Let g be sufficiently smooth, where g maps $[u_0, u_1]$ onto $[u_0, u_1]$. Then, according to our proposed method, $g(u)$ must satisfy an ordinary differential equation (for $\alpha_1, \alpha_2 > 0$),

$$g'(u) = \frac{\alpha_2 g(u)}{\sqrt{c^2 g^2(u)\alpha_1^2\alpha_2^2 - \alpha_1^2}}, \quad (27)$$

where $g(u_0) = u_0$ and $g(u_1) = u_1$, and,

$$\frac{ds}{du} = \sqrt{g'^2(u)\alpha_1^2 + g^2(u)\alpha_2^2}, \quad (28)$$

$$\frac{dA}{du^2} = g(u)g'(u)\alpha_1\alpha_2. \quad (29)$$

Equation (27) has the closed-form solution,

$$\frac{\alpha_1}{\alpha_2}\left\{\sqrt{c^2 g^2(u)\alpha_2^2 - 1} + \arctan\left(\frac{1}{\sqrt{c^2 g^2(u)\alpha_2^2 - 1}}\right)\right\} = u + D, \quad (30)$$

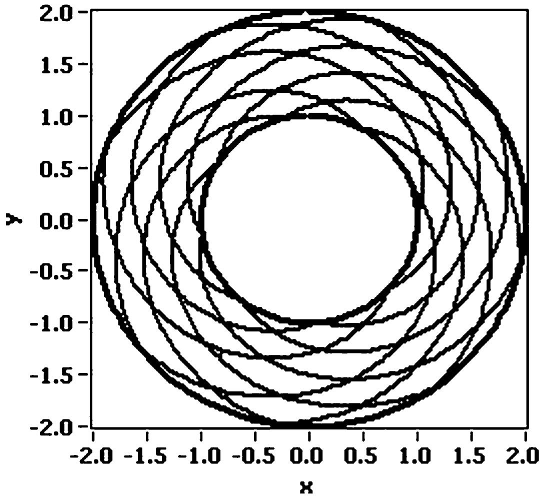

Fig. 3. A low-discrepancy curve in the annulus.

where D is the unknown constant of integration. Using boundary conditions that arise from the domain and image constraints of the mapping, $g(u_0) = u_0$ and $g(u_1) = u_1$, it follows that,

$$\frac{\alpha_1}{\alpha_2}\left\{\sqrt{c^2 u_0^2\alpha_2^2 - 1} + \arctan\left(\frac{1}{\sqrt{c^2 u_0^2\alpha_2^2 - 1}}\right)\right\} = u_0 + D, \quad (31)$$

$$\frac{\alpha_1}{\alpha_2}\left\{\sqrt{c^2 u_1^2\alpha_2^2 - 1} + \arctan\left(\frac{1}{\sqrt{c^2 u_1^2\alpha_2^2 - 1}}\right)\right\} = u_1 + D. \quad (32)$$

We solve for c, D and then use Eq. (30) as an implicit definition of the required map. In general, one may solve the nonlinear differential equations numerically using shooting methods.[21]

Figure 3 shows a resulting low-discrepancy curve filling the given ring for the specific case, where $u_0 = 1, u_1 = 2$. The parameters α_1, α_2, and c were chosen appropriately by numerical experimentation.

5.2. Covering the surface of a torus

The torus is an important nontrivial surface that appears in robotics, especially when working with configuration spaces of mechanical systems. We will use our algorithm to generate a low-discrepancy curve for this surface. We use the map $u \mapsto u$ and $v \mapsto g(v)$. Given the $\mathbb{R}^3$ embedding of a torus $(b < a)$,

$$x(u, v) = ((a + b\cos(2\pi g(v)))\cos(2\pi u), (a + b\cos(2\pi g(v)))\sin(2\pi u), b\sin(2\pi g(v))), \quad (33)$$

$$ds^2 = 4\pi^2(a + b\cos(2\pi g(v)))^2 du^2 + 4\pi^2 b^2 g'^2(v) dv^2, \quad (34)$$

$$dA^2 = 16\pi^4 b^2(a + b\cos(2\pi g(v)))^2 g'^2(v) du^2 dv^2. \quad (35)$$

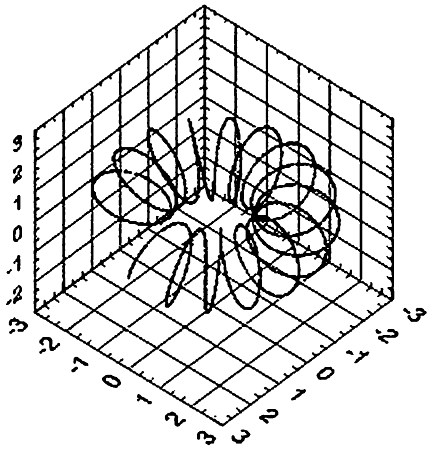

Fig. 4. Low-discrepancy curve filling the surface of a torus.

The function g maps [0, 1] onto [0, 1] and is sufficiently smooth. Constant ratio of $\frac{ds}{du}$ and $\frac{dA}{du^2}$ in α-direction can be achieved if the following equation holds true (c is a constant, $\alpha_1 > 0$):

$$\frac{(a + b\cos(2\pi g(v)))^2\alpha_1^2 + b^2 g'^2(v)\alpha_2^2}{4\pi^2 b^2 g'^2(v)(a + b\cos(2\pi g(v)))^2\alpha_1^2\alpha_2^2} = c \Rightarrow$$
$$g'(v) = \frac{(a + b\cos(2\pi g(v)))\alpha_1}{\sqrt{4c\pi^2 b^2(a + b\cos(2\pi g(v)))^2\alpha_1^2\alpha_2^2 - b^2\alpha_2^2}}. \tag{36}$$

The boundary conditions are $g(0) = 0$ and $g(1) = 1$. A solution of Eq. (36) guarantees $g'(0) = g'(1)$. Here, the parameter c is chosen numerically with the aid of a shooting method. Figure 4 depicts part of the resulting low-discrepancy curve lying on the surface of a torus. Because of $g'(0) = g'(1)$, the curve is smooth.

5.3. *Covering the surface of a sphere*

We wish to cover a part of the sphere given as an abstract surface by $ds^2 = 4\pi^2\sin^2(\pi g(v))du^2 + \pi^2 dv^2$, where (u, v) is in $[0, 1] \times [v_0, v_1]$ with $v_0 < v_1$ in (0, 1). A Euclidean embedding of this surface is given by,

$$x(u, v) = (\sin(2\pi u)\sin(\pi g(v)),$$
$$\cos(2\pi u)\sin(\pi g(v)), \cos(\pi g(v))). \tag{37}$$

The function $g(v)$ is smooth and maps $[v_0, v_1]$ onto $[v_0, v_1]$. According to Definition 1 and Eq. (22), we have,

$$g'(v) = \frac{2\sin(\pi g(v))\alpha_1}{\sqrt{4c^2\pi^2\sin^2(\pi g(v))\alpha_1^2\alpha_2^2 - \alpha_2^2}}. \tag{38}$$

The boundary conditions are $g(v_0) = v_0$ and $g(v_1) = v_1$. Figure 5 depicts the resulting low-discrepancy curve.

6. Discussion and Open Questions

6.1. *Definitions of fair reparametrizations*

Broadly speaking, there are three major steps in the procedure outlined above:

1. Define a criterion, according to which the curve optimally covers space. This is a *fairness* requirement.
2. Define a map that carries the unit square to an abstract surface.
3. Based on the previous two steps, solve a differential equation whose solution yields the map that respects our stated criterion for fairness.

In this paper, we have defined fairness according to the requirement that any two arbitrary segments of the curve, if they have the same length, must cover the same amounts of area or volume. It is easy to visualize this requirement for a curve in the plane, where it makes sense intuitively. However, in higher dimensions, this requirement alone may not be sufficient in the sense that the resulting family of solutions may be large. One may wish to utilize additional definitions of fairness.

For instance, with a three-manifold, in addition to the basic fairness requirement in Eq. (26), we would require that the volume is not affected, $\sqrt{\det(g)}dx_1dx_2dx_3 = const$. and the curve is fair in the sense of being distributed in accordance with the surface element, i.e., $\frac{dl}{L} = \frac{da}{A(S)}$ where $da = \sqrt{(g_{11}g_{22} - g_{12}^2)}dx_1dx_2 + \sqrt{(g_{11}g_{33} - g_{13}^2)}dx_1dx_3 + \sqrt{(g_{22}g_{33} - g_{23}^2)}dx_2dx_3$ is the area element.

In general, this sort of engineering of fairness requirements would be difficult to factor into traditional techniques for generating low-discrepancy sequences. On the other hand, due to the geometric nature of our problem formulation, such requirements fit very naturally into our algorithmic framework.

6.2. *On the class of abstract surfaces addressed by this approach*

In this paper, for the purposes of exposition, we have considered as examples some standard surfaces that commonly appear in robotics. A variety of other abstract surfaces may be covered by following the same approach. Can we characterize the class of surfaces for which this approach is applicable? We do not yet have a general answer to this question. However, in this section, we study this question from a different direction—by asking what sort of surfaces can be reached by our *fair* maps from the unit cube? Specifically, we study the solutions of a partial differential equation that corresponds to a generalization of the example in Section 5.1. We will derive an expression for a subset of the class of surfaces that are realizable, and this will show that the region of applicability of the algorithms presented here is large.

Consider a generalization to the surface defined in Section 5.1—an annulus defined in terms of the intersection of two planar surfaces with star convex, but otherwise arbitrarily shaped boundaries. Instead of $x(u, v) = (g(u)\cos v, g(u)\sin v)$, use the more general

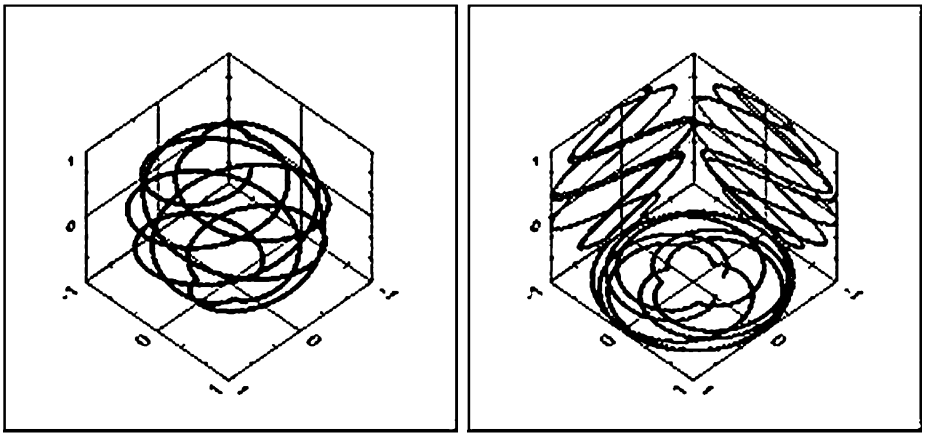

Fig. 5. A low-discrepancy curve on the surface of a sphere. The figure on the right shows the three two-dimensional projections along the axis planes.

parametrization $x(u, v) = (M(u, v), N(u, v))$. Then our procedure for determining the reparametrization would generate a partial differential equation,

$$Q^2\alpha_1^2\alpha_2^2(M_uN_v - M_vN_u)^2 = \left(M_u^2 + N_u^2\right)\alpha_1^2 + 2(M_uM_v + N_uN_v)\alpha_1\alpha_2 + \left(M_v^2 + N_v^2\right)\alpha_2^2.$$

For the purposes of the present analysis, we ignore some of the constants and focus on the structural form. We will analytically characterize the solutions of the following partial differential equation:

$$2(M_uN_v - M_vN_u)^2 = (M_u + M_v)^2 + (M_v + N_v)^2, \quad (39)$$

$$M_uN_v - M_vN_u = 1. \quad (40)$$

Define new variables,

$$m(u, v) = M(u + v, u - v), \quad (41)$$

$$n(u, v) = N(u + v, u - v). \quad (42)$$

From this,

$$M_u(u + v, u - v)N_v(u + v, u - v) - M_v(u + v, u - v)N_u(u + v, u - v) = \frac{1}{2}(-m_u(u, v)n_v(u, v) + m_v(u, v)n_u(u, v)), \quad (43)$$

$$M_u(u + v, u - v) + M_v(u + v, u - v) = m_u(u, v), \quad (44)$$

$$N_u(u + v, u - v) + N_v(u + v, u - v) = n_u(u, v). \quad (45)$$

The partial differential Eq. (39), can be rewritten as,

$$m_u^2 + n_u^2 = 2, \quad (46)$$

$$m_un_v - m_vn_u = -2. \quad (47)$$

Picking the positive solutions,

$$m_u = \sqrt{2 - n_u^2}, \quad (48)$$

$$m_v = \frac{2 + n_v\sqrt{2 - n_u^2}}{n_u}, \quad (49)$$

with $m_{uv} = m_{vu}$,

$$-\frac{n_un_{uu}}{\sqrt{2 - n_u^2}} = \frac{\left(n_{vu}\sqrt{2 - n_u^2} - \frac{n_vn_un_{uu}}{\sqrt{2 - n_u^2}}\right)n_u - \left(2 + n_v\sqrt{2 - n_u^2}\right)n_{uu}}{n_u^2}, \quad (50)$$

$$\left(\sqrt{2 - n_u^2} + n_v\right)n_{uu} = n_un_{uv}. \quad (51)$$

In order to solve this nonlinear equation, we may use the method of Legendre transformations,[23] defined in terms of the following equations:

$$\begin{aligned} n(u, v) + w(\zeta, \eta) &= u\zeta + n\eta \\ n_u &= \zeta \\ n_v &= \eta \\ n_{uu} &= Jw_{\eta\eta} \\ n_{uv} &= -Jw_{\zeta\eta} \\ w_{\eta\eta} &= J^{-1}n_{uu} \\ w_{\zeta\eta} &= -J^{-1}n_{uv} \\ J = n_{uu}n_{vv} - n_{uv}^2 &= \frac{1}{w_{\zeta\zeta}w_{\eta\eta} - w_{\zeta\eta}^2} \\ w_\zeta &= u \\ w_\eta &= v. \end{aligned} \quad (52)$$

This yields the simpler partial differential equation,

$$\left(\sqrt{2 - \zeta^2} + \eta\right)w_{\eta\eta} = -\zeta w_{\eta\zeta}. \quad (53)$$

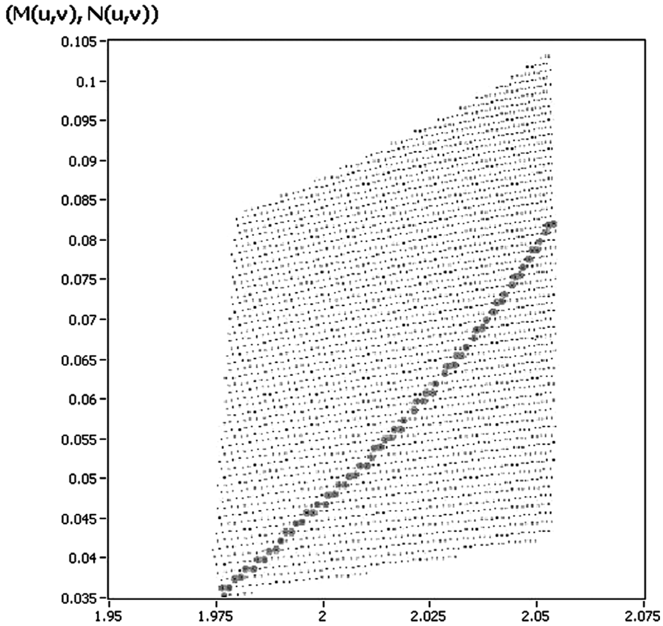

Fig. 6. An example of the image of a rectangular patch and a line, under the map $x(u, v) = (M(u, v), N(u, v))$, with $\Phi(x) = x^2$, $\Psi(x) = x$.

The solution of this equation may be computed by a symbolic mathematics package, or other means, as

$$w_\eta(\zeta, \eta) = \Psi\left(\frac{\eta}{\zeta} + \sin^{-1}\left(\frac{\zeta}{\sqrt{2}}\right) + \frac{\sqrt{2-\zeta^2}}{\zeta}\right), \quad (54)$$

$$w(\zeta, \eta) = \Phi(\zeta) + \int \Psi\left(\frac{\eta}{\zeta} + \sin^{-1}\left(\frac{\zeta}{\sqrt{2}}\right) + \frac{\sqrt{2-\zeta^2}}{\zeta}\right) d\eta. \quad (55)$$

The functions $\Phi(\bullet)$ and $\Psi(\bullet)$, in Eq. (55), parameterize the solutions of the original partial differential Eq. (39). To the extent that these functions may be freely chosen, we have available a rich class of mappings that are expressible with maps, $x(u, v) = (M(u, v), N(u, v))$, from the unit square to an abstract surface.

As one specific illustration, consider the case of $\Phi(x) = x^2$, $\Psi(x) = x$. Figure 6 depicts this solution as the image of a rectangular patch. The point of the above argument is to establish that a variety of fairly sophisticated surfaces can be reached by suitable choices of $\Phi(\bullet)$ and $\Psi(\bullet)$. Formally characterizing this complete class is an open question for future work. However, it is clear that, given a suitably flexible reparametrization, the algorithms in this paper can be used in a variety of application instances to achieve incremental and efficient space coverage using a single continuous curve.

7. Conclusions

We have provided an algorithmic technique to generate continuous curves that efficiently and uniformly cover an abstract surface. In doing so, we take an approach that is more general than prior work based on constructing and selecting points from lattices and grids. Our approach is incremental, well suited to search problems in abstract spaces, and applicable to a variety of problems where the abstract surface is described by a parametrization and metric. We have demonstrated the applicability of our proposed approach, using examples that are of relevance to robotics.

In our current and future work, we are trying to extend these ideas in several directions. From an applications standpoint, we are interested in the use of similar ideas to tackle search and coverage problems in abstract spaces, including shape spaces,[22] parameter spaces, and configuration spaces of complex dynamical systems.

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