# NORMAL-ORDER REDUCTION GRAMMARS 

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#### Abstract

We present an algorithm which, for given $n$, generates an unambiguous regular tree grammar defining the set of combinatory logic terms, over the set $\{S, K\}$ of primitive combinators, requiring exactly $n$ normal-order reduction steps to normalize. As a consequence of Curry and Feys's standardization theorem, our reduction grammars form a complete syntactic characterization of normalizing combinatory logic terms. Using them, we provide a recursive method of constructing ordinary generating functions counting the number of $S K$-combinators reducing in $n$ normal-order reduction steps. Finally, we investigate the size of generated grammars, giving a primitive recursive upper bound.


## 1. Introduction

Since the time of the pioneering works of Moses Schönfinkel [16] and Haskell Curry [8, combinatory logic is known as a powerful, yet extremely simple in structure, formalism expressing the notion of computability. With the dawn of functional programming languages in the early 1970s, combinatory logic, with its standard normal-order reduction scheme [9, is used as a practical implementation of lazy semantics in languages such as SASL [17] or its successor Miranda 18. Lack of bound variables in the language resolves the intrinsic problem of substitution in $\lambda$-calculus, making the reduction relation a simple computational step and so, in consequence, the leading workhorse in implementing call-by-need reduction schemes.
Surprisingly, little is known about the combinatorial properties of normal-order reduction and, in particular, its behaviour in the 'typical' case of large random combinators. With the growing popularity of random software testing (see, e.g. [15]) 'typical' properties of random $\lambda$-terms and combinators became of immense practical importance. In this approach to software verification, large random terms are generated and used to check the programmer-declared function invariants, making it crucial to understand and exploit the semantic properties of so generated terms.
State-of-the-art research in this field includes counting and generating $\lambda$-terms (see e.g. [13] [14] [12]), their restricted classes [6, investigating their asymptotic properties [10] [4] as well as the asymptotic properties of combinatory logic 5 .

Main tools used in this line of research include formal power series and generating functions. Interested in a particular counting sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$ corresponding to a set of terms $A$, we construct a suitable generating function, which treated as a complex function in one variable $z$ yields a Taylor series expansion around $z=0$ with coefficients forming our sequence $\left(a_{n}\right)_{n \in \mathbb{N}}$. Methods of analytic combinatorics [11] allow us to derive, sometimes surprisingly accurate, asymptotic approximations of the growth rate of $\left(a_{n}\right)_{n \in \mathbb{N}}$ and, in consequence, use them to study the asymptotic behaviour of $A$.

Finding appropriate generating functions plays therefore an important role in the process of investigating properties of 'typical' terms. In [5], authors investigated the asymptotic density of weakly normalizing terms in the set of all combinators, showing that a 'typical' combinator cannot have a trivial $0-1$ asymptotic probability of normalization. The result was obtained by constructing large classes of terms with and without the normalization property. Though sufficient for the purpose of showing the non-trivial behaviour of normalization, their classes reveal the combinatorial structure of just a small asymptotic portion of normalizing terms.

[^0]In this paper we give a complete combinatorial characterization of normalizing combinatory logic terms over the set $\{S, K\}$ of primitive combinators. We construct a recursive family $\left\{R_{n}\right\}_{n \in \mathbb{N}}$ of regular tree grammars defining combinators reducing in exactly $n$ normal-order reductions. By Curry and Feys's standardization theorem [9], normal-order evaluation of normalizing combinators leads to their normal forms, hence our normal-order reduction grammars form a complete partition of normalizing combinators. Our approach is algorithmic in nature and provides fully automated methods for constructing $\left\{R_{n}\right\}_{n \in \mathbb{N}}$ as well as their corresponding ordinary generating functions.

The paper is organized as follows. In Sections 1.1 and 1.2 we give preliminary definitions and notational conventions. In Section 1.3 we explain our pseudo-code notation and related implementation. In Section 2 we present a high-level overview on the algorithm. In Section 3 we analyse the algorithm giving proofs of soundness 3.2, completeness 3.3 and unambiguity 3.4 In Section [3.5 we give a recursive construction of ordinary generating functions corresponding to $\left\{R_{n}\right\}_{n \in \mathbb{N}}$. In Section 3.6 we discuss some consequences and applications of normal-order reduction grammars. Finally, in Section 3.7 we investigate the size of the generated grammars.
1.1. Combinatory Logic. We consider the set of terms over primitive combinators $S$ and $K$. In other words, the set $\mathcal{C}$ of combinatory logic terms defined as $\mathcal{C}:=S|K| \mathcal{C} \mathcal{C}$. We follow standard notational conventions (see e.g. [2]) - we omit outermost parentheses and drop parentheses from left-associated terms, e.g. instead of $((S K)(K K))$ we write $S K(K K)$. We use $\rightarrow_{w}$ to denote the normal-order reduction relation (reduce the leftmost outermost redex) to which we usually refer briefly as the reduction relation. We use lower case letters $x, y, z, \ldots$ to denote combinatory logic terms. For an introduction to combinatory logic we refer the reader to [2], [9].
1.2. Regular tree grammars. In order to characterize terms normalizing in $n$ steps we use regular tree grammars (see e.g. [7]), a generalization of regular word grammars. A regular tree grammar $G=(S, N, \mathcal{F}, P)$ consists of an axiom $S$, a set $N$ of non-terminal symbols such that $S \in N$, a set of terminal symbols $\mathcal{F}$ with corresponding arities and a finite set of production rules $P$ of the form $\alpha \rightarrow \beta$ where $\alpha \in N$ is a non-terminal and $\beta \in T_{\mathcal{F}}(N)$ is a term in the corresponding term algebra $T_{\mathcal{F}}(N)$, i.e. the set of directed trees built upon terminals $\mathcal{F}$ according to their associated arities. To build terms of grammar $G$, we start with the axiom $S$ and use the corresponding derivation relation, denoted by $\rightarrow$, as defined through the set of production rules $P$.

Example 1. Consider the following regular tree grammar defined as $B=(S, N, \mathcal{F}, P)$ where $S:=\mathcal{B}, N:=\{\mathcal{B}\}, \mathcal{F}:=\{\bullet, \circ(\cdot, \cdot)\}$, and $P$ consists of the two following rules:

$$
\left\{\begin{array}{l}
\mathcal{B} \rightarrow \circ(\mathcal{B}, \mathcal{B}) \\
\mathcal{B} \rightarrow \bullet
\end{array}\right.
$$

Note that $B$ defines the set of terms isomorphic to plane binary trees where leafs correspond to the nullary constant - and inner nodes correspond to the binary terminal $\circ(\cdot, \cdot)$.

In our endeavour, we are going to recursively construct regular tree grammars generating sets of combinatory logic terms. We set a priori their axioms and both terminal and nonterminal symbols, leaving the algorithm to define the remaining production rules. And so, the $n$th grammar $R_{n}$ will have:
(i) an axiom $S=R_{n}$,
(ii) a set $\mathcal{F}$ of terminal symbols consisting of two nullary constants $S, K$ and a single binary application operator,
(iii) a set of non-terminal symbols $N=\{\mathcal{C}\} \cup\left\{R_{0}, \ldots, R_{n}\right\}$ where $\mathcal{C}$ denotes the axiom of the set of all combinatory logic terms, as defined in the previous section.
In other words, the grammar $R_{n}$ defining terms normalizing in $n$ steps, will reference all previous grammars $R_{0}, \ldots, R_{n-1}$ and the set of all combinatory logic terms $\mathcal{C}$.

Throughout the paper, we adopt the following common definitions and notational conventions regarding trees. We use lower case letters $\alpha, \beta, \gamma, \delta, \ldots$ to denote trees, i.e. elements of the term algebra $T_{\mathcal{F}}(N)$ where $N=\{\mathcal{C}\} \cup\left\{R_{0}, \ldots, R_{n}\right\}$ for some $n$. Whenever we want to use a combinator without specifying its type, we use capital letters $X, Y, \ldots$. We define the size of $\alpha$ as the number of applications in $\alpha$. We say that $\alpha$ is normal if either $\alpha$ is of size 0 , or $\alpha=X \alpha_{1} \ldots \alpha_{m}$, for some $m \geq 1$, where all $\alpha_{1}, \ldots, \alpha_{m}$ are normal. In the latter case we say moreover that $\alpha$ is complex. Since we are going to work exclusively with normal trees, we assume that all trees are henceforth normal. We say that a complex $\alpha$ is of length $m$ if $\alpha$ is in form of $X \alpha_{1} \ldots \alpha_{m}$. Otherwise, if $\alpha$ is not complex, we say that it is of length 0 . The degree of $\alpha$, denoted as $\rho(\alpha)$, is the minimum natural number $n$ such that $\alpha$ does not contain references to any $R_{i}$ for $i \geq n$. In particular, if $\alpha$ does not reference any reduction grammar, its degree is equal to 0 . We use $L_{G}(\alpha)$ to denote the language of $\alpha$ in grammar $G$. Since $R_{n}$ does not reference grammars of greater index, we have $L_{R_{\rho(\alpha)-1}}(\alpha)=L_{R_{n}}(\alpha)$ for arbitrary $n \geq \rho(\alpha)$. And so, for convenience, we use $L(\alpha)$ to denote the language of $\alpha$ in grammar $R_{\rho(\alpha)-1}$ if $\rho(\alpha)>0$. Otherwise, if $\rho(\alpha)=0$ we assume that $L(\alpha)$ denotes the language of $\alpha$ in grammar $\mathcal{C}$. Finally, we say that two normal trees are similar if both start with the same combinator $X$ and are of equal length.

Example 2. Consider the following trees:
(i) $\alpha=S\left(K R_{1}\right) \mathcal{C}$, and
(ii) $\beta=K(\mathcal{C} S) R_{0}$.

Note that both $\alpha$ and $\beta$ are of size 3 and of equal length 2 , although they are not similar since both start with different combinators. Moreover, only $\alpha$ is normal as $\beta$ has a subtree $\mathcal{C} S$, which is of positive size, but does not start with a combinator. Since $\alpha$ contains a reference to $R_{1}$ and no other reduction grammar, its degree is equal to 2 , whereas the degree of $\beta$ is equal to 1 .

A crucial observation, which we are going to exploit in our construction, is the fact that normal trees preserve length of generated terms. In other words, if $\alpha$ is of length $m \geq 1$, then any term $x \in L(\alpha)$ is of length $m$ as well, i.e. $x=X x_{1} \ldots x_{m}$.
1.3. Pseudo-codes and implementation. We state our algorithm using functional pseudocodes formalising key design subroutines. The adopted syntax echoes basic Haskell notation and build-in primitives, though we use certain abbreviations making the overall presentation more comprehensible. And so, we use the following data structure representing normal trees.

```
-- | Normal trees.
data Tree = S | K | C | R Int
    | App Tree Tree
```

In our subroutines, we use the following 'syntactic sugar' abbreviating the structure of normal trees.

```
-- | Syntactic sugar.
X a_1 ... a_m := App X (App a_1 (... App a_{m-1} a_m) ...)
```

Moreover, we allow the use of this abbreviated notation in pattern matching, meaning that by writing ( X a_1 $\ldots$ a_m) we expect a complex tree of length $m$ for some $m \in \mathbb{N}$. If multiple arguments are supposed to share the same length, we use the same natural number $m$, e.g. ( $X \quad$ a_1 ... a_m) and ( $X$ b_1 ... b_m). A working Haskell implementation of our algorithm is available at [3].

## 2. Algorithm

The key idea used in the construction of reduction grammars is to generate new productions in $R_{n+1}$ based on the productions in $R_{n}$. Necessarily, any term normalizing in $n+1$ steps reduces directly to a term normalizing in $n$ steps, hence their syntactic structure should be
closely related. As the base of our inductive construction, we use the set of normal forms $R_{0}$ given by

$$
R_{0}:=S|K| S R_{0}\left|K R_{0}\right| S R_{0} R_{0} .
$$

Clearly, primitive combinators $S$ and $K$ are in normal form. If we take a normal form $x$, then both $S x$ and $K x$ are again normal since we did not create any new redex. For the same reason, any term $S x_{1} x_{2}$ where $x_{1}$ and $x_{2}$ are normal forms, is itself in normal form. And so, with the above grammar we have captured exactly all redex-free terms.

Let us consider productions of $R_{0}$. Note that from both the cases of $S R_{0}$ and $K R_{0}$ we can abstract a more general rule - if $x$ reduces in $n$ steps, then $S x$ and $K x$ reduce in $n$ steps as well, since after reducing $x$ we have no additional redexes left to consider. It follows that any $R_{n}$ should contain productions $S R_{n}$ and $K R_{n}$. Similarly, from the case of $S R_{0} R_{0}$ we can abstract a more general rule - if $S x_{1} x_{2}$ reduces in $n$ steps, then both $x_{1}$ and $x_{2}$ must reduce in total of $n$ steps. The normal-order reduction of $S x_{1} x_{2}$ proceeds to normalize $x_{1}$ and $x_{2}$ sequentially. As there is no head redex, after $n$ steps we obtain a term in normal form. And so, $R_{n}$ should also contain productions $S R_{i} R_{n-i}$ for $i \in\{0, \ldots, n\}$.
As we have noticed, all the above productions do not contain head redexes and hence do not increase the total amount of required reduction steps to normalize. Formalizing the above observations, we say that $\alpha$ is short if either $\alpha=X \alpha_{1}$ or $\alpha=S \alpha_{1} \alpha_{2}$. Otherwise, $\alpha$ is said to be long. Hence, we can set a priori the short productions of $R_{n}$ for $n \geq 1$ and continue to construct the remaining long productions. Naturally, as we consider terms over two primitive combinators $S$ and $K$, we distinguish two types of long productions, i.e. S- and K-Expansions.
2.1. K-Expansions. Let us consider a production $\alpha=X \alpha_{1} \ldots \alpha_{m}$ where $m \geq 0$. The set K-Expansions $(\alpha)$ is defined as

$$
\left\{K\left(X \alpha_{1} \ldots \alpha_{k}\right) \mathcal{C} \alpha_{k+1} \ldots \alpha_{m} \mid k \in\{0, \ldots, m-1\}\right\} .
$$

Proposition 3. Let $x \in L\left(K\left(X \alpha_{1} \ldots \alpha_{k}\right) \mathcal{C} \alpha_{k+1} \ldots \alpha_{m}\right)$. If $x \rightarrow_{w} y$, then $y \in L\left(X \alpha_{1} \ldots \alpha_{m}\right)$.
Proof. Let $x=K\left(X x_{1} \ldots x_{k}\right) z x_{k+1} \ldots x_{m}$. Consider its direct reduct $y=X x_{1} \ldots x_{k} x_{k+1} \ldots x_{m}$. Clearly, $x_{i} \in L\left(\alpha_{i}\right)$ for $i \in\{1, \ldots, m\}$ which finishes the proof.

In other words, the set K-Expansions $(\alpha)$ has the property that any K-Expansion of $\alpha$ generates terms that reduce in one step to terms generated by $\alpha$. If we compute the sets K-Expansions $(\alpha)$ for all productions $\alpha \in R_{n}$, we have almost constructed all of the long $K$-productions in $R_{n+1}$. What remains is to include the production $K R_{n} \mathcal{C}$ as any term $x \in$ $L\left(K R_{n} \mathcal{C}\right)$ reduces directly to $y \in L(\alpha)$ for some production $\alpha \in R_{n}$.

We use the following subroutine computing the set of K-Expansions of a given production.

```
-- | Returns K-Expansions of the given production.
kExpansions :: Tree -> [Tree]
kExpansions p = case p of
    (K a_1 ... a_m) -> kExpansions, K [a_1,...,a_m]
    (S a_1 ... a_m) -> kExpansions' S [a_1,...,a_m]
    where
                kExpansions' _ [] = []
            kExpansions' h [x_1,..., x_k] = K h C x_1 ... x_k
                : kExpansions' (App h x_1) [x_2,..., x_k]
```

2.2. S-Expansions. Let us consider a production $\alpha=X \alpha_{1} \ldots \alpha_{m}$ where $m \geq 0$. We would like to define the set $\operatorname{S-Expansions}(\alpha)$ similarly to $\operatorname{K-Expansions}(\alpha)$, i.e. in such a way that any term generated by an S-Expansion of $\alpha$ reduces in a single step to some $y \in L(\alpha)$. Unfortunately, defining and computing such a set is significantly more complex than the corresponding K-Expansions $(\alpha)$.

Let $q=X x_{1} \ldots x_{k} z(y z)$. Suppose that $q \in L(\alpha)$ for some production $\alpha \in R_{n}$. Evidently, $S\left(X x_{1} \ldots x_{k}\right) y z \rightarrow_{w} q$ and so we would like to guarantee that $q \in L(\beta)$ for some $\beta \in \operatorname{S-Expansions}(\alpha)$. Assume that $\alpha=X \alpha_{1} \ldots \alpha_{k} \gamma \delta$ where $z \in L(\gamma)$ and $y z \in L(\delta)$. Unfortunately, in order to guarantee that we capture all terms reducing to $\alpha$ via an $S$-redex and nothing more, we cannot use both $\gamma$ and $\delta$ directly. We require an additional 'rewriting' operation that would extract the important sublanguages of $\gamma$ and $\delta$ so that we can operate on them, instead of $\gamma$ and $\delta$.

Hence, let us consider the following rewriting relation $\triangleright$, extending the standard derivation relation:

$$
\alpha \triangleright \beta \Leftrightarrow \alpha \rightarrow \beta \vee\left(\alpha=\mathcal{C} \wedge \exists_{n \in \mathbb{N}} \beta=R_{n}\right)
$$

We use $\unrhd$ to denote the transitive-reflexive closure of $\triangleright$. The important property of $\unrhd$ is the fact that if $\alpha \unrhd \beta$, then $L(\beta) \subseteq L(\alpha)$. To denote the fact that $\alpha$ does not rewrite to $\beta$ and vice versa, we use the symbol $\alpha \| \beta$. In such case we say that $\alpha$ and $\beta$ are non-rewritable. Otherwise, if one of them rewrites to the other, meaning that $\alpha$ and $\beta$ are rewritable, we use the symbol $\alpha \triangleright \triangleleft \beta$.
2.2.1. Mesh Set. In the endeavour of finding appropriate S-Expansions rewritings, we need to find common meshes of given non-rewritable trees $\alpha \| \beta$. In other words, a complete partition of $L(\alpha) \cap L(\beta)$ using all possible trees $\gamma$ such that $\alpha, \beta \unrhd \gamma$. For this purpose, we use the following pseudo-code subroutines.

```
-- | Given X }\mp@subsup{\alpha}{1}{\ldots}.\ldots\mp@subsup{\alpha}{m}{}\mathrm{ and X }\mp@subsup{\beta}{1}{\ldots}\ldots\mp@subsup{\beta}{m}{}\mathrm{ computes
-- the family {}\mp@subsup{\gamma}{1}{},\ldots,\mp@subsup{\gamma}{m}{}}\mathrm{ of tree meshes.
mesh :: [Tree] -> [Tree] -> [[Tree]]
mesh (x : xs) (y : ys)
    | x 'rew' y = [y] : mesh xs ys -- case when x \unrhd y
    | y 'rew' x = [x] : mesh xs ys -- case when y \unrhd x
    | otherwise = meshSet x y : mesh xs ys -- case when x || y
mesh [] [] = []
```

The function Mesh, when given two similar productions $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$, constructs a family $\left\{\gamma_{i}\right\}_{i=1}^{m}$ where each $\gamma_{i}$ depends on the comparison of corresponding arguments. In the case when $x$ rewrites to $y$ (denoted as x 'rew' y in the pseudo-code) the singleton $\{y\}$ is constructed. Similarly, when $y \unrhd x$, the singleton $\{x\}$ is constructed. Otherwise, when $x$ and $y$ are both non-rewritable, $\gamma_{i}$ is computed using the MESHSET subroutine.

```
-- | Returns the mesh set of given trees.
meshSet :: Tree -> Tree -> [Tree]
meshSet (X a_1 ... a_m) (X b_1 ... b_m) =
    cartesian X [mesh a_i b_i | i <- [1..m]]
meshSet (R k) b @ (X b_1 ... b_m) =
    nub $ concatMap (\p -> meshSet p b) $ productions (R k)
meshSet b @ (X b_1 ... b_m) (R k) =
    nub $ concatMap (\p -> meshSet b p) $ productions (R k)
meshSet _ _ = []
```

When given two similar trees $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$, MeshSet computes meshes $\gamma_{1}, \ldots, \gamma_{m}$ of corresponding arguments $\alpha_{i}$ and $\beta_{i}$ using the subroutine Mesh. Next, argument meshes $\left\{\gamma_{i}\right\}_{i=1}^{m}$ are used to construct meshes for $\alpha$ and $\beta$, using the subroutine CARTESIAN which computes the Cartesian product $\{X\} \times \gamma_{1} \times \cdots \times \gamma_{m}$ using term application. In the case when one of MeshSet's argument is a reduction grammar $R_{k}$ and the other $\alpha$ is complex, MeshSet computes recursively mesh sets of $\alpha$ and each production $\delta \in R_{k}$, outputting their set-theoretic union. In any other case, MeshSet returns the empty set.

Example 4. Let $\alpha=K \mathcal{C} R_{0} S$ and $\beta=K S\left(S R_{0} \mathcal{C}\right) S$. Consider MeshSet $(\alpha, \beta)$. Both $\alpha$ and $\beta$ are similar and complex, hence MeshSet proceeds directly to construct mesh sets of corresponding arguments of $\alpha$ and $\beta$. Since $\mathcal{C} \unrhd S$, we get $\gamma_{1}=\{S\}$. Then, as both $R_{0}$ and $S R_{0} \mathcal{C}$ are non-rewritable, $\gamma_{2}=\operatorname{MeshSet}\left(R_{0}, S R_{0} \mathcal{C}\right)$. It follows that $\operatorname{MeshSet}\left(R_{0}, S R_{0} \mathcal{C}\right)$ is equal to $\bigcup_{\delta \in R_{0}} \operatorname{MeshSet}\left(\delta, S R_{0} \mathcal{C}\right)$. Further inspection reveals that $\operatorname{MeshSet}\left(R_{0}, S R_{0} \mathcal{C}\right)=\left\{S R_{0} R_{0}\right\}$ and thus $\gamma_{2}=\left\{S R_{0} R_{0}\right\}$. Finally, $\gamma_{3}=\{S\}$ as $S$ rewrites trivially to itself. Since each $\gamma_{i}$ is a singleton, it follows that

$$
\operatorname{MeshSet}(\alpha, \beta)=\left\{K S\left(S R_{0} R_{0}\right) S\right\} .
$$

We leave the analysis of MeshSet until we fully define the construction of reduction grammars $\left\{R_{n}\right\}_{n \in \mathbb{N}}$.
2.2.2. Rewriting Set. Consider again our previous example of $q=X x_{1} \ldots x_{k} z(y z) \in L(\alpha)$ where $\alpha=X \alpha_{1} \ldots \alpha_{k} \gamma \delta$ such that both $z \in L(\gamma)$ and $y z \in L(\delta)$. In order to capture terms reducing to $\alpha$ via an $S$-redex, we need to find all pairs of trees $\eta, \zeta$ such that $\gamma \unrhd \zeta$ and $\delta \unrhd \eta \zeta$. Since such pairs of trees follow exactly the structure of $z(y z)$ we can use them to define the set S-Expansions $(\alpha)$. And so, to find such rewriting pairs, we use the following RewritingSet pseudo-code subroutine.

```
-- | Given \alpha and \beta computes their rewriting set.
rewritingSet :: Tree -> Tree -> [Tree]
rewritingSet a S = []
rewritingSet a K = []
rewritingSet a C = [C a]
rewritingSet a (R k) =
    nub $ concatMap (\p -> rewritingSet a p) $ productions (R k)
rewritingSet a (X b_1 ... b_m)
    | a 'rew' b_m => [X b_1 ... b_m]
    | b_m 'rew' a => [X b_1 ... b_{m-1} a]
    | otherwise =>
                cartesian (X b_1 ... b_{m-1}) [meshSet a b_m]
```

The outcome of RewritingSet $(\alpha, \beta)$ depends on $\beta$ 's structure. If $\beta$ is a primitive combinator $S$ or $K$, RewritingSet returns the empty set. If $\beta=\mathcal{C}$, a singleton $\{\mathcal{C} \alpha\}$ is returned. When $\beta=R_{k}$ for some $k \in \mathbb{N}$, RewritingSet computes recursively the rewriting sets of $\alpha$ and $\gamma \in R_{k}$, outputting their set-theoretic union. Otherwise when $\beta=X \beta_{1} \ldots \beta_{m}$, RewritingSet determines whether $\alpha \triangleright \triangleleft \beta_{m}$. If $\alpha \unrhd \beta_{m}$, a singleton $\left\{X \beta_{1}, \ldots, \beta_{m}\right\}$ is returned. Conversely, in the case of $\beta_{m} \unrhd \alpha$, RewritingSet returns $\left\{X \beta_{1}, \ldots, \beta_{m-1} \alpha\right\}$. Finally if $\alpha$ and $\beta_{m}$ are non-rewritable, RewritingSet invokes the Cartesian subroutine computing the Cartesian product of $\left\{X \beta_{1}, \ldots, \beta_{m-1}\right\} \times \operatorname{MeshSet}\left(\alpha, \beta_{m}\right)$ using term application, passing afterwards its result as the computed rewriting set.
Example 5. Let us consider the rewriting set RewritingSet ( $S, R_{0}$ ). Since $\beta=R_{0}$, we know that RewritingSet $\left(S, R_{0}\right)=\bigcup_{\gamma \in R_{0}}$ RewritingSet $(S, \gamma)$. It follows therefore that in order to compute RewritingSet $\left(S, R_{0}\right)$, we have to consider rewriting sets involving productions of $R_{0}$. Note that both productions $S$ and $K$ do not contribute new trees. It remains to consider productions $S R_{0}, K R_{0}$ and $S R_{0} R_{0}$. Evidently, each of them is complex and has $R_{0}$ as its final argument. Hence, their corresponding rewriting sets are $S S, K S$ and $S R_{0} S$, respectively. And so, we obtain that

$$
\text { RewritingSet }\left(S, R_{0}\right)=\left\{S S, K S, S R_{0} S\right\} .
$$

Similarly to the case of MeshSet, we postpone the analysis until we define the construction of $\left\{R_{n}\right\}_{n \in \mathbb{N}}$.

Equipped with the notion of mesh and rewriting sets, we are ready to define the set of SExpansions. And so, let $\alpha=X \alpha_{1} \ldots \alpha_{m}$ where $m \geq 0$. The set S-Expansions $(\alpha)$ is defined
as

$$
\left\{S\left(X \alpha_{1} \ldots \alpha_{k}\right) \varphi_{l} \varphi_{r} \alpha_{k+3} \ldots \alpha_{m} \mid k \in\{0, \ldots, m-2\}\right\}
$$

where $\left(\varphi_{l} \varphi_{r}\right) \in \operatorname{RewritingSET}\left(\alpha_{k+1}, \alpha_{k+2}\right)$. We use the following subroutine computing the set of S-Expansions for a given $\alpha$.

```
-- | Returns S-Expansions of the given production.
sExpansions :: Tree -> [Tree]
sExpansions \(p=c a s e p\) of
    (K a_1 ... a_m) -> sExpansions' K [a_1,..., a_m]
    (S a_1 ... a_m) -> sExpansions' \(S\) [a_1,..., a_m]
    where
        sExpansions, _ [] = []
        sExpansions, \(\quad\left[\_\right]=[]\)
        sExpansions, \(h\) [ \(\left.x_{-} 1, x_{-} 2, \ldots, x_{-} k\right]=\)
                \(\operatorname{map}\left(\backslash(A p p l r)->S h 1 r x_{-} 3 \ldots x_{-}\right)\)
                (rewritingSet \(x_{-} 1 x_{-}\)) ++
                sExpansions, (App h x_1) [x_2,..., \(x_{\_} m\) ]
```

Proposition 6. Let $x \in L\left(S\left(X \alpha_{1} \ldots \alpha_{k}\right) \varphi_{l} \varphi_{r} \alpha_{k+3} \ldots \alpha_{m}\right)$. If $x \rightarrow_{w} y$, then $y \in L\left(X \alpha_{1} \ldots \alpha_{k} \varphi_{r}\left(\varphi_{l} \varphi_{r}\right) \alpha_{k+3} \ldots \alpha_{m}\right)$.
Proof. Let $x=S\left(X x_{1} \ldots x_{k}\right) w z x_{k+3} \ldots x_{m}$. Let us consider its direct reduct $y$ in form of $X x_{1} \ldots x_{k} z(w z) x_{k+3} \ldots x_{m}$. Clearly, $x_{i} \in L\left(\alpha_{i}\right)$ for $i$ in proper range. Moreover, both $w \in L\left(\varphi_{l}\right)$ and $z \in L\left(\varphi_{r}\right)$, which finishes the proof.
2.3. Algorithm pseudo-code. With the complete and formal definitions of both S- and KExpansions we are ready to give the main algorithm Reduction Grammar, which for given $n \in \mathbb{N}$ constructs the grammar $R_{n}$.

```
-- | Given }n\in\mathbb{N}\mathrm{ constructs }\mp@subsup{R}{n}{
reductionGrammar :: Integer -> [Tree]
reductionGrammar 0 = [S, K, S (R 0), K (R 0), S (R 0) (R 0)]
reductionGrammar n = [S (R n), K (R n)]
    ++ [S (R $ n-i) R_i | i <- [0..n]]
    ++ [K (R $ n-1) C]
    ++ concatMap kExpansions (reductionGrammar $ n-1)
    ++ concatMap sExpansions (reductionGrammar $ n-1)
```

Example 7. Let us consider $\alpha=S S S R_{0}$. Since $\alpha \in \operatorname{S-Expansions}\left(S R_{0} R_{0}\right)$ we get $\alpha \in R_{1}$. Note that S-Expansions $(\alpha)$ contains $\beta_{1}=S(S S) S S$ and $\beta_{2}=S(S S) K S$. It follows that $\beta_{1}, \beta_{2} \in R_{2}$.

## 3. Analysis

3.1. Tree potential. Most of our proofs in the following sections are using inductive reasoning on the underlying tree structure. Unfortunately, in certain cases most natural candidates for induction such as tree size fail due to self-referencing productions, i.e. productions of $R_{n}$ which explicitly use the non-terminal symbol $R_{n}$. In order to remedy such problems, we introduce the notion of tree potential $\pi(\alpha)$, defined inductively as

$$
\begin{gathered}
\pi(S)=\pi(K)=\pi(\mathcal{C})=0 \\
\pi\left(X \alpha_{1} \ldots \alpha_{m}\right)=m+\sum_{i=1}^{m} \pi\left(\alpha_{i}\right) \\
\pi\left(R_{n}\right)=1+\max _{\gamma \in \Phi\left(R_{n}\right)} \pi(\gamma)
\end{gathered}
$$

where $\Phi\left(R_{n}\right)$ denotes the set of productions of $R_{n}$ which do not use the non-terminal symbol $R_{n}$. Note that such a definition of potential is almost identical to the notion of tree size. The potential of $\alpha$ is the sum of $\alpha$ 's size and the weighted sum of all non-terminal grammar symbols occurring in $\alpha$.

Immediately from the definition we get $\pi\left(R_{0}\right)=1$. Moreover, $\pi\left(R_{n+1}\right)>\pi\left(R_{n}\right)$ for any $n \in \mathbb{N}$. Indeed, let $\alpha \in R_{n}$ be the witness of $R_{n}$ 's potential. Clearly, $(K \alpha \mathcal{C}) \in \Phi\left(R_{n+1}\right)$ and so $R_{n+1}$ has necessarily greater potential. Moreover, $\pi(\alpha)>\pi(\beta)$ if $\beta$ is a subtree of $\alpha$. It follows that the notion of tree potential is a good candidate for the intuitive tree complexity measure.
3.2. Soundness. In this section we are interested in the soundness of Reduction Grammar. In particular, we prove that it is computable, terminates on all legal inputs and, for given $n$, constructs a reduction grammar $R_{n}$ generating only terms that require exactly $n$ steps to normalize.

Let us start with showing that the rewriting relation is decidable.
Proposition 8. It is decidable to check whether $\alpha \unrhd \beta$.
Proof. Induction over $n=\pi(\alpha)+\pi(\beta)$. If $\alpha=X$, then the only tree $\alpha$ rewrites to is $X$. On the other hand, if $\alpha=\mathcal{C}$, then $\alpha$ rewrites to any $\beta$. And so, it is decidable to check whether $\alpha \unrhd \beta$ in case $n=0$. Now, let us assume that $n>0$. We have two remaining cases to consider.
(i) If $\alpha=X \alpha_{1} \ldots \alpha_{m}$, then $\alpha \unrhd \beta$ if and only if $\beta=X \beta_{1} \ldots \beta_{m}$ and $\alpha_{i} \unrhd \beta_{i}$ for all $i \in$ $\{1, \ldots, m\}$. Since the total potential of $\pi\left(\alpha_{i}\right)+\pi\left(\beta_{i}\right)$ is less than $n$, we can use the induction hypothesis to decide whether all arguments of $\alpha$ rewrite to the respective arguments of $\beta$. It follows that we can decide whether $\alpha \unrhd \beta$.
(ii) If $\alpha=R_{k}$, then clearly $\alpha \unrhd \beta$ if and only if $\beta=R_{k}$ or there exists a production $\gamma \in R_{k}$ such that $\gamma \unrhd \beta$. Let us assume that $\gamma$ is a production of $R_{k}$. Note that if $\gamma \unrhd \beta$, then $\gamma$ and $\beta$ are similar. And so, since similarity is decidable, we can rephrase our previous observation as $\alpha \unrhd \beta$ if and only if $\beta=R_{k}$ or there exists a production $\gamma \in R_{k}$ such that $\gamma$ is similar to $\beta$ and $\gamma \unrhd \beta$. Checking whether $\beta=R_{k}$ is trivial, so let us assume the other option and start with the case when $\gamma$ is a short production referencing $R_{k}$.

If $\gamma=X R_{k}$ is similar to $\beta=X \beta_{1}$, we know that $\gamma \unrhd \beta$ if and only if $R_{k} \unrhd \beta_{1}$. Since $\pi\left(R_{k}\right)+\pi\left(\beta_{1}\right)<n$, we know that checking whether $R_{k} \unrhd \beta_{1}$ is decidable, hence so is $\gamma \unrhd \beta$.

Let us assume w.l.o.g. that $\gamma=S R_{k} R_{0}$. Clearly, $\beta=S \beta_{1} \beta_{2}$. And so, $\gamma \unrhd \beta$ if and only if $R_{k} \unrhd \beta_{1}$ and $R_{0} \unrhd \beta_{2}$. Notice that $\pi\left(R_{k}\right)+\pi\left(\beta_{1}\right)<n$ as well as $\pi\left(R_{0}\right)+\pi\left(\beta_{2}\right)<n$. Using the induction hypothesis to both, we get that checking $R_{k} \unrhd \beta_{1}$ and $R_{0} \unrhd \beta_{2}$ is decidable, hence so is $\alpha \unrhd \beta$.

Finally, if $\gamma$ is a long production we can rewrite it as $\gamma=X \gamma_{1} \ldots \gamma_{m}$, and so reduce this case to the previous one when both trees are complex, as $\pi(\gamma)$ is necessarily smaller than $n$.

Proposition 9. Let $\alpha, \beta$ be two trees. Then, both $\alpha \unrhd \gamma$ and $\beta \unrhd \gamma$ for arbitrary $\gamma \in$ $\operatorname{MeshSet}(\alpha, \beta)$.

Proof. Induction over $n=\pi(\alpha)+\pi(\beta)$. Let $M=\operatorname{MeshSET}(\alpha, \beta)$. Clearly, it suffices to consider such $\alpha, \beta$ that $M \neq \emptyset$.

Let us assume that both $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. If $\alpha_{i} \triangleright \triangleleft \beta_{i}$ for all $i \in$ $\{1, \ldots, m\}$, then $M$ consists of a single tree $\gamma=X \gamma_{1} \ldots \gamma_{m}$ for which $\alpha_{i}, \beta_{i} \unrhd \gamma_{i}$. Evidently, our
claim holds. Suppose that there exists an $i \in\{1, \ldots, m\}$ such that $\alpha_{i} \| \beta_{i}$. Since $\pi\left(\alpha_{i}\right)+\pi\left(\beta_{i}\right)<$ $n$, we can apply the induction hypothesis to $\operatorname{MeshSet}\left(\alpha_{i}, \beta_{i}\right)$. The set $M^{\prime}=\operatorname{MeshSet}\left(\alpha_{i}, \beta_{i}\right)$ cannot be empty and so let $\delta_{i}$ be an arbitrary mesh in $M^{\prime}$. We know that $\alpha_{i}, \beta_{i} \unrhd \delta_{i}$. And so, if we consider an arbitrary $\gamma=X \gamma_{i} \ldots \gamma_{m} \in M$, we get $\alpha_{i}, \beta_{i} \unrhd \gamma_{i}$ for all $i \in\{1, \ldots, m\}$, which implies our claim.

What remains is to consider the case when either $\alpha=R_{k}$ and $\beta$ is complex or, symmetrically, $\beta=R_{k}$ and $\alpha$ is complex. Let us assume w.l.o.g. the former case. From the definition, $\operatorname{MeshSet}\left(R_{k}, \beta\right)$ depends on the union of $\operatorname{MeshSet}(\gamma, \beta)$ for $\gamma \in R_{k}$. Clearly, $R_{k}$ rewrites to any of its productions. Let $\gamma \in R_{k}$ be a production referencing $R_{k}$. We have to consider two cases based on the structure of $\gamma$.
(i) Let $\gamma=X R_{k}$. Then, $\pi(\gamma)=\pi\left(R_{k}\right)+1$ and so we cannot use the induction hypothesis to $\operatorname{MeshSet}(\gamma, \beta)$ directly. Note however, that we can assume that $\beta=X \beta_{1}$, since otherwise $\operatorname{MeshSet}(\gamma, \beta)$ would be empty. Therefore, we know that $\operatorname{MeshSet}\left(R_{k}, \beta_{1}\right) \neq \emptyset$ to which we can now use the induction hypothesis, as $\pi\left(R_{k}\right)+\pi\left(\beta_{1}\right)<n$. Immediately, we get that $R_{k}, \beta \unrhd \gamma$.
(ii) W.l.o.g. let $\gamma=S R_{k} R_{0}$. Then, $\pi(\gamma)=3+\pi\left(R_{k}\right)$. Again, we cannot directly use the induction hypothesis. Note however, that we can assume that $\beta=S \beta_{1} \beta_{2}$. And so we get $\pi\left(R_{k}\right)+\pi\left(\beta_{1}\right)<n$ and $\pi\left(R_{0}\right)+\pi\left(\beta_{2}\right)<n$. Using the induction hypothesis to both parts we conclude that $R_{k}, \beta \unrhd \gamma$ in this case as well.
To finish the proof we need to show that our claim holds for all $\gamma \in R_{k}$ which do not reference $R_{k}$. Indeed, any such production has necessarily smaller potential than $R_{k}$, and so, we can use the induction hypothesis directly to the resulting mesh set. Evidently, our claim holds.

In other words, $\operatorname{MeshSet}(\alpha, \beta)$ is in fact a set of meshes, i.e. trees generating a joint portion of $L(\alpha)$ and $L(\beta)$. Note, that along the lines of proving the above proposition, we have also showed that indeed $\operatorname{MeshStet}(\alpha, \beta)$ terminates on all legal inputs, as the number of recursive calls cannot exceed $2(\pi(\alpha)+\pi(\beta))$ - in the worst case, every second recursive call decreases the total potential sum of its inputs.

Proposition 10. Let $\alpha, \beta$ be two trees. Then, $\alpha \unrhd \varphi_{r}$ and $\beta \unrhd \varphi_{l} \varphi_{r}$ for arbitrary $\varphi_{l} \varphi_{r} \in$ RewritingSet $(\alpha, \beta)$.

Proof. We can assume that Rewriting $\operatorname{Set}(\alpha, \beta) \neq \emptyset$, as otherwise our claim trivially holds. Let $\varphi_{l} \varphi_{r} \in \operatorname{RewritingSet}(\alpha, \beta)$. Based on the structure of $\beta$, we have to three cases to consider.
(i) If $\beta=\mathcal{C}$, then $\varphi_{l} \varphi_{r}=\mathcal{C} \alpha$. Clearly, $\alpha \unrhd \alpha$ and $\mathcal{C} \unrhd \mathcal{C} \alpha$.
(ii) If $\beta=X \beta_{1} \ldots \beta_{m}$, then we have again exactly three possibilities. Both cases when $\alpha \triangleright \triangleleft$ $\beta_{m}$ are trivial, so let us assume that $\alpha \| \beta_{m}$. It follows that there exists such a $\gamma \in$ $\operatorname{MeshSet}\left(\alpha, \beta_{m}\right)$ that $\varphi_{l} \varphi_{r}=X \beta_{1} \ldots \beta_{m-1} \gamma$. Due to Proposition (9) we know that $\alpha, \beta_{m} \unrhd$ $\gamma$ and so directly that $\alpha \unrhd \varphi_{r}$ and $\beta \unrhd \varphi_{l} \varphi_{r}$.
(iii) Finally, suppose that $\beta=R_{n}$. Then, there exists a production $\gamma \in R_{n}$ such that $\varphi_{l} \varphi_{r} \in$ Rewriting $\operatorname{Set}(\alpha, \gamma)$. Note however, that in this case $\gamma=X \gamma_{1} \ldots \gamma_{m}$ and so we can reduce this case to the already considered case above.

Now we are ready to give the anticipated soundness theorem.
Theorem 11 (Soundness). If $x \in L\left(R_{n}\right)$, then $x$ reduces in $n$ steps.
Proof. Induction over pairs $(n, m)$ where $m$ denotes the length of a minimal, in terms of length, derivation $\Sigma$ of $x \in L\left(R_{n}\right)$. Let $n=0$ and so $x \in L\left(R_{0}\right)$. If $m=1$, then $x \in\{S, K\}$ hence $x$ is already in normal form. Suppose that $m>1$. Clearly, $x \notin\{S, K\}$. Let $R_{0} \rightarrow \alpha$ be the first production rule used in derivation $\Sigma$. Using the induction hypothesis to the reminder of the derivation, we know that $x$ does not contain any nested redexes. Moreover, $\alpha$ avoids any head redexes and so we get that $x$ is in normal form.

Let $n>0$. We have to consider several cases based on the choice of the first production rule $R_{n} \rightarrow \alpha$ used in the derivation $\Sigma$.
(i) $\alpha=S R_{n}$ or $\alpha=K R_{n}$. Using the induction hypothesis we know that $x=X y$ where $y$ reduces in $n$ steps. Clearly, so does $x$.
(ii) $\alpha=S R_{n-i} R_{i}$ for some $i \in\{0, \ldots, n\}$. Then, $x=S y z$ where $y \in L\left(R_{n-i}\right)$ and $z \in L\left(R_{i}\right)$. Note that both their derivations are in fact shorter than the derivation of $x$ and thus applying the induction hypothesis to both $y$ and $z$ we know that they reduce in $n-i$ and $i$ steps, respectively. Following the normal-order reduction strategy, we note that $y$ and $z$ and reduce sequentially in $x$. Since $x$ does not contain a head redex itself, we reduce it in total of $n$ reductions.
(iii) $\alpha=K R_{n-1} \mathcal{C}$. Directly from the induction hypothesis we know that $x=K y z$ where $y$ reduces in $n-1$ steps. And so $x \rightarrow_{w} y$, implying that $x$ reduces in $n$ steps.
(iv) $\alpha=K\left(X \alpha_{1} \ldots \alpha_{k}\right) \mathcal{C} \alpha_{k+1} \ldots \alpha_{m}$. Let $x \in L(\alpha)$. Clearly, $x$ has a head redex and so let $x \rightarrow_{w} y$. Using Proposition 3, we know that $y \in L\left(X \alpha_{1} \ldots \alpha_{m}\right)$. Moreover, by the construction of $R_{n}$ we get $\alpha \in \mathrm{K}$-Expansions $\left(X \alpha_{1} \ldots \alpha_{m}\right)$ and therefore $y \in L\left(R_{n-1}\right)$. It follows that $y$ reduces in $n-1$ steps and so $x$ in $n$ steps.
(v) $\alpha=S\left(X \alpha_{1} \ldots \alpha_{k}\right) \varphi_{l} \varphi_{r} \alpha_{k+3} \ldots \alpha_{m}$. Let $x \in L(\alpha)$. Clearly, $x$ has a head redex and so let $x \rightarrow_{w} y$. Due to Proposition 6 we get that $y \in L\left(X \alpha_{1} \ldots \alpha_{k} \varphi_{r}\left(\varphi_{l} \varphi_{r}\right) \alpha_{k+3} \ldots \alpha_{m}\right)$. In order to show that $x$ reduces in $n$ steps it suffices to show that $y \in L\left(R_{n-1}\right)$. Let us consider $\beta$ such that $\alpha \in \operatorname{S-Expansions}(\beta)$. From the structure of $\alpha$ we can rewrite it as $\beta=X \alpha_{1} \ldots \alpha_{k} \alpha_{k+1} \alpha_{k+2} \ldots \alpha_{m}$. Moreover, from Proposition 10 we know that $\alpha_{k+1} \unrhd \varphi_{r}$ and $\alpha_{k+2} \unrhd \varphi_{l} \varphi_{r}$. Clearly, $y \in L(\beta)$, which finishes the proof.

Combining the above result with the fact that each normalizing combinatory logic term reduces in a determined number of normal-order reduction steps, gives us the following corollary.

Corollary 12. If $L\left(R_{n}\right) \cap L\left(R_{m}\right) \neq \emptyset$, then $n=m$.
3.3. Completeness. In this section we are interested in the completeness of Reduction Grammar. In other words, we show that every term normalizing in exactly $n$ steps is generated by $R_{n}$.

W start with some auxiliary lemmas showing the completeness of MeshSet and, in consequence, RewritingSet.

Lemma 13. Let $\alpha, \beta$ be two non-rewritable trees. Let $x$ be a term. Then, $x \in L(\alpha) \cap L(\beta)$ if and only if there exists a mesh $\gamma \in \operatorname{MeshSet}(\alpha, \beta)$ such that $x \in L(\gamma)$.

Proof. It suffices to show the necessary part, the sufficiency is clear from Proposition 0 We show this result using induction over the size $|x|$ of $x$. Let $x \in L(\alpha) \cap L(\beta)$. Let us start with noticing that $|\alpha|+|\beta|>0$. Moreover, there are only two cases where $x \in L(\alpha) \cap L(\beta)$, i.e. when either $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$ or when exactly one of them is equal to some $R_{n}$ and the other is complex. And so, let us consider these cases separately.
(i) Suppose that $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. It follows that we can rewrite $x$ as $X x_{1} \ldots x_{m}$ such that $x_{i} \in L\left(\alpha_{i}\right) \cap L\left(\beta_{i}\right)$. Clearly, if all $\alpha_{i} \triangleright \triangleleft \beta_{i}$, then there exists a mesh $\gamma$ such that $x \in L(\gamma)$. Let us assume that some $\alpha_{i}$ and $\beta_{i}$ are non-rewritable. Then, using the induction hypothesis we find a mesh $\gamma_{i} \in \operatorname{MeshSet}\left(\alpha_{i}, \beta_{i}\right)$ such that $x_{i} \in L\left(\gamma_{i}\right)$. Immediately, we get that there exists a mesh in $\operatorname{MeshSet}(\alpha, \beta)$ which generates $x$.
(ii) Let us assume w.l.o.g. that $\alpha=R_{n}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. Since $x \in L\left(R_{n}\right)$, there must be such a production $\gamma \in R_{n}$ that $x \in L(\gamma)$. Although the size of $x$ does not decrease, note that we can reduce this case to the one considered above since both $\gamma$ and $\beta$ are complex. Clearly, it follows that we can find a suiting mesh $\delta \in \operatorname{MeshSet}(\gamma, \beta)$ such that $x \in L(\delta)$. Immediately, we get $\delta \in \operatorname{MeshSet}(\alpha, \beta)$ which finishes the proof.

Lemma 14. Let $\alpha, \beta$ be two trees. Let $x, y x$ be two terms. Then, $x \in L(\alpha)$ and $y x \in L(\beta)$ if and only if there exists such a $\varphi_{l} \varphi_{r} \in \operatorname{RewritingSet}(\alpha, \beta)$ that $x \in L\left(\varphi_{r}\right)$ and $y x \in L\left(\varphi_{l} \varphi_{r}\right)$.

Proof. Due to Proposition 10 the sufficiency part is clear. What remains is to show the necessary part. Let $x \in L(\alpha)$ and $y x \in L(\beta)$. Consider the structure of $\beta$. If $\beta=\mathcal{C}$, then $\mathcal{C} \alpha \in$ RewritingSet $(\alpha, \beta)$ and so $\varphi_{l}=\mathcal{C}, \varphi_{r}=\alpha$. Clearly, our claim holds. Now, consider the case when $\beta=X \beta_{1} \ldots \beta_{m}$. Based on the rewritability of $\alpha$ and $\beta_{m}$ we distinguish three subcases.
(i) If $\alpha \unrhd \beta_{m}$, then $X \beta_{1} \ldots \beta_{m} \in \operatorname{RewritingSet}(\alpha, \beta)$. Since $y x \in L(\beta)$, we get $x \in L\left(\beta_{m}\right)$ and in consequence $x \in L\left(\varphi_{r}\right)$.
(ii) If $\beta_{m} \unrhd \alpha$, then $X \beta_{1} \ldots \beta_{m-1} \alpha \in \operatorname{RewritingSet}(\alpha, \beta)$. Since $\beta_{m} \unrhd \alpha$, we know that $L(\alpha) \subseteq L\left(\beta_{m}\right)$ and so $y x \in L\left(X \beta_{1} \ldots \beta_{m-1} \alpha\right)$.
(iii) If $\alpha \| \beta_{m}$, then we know that $x \in L(\alpha) \cap L\left(\beta_{m}\right)$. If not, then $y x$ could not be a term of $L(\beta)$. And so, using Lemma 13 we find a mesh $\gamma \in \operatorname{MeshSet}\left(\alpha, \beta_{m}\right)$ such that $x \in L(\gamma)$. We know that $X \beta_{1} \ldots \beta_{m-1} \gamma \in \operatorname{Rewriting} \operatorname{Set}(\alpha, \beta)$. Clearly, it is the tree we were looking for.
It remains to consider the case when $\beta=R_{k}$. Note however, that it can be reduced to the case when $\beta=X \beta_{1} \ldots \beta_{m}$. Indeed, since $x \in L\left(R_{k}\right)$, then there exists a production $\gamma \in R_{k}$ such that $x \in L(\gamma)$. From the previous arguments we know that we can find a tree satisfying our claim.

Using the above completeness results for MeshSet and RewritingSet, we are ready to give the anticipated completeness result of $\left\{R_{n}\right\}_{n \in \mathbb{N}}$.
Theorem 15 (Completeness). If $x$ reduces in $n$ steps, then $x \in L\left(R_{n}\right)$.
Proof. Induction over pairs $(n, s)$ where $s$ denotes the size of $x$. The base case $n=0$ is clear due to the completeness of $R_{0}$. Let $n>0$.

Let us start with considering short terms. Let $x=X y$ be a term of size $s$. Since $x$ has no head redex, $y$ must reduce in $n$ steps as well. Now, we can apply the induction hypothesis to $y$ and deduce that $y \in L\left(R_{n}\right)$. It follows that $x \in L\left(X R_{n}\right)$. Clearly, $X R_{n}$ is a production of $R_{n}$ and so $x \in L\left(R_{n}\right)$. Now, assume that $x=$ Syz. Since $x$ reduces in $n$ steps and does not contain a head redex, there exists such an $i \in\{0, \ldots, n\}$ that $y$ reduces in $i$ steps and $z$ reduces in $n-i$ steps. Applying the induction hypothesis to both $y$ and $z$, we get that $y \in L\left(R_{i}\right)$ whereas $z \in L\left(R_{n-i}\right)$. Immediately, we get that $x \in L\left(R_{n}\right)$ as $S R_{i} R_{n-i} \in R_{n}$.

What remains is to consider long terms. Let $x=K x_{1} x_{2}$. Note that $x_{1}$ must reduce in $n-1$ steps, as $x \rightarrow_{w} x_{1}$. And so, from the induction hypothesis we get that $x_{1} \in L\left(R_{n-1}\right)$. Now we have $x \in L\left(K R_{n-1} \mathcal{C}\right)$ and hence $x \in L\left(R_{n}\right)$ as $K R_{n-1} \mathcal{C}$ is a production of $R_{n}$.

Now, let $x=K x_{1} \ldots x_{m}$ for $m \geq 3$. Since $x$ has a head redex, we know that $x \rightarrow_{w} y=$ $x_{1} x_{3} \ldots x_{m}$, which itself reduces in $n-1$ steps. Let us rewrite $y$ as $X y_{1} \ldots y_{k} x_{3} \ldots x_{m}$ where $x_{1}=X y_{1} \ldots y_{k}$. We know that there exists a production $\alpha \in R_{n-1}$ such that $y \in L(\alpha)$. Let $\alpha=$ $X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}} \alpha_{3} \ldots \alpha_{m}$. Clearly, there exists a $\beta=K\left(X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}}\right) \mathcal{C} \alpha_{3} \ldots \alpha_{m} \in \operatorname{K-Expansions}(\alpha)$. We claim that $x \in L(\beta)$. Indeed, $y \in L(\alpha)$ implies that $y_{i} \in L\left(\overline{\alpha_{i}}\right)$ and $x_{j} \in L\left(\alpha_{j}\right)$ for any $i$ and $j$ in proper ranges. Since $x_{2} \in L(\mathcal{C})$, we conclude that $x \in L(\beta)$ and hence $x \in L\left(R_{n}\right)$.

Let $x=S x_{1} \ldots x_{m}$ for $m \geq 3$. Since $x$ has a head redex $x \rightarrow_{w} y=x_{1} x_{3}\left(x_{2} x_{3}\right) x_{4} \ldots x_{m}$ which reduces in $n-1$ steps. Again, let us rewrite $y$ as $X y_{1} \ldots y_{k} x_{3}\left(x_{2} x_{3}\right) x_{4} \ldots x_{m}$ where $x_{1}=$ $X y_{1} \ldots y_{k}$. Now, since $y \in L\left(R_{n-1}\right)$, there exists a production $\alpha=X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}} \alpha_{3} \gamma \alpha_{4} \ldots \alpha_{m} \in$ $R_{n-1}$ such that $y \in L(\alpha)$. We claim that there must be a production $\beta \in \operatorname{S-Expansions}(\alpha)$ such that $x \in L\left(R_{n}\right)$. If so, the proof would be complete. Notice that $x_{3} \in L\left(\alpha_{3}\right)$ and $x_{2} x_{3} \in L(\gamma)$. Using Lemma 14 we know that there exists a tree $\varphi_{l} \varphi_{r} \in \operatorname{Rewriting} \operatorname{Set}\left(\alpha_{3}, \gamma\right)$ such that $x_{3} \in L\left(\varphi_{r}\right)$ and $\left(x_{2} x_{3}\right) \in L\left(\varphi_{l} \varphi_{r}\right)$. And so $y \in L\left(X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}} \varphi_{r}\left(\varphi_{l} \varphi_{r}\right) \alpha_{4} \ldots \alpha_{m}\right)$. Moreover, due to the fact that $\varphi_{l} \varphi_{r} \in \operatorname{RewritingSet}\left(\alpha_{3}, \gamma\right)$, we know that the tree $\beta=$ $S\left(X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}}\right) \varphi_{l} \varphi_{r} \alpha_{4} \ldots \alpha_{m} \in \operatorname{S-ExpAnsions}(\alpha)$ and so also $\beta \in R_{n}$. Since $x_{2} \in L\left(\varphi_{l}\right)$, we get that $x \in L(\beta)$.
3.4. Unambiguity. In this section we show that reduction grammars are in fact unambiguous, i.e. every term $x \in L\left(R_{n}\right)$ has exactly one derivation. Due to the mutual recursive nature of MeshSet, RewritingSet and ReductionGrammar, we split the proof into two separate parts. In the following lemma, we show that MeshSet returns unambiguous meshes under the assumption that $R_{0}, \ldots, R_{n}$ up to some $n$ are themselves unambiguous. In the corresponding theorem we use inductive reasoning which supplies the aforementioned assumption and thus, as a consequence, allows us to prove the main result.

Lemma 16. Let $\alpha, \beta$ be two trees such that $\gamma, \bar{\gamma} \in \operatorname{MeshSet}(\alpha, \beta)$ where in addition $\rho(\alpha), \rho(\beta) \leq$ $r+1$. If $R_{0}, \ldots, R_{r}$ are unambiguous and $L(\gamma) \cap L(\bar{\gamma}) \neq \emptyset$, then $\gamma=\bar{\gamma}$.

Proof. Induction over $n=\pi(\alpha)+\pi(\beta)$. Let $x \in L(\gamma) \cap L(\bar{\gamma})$. We can assume that $|\operatorname{MeshSet}(\alpha, \beta)|$ is greater than 1 as the case for $|\operatorname{MeshSet}(\alpha, \beta)|=1$ is trivial. In consequence, the base case $n=0$ is clear as the resulting MeshSet for two trees of potential 0 has to be necessarily empty. Hence, we have to consider two cases based on the structure of $\alpha$ and $\beta$.
(i) Let $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. Clearly, $x$ is in form of $x=X x_{1} \ldots x_{m}$. Let $\alpha_{i} \|$ $\beta_{i}$ be an arbitrary non-rewritable pair of arguments in $\alpha, \beta$. It follows that $x_{i} \in L\left(\alpha_{i}\right) \cap L\left(\beta_{i}\right)$ and so, due to Lemma [13, there exists a mesh $\delta \in \operatorname{MeshSet}\left(\alpha_{i}, \beta_{i}\right)$ such that $x_{i} \in L(\delta)$. Let $M_{i}=\operatorname{MeshSet}\left(\alpha_{i}, \beta_{i}\right)$. Since $\pi\left(\alpha_{i}\right)+\pi\left(\beta_{i}\right)<n$ we can use the induction hypothesis to $M_{i}$ and immediately conclude that $\delta$ is the only mesh in $M_{i}$ generating $x_{i}$. And so, we know that $\gamma$ and $\bar{\gamma}$ are equal on the non-rewritable arguments of $\alpha, \beta$. Note that if $\alpha_{i} \bowtie \triangleleft \beta_{i}$, then both contribute a single mesh at position $i$. Immediately, we get that both $\gamma$ and $\bar{\gamma}$ are also equal on the rewritable arguments of $\alpha$ and $\beta$, hence finally $\gamma=\bar{\gamma}$.
(ii) W.l.o.g. let $\alpha=R_{k}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. Clearly, as $\rho(\alpha) \leq r+1$, we know that $R_{k}$ is unambiguous. From the definition of MeshSet there exist productions $\delta, \bar{\delta} \in R_{k}$ such that $\gamma \in \operatorname{MeshSet}(\delta, \beta)$ and $\bar{\gamma} \in \operatorname{MeshSet}(\bar{\delta}, \beta)$. We claim that $\gamma=\bar{\gamma}$ as otherwise $\delta, \bar{\delta}$ would generate a common term. Suppose that $\gamma \neq \bar{\gamma}$. From Lemma 13 we know that $L(\gamma) \subseteq L(\delta)$ and $L(\bar{\gamma}) \subseteq L(\bar{\delta})$. Since $x \in L(\gamma) \cap L(\bar{\gamma})$, we get that $x \in L(\delta) \cap L(\bar{\delta})$ and therefore a contradiction with the fact that $R_{k}$ is unambiguous. It follows that $\gamma=\bar{\gamma}$, which finishes the proof.

Theorem 17 (Unambiguity). Let $\alpha, \beta \in R_{n}$. If $L(\alpha) \cap L(\beta) \neq \emptyset$, then $\alpha=\beta$.
Proof. Induction over $n$. Let $x \in L(\alpha) \cap L(\beta)$. Note that if $x \in L(\alpha) \cap L(\beta)$, then both $\alpha, \beta$ must be similar. We can therefore focus on similar productions of $R_{n}$. For that reason, we immediately notice that $R_{0}$ satisfies our claim.

Let $n>0$. Since $R_{n}$ does not contain combinators as productions, we can rewrite both $\alpha$ as $X \alpha_{1} \ldots \alpha_{m}$ and $\beta$ as $X \beta_{1} \ldots \beta_{m}$. Let us consider several cases based on their common structure.
(i) Let $X=K$. If $m=1$, then $\alpha$ and $\beta$ are equal as there is exactly one short $K$-production in $R_{n}$. If $m=2$, then again $\alpha=\beta$, since there is a unique $K$-production $K R_{n-1} \mathcal{C}$ of length two in $R_{n}$. If $m>2$, then both are K-Expansions of some productions in $R_{n-1}$. And so

$$
\begin{aligned}
& \alpha=K\left(X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}}\right) \mathcal{C} \alpha_{3} \ldots \alpha_{m} \in \operatorname{K-EXPANSIONs}(\gamma), \\
& \beta=K\left(X \overline{\beta_{1}} \ldots \overline{\beta_{k}}\right) \mathcal{C} \beta_{3} \ldots \beta_{m} \in \operatorname{K-ExPANSIONS}(\delta),
\end{aligned}
$$

where

$$
\begin{gathered}
\gamma=X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}} \alpha_{3} \ldots \alpha_{m}, \\
\delta=X \overline{\beta_{1}} \ldots \overline{\beta_{k}} \beta_{3} \ldots \beta_{m} .
\end{gathered}
$$

Since $x \in L(\alpha) \cap L(\beta)$, we can assume that $x$ is in form of $K\left(X y_{1} \ldots y_{k}\right) x_{2} x_{3} \ldots x_{m}$ where $y_{i} \in L\left(\overline{\alpha_{i}}\right) \cap L\left(\overline{\beta_{i}}\right)$ and $x_{j} \in L\left(\alpha_{j}\right) \cap L\left(\beta_{j}\right)$. It follows that we can use the induction hypothesis to $\gamma, \delta \in R_{n-1}$ obtaining $\overline{\alpha_{i}}=\overline{\beta_{i}}$ and $\alpha_{j}=\beta_{j}$. Immediately, we get $\alpha=\beta$.
(ii) Let $X=S$. If $m=1$, then $\alpha$ and $\beta$ are equal due to the fact that there is exactly one $S$-production of length one in $R_{n}$. If $m=2$, then $\alpha, \beta$ are in form of $\alpha=S R_{i} R_{n-i}$ and $\beta=S R_{j} R_{n-j}$. Hence, $x=S x_{1} x_{2}$ for some terms $x_{1}, x_{2}$. Since $x_{1} \in L\left(R_{i}\right) \cap L\left(R_{j}\right)$ and $x_{2} \in L\left(R_{n-i}\right) \cap L\left(R_{n-j}\right)$, we know that $i=j$ due to Corollary 12 and thus $\alpha=\beta$. It remains to consider long $S$-productions. Let

$$
\begin{gathered}
\alpha=S\left(X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}}\right) \varphi_{l} \varphi_{r} \alpha_{4} \ldots \alpha_{m} \in \operatorname{S-EXPANSIONS}(\gamma), \\
\beta=S\left(X \overline{\beta_{1}} \ldots \overline{\beta_{k}}\right) \overline{\varphi_{l} \varphi_{r}} \beta_{4} \ldots \beta_{m} \in \operatorname{S-ExPANSIONS}(\delta),
\end{gathered}
$$

where

$$
\begin{gathered}
\gamma=X \overline{\alpha_{1}} \ldots \overline{\alpha_{k}} \alpha_{2} \alpha_{3} \alpha_{4} \ldots \alpha_{m}, \\
\delta=X \overline{\beta_{1}} \ldots \overline{\beta_{k}} \beta_{2} \beta_{3} \beta_{4} \ldots \beta_{m} .
\end{gathered}
$$

It follows that we can rewrite $x$ as $S\left(X y_{1} \ldots y_{k}\right) w z x_{4} \ldots x_{m}$. Let us focus on the reduct $x \rightarrow_{w} y=X y_{1} \ldots y_{k} z(w z) x_{4} \ldots x_{m}$. Evidently, $y \in L(\gamma) \cap L(\delta)$ and so according to the induction hypothesis we know that $\gamma=\delta$, in particular $\alpha_{2}=\beta_{2}$ and $\alpha_{3}=\beta_{3}$. Hence, both $\varphi_{l} \varphi_{r}$ and $\overline{\varphi_{l} \varphi_{r}}$ are elements of the same RewritingSet. If we could guarantee that $\varphi_{l} \varphi_{r}=\overline{\varphi_{l} \varphi_{r}}$, then immediately $\alpha=\beta$ and the proof is finished. From the construction of the RewritingSet we have two cases left to consider.
(i) If $\alpha_{3}=X \gamma_{1} \ldots \gamma_{m}$, then both $\varphi_{l} \varphi_{r}$ and $\overline{\varphi_{l} \varphi_{r}}$ are either in form of $X \gamma_{1} \ldots \gamma_{m-1} \varphi_{r}$ or $X \gamma_{1} \ldots \gamma_{m-1} \overline{\varphi_{r}}$. It follows that $\varphi_{l}=\overline{\varphi_{l}}$. It remains to show that $\varphi_{r}=\overline{\varphi_{r}}$. Note that $\rho\left(\alpha_{2}\right), \rho\left(\alpha_{3}\right) \leq n$ since both $\gamma, \delta \in R_{n-1}$. Moreover, from the induction hypothesis we know that $R_{0}, \ldots, R_{n-1}$ are unambiguous. And so, since $z \in L\left(\varphi_{r}\right) \cap L\left(\overline{\varphi_{r}}\right)$, we can use Lemma 16 to conclude that $\varphi_{r}=\overline{\varphi_{r}}$.
(ii) If $\alpha_{3}=R_{k}$, then necessarily there exist such productions $\eta, \bar{\eta} \in R_{k}$ that $\varphi_{l} \varphi_{r} \in$ RewritingSet $\left(\alpha_{2}, \eta\right)$ whereas $\overline{\varphi_{l} \varphi_{r}} \in \operatorname{RewritingSet}\left(\alpha_{2}, \bar{\eta}\right)$. Due to Proposition 10, we know that $L\left(\varphi_{l} \varphi_{r}\right) \subseteq L(\eta)$ and $L\left(\overline{\varphi_{l} \varphi_{r}}\right) \subseteq L(\bar{\eta})$. It implies that $w z \in$ $L(\eta) \cap L(\bar{\eta})$, however, since $k<n$, we know from the induction hypothesis that $R_{k}$ is unambiguous. Hence $\eta=\bar{\eta}$. Finally, it means that we can reduce this case to one of the previous cases when $\alpha_{3}$ is complex, concluding that $\varphi_{l} \varphi_{r}=\overline{\varphi_{l} \varphi_{r}}$.
3.5. Generating functions. Fix an arbitrary normal-order reduction grammar $R_{n}$. Let us consider the counting sequence $\left\{r_{n, k}\right\}_{k \in \mathbb{N}}$ where $r_{n, k}$ denotes the number of $S K$-combinators of size $k$ reducing in $n$ normal-order reduction steps. Suppose we associate with it a formal power series $R_{n}(z)$ defined as

$$
R_{n}(z)=\sum_{k=0}^{\infty} r_{n, k} z^{k}
$$

In the following theorem we present a recursive method of computing the closed-form solution of $R_{n}(z)$ using the regular tree grammars $R_{0}, \ldots, R_{n}$ and the inductive use of the Symbolic Method developed by Flajolet and Sedgewick [11].

Theorem 18. For each $n \geq 0$, the ordinary generating function $R_{n}(z)$ corresponding to the sequence $\left\{r_{n, k}\right\}_{k \in \mathbb{N}}$ has a computable closed form solution.

Proof. Induction over $n$. Let us start with giving previously computed closed-form solutions for $C(z)$, i.e. the generating function corresponding to the set of all $S K$-combinators, and $R_{0}(z)$ [5]:

$$
\begin{equation*}
C(z)=\frac{1-\sqrt{1-8 z}}{2 z} \quad R_{0}(z)=\frac{1-2 z-\sqrt{1-4 z-4 z^{2}}}{2 z^{2}} . \tag{1}
\end{equation*}
$$

Clearly, both $C(z)$ and $R_{0}(z)$ are computable.
Now, suppose that $n \geq 1$. Recall that in its construction, $R_{n}$ might depend on previous reduction grammars $R_{0}, \ldots, R_{n-1}$, the set $\mathcal{C}$ of all $S K$-combinators and itself, via self-referencing
productions. Due to Theorem [17, $R_{n}$ is unambiguous and so we can express its generating function $R_{n}(z)$ as the unique solution of

$$
\begin{equation*}
R_{n}(z)=\sum_{\alpha \in R_{n}} z^{k(\alpha)} C(z)^{c(\alpha)} \prod_{i=0}^{n} R_{i}(z)^{r_{i}(\alpha)} \tag{2}
\end{equation*}
$$

where $k(\alpha), c(\alpha)$ and $r_{i}(\alpha)$ denote respectively, the number of applications, the number of non-terminal symbols $C$ and the number of non-terminal symbols $R_{i}$ in $\alpha$.

Note that $R_{n}$ has exactly four self-referencing productions, i.e. $S R_{n}, K R_{n}, S R_{0} R_{n}$ and $S R_{n} R_{0}$. It means that by converting them into appropriate functional equations, we can further rewrite (2) as

$$
\begin{equation*}
R_{n}(z)=2 z R_{n}(z)+2 z^{2} R_{0}(z) R_{n}(z)+\sum_{\alpha \in \Phi\left(R_{n}\right)} z^{k(\alpha)} C(z)^{c(\alpha)} \prod_{i=0}^{n-1} R_{i}(z)^{r_{i}(\alpha)}, \tag{3}
\end{equation*}
$$

where $\Phi\left(R_{n}\right)$ denotes the set of productions $\alpha \in R_{n}$ which do not reference $R_{n}$. By the induction hypothesis, we can compute the closed-form solutions for $R_{0}(z), \ldots, R_{n-1}(z)$ turning (3) into a linear equation in $R_{n}(z)$. Simplifying (1) for $R_{0}(z)$, we derive the final closed-form solution

$$
R_{n}(z)=\frac{1}{\sqrt{1-4 z-4 z^{2}}} \sum_{\alpha \in \Phi\left(R_{n}\right)} z^{k(\alpha)} C(z)^{c(\alpha)} \prod_{i=0}^{n-1} R_{i}(z)^{r_{i}(\alpha)}
$$

3.6. Other applications. In this section we highlight some interesting consequences of the existence of normal-order reduction grammars. In particular, we prove that terms reducing in $n$ steps have necessarily bounded length. Moreover, we show that the problem of deciding whether a given term reduces in $n$ steps, can be done in memory independent of the size of the term.

Proposition 19. If $\alpha \in R_{n}$, then $\alpha$ has length at most $2 n+2$.
Proof. Induction over $n$. The base case $n=0$ is clear from the shape of $R_{0}$. Fix $n>0$. Let us consider long productions in $R_{n}$. If $\beta$ is a K-Expansion of some $X \alpha_{1} \ldots \alpha_{m} \in R_{n-1}$, then

$$
\beta=K\left(X \alpha_{1} \ldots \alpha_{k}\right) \mathcal{C} \alpha_{k+1} \ldots \alpha_{m} \quad \text { for } \quad 0 \leq k \leq m-1 .
$$

Since setting $k=0$ maximizes the length of $\beta$, we note that $\beta$ is of length $m+2$ and so by the induction hypothesis at most $2 n+2$. Now, let us consider the case when $\beta$ is a S-Expansion of some $X \alpha_{1} \ldots \alpha_{m} \in R_{n-1}$. Then,

$$
\beta=S\left(X \alpha_{1} \ldots \alpha_{k}\right) \varphi_{l} \varphi_{r} \alpha_{k+3} \ldots \alpha_{m} \quad \text { for } \quad 0 \leq k \leq m-2 .
$$

where in addition $\left(\varphi_{l} \varphi_{r}\right) \in \operatorname{Rewriting} \operatorname{Set}\left(\alpha_{k+1}, \alpha_{k+2}\right)$. Again, setting $k=0$ maximizes the length of $\beta$. It follows that $\beta$ is of length at most $m+1$ and so also at most $2 n+1$.

In other words, terms reducing in $n$ steps cannot be too long as their length is tightly bounded by $2 n+2$. Now, let us consider the following two problems.

## Problem: N-STEP-REDUCIBLE

Input: A combinatory logic term $x \in L(\mathcal{C})$.
Output: YES if and only if $x$ reduces in $n$ steps.

## Problem: Reduces-in-n-steps

Input: A combinatory logic term $x \in L(\mathcal{C})$ and a number $n \in \mathbb{N}$.
Output: yes if and only if $x$ reduces in $n$ steps.
Since $n$ in not a part of the input, we can compute $R_{n}$ in constant time and memory. Using $R_{n}$ we build a bottom-up tree automaton recognizing $L\left(R_{n}\right)$ [7] and use it to check whether
$x \in L\left(R_{n}\right)$ in time $O(|x|)$, without using additional memory. On the other hand, the Naive algorithm requires $O(|x|)$ time and additional memory. At each reduction step, the considered term doubles at most in size, as $S x y z \rightarrow_{w} x z(y z)$. In order to find the next redex we spend up to linear time in the current size of $x$, therefore both size and time are bounded by

$$
\begin{aligned}
|x|+2|x|+4|x|+\cdots+2^{n}|x| & =|x|\left(1+2+4+\cdots+2^{n}\right) \\
& =|x|\left(2^{n+1}-1\right)=O(|x|)
\end{aligned}
$$

As a natural extension, we get the following corollary.
Corollary 20. The REDUCES-IN-N-STEPS problem is decidable in space depending exclusively on $n$, independently of $|x|$.
3.7. Upper bound. In this section we focus on the upper bound on the number of productions in $R_{n}$. We show that there exists a primitive recursive function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that $\left|R_{n}\right| \leq f(n)$.

Following the scheme of the soundness proofs in Section 3.2, we construct suitable upper bounds using the notions of tree potential and degree. In the end of this section, we show that these values are in fact bounded in each $R_{n}$, thus giving the desired upper bound.
Lemma 21. Let $\alpha, \beta$ be two trees of degree at most $n$ such that their total potential $\pi(\alpha)+\pi(\beta)$ is equal to $p$. Then, the number of distinct meshes in $\operatorname{MeshSet}(\alpha, \beta)$ is bounded by $\left|R_{n}\right|^{e p!}$.

Proof. Induction over the total potential $p$. Consider the following primitive recursive function $f_{n}: \mathbb{N} \rightarrow \mathbb{N}$.

$$
f_{n}(k)= \begin{cases}1 & \text { if } k=0 \\ \left(\left|R_{n}\right| \cdot f_{n}(k-1)\right)^{k} & \text { otherwise }\end{cases}
$$

We claim that $|\operatorname{MeshSet}(\alpha, \beta)| \leq f_{n}(p)$. Note that it suffices to consider such $\alpha, \beta$ that $|\operatorname{MeshSet}(\alpha, \beta)|>1$ since $f_{n}$ is an increasing function attaining positive values for any given input. It follows that the base case $p=0$ is clear, as if $\pi(\alpha)+\pi(\beta)=0$, then $\operatorname{MeshSet}(\alpha, \beta)$ is necessarily empty. Now, let us assume that $p>0$. From the construction of the common mesh set $M$ of $\alpha$ and $\beta$, we can distinguish two cases left to consider.
(i) Suppose that $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. In order to maximize the size of $M$, we can furthermore assume that none of the pairs $\alpha_{i}, \beta_{i}$ are rewritable. And so, the total number of meshes in $M$ is equal to the product of all meshes in corresponding mesh sets for $\alpha_{i}$ and $\beta_{i}$. The degree of $\alpha_{i}$ and $\beta_{i}$ is still at most $n$, however $\pi\left(\alpha_{i}\right)+\pi\left(\beta_{i}\right) \leq p-2$. Hence, using the induction hypothesis we get $\left|\operatorname{MeshSet}\left(\alpha_{i}, \beta_{i}\right)\right| \leq f_{n}(p-2)$. Since both $\alpha, \beta$ are of length $m \leq p$ we can furthermore state that

$$
\begin{aligned}
|M| & \leq\left(f_{n}(p-2)\right)^{m} \leq\left(f_{n}(p-2)\right)^{p} \\
& \leq\left(f_{n}(p-1)\right)^{p} \leq\left(\left|R_{n}\right| \cdot f_{n}(p-1)\right)^{p} \\
& =f_{n}(p) .
\end{aligned}
$$

(ii) Let us assume w.l.o.g. that $\alpha=R_{i}$ and $\beta$ is complex. In order to maximize the total number of meshes in $M$, we can moreover assume that all productions $\gamma \in R_{i}$ are similar to $\beta$ and generate disjoint sets of meshes. We claim that $\operatorname{MeshSet}(\gamma, \beta) \leq f_{n}(p-1)$. Clearly, if $\gamma$ does not reference $R_{i}$, then our claim is trivially true. Suppose that $\gamma$ is a self-referencing production. If $\gamma=X R_{i}$, then $\beta$ is in form of $X \beta_{1}$. From the construction of $M$, we get that

$$
|\operatorname{MeshSet}(\gamma, \beta)|=\left|\operatorname{MeshSet}\left(R_{i}, \beta_{1}\right)\right| .
$$

As $\pi\left(R_{i}\right)+\pi\left(\beta_{1}\right) \leq p-1$, we can apply the induction hypothesis to $\operatorname{MeshSet}\left(R_{i}, \beta_{1}\right)$ and immediately obtain $|\operatorname{MESHSET}(\gamma, \beta)| \leq f_{n}(p-1)$. Now, suppose w.l.o.g. that $\gamma=S R_{i} R_{0}$ and hence $\beta=S \beta_{1} \beta_{2}$. Again, from the construction of $M$ we know that

$$
|\operatorname{MeshSet}(\gamma, \beta)|=\left|\operatorname{MeshSET}\left(R_{i}, \beta_{1}\right)\right| \cdot\left|\operatorname{MeshSet}\left(R_{0}, \beta_{2}\right)\right| .
$$

Due to the fact that both $\pi\left(R_{i}\right)+\pi\left(\beta_{1}\right) \leq p-2$ and $\pi\left(R_{0}\right)+\pi\left(\beta_{2}\right) \leq p-2$, we can use the induction hypothesis and immediately get that

$$
\begin{aligned}
|\operatorname{MeshSET}(\gamma, \beta)| & =\left|\operatorname{MeshSET}\left(R_{i}, \beta_{1}\right)\right| \cdot\left|\operatorname{MeShSEt}\left(R_{0}, \beta_{2}\right)\right| \\
& \leq f_{n}(p-2) f_{n}(p-2)
\end{aligned}
$$

Note that $\left(f_{n}(p-2)\right)^{2} \leq f_{n}(p-1)$ for $p \geq 2$ and, in consequence, $|\operatorname{MeShSET}(\gamma, \beta)| \leq$ $f_{n}(p-1)$. Indeed, if $p=2$, then $\left(f_{n}(p-2)\right)^{2}=1 \leq f_{n}(1)=\left|R_{n}\right|$. Otherwise if $p>2$, then

$$
\begin{aligned}
f_{n}(p-1) & =\left(\left|R_{n}\right| \cdot f_{n}(p-2)\right)^{p-1} \\
& =\left(\left|R_{n}\right|^{p-1}\left(f_{n}(p-3)\right)^{p-2}\right)^{p-1} \\
& \geq\left(\left|R_{n}\right|^{p-2}\left(f_{n}(p-3)\right)^{p-2}\right)^{p-1} \\
& =\left(\left|R_{n}\right| \cdot f_{n}(p-3)\right)^{(p-1)(p-2)}
\end{aligned}
$$

As $2(p-2) \leq(p-1)(p-2)$ for $p>2$, we finally obtain

$$
\begin{aligned}
\left(\left|R_{n}\right| \cdot f_{n}(p-3)\right)^{(p-1)(p-2)} & \geq\left(\left|R_{n}\right| \cdot f_{n}(p-3)\right)^{2(p-2)} \\
& =\left(f_{n}(p-2)\right)^{2}
\end{aligned}
$$

We know therefore that $\operatorname{MeshSet}(\gamma, \beta) \leq f_{n}(p-1)$ for each $\gamma \in R_{i}$. Finally, using the fact that $\left|R_{i}\right| \leq\left|R_{n}\right|$, we get

$$
\begin{aligned}
|M| & \leq\left|R_{n}\right| \cdot f_{n}(p-1) \\
& \leq\left(\left|R_{n}\right| \cdot f_{n}(p-1)\right)^{p} \\
& =f_{n}(p)
\end{aligned}
$$

And so, we know that $|\operatorname{MeshSet}(\alpha, \beta)| \leq f_{n}(p)$. Solving the recurrence for $f_{n}(p)$, using e.g. Mathematica $\circledR$ [19], we obtain the following closed form expression

$$
f_{n}(p)=\left|R_{n}\right|^{e p \Gamma(p, 1)}
$$

where

$$
\Gamma(s, x)=(s-1)!e^{-x} \sum_{k=0}^{s-1} \frac{x^{k}}{k!}
$$

is the upper incomplete gamma function (see e.g. [1]). Simplifying the above expression in the case $x=1$ and using the observation that $\sum_{k=0}^{s-1} \frac{1}{k!} \leq e$ for arbitrary $s$, we finally obtain the anticipated upper bound

$$
f_{n}(p) \leq\left|R_{n}\right|^{e p!}
$$

Lemma 22. Let $\alpha, \beta$ be two trees of degree at most $n$ such that their total potential $\pi(\alpha)+$ $\pi(\beta)$ is equal to $p$. Then, the number of distinct trees in $\operatorname{RewritingSet}(\alpha, \beta)$ is bounded by $\left|R_{n}\right|^{1+e p!}$.

Proof. If $\mid$ REWRITINGSET $(\alpha, \beta) \mid \leq 1$, then our claim is trivially true. Let us focus therefore on the remaining cases when either $\beta=X \beta_{1} \ldots \beta_{m}$ and both $\beta_{m}$ and $\alpha$ are non-rewritable, or $\beta=R_{i}$.

First, consider the former case. Note that the resulting rewriting set is of equal size as $\operatorname{MeshSet}\left(\alpha, \beta_{m}\right)$. Since $\pi(\alpha)+\pi\left(\beta_{m}\right) \leq p-1$, we can use Lemma 21 to deduce that

$$
|\operatorname{REWRItinGSET}(\alpha, \beta)|=\left|\operatorname{MESHSET}\left(\alpha, \beta_{m}\right)\right| \leq\left|R_{n}\right|^{e(p-1)!}<\left|R_{n}\right|^{1+e p!}
$$

Now, let us consider the latter case. In order to maximize the resulting rewriting set we assume that each production $\gamma \in R_{i}$ generates a disjoint set of trees. We claim that each production $\gamma$ contributes at most $\left|R_{n}\right|^{e p!}$ new trees to the resulting rewriting set and therefore $|\operatorname{REWRITINGSET}(\alpha, \beta)| \leq\left|R_{n}\right|^{1+e p!}$, as there are at most $\left|R_{n}\right|$ productions in $R_{i}$. Indeed,
consider an arbitrary $\gamma \in R_{i}$. Evidently, if $|\operatorname{RewritingSet}(\alpha, \gamma)| \leq 1$, then our claim is true. Hence, let us assume that $|\operatorname{Rewriting} \operatorname{Set}(\alpha, \gamma)|>1$. It follows that $\gamma$ is complex. Let us rewrite it as $X \gamma_{1} \ldots \gamma_{m}$. Note that as in the previous case, the resulting rewriting set is of equal size as $\operatorname{MeshSet}\left(\alpha, \gamma_{m}\right)$. Since $\pi(\alpha)+\pi\left(\gamma_{m}\right) \leq p-1$ we use Lemma 21 and get

$$
|\operatorname{RewritingSet}(\alpha, \gamma)|=\left|\operatorname{MeShSet}\left(\alpha, \gamma_{m}\right)\right| \leq\left|R_{n}\right|^{e(p-1)!}<\left|R_{n}\right|^{e p!}
$$

Lemma 23. Let $\alpha, \beta$ be two trees of total potential $\pi(\alpha)+\pi(\beta)$ equal to $p$. Then, each mesh in $\operatorname{MeshSet}(\alpha, \beta)$ has potential bounded by $p!(1+e)$.

Proof. Induction over total potential $p$. Again, it suffices to consider such $\alpha, \beta$ that $\operatorname{MeshSet}(\alpha, \beta)$ is not empty. Immediately, the base case $p=0$ is clear. Let us assume that $p>0$. Consider the following primitive recursive function $f: \mathbb{N} \rightarrow \mathbb{N}$.

$$
f(k)= \begin{cases}1 & \text { if } k=0 \\ k \cdot(f(k-1)+1) & \text { otherwise }\end{cases}
$$

Let $\gamma \in \operatorname{MeshSet}(\alpha, \beta)$. We claim that $\pi(\gamma) \leq f(p)$. Note that $f$ is an increasing function attaining positive values for any input. We have two cases to consider.
(i) Suppose that $\alpha=X \alpha_{1} \ldots \alpha_{m}$ and $\beta=X \beta_{1} \ldots \beta_{m}$. Note that $\pi\left(\alpha_{i}\right)+\pi\left(\beta_{i}\right) \leq p-2$ for each pair of corresponding arguments $\alpha_{i}, \beta_{i}$. Using the induction hypothesis to pairs $\alpha_{i}, \beta_{i}$ and the fact that $\gamma \in \operatorname{MeshSet}(\alpha, \beta)$ is similar to both $\alpha$ and $\beta$, we bound $\gamma$ 's potential by

$$
\pi(\gamma) \leq m \cdot f(p-2)+m \leq p \cdot(f(p-2)+1) \leq f(p)
$$

(ii) Assume w.l.o.g. that $\alpha=R_{i}$ and $\beta$ is complex. It follows that $\gamma \in \operatorname{MeshSet}(\delta, \beta)$ for some $\delta \in R_{i}$. If $\delta$ does not reference $R_{i}$, then clearly $\pi(\delta) \leq \pi\left(R_{i}\right)-1$ and therefore $\pi(\gamma) \leq f(p-1)$. Now, suppose that $\delta$ is a self-referencing production of $R_{i}$.

If $\delta=X R_{i}$, then $\beta$ is in form of $X \beta_{1}$ and similarly $\gamma=X \gamma_{1}$. It follows that $\pi(\delta)=$ $\pi\left(R_{i}\right)+1$ and therefore $\pi(\delta)+\pi(\beta)=p+1$. Note however that $\pi\left(\gamma_{1}\right) \leq f(p-1)$ as $\pi\left(R_{i}\right)+\pi\left(\beta_{1}\right) \leq p-1$. Due to that, $\pi(\gamma)=1+f(p-1) \leq f(p)$.

Let us assume w.l.o.g. that $\delta=S R_{i} R_{0}$. Immediately, $\beta$ is in form of $S \beta_{1} \beta_{2}$ whereas $\gamma=S \gamma_{1} \gamma_{2}$. Moreover, $\pi(\delta)=\pi\left(R_{i}\right)+3$. Note however that both $\pi\left(R_{i}\right)+\pi\left(\beta_{1}\right) \leq p-2$ and $\pi\left(R_{0}\right)+\pi\left(\beta_{2}\right) \leq p-2$. We can therefore use the induction hypothesis and conclude that

$$
\pi(\gamma)=2+\pi\left(\gamma_{1}\right)+\pi\left(\gamma_{2}\right) \leq 2+2 \cdot f(p-2) .
$$

Since $\pi(\delta) \geq 4$, we know that $p \geq 3$ and so we can further bound $\pi(\gamma)$ by

$$
\begin{aligned}
\pi(\gamma) & =2(1+f(p-2)) \\
\leq & (p-1)(1+f(p-2)) \\
& =f(p-1) \leq f(p) .
\end{aligned}
$$

Finally, we know that $\pi(\gamma) \leq f(p)$. What remains is to solve the recursion, using e.g. Mathematica ${ }^{\circledR}$ [19], for $f$ and give its closed form solution. It follows that

$$
\begin{gathered}
f(p)=\Gamma(1+p)+e p \Gamma(p, 1) \\
\leq p!+e p! \\
=p!(1+e)
\end{gathered}
$$

where

$$
\Gamma(n)=(n-1)!
$$

Lemma 24. Let $\alpha, \beta$ be two trees of potential $\pi(\alpha)+\pi(\beta)=p$. Then, each tree in RewritingSet $(\alpha, \beta)$ has potential bounded by $p!(1+e)+p$.

Proof. Let $\gamma$ be an arbitrary tree in RewritingSet $(\alpha, \beta)$. Based on the structure of $\beta$ we have several cases to consider. If $\beta=\mathcal{C}$, then $\gamma=\mathcal{C} \alpha$ and so $\pi(\gamma)=\pi(\alpha)+1=p+1$. Note that $1<p!(1+e)$ for any $p$ and thus our bound holds.

If $\beta=X \beta_{1} \ldots \beta_{m}$, then $\pi(\alpha)+\pi\left(\beta_{m}\right) \leq p-1$. In both cases when $\alpha \triangleright \triangleleft \beta_{m}$ the resulting tree has potential bounded by $p$ and so also by $p!(1+e)+p$. Let us assume that $\alpha \| \beta_{m}$. We can therefore rewrite $\gamma$ as $X \gamma_{1} \ldots \gamma_{m}$. Using Lemma 23, we know that $\pi\left(\gamma_{m}\right) \leq(p-1)!(1+e)$. Moreover, both $\alpha$ and $\beta$ are similar to $\gamma$. Let us rewrite them as $X \alpha_{1} \ldots \alpha_{m}$ and $X \beta_{1}, \ldots, \beta_{m}$, respectively. Note that for each $i<m, \gamma_{i}$ is equal to $\alpha_{i}$ or $\beta_{i}$. It follows that we can bound the potential of $X \gamma_{1} \ldots \gamma_{m-1}$ by $p-1$ and hence $\gamma$ 's potential by $(p-1)!(1+e)+p$.

Now, if $\beta=R_{i}$, then $\gamma \in \operatorname{Rewriting} \operatorname{Set}(\alpha, \delta)$ for some $\delta \in R_{i}$. Clearly, if $\delta$ does not reference $R_{i}$, we know that $\pi(\delta) \leq \pi\left(R_{i}\right)-1 \leq p-1$. Moreover, $\delta$ is complex, as otherwise RewritingSet $(\alpha, \delta)=\emptyset$. Using our previous argumentation, we can therefore conclude that $\pi(\gamma) \leq(p-1)!(1+e)+p$. Suppose that $\delta$ is a self-referencing production of $R_{i}$. If $\delta=X R_{i}$, then $\alpha$ is in form of $X \alpha_{1}$ and $\gamma=X \gamma_{1}$. Immediately, $\pi(\alpha)+\pi(\delta)=p+1$. If $R_{i} \triangleright \triangleleft \alpha_{1}$, then $\gamma$ has potential bounded by $p$. Therefore, let us assume that $R_{i} \| \alpha_{1}$. Since $\pi\left(R_{i}\right)+\pi\left(\alpha_{1}\right)=p-1$, we know from Lemma 23 that $\pi\left(\gamma_{1}\right) \leq(p-1)!(1+e)$. It follows immediately that $\pi(\gamma) \leq$ $(p-1)!(1+e)+1 \leq p!(1+e)+p$.
Finally, suppose that $\delta=S \delta_{1} \delta_{2}$ and so $\alpha=S \alpha_{1} \alpha_{2}$. Immediately, $\gamma=S \gamma_{1} \gamma_{2}$. Again, if $\delta_{2} \triangleright \triangleleft \alpha_{2}$, we can bound $\gamma$ 's potential by $p$. Hence, let us assume that $\delta_{2} \| \alpha_{2}$. Clearly, $\pi(\alpha)+\pi(\delta)=p+3$. Note however that $\pi\left(\alpha_{1}\right)+\pi\left(\delta_{1}\right) \leq p-2$ and $\pi\left(\alpha_{2}\right)+\pi\left(\delta_{2}\right) \leq p-2$, as both $\delta_{1}$ and $\delta_{2}$ are non-terminal reduction grammar symbols of positive potential. Using Lemma 23 to $\operatorname{MeshSet}\left(\alpha_{2}, \delta_{2}\right)$ we conclude that $\pi\left(\gamma_{2}\right) \leq(p-2)!(1+e)$. It follows that $\pi(\gamma) \leq(p-2)!(1+e)+p \leq p!(1+e)+p$.
Lemma 25. There exists a primitive recursive function $\psi: \mathbb{N} \rightarrow \mathbb{N}$ such that $\pi\left(R_{n}\right) \leq \psi(n)$.
Proof. Consider the following function $\psi: \mathbb{N} \rightarrow \mathbb{N}$ :

$$
\psi(k)= \begin{cases}1 & \text { if } k=0 \\ 4(\psi(k-1)+2)!+2 \psi(k-1)+5 & \text { otherwise }\end{cases}
$$

Clearly, $\psi$ is an increasing primitive recursive function. We show that $\psi(n)$ bounds the potential of $R_{n}$ using induction over $n$. Since $\pi\left(R_{0}\right)=\psi(0)=1$, the base case is clear. Let $n>0$. In order to prove our claim, we have to check that $\pi(\alpha) \leq \psi(n)-1$ for all productions $\alpha \in R_{n}$ which do not reference $R_{n}$.
(i) Suppose that $\alpha=S R_{n-i} R_{i}$. Clearly, the potential of $\alpha$ is equal to $2+\pi\left(R_{n-i}\right)+\pi\left(R_{i}\right)$. Using the induction hypothesis, we know moreover that

$$
\begin{aligned}
\pi(\alpha) & \leq 2+\psi(n-i)+\psi(i) \\
& \leq 2+2 \psi(n-1) \\
& \leq \psi(n)-1
\end{aligned}
$$

(ii) Let $\alpha=K R_{n-1} \mathcal{C}$. Due to the fact that $\pi(\alpha)=2+\pi\left(R_{n-1}\right)$, we use the induction hypothesis and immediately obtain

$$
\pi(\alpha) \leq 2+\psi(n-1) \leq \psi(n)-1
$$

(iii) Suppose that $\alpha \in \operatorname{K}$ - Expansions( $\beta$ ) for some $\beta \in R_{n-1}$. Note that $\pi(\beta) \leq \psi(n-1)+3$ as the productions of greatest potential in $R_{n-1}$ are exactly $S R_{n-1} R_{0}$ and $S R_{0} R_{n-1}$. Since $\pi(\alpha)=2+\pi(\beta)$, we get

$$
\pi(\alpha) \leq 5+\psi(n-1) \leq \psi(n)-1
$$

(iv) Finally, let $\alpha \in \operatorname{S-Expansions}(\beta)$ for some $\beta \in R_{n-1}$. Again, $\pi(\beta) \leq \pi\left(R_{n-1}\right)+3$ and hence from the induction hypothesis $\pi(\beta) \leq \psi(n-1)+3$. Let us rewrite $\alpha$ as $S\left(X \beta_{1} \ldots \beta_{k}\right) \varphi_{l} \varphi_{r} \beta_{k+3} \ldots \beta_{m}$ where $\beta=X \beta_{1} \ldots \beta_{m}$. Note that $\pi(\alpha) \leq \pi(\beta)+\pi\left(\varphi_{l}\right)+$ $\pi\left(\varphi_{r}\right)+1$. Moreover, as $\pi\left(\varphi_{l} \varphi_{r}\right)=1+\pi\left(\varphi_{l}\right)+\pi\left(\varphi_{r}\right)$, we get $\pi(\alpha) \leq \pi(\beta)+\pi\left(\varphi_{l} \varphi_{r}\right)$. Since
$\pi\left(\beta_{k+1} \beta_{k+2}\right) \leq \pi(\beta)-1$ and thus, $\pi\left(\beta_{k+1} \beta_{k+2}\right) \leq \psi(n-1)+2$, we can use Lemma 24 to obtain

$$
\pi\left(\varphi_{l} \varphi_{r}\right) \leq(\psi(n-1)+2)!(1+e)+\psi(n-1)+2
$$

It follows therefore that

$$
\begin{aligned}
\pi(\alpha) & \leq \pi(\beta)+\pi\left(\varphi_{l} \varphi_{r}\right) \\
& \leq(\psi(n-1)+2)!(1+e)+2 \psi(n-1)+5 \\
& \leq \psi(n)-1
\end{aligned}
$$

where the last inequality follows from the fact that

$$
(3-e)(\psi(n-1)+2)!\geq \frac{1}{5}(\psi(n-1)+2)!\geq \frac{6}{5} \geq 0
$$

Theorem 26. There exists a primitive recursive function $\chi: \mathbb{N} \rightarrow \mathbb{N}$ such that the number $\left|R_{n}\right|$ of productions in $R_{n}$ is bounded by $\chi(n)$.
Proof. Consider $R_{n}$ for some $n>0$. Note that $R_{n}$ consists of:
(i) two productions $S R_{n}$ and $K R_{n}$,
(ii) $n+1$ short $S$-productions in form of $S R_{n-i} R_{i}$,
(iii) an additional $K$-production $K R_{n-1} \mathcal{C}$,
(iv) K-EXPANSIOns $(\alpha)$ for each $\alpha \in R_{n-1}$ and
(v) $\operatorname{S-ExPANSIONS}(\alpha)$ for each $\alpha \in R_{n-1}$.

It suffices therefore to bound the number of K - and S -Expansions, as the number of other productions in $R_{n}$ is clear. Let us start with K-Expansions. Suppose that $\alpha$ is of length $m$. Clearly, $\mid \mathrm{K}$-Expansions $(\alpha) \mid=m$. Using Proposition 19, we know that that each production $\alpha \in R_{n-1}$ is of length at most $2 n$. It follows that there are at most $2 n \cdot\left|R_{n-1}\right|$ K-Expansions in $R_{n}$. Now, let us consider S-Expansions. In order to bound the number of S-Expansions in $R_{n}$, we assume that each production $\alpha \in R_{n-1}$ is of length $2 n$ and moreover each REWRITINGSET of appropriate portions of $\alpha$ generates a worst-case set of trees. And so, assuming that $\alpha$ is of length $2 n$ we can rewrite it as $X \alpha_{1} \ldots \alpha_{2 n}$. Let $\psi$ denote the upper bound function on the potential of $R_{n-1}$ from Lemma 25. Evidently, $\pi(\alpha) \leq \psi(n-1)+3$. Now, using Lemma 22 we know that each RewritingSet $\left(\alpha_{i}, \alpha_{i+1}\right)$ contributes at most

$$
\left|R_{n-1}\right|^{1+e(\psi(n-1)+3)!}
$$

new S-Expansions. As there are at most $2 n-1$ pairs of indices $(i, i+1)$ yielding REWRITIngSets, we get that the number of S-Expansions in $R_{n}$ is bounded by

$$
(2 n-1) \cdot\left|R_{n-1}\right| \cdot\left|R_{n-1}\right|^{1+e(\psi(n-1)+3)!} \leq(2 n-1) \cdot\left|R_{n-1}\right|^{2+3(\psi(n-1)+3)!}
$$

Finally, since $\left|R_{0}\right|=5$, we combine the above observations and get the following primitive recursive upper bound on $\left|R_{n}\right|$.

$$
\chi(k)= \begin{cases}5 & \text { if } k=0 \\ 4+k+2 k \cdot \chi(k-1) \\ +(2 k-1) \cdot \chi(k-1)^{2+3(\psi(k-1)+3)!} & \text { otherwise }\end{cases}
$$

## 4. Conclusion

We gave a complete syntactic characterization of normal-order reduction for combinatory logic over the set of primitive combinators $S$ and $K$. Our characterization uses regular tree grammars and therefore exhibits interesting algorithmic applications, including the computation of corresponding generating functions. We investigated the complexity of the generated reduction grammars, giving a primitive recursive upper bound on the number of their productions. We
emphasize the fact that although the size of $R_{n}$ is bounded by a primitive recursive function of $n$, it seems to be enormously overestimated. Our computer implementation of the REDUCTION GRAMMAR algorithm [3] suggests that the first few numbers in the sequence $\left\{\left|R_{n}\right|\right\}_{n \in \mathbb{N}}$ are in fact

$$
5,12,75,625,5673,53164,508199, \ldots
$$

The upper bound $\chi(1)$ on the size of $R_{1}$ is already of order $6 \cdot 10^{84549}$, whereas the actual size of $R_{1}$ is equal to 12 . Naturally, we conjecture that $\left\{R_{n}\right\}_{n \in \mathbb{N}}$ grows much slower than $\{\chi(n)\}_{n \in \mathbb{N}}$, although the intriguing problem of giving better approximations on the size of $R_{n}$ for large $n$ is still open.

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