# Coherence for Monoidal Endofunctors 

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#### Abstract

The goal of this paper is to prove coherence results with respect to relational graphs for monoidal endofunctors, i.e. endofunctors of a monoidal category that preserve the monoidal structure up to a natural transformation that need not be an isomorphism. These results are proved first in the absence of symmetry in the monoidal structure, and then with this symmetry. In the later parts of the paper the coherence results are extended to monoidal endofunctors in monoidal categories that have diagonal or codiagonal natural transformations, or where the monoidal structure is given by finite products or coproducts. Monoidal endofunctors are interesting because they stand behind monoidal monads and comonads, for which coherence will be proved in a sequel to this paper.


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## 1 Introduction

A monoidal functor is a functor between monoidal categories that preserves the monoidal structure up to a natural transformation that need not be an isomorphism (see Section 3 below; this notion stems from [5], Section II.1). Coherence results for monoidal functors were obtained long ago in [6] and [12]. In [6] one can find a result for such functors between symmetric monoidal categories in the absence of unit objects, while in [12] unit objects are allowed, and nonsymmetric monoidal categories are considered too.

To get coherence with the unit objects, [12] introduces implicitly graphs that connect occurrences of the generating functor (see the beginning of the next section below). The standard graphs, which stem from [8], and earlier work of Mac Lane and Kelly, connect occurrences of generating objects.

Our goal in this paper is first to extend these old coherence results to the situation where we have not a monoidal functor between two categories, but an
endofunctor of a single monoidal category. This involves matters that go beyond [12], where application of functors cannot be iterated. Monoidal endofunctors are interesting because they stand behind monoidal monads and comonads, and the present paper lays the ground for a study of coherence in these monads and comonads.

Monoidal monads stem from [9] and [10]. More recent papers on monoidal comonads are [14], [1] and [16]. We will prove coherence results for monoidal monads and comonads in a sequel to this paper [4]. In the present paper, and in that sequel, we understand coherence with respect to graphs that are like those of [12]. Coherence states that there is a faithful functor from a freely generated categorial structure, for which we prove coherence, into the category whose arrows are such graphs. We obtain thereby a characterization of the freely generated categorial structure in terms of graphs. Such coherence results give very useful procedures for deciding whether a diagram of canonical arrows commutes. (A general treatment of coherence in this spirit may be found in [3].)

One finds in [15] a notion of monad inspired by [9] and [10], at the basis of which one finds the notions of left and right monoidal endofunctors, for which we are also going to prove coherence. We prove our results first in the absence of symmetry, and then with symmetry. In the later part of the paper we extend our coherence results to monoidal endofunctors in monoidal categories that have diagonal or codiagonal natural transformations (we call these monoidal categories relevant categories), or where the monoidal structure is given by finite products or coproducts.

Some of our coherence results may be understood as basic coherence results for equations between deductions in modal logic. In this paper we find systems that may be understood as fragments of $K$ with the necessity operator $\square$ primitive; in the sequel, with comonads, we will find fragments of $S 4$ with primitive.

## 2 Endofunctors in monoidal categories

In this section we deal with coherence for monoidal categories with endofunctors for which we do not assume yet that they are monoidal. This is a basic auxiliary result, which we will need later.

A monoidal category is, as usual, a category with a biendofunctor $\otimes$, a special unit object $I$, and the natural isomorphisms whose components are the arrows

$$
\begin{aligned}
& a_{A, B, C}:(A \otimes B) \otimes C \rightarrow A \otimes(B \otimes C), \\
& l_{A}: I \otimes A \rightarrow A, \\
& r_{A}: A \otimes I \rightarrow A
\end{aligned}
$$

which satisfy Mac Lane's coherence equations (see [13], Section VII.1; our notation comes from [5], Section II.1).

Let $\mathcal{E}$ be the free monoidal category with a family of endofunctors; freedom means here and later free generation by two arbitrary sets, one of which is conceived as the set of generating objects, and the other as the set of generating functors $\left\{E^{i} \mid i \in \mathcal{I}\right\}$. We call the generating objects and the generating functors collectively generators. If $\mathcal{I}$ is empty, then $\mathcal{E}$ is just the free monoidal category generated by a set of generating objects.

The category $\mathcal{E}$ is made of syntactical material. Its objects are propositional formulae built with the binary connective $\otimes$, the unary connectives $E^{i}$ and the nullary connective $I$ out of the generating objects, which we take to be the propositional letters $p, q, r, \ldots$ An object of $\mathcal{E}$ is atomic when it is a generating object or of the form $E^{i} A$. An object of $\mathcal{E}$ is diversified on generating objects when every generating object occurs in it at most once. We define analogously diversification on generating functors, and we say that an object is diversified when it is diversified both on generating objects and on generating functors. For $E^{i} A$ a subformula of an object $B$ of $\mathcal{E}$, the scope in $B$ of the outermost occurrence of $E^{i}$ in $E^{i} A$ is the set of all the generators in $A$.

The arrows of $\mathcal{E}$ are equivalence classes of arrow terms made out of the primitive arrow terms $\mathbf{1}_{A}, a_{A, B, C}, l_{A}, r_{A}$, with the operations $\circ$, $\otimes$ and $E^{i}$ so that the equations assumed for defining a monoidal category together with functorial equations for $E^{i}$, for each $i \in \mathcal{I}$, are satisfied (cf. [3], Chapter 2). We take for granted the superscripts $i$ of $E^{i}$ and omit them, except when they are essential. (We do the same later with $\psi, \psi_{0}, \psi^{L}$ and $\psi^{R}$.) The existence of free structures like $\mathcal{E}$ is guaranteed by the purely equational definition of these structures.

Every arrow term $f$ of $\mathcal{E}$ is equal to an arrow term $f_{n} \circ \ldots \circ f_{1}$, called developed, which is $\mathbf{1}_{A}$ if $n=0$, and if $n \geq 1$, then for every $j \in\{1, \ldots, n\}$ in $f_{j}$ we have exactly one occurrence of $a, l$ or $r$, and no occurrence of $\circ$; such an $f_{i}$ is called a factor. The subterm $a_{A, B, C}$, or $l_{A}$, or $r_{A}$, of a factor is its head (see [3], Section 2.7). A factor with head $a_{A, B, C}$ is an $a$-factor, and analogously in other cases. We are going to prove the following theorem.
$\mathcal{E}$-Coherence. The category $\mathcal{E}$ is a preorder.
Proof. Suppose we have two arrow terms $f, g: A \rightarrow B$ of $\mathcal{E}$. To show that $f=g$, we proceed by induction on the number $n$ of occurrences of $E$ in $A$, which is equal to this number in $B$. In the basis, when $n=0$, we have Mac Lane's coherence result for monoidal categories (see [13], Section VII.2, or [3], Chapter 4).

When $n>0$, take a single arbitrary occurrence of $E^{i}$ in $A$, and replace it by $E^{j}$ such that $j$ is not an index of any $E$ in $A$. (If all the generating functors occur in $A$, then we enlarge for the sake of the proof the set of generating functors with a new functor $E^{j}$, which functions just as a placeholder.) Make this replacement at the appropriate place in $B$, and in the arrow terms $f$ and $g$ so as to obtain the arrow terms $f^{\prime}, g^{\prime}: A^{\prime} \rightarrow B^{\prime}$ of $\mathcal{E}$. By naturality and functorial equations
$f^{\prime}$ is equal to a developed arrow term $f_{2} \circ f_{1}$ such that no head of a factor of $f_{1}: A^{\prime} \rightarrow C$ is in the scope of $E^{j}$ and all the heads of the factors of $f_{2}: C \rightarrow B^{\prime}$ are in the scope of $E^{j}$. The object $C$ is completely determined by $A^{\prime}$ and $B^{\prime}$. Analogously we have $g^{\prime}=g_{2} \circ g_{1}$, with $g_{1}: A^{\prime} \rightarrow C$ and $g_{2}: C \rightarrow B^{\prime}$. Since $E^{j} D$ in $f_{1}$ and $g_{1}$ amounts to a generating object (it is a parameter), and since in $f_{2}$ and $g_{2}$ only what is within the scope of $E^{j}$ counts, by the induction hypothesis we have $f_{1}=g_{1}$ and $f_{2}=g_{2}$, and so $f^{\prime}=g^{\prime}$, from which $f=g$ follows by substitution.

## 3 Monoidal and locally monoidal endofunctors

An endofunctor may preserve the monoidal structure of a category up to a natural transformation either globally or locally. We have global preservation when our endofunctor $E$ is a monoidal functor in the sense of [5] (Section II.1; see also [13], Section XI.2). This means that in our monoidal category we have a natural transformation whose components are the arrows

$$
\psi_{A, B}: E A \otimes E B \rightarrow E(A \otimes B)
$$

and we have also the arrow $\psi_{0}: I \rightarrow E I$; the monoidal structure is preserved up to $\psi$ and $\psi_{0}$, which means that the following equations hold:
$(\psi a) \quad E a_{A, B, C} \circ \psi_{A \otimes B, C} \circ\left(\psi_{A, B} \otimes \mathbf{1}_{E C}\right)=\psi_{A, B \otimes C} \circ\left(\mathbf{1}_{E A} \otimes \psi_{B, C}\right) \circ a_{E A, E B, E C}$,

$$
\begin{align*}
& E l_{A} \circ \psi_{I, A} \circ\left(\psi_{0} \otimes \mathbf{1}_{E A}\right)=l_{E A} \\
& E r_{A} \circ \psi_{A, I} \circ\left(\mathbf{1}_{E A} \otimes \psi_{0}\right)=r_{E A}
\end{align*}
$$

The global character of the preservation is manifested in $(\psi a)$ by $E$ from the left-hand side falling on every index of $a$ on the right-hand side. (The notation with $\psi$ stems from [9] and [10].)

We have local preservation with the following three notions of endofunctor suggested by [9] and [10]. Monoidal functors need not be endofunctors, but the notions we are going to consider now are tied to endofunctors only.

We say that an endofunctor $E$ of a monoidal category is left monoidal when we have a natural transformation whose components are the arrows

$$
\psi_{A, B}^{L}: E A \otimes B \rightarrow E(A \otimes B)
$$

and the monoidal structure is preserved up to $\psi^{L}$, which means that the following equations hold:

$$
\begin{aligned}
& \left(\psi^{L} a\right) \quad E a_{A, B, C} \circ \psi_{A \otimes B, C}^{L} \circ\left(\psi_{A, B}^{L} \otimes \mathbf{1}_{C}\right)=\psi_{A, B \otimes C}^{L} \circ a_{E A, B, C}, \\
& \left(\psi^{L} r\right) \quad E r_{A} \circ \psi_{A, I}^{L}=r_{E A} .
\end{aligned}
$$

The local character of the preservation is manifested in $\left(\psi^{L} a\right)$ by $E$ from the left-hand side falling on a single index of $a$ on the right-hand side.

We say, analogously, that $E$ is right monoidal when we have a natural transformation whose components are the arrows

$$
\psi_{A, B}^{R}: A \otimes E B \rightarrow E(A \otimes B)
$$

and the monoidal structure is preserved up to $\psi^{R}$, which means that the following equations hold:

$$
\begin{aligned}
& \left(\psi^{R} a\right) \quad E a_{A, B, C} \circ \psi_{A \otimes B, C}^{R}=\psi_{A, B \otimes C}^{R} \circ\left(\mathbf{1}_{A} \otimes \psi_{B, C}^{R}\right) \circ a_{A, B, E C}, \\
& \left(\psi^{R} l\right)
\end{aligned} \quad E l_{A} \circ \psi_{I, A}^{R}=l_{E A} .
$$

We say that $E$ is locally monoidal when it is both left and right monoidal, and we have moreover the equation
$\left(\psi^{L} \psi^{R} a\right) \quad E a_{A, B, C} \circ \psi_{A \otimes B, C}^{L} \circ\left(\psi_{A, B}^{R} \otimes \mathbf{1}_{C}\right)=\psi_{A, B \otimes C}^{R} \circ\left(\mathbf{1}_{A} \otimes \psi_{B, C}^{L}\right) \circ a_{A, E B, C}$.
We define, as we defined the category $\mathcal{E}$ in the preceding section, the free monoidal categories with a family of monoidal endofunctors, a family of left monoidal endofunctors, a family of right monoidal endofunctors, or a family of locally monoidal endofunctors, which we call respectively $\mathcal{M}, \mathcal{L}^{L}, \mathcal{L}^{R}$ and $\mathcal{L}$. All these categories have the same propositional formulae as objects (provided the sets of generators are the same). Then it is easy to see that the category $\mathcal{L}^{R}$ is an isomorphic, mirror image, of $\mathcal{L}^{L}$. For the categories $\mathcal{M}, \mathcal{L}^{L}, \mathcal{L}^{R}$ and $\mathcal{L}$, we define the notions of developed arrow term, factor and head of a factor analogously to what we had for $\mathcal{E}$ in the preceding section.

We define a functor $G$ from $\mathcal{M}$ to the category Fun of functions between finite ordinals by stipulating that $G A$, for $A$ a propositional formula, is the number of occurrences of $E$ in $A$ (i.e. the number of all $E^{i}$ 's in $A$, for every $i$ ), while $G f$ for an arrow term $f$ of $\mathcal{M}$ is defined inductively on the complexity of $f$. We have that $G \mathbf{1}_{A}, G a_{A, B, C}, G l_{A}$ and $G r_{A}$ are identity functions, while for the remaining primitive arrow terms we have the clause corresponding to the following picture:

and $G \psi_{0}$ is the empty function from $\varnothing$, which is $G I$, to $\{\varnothing\}$, which is $G E I$ (in our picture we obtain a crossing if there is an $E$ in $A$, different or not from the $E$ in the picture). We also have clauses corresponding to the following pictures:

and $G(g \circ f)$ is the composition of the functions $G f$ and $G g$. It is easy to verify by induction on the length of derivation that $G$ so defined on the arrow terms of $\mathcal{M}$ induces a functor from $\mathcal{M}$ to Fun.

Intuitively, with $G f: G A \rightarrow G B$ we note from which occurrences of $E$ in $A$ the occurrences of $E$ in $B$ originate. We call $G f$ a graph. For example, for the two sides of the equation $(\psi a)$ we have the following pictures:

and for the two sides of $(\psi l)$ we have the following pictures:


The functors from $\mathcal{L}^{L}, \mathcal{L}^{R}$ and $\mathcal{L}$ to Fun analogous to $G$, which we all call $G$, are defined as $G$ save for the clauses corresponding to the following pictures:

(this means that $G \psi_{A, B}^{L}$ is an identity arrow). The target category of $G$ for $\mathcal{L}^{R}$ and $\mathcal{L}$ is the subcategory of Fun of bijections between finite ordinals, and for $\mathcal{L}^{L}$ this is the discrete subcategory of $F u n$, with all arrows just identity arrows.

If $\mathcal{K}$ is a category like $\mathcal{M}, \mathcal{L}^{L}, \mathcal{L}^{R}$ or $\mathcal{L}$, then we call $\mathcal{K}$-Coherence the proposition that $G$ from $\mathcal{K}$ to $F u n$, or a category like $F u n$, is a faithful functor. Since the image of $\mathcal{L}^{L}$ under $G$ is a discrete category, $\mathcal{L}^{L}$-Coherence amounts to the proposition that $\mathcal{L}^{L}$ is a preorder, and since $\mathcal{L}^{L}$ and $\mathcal{L}^{R}$ are isomorphic, $\mathcal{L}^{R_{-}}$ Coherence amounts too to the proposition that $\mathcal{L}^{R}$ is a preorder. (Our notion
of $\mathcal{K}$-coherence is a standard notion of coherence, which stems from Mac Lane's coherence results for monoidal and symmetric monoidal categories; see [13], [3] and references therein.)

Note that with this understanding of coherence we cannot expect that $\psi_{A, B}$ and $\psi_{0}$ be isomorphisms. With the natural $G$ image (now not in Fun) of the inverses of $\psi_{A, B}$ and $\psi_{0}$, we have

neither of which corresponds to an identity arrow. The reasons for this failure of isomorphism are similar to the reasons for the failure of the isomorphism of distribution investigated in [3].

Instead of formulating our coherence results in terms of $G$ and graphs, we could have formulations based on diversified objects (see the preceding section). For example, $\mathcal{M}$-Coherence, which we are going to prove in the next section, is equivalent to the proposition that for all arrow terms $f, g: A \rightarrow B$ of $\mathcal{M}$ with $B$ diversified we have $f=g$ in $\mathcal{M}$.

## $4 \mathcal{M}$-Coherence

The category $\mathcal{M}$ is equivalent to its strictification $\mathcal{M}^{\text {str }}$, where

$$
\begin{array}{ll}
(A \otimes B) \otimes C=A \otimes(B \otimes C), & a_{A, B, C}=\mathbf{1}_{A \otimes B \otimes C} \\
I \otimes A=A=A \otimes I, & l_{A}=\mathbf{1}_{A}=r_{A}
\end{array}
$$

The preordered groupoid subcategory of $\mathcal{M}$ over which we make the strictification is the category $\mathcal{E}$ of Section 2 (see [3], Section 3.2, which, together with Section 3.1, provides a general treatment of strictification, where references to earlier approaches may be found).

Let $H$ be the functor from $\mathcal{M}^{\text {str }}$ to $\mathcal{M}$, and $H^{\prime}$ the functor in the opposite direction, by which $\mathcal{M}^{\text {str }}$ and $\mathcal{M}$ are equivalent categories. We will show that the composite functor $G H$ from $\mathcal{M}^{s t r}$ to Fun is faithful. This implies that $G$ from $\mathcal{M}$ to $F u n$ is faithful, i.e. $\mathcal{M}$-Coherence, in the following manner. Suppose that $G f=G g$; then, since for every arrow $h$ of $\mathcal{M}$ we have $G H H^{\prime} h=G h$, we obtain $G H H^{\prime} f=G H H^{\prime} g$, and by the faithfulness of $G H$, we obtain $H^{\prime} f=H^{\prime} g$, from which we obtain $H H^{\prime} f=H H^{\prime} g$, and hence $f=g$ in $\mathcal{M}$.

Proposition 1. The functor $G H$ from $\mathcal{M}^{\text {str }}$ to Fun is faithful.

Proof. Every arrow term $f$ of $\mathcal{M}^{\text {str }}$ is equal to a developed arrow term $f_{n} \circ \ldots \circ f_{1}$, such that each $f_{j}$ is either a $\psi$-factor or a $\psi_{0}$-factor. By applying naturality and functorial equations, and the equations

$$
\psi_{I, A} \circ\left(\psi_{0} \otimes \mathbf{1}_{E A}\right)=\mathbf{1}_{E A}=\psi_{A, I} \circ\left(\mathbf{1}_{E A} \otimes \psi_{0}\right)
$$

which are $(\psi l)$ and $(\psi r)$ strictified, we obtain from a developed arrow term an arrow term equal to it, of the form $h \circ g_{m} \circ \ldots \circ g_{1}$, where in $h$ we have no occurrence of $\psi$, and $\circ$ may occur only in subterms of $h$ of the form $E \ldots E \psi_{0} \circ \ldots \circ \psi_{0}$ : $I \rightarrow E \ldots E I$; for $g_{m} \circ \ldots \circ g_{1}$ we assume that it is developed without $\psi_{0}$-factors.

If $m \geq 1$, then each $g_{j}$ is a $\psi$-factor, and we assign to $g_{j}$ a finite ordinal $\tau\left(g_{j}\right)$ obtained by applying the function $G H\left(g_{m} \circ \ldots \circ g_{j+1}\right)$ to the number $\kappa\left(g_{j}\right)$ of occurrences of $E$ in $g_{j}$ to the left of $\psi$. Intuitively, this is the place where the contracted $E$ of $g_{j}$ will end up in the codomain of $g_{m}$. For example, with $g_{2} \circ g_{1}$ being

$$
E^{1} \psi_{p, E^{1}(p \otimes q)}^{2} \circ E^{1}\left(\mathbf{1}_{E^{2} p} \otimes E^{2} \psi_{p, q}^{1}\right)
$$

$\kappa\left(g_{1}\right)=3$ and $\tau\left(g_{1}\right)=2$, which is clear from the following picture:


It is not difficult to see that for the $\psi$-factors $g_{i}$ and $g_{i+1}$ such that $\tau\left(g_{i+1}\right)=$ $k<l=\tau\left(g_{i}\right)$ we have, by naturality and functorial equations, that $g_{i+1} \circ g_{i}=$ $g_{i+1}^{\prime} \circ g_{i}^{\prime}$ for some $\psi$-factors $g_{i}^{\prime}$ and $g_{i+1}^{\prime}$ such that $\tau\left(g_{i}^{\prime}\right)=k$ and $\tau\left(g_{i+1}^{\prime}\right)=l$. So $g_{m} \circ \ldots \circ g_{1}$ is equal to $g_{m}^{\prime} \circ \ldots \circ g_{1}^{\prime}$ such that $\tau\left(g_{i+1}^{\prime}\right) \geq \tau\left(g_{i}^{\prime}\right)$. If $\tau\left(g_{i+1}^{\prime}\right)=\tau\left(g_{i}^{\prime}\right)$, then they can be permuted by the equation $(\psi a)$ strictified. (Note that with five applications of that equation we may permute also the rightmost two factors of $\left.\psi_{p \otimes q, r \otimes s} \circ\left(\mathbf{1}_{E(p \otimes q)} \otimes \psi_{r, s}\right) \circ\left(\psi_{p, q} \otimes \mathbf{1}_{E r \otimes E s}\right).\right)$ We take $h \circ g_{m}^{\prime} \circ \ldots \circ g_{1}^{\prime}$ to be a normal form of $f$. As an arrow term, this normal form is not unique, because we may have differences based on the last mentioned permutations or on equations like $E \mathbf{1}_{A}=\mathbf{1}_{E A}$.

We may show however that if $G H f=G H f^{\prime}$ and $f$ and $f^{\prime}$ are in normal form, then $f$ and $f^{\prime}$ differ from each other only with respect to what is mentioned in the preceding sentence. In $f$ let a block $\vec{f}_{i}$ be a composition of $\psi$-factors $f_{i_{k}} \circ \ldots \circ f_{i_{1}}$ such that $\tau\left(f_{i_{k}}\right)=\ldots=\tau\left(f_{i_{1}}\right)=l$. We stipulate then that $\tau\left(\vec{f}_{i}\right)=l$. Let $f$ be $h \circ \vec{f}_{n} \circ \ldots \circ \vec{f}_{1}$ such that for $i, j \in\{1, \ldots, n\}$ if $i<j$, then $\tau\left(\vec{f}_{i}\right)<\tau\left(\vec{f}_{j}\right)$. The arrangement of these blocks is strictly increasing. Let analogously $f^{\prime}$ be $h^{\prime} \circ \vec{f}_{n}^{\prime} \circ \ldots \circ \vec{f}_{1}^{\prime}$. From $G H f=G H f^{\prime}$ we conclude first that $n=n^{\prime}$, and we proceed by induction on $n$. If $n=0$, then we conclude easily
that $h=h^{\prime}$. If $n>0$, then we conclude that $\tau\left(\overrightarrow{f_{1}}\right)=\tau\left(\overrightarrow{f_{1}^{\prime}}\right)$, that $G H \overrightarrow{f_{1}}=G H \overrightarrow{f_{1}^{\prime}}$ and that $G H\left(h \circ \overrightarrow{f_{n}} \circ \ldots \circ \overrightarrow{f_{2}}\right)=G H\left(h^{\prime} \circ \overrightarrow{f_{n}^{\prime}} \circ \ldots \circ \overrightarrow{f_{2}^{\prime}}\right)$. From this we conclude that $\vec{f}_{1}=\vec{f}_{1}^{\prime}$, and by the induction hypothesis $h \circ \vec{f}_{n} \circ \ldots \circ \overrightarrow{f_{2}}=h^{\prime} \circ \overrightarrow{f_{n}^{\prime}} \circ \ldots \circ \overrightarrow{f_{2}^{\prime}}$. So $f=g$ in $\mathcal{M}^{s t r}$.

Hence we have $\mathcal{M}$-Coherence.
Note that $\mathcal{M}$ is not a preorder. The following two arrows:

$$
\left(\psi_{0} \otimes \mathbf{1}_{E I}\right) \circ l_{E I}^{-1},\left(\mathbf{1}_{E I} \otimes \psi_{0}\right) \circ r_{E I}^{-1}: E I \rightarrow E I \otimes E I
$$

have different $G$ images, and are different in $\mathcal{M}$; this counterexample for preorder is from [12] (Section 0). Another counterexample is given by the two arrows

$$
\left(\mathbf{1}_{E A} \otimes\left(E l_{B^{\circ}} \psi_{I, B}\right)\right) \circ a_{E A, E I, E B},\left(E r_{A^{\circ}} \psi_{A, I}\right) \otimes \mathbf{1}_{E B}:(E A \otimes E I) \otimes E B \rightarrow E A \otimes E B
$$

This counterexample shows that graphs are essential for coherence even in the absence of $\psi_{0}$.

Let, however, $\mathcal{M}^{-}$be the category defined like $\mathcal{M}$ save that we reject $I$ and everything that involves it-namely, $l, r$ and $\psi_{0}$. The category $\mathcal{M}^{-}$is a preorder, and graphs are irrelevant for its coherence. When we try to determine whether there is an arrow of $\mathcal{M}^{-}$of a given type (i.e. with a given source and target), we find that if there is such an arrow it must be unique.

This will become clear with the following example. Suppose we want to determine whether there is an arrow

$$
f: E(E p \otimes E(E q \otimes E p)) \rightarrow E E(p \otimes E(q \otimes p))
$$

We diversify first the propositional letters and the occurrences of $E$ in the target, and the question is then whether we have an arrow

$$
f^{\prime}: E(E p \otimes E(E q \otimes E r)) \rightarrow E^{1} E^{2}\left(p \otimes E^{3}(q \otimes r)\right)
$$

for some superscripts assigned to the occurrences of $E$ in the source. The leftmost $E$ in the source must be $E^{1}$. Since this $E$ has $\{p, q, r\}$ in its scope as $E^{1}$ in the target, we are done with $E^{1}$. The leftmost of the remaining $E^{\prime}$ s in the source must be $E^{2}$. Since this $E$ has only $\{p\}$ in its scope, while $E^{2}$ in the target has $\{p, q, r\}$, we take as $E^{2}$ the leftmost of the remaining $E$ 's in the source in whose scope we find $\{q, r\}$. By iterating this procedure we find the arrow
$E^{1} E^{2}\left(\mathbf{1}_{p} \otimes \psi_{a, r}^{3}\right) \circ E^{1} \psi_{p, E^{3} q \otimes E^{3} r}^{2}: E^{1}\left(E^{2} p \otimes E^{2}\left(E^{3} q \otimes E^{3} r\right)\right) \rightarrow E^{1} E^{2}\left(p \otimes E^{3}(q \otimes r)\right)$.

## $5 \quad \mathcal{L}^{L}, \mathcal{L}^{R}$ and $\mathcal{L}$-Coherence

To prove $\mathcal{L}^{L}$-Coherence, which as we said towards the end of Section 3 amounts to $\mathcal{L}^{L}$ being a preorder, we proceed as for $\mathcal{M}$-Coherence. We introduce the
strictification of $\mathcal{L}^{L}$ and in the proof of the faithfulness of $G H$ we have a normal form that is a simplified version of the normal form of the preceding proof for $\mathcal{M}^{s t r}$. The $h$ part of the normal form with $\psi_{0}$-factors does not exist, and instead of the $\psi$-factors part we have a $\psi^{L}$-factors part. The blocks are now of length 1, i.e. single $\psi^{L}$-factors, according to the equation $\left(\psi^{L} a\right)$ strictified, and $\psi^{L_{-}}$ factors with $\psi_{A, I}^{L}$ are identity arrows by the equation $\left(\psi^{L} r\right)$ strictified. We may prove $\mathcal{L}^{R}$-Coherence either directly in the same manner, or just appeal to the isomorphism of $\mathcal{L}^{L}$ and $\mathcal{L}^{R}$.

To prove $\mathcal{L}$-Coherence we proceed again as before. The $h$ part of the normal form does not exist again, and we have $\psi^{L}$-factors and $\psi^{R}$-factors. A block is either a single $\psi^{L}$-factor, or a single $\psi^{R}$-factor, or a pair of factors made of one $\psi^{L}$-factor and one $\psi^{R}$-factor, which may be permuted according to the equation $\left(\psi^{L} \psi^{R} a\right)$ strictified.

## 6 Coherence with linear endofunctors

A symmetric monoidal category is, as usual, a monoidal category with the natural isomorphism whose components are the arrows

$$
c_{A, B}: A \otimes B \rightarrow B \otimes A,
$$

which satisfy Mac Lane's coherence conditions (see [13], Section XI.1).
A linear endofunctor in a symmetric monoidal category is a monoidal endofunctor $E$ that preserves $c$ globally; i.e. we have the equation
$(\psi c) \quad E c_{A, B} \circ \psi_{A, B}=\psi_{B, A} \circ c_{E A, E B}$.
(We use linear instead of symmetric monoidal for the sake of brevity; linear comes from the connection with the structural fragment of linear logic, whose name comes from linear algebra.)

A locally linear endofunctor in a symmetric monoidal category may be defined as a locally monoidal endofunctor $E$ that satisfies

$$
\left(\psi^{L} \psi^{R} c\right) \quad E c_{A, B} \circ \psi_{A, B}^{L}=\psi_{B, A}^{R} \circ c_{E A, B} .
$$

An alternative, simpler, definition is that it is either a left monoidal or a right monoidal endofunctor in a symmetric monoidal category. If it is left monoidal, then from $\left(\psi^{L} \psi^{R} c\right)$ we obtain the definition of $\psi^{R}$ in terms of $\psi^{L}$ and $c$, and we derive $\left(\psi^{R} a\right),\left(\psi^{R} l\right)$ and $\left(\psi^{L} \psi^{R} a\right)$.

Let $\mathcal{M}_{c}$ and $\mathcal{L}_{c}$ be the free symmetric monoidal categories with a family of respectively linear or locally linear endofunctors; these categories are defined analogously to $\mathcal{M}$ and $\mathcal{L}$. We define the functors $G$ from $\mathcal{M}_{c}$ and $\mathcal{L}_{c}$ to Fun by stipulating first that $G A$ is the number of occurrences of generators in $A$. Up to now we took $G A$ to be just the number of occurrences of generating functors in $A$, but we could as well have counted also occurrences of generating objects;
this was however superfluous up to now. The remainder of the definitions of the new functors $G$ is analogous to the definitions of $G$ from $\mathcal{M}$ and $\mathcal{L}$ to Fun, save that we add the clause corresponding to the picture


The category $\mathcal{M}_{c}$ is equivalent to its strictification $\mathcal{M}_{c}^{\text {str }}$, as $\mathcal{M}$ is equivalent to $\mathcal{M}^{\text {str }}$ (see Section 4), and as before we prove the following proposition, which entails $\mathcal{M}_{c}$-Coherence.

Proposition 2. The functor $G H$ from $\mathcal{M}_{c}^{\text {str }}$ to Fun is faithful.
Proof. We introduce the following abbreviation in $\mathcal{M}_{c}^{\text {str }}$ :
$\Psi_{A_{1}, A_{2} ; B}={ }_{d f}\left(\psi_{A_{1}, A_{2}} \otimes \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{E A_{1}} \otimes c_{B, E A_{2}}\right): E A_{1} \otimes B \otimes E A_{2} \rightarrow E\left(A_{1} \otimes A_{2}\right) \otimes B$.
We obtain from a developed arrow term of $\mathcal{M}_{c}^{\text {str }}$ a $\Psi$-developed arrow term by replacing the head $\psi_{A_{1}, A_{2}}$ of every $\psi$-factor by $\Psi_{A_{1}, A_{2} ; I}$; this replacement is justified by the equation $\psi_{A_{1}, A_{2}}=\Psi_{A_{1}, A_{2} ; I}$ of $\mathcal{M}_{c}^{\text {str }}$. Every $\Psi$-developed arrow term is equal in $\mathcal{M}_{c}^{\text {str }}$ to an arrow term of the form $h \circ g_{m} \circ \ldots \circ g_{1}$, where in $h$ we have no occurrences $\Psi$ and $c$, while occurrences of $\circ$ are restricted as in the proof of Proposition 1; for $g_{m} \circ \ldots \circ g_{1}$ we suppose that it is $\Psi$-developed without $\psi_{0}$-factors; i.e. it has only $\Psi$-factors and $c$-factors.

If $m \geq 1$, and $g_{i}$ is $\Psi$-factor, then we assign to $g_{i}$ a finite ordinal $\tau\left(g_{i}\right)$ exactly as we did in the proof of Proposition 1. We then proceed in principle as in that proof to obtain a normal form. When $\tau\left(g_{i+1}\right)<\tau\left(g_{i}\right)$ we proceed exactly as before. Here are the new cases we have to consider.

Suppose we have the $\Psi$-factors $g_{i}$ and $g_{i+1}$ such that $\tau\left(g_{i+1}\right)=\tau\left(g_{i}\right)$. Then we may have the opportunity to apply the following equations of $\mathcal{M}_{c}^{\text {str }}$ from left to right:

```
\((\Psi \Psi 1) \quad\left(\Psi_{A_{1} \otimes A_{3}, A_{2} ; B_{1}} \otimes \mathbf{1}_{B_{2}}\right) \circ \Psi_{A_{1}, A_{3} ; B_{1} \otimes E A_{2} \otimes B_{2}}=\)
    \(\left(E\left(\mathbf{1}_{A_{1}} \otimes c_{A_{2}, A_{3}}\right) \otimes \mathbf{1}_{B_{1} \otimes B_{2}}\right) \circ \Psi_{A_{1} \otimes A_{2}, A_{3} ; B_{1} \otimes B_{2}} \circ\left(\Psi_{A_{1}, A_{2} ; B_{1}} \otimes \mathbf{1}_{B_{2} \otimes E A_{3}}\right)\),
\((\Psi \Psi 2) \quad\left(\Psi_{A_{1}, A_{2} \otimes A_{3} ; B_{1}} \otimes \mathbf{1}_{B_{2}}\right) \circ\left(\mathbf{1}_{E A_{1} \otimes B_{1}} \otimes \Psi_{A_{2}, A_{3} ; B_{2}}\right)=\)
    \(\Psi_{A_{1} \otimes A_{2}, A_{3} ; B_{1} \otimes B_{2}} \circ\left(\Psi_{A_{1}, A_{2} ; B_{1}} \otimes \mathbf{1}_{B_{2} \otimes E A_{3}}\right)\).
```

We call a $c$-factor atomized when in its head $c_{A, B}$ the objects $A$ and $B$ are atomic (see Section 2). By the strictified version of Mac Lane's hexagonal coherence condition for symmetric monoidal categories (see [13], Section XI.1), and by $c_{A, I}=l_{A}^{-1} \circ r_{A}=\mathbf{1}_{A}$, we may assume that all our $c$-factors are atomized. Suppose we have an atomic $c$-factor $g_{i}$ and a $\Psi$-factor $g_{i+1}$. Then we may have the opportunity to apply either the naturality and functorial equations, or the equation $(\psi c)$, or the following equations of $\mathcal{M}_{c}^{s t r}$ :

$$
\begin{align*}
& \Psi_{A_{1}, A_{2} ; B_{1} \otimes B_{2}} \circ\left(c_{B_{1}, E A_{1}} \otimes \mathbf{1}_{B_{2} \otimes E A_{2}}\right)= \\
& \left(c_{B_{1}, E\left(A_{1} \otimes A_{2}\right)} \otimes \mathbf{1}_{B_{2}}\right) \circ\left(\mathbf{1}_{B_{1}} \otimes \Psi_{A_{1}, A_{2} ; B_{2}}\right), \\
& \left(\mathbf{1}_{B_{1}} \otimes \Psi_{A_{1}, A_{2} ; B_{2}}\right) \circ\left(c_{E A_{1}, B_{1}} \otimes \mathbf{1}_{B_{2} \otimes E A_{2}}\right)= \\
& \quad\left(c_{E\left(A_{1} \otimes A_{2}\right), B_{1}} \otimes \mathbf{1}_{B_{2}}\right) \circ \Psi_{A_{1}, A_{2} ; B_{1} \otimes B_{2}},
\end{align*}
$$

in order to obtain $g_{i+1} \circ g_{i}=g_{i+1}^{\prime} \circ g_{i}^{\prime}$ for a $\Psi$-factor $g_{i}^{\prime}$ and a $c$-factor $g_{i+1}^{\prime}$. Otherwise, we must have the opportunity to apply the equations

$$
\begin{array}{ll}
(\Psi c 3) & \left(\Psi_{A_{1}, A_{2} ; B_{1}} \otimes \mathbf{1}_{B_{2}}\right) \circ\left(\mathbf{1}_{E A_{1} \otimes B_{1}} \otimes c_{B_{2}, E A_{2}}\right)=\Psi_{A_{1}, A_{2} ; B_{1} \otimes B_{2}}, \\
(\Psi c 4) & \Psi_{A_{1}, A_{2} ; B_{1} \otimes B_{2}} \circ\left(\mathbf{1}_{E A_{1} \otimes B_{1}} \otimes c_{E A_{2}, B_{2}}\right)=\Psi_{A_{1}, A_{2} ; B_{1}} \otimes \mathbf{1}_{B_{2}},
\end{array}
$$

which follow from the definition of $\Psi$, in order to obtain $g_{i+1} \circ g_{i}=g$ for a $\Psi$-factor $g$. By applying all these reductions we reach our normal form, which looks as follows.

Let a block $\vec{f}_{i}$ be a composition of $\Psi$-factors $f_{i_{k}} \circ \ldots \circ f_{i_{1}}$, all with the same $\tau$ value, and such that $f_{i_{j+1}} \circ f_{i_{j}}$ is never of the form of the left-hand side of $(\Psi \Psi 1)$ and ( $\Psi \Psi 2$ ). Our normal form is $h \circ g \circ \vec{f}_{n} \circ \ldots \circ \vec{f}_{1}$ such that $n \geq 0$ and the arrangement of the blocks is strictly increasing (see the proof of Proposition 1 in Section 4); the arrow term $g$ has no occurrence of $\psi$ and $\psi_{0}$ (but $c$ may occur), and $h$ has no occurrence of $\psi$ and $c$ (but $\psi_{0}$ may occur); 。 may occur in $h$ only as specified in the proof of Proposition 1.

The last part of the proof is obtained with slight modifications of the last part of the proof of Proposition 1. We have the same kind of induction, but in the basis we do not have just $h=h^{\prime}$, but $h \circ g=h^{\prime} \circ g^{\prime}$. That $h=h^{\prime}$ follows as before, while $g=g^{\prime}$ follows by a coherence result generalizing Mac Lane's symmetric monoidal coherence (see [13], Section XI.1, or [3], Chapter 5) as $\mathcal{E}$ Coherence of Section 2 generalizes Mac Lane's monoidal coherence. This result is proved analogously to $\mathcal{E}$-Coherence.

From this proposition we infer $\mathcal{M}_{c}$-Coherence.
To prove $\mathcal{L}_{c}$-Coherence we proceed as for $\mathcal{M}_{c}$-Coherence. We introduce the strictification $\mathcal{L}_{c}^{s t r}$ of $\mathcal{L}_{c}$ and we prove the following proposition, from which we will infer $\mathcal{L}_{c}$-Coherence.

Proposition 3. The functor GH from $\mathcal{L}_{c}^{s t r}$ to Fun is faithful.
Proof. We have a normal form for the arrow terms of $\mathcal{L}_{c}^{s t r}$ which is a modification of the normal form of the proof of Proposition 2. For this normal form we have the following abbreviations in $\mathcal{L}_{c}^{s t r}$ :

$$
\begin{gathered}
\Psi_{A_{1}, A_{2} ; B}^{L}={ }_{d f}\left(\psi_{A_{1}, A_{2}}^{L} \otimes \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{E A_{1}} \otimes c_{B, A_{2}}\right): \\
E A_{1} \otimes B \otimes A_{2} \rightarrow E\left(A_{1} \otimes A_{2}\right) \otimes B, \\
\Psi_{A_{1}, A_{2} ; B}^{R}={ }_{d f}\left(\psi_{A_{1}, A_{2}}^{R} \otimes \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{A_{1}} \otimes c_{B, E A_{2}}\right): \\
A_{1} \otimes B \otimes E A_{2} \rightarrow E\left(A_{1} \otimes A_{2}\right) \otimes B,
\end{gathered}
$$

which are both obtained from the definition of $\Psi_{A_{1}, A_{2} ; B}$ by adding the superscripts $L$ or $R$ to $\psi$ and deleting some occurrences of $E$ in the subscripted indices.

The $h$ part of the normal form with $\psi_{0}$-factors does not exist now, and instead of $\Psi$-factors we have $\Psi^{L}$-factors and $\Psi^{R}$-factors, which we call collectively $\Psi$-factors. For a $\Psi$-factor $g_{j}$ we define $\tau\left(g_{j}\right)$ as before, and we proceed as before when $\tau\left(g_{i+1}\right)<\tau\left(g_{i}\right)$ for the $\Psi$-factors $g_{i}$ and $g_{i+1}$.

When we have $\tau\left(g_{i+1}\right)=\tau\left(g_{i}\right)$, then we may apply one of the following equations of $\mathcal{L}_{c}^{s t r}$ from left to right:

$$
\begin{aligned}
& \left(\Psi^{L} \Psi^{L}\right) \quad\left(\Psi_{A_{1} \otimes A_{3}, A_{2} ; B_{1}}^{L} \otimes \mathbf{1}_{B_{2}}\right) \circ \Psi_{A_{1}, A_{3} ; B_{1} \otimes A_{2} \otimes B_{2}}^{L}= \\
& \left(E\left(\mathbf{1}_{A_{1}} \otimes c_{A_{2}, A_{3}}\right) \otimes \mathbf{1}_{B_{1} \otimes B_{2}}\right) \circ \Psi_{A_{1} \otimes A_{2}, A_{3} ; B_{1} \otimes B_{2}}^{L} \circ\left(\Psi_{A_{1}, A_{2} ; B_{1}}^{L} \otimes \mathbf{1}_{B_{2} \otimes A_{3}}\right) \\
& \left(\Psi^{L} \Psi^{R} 1\right) \quad \Psi_{A_{1} \otimes A_{2}, A_{3} ; B_{1} \otimes B_{2}}^{L} \circ\left(\Psi_{A_{1}, A_{2} ; B_{1}}^{R} \otimes \mathbf{1}_{B_{2} \otimes A_{3}}\right)= \\
& \left(\Psi_{A_{1}, A_{2} \otimes A_{3} ; B_{1}}^{R} \otimes \mathbf{1}_{B_{2}}\right) \circ\left(\mathbf{1}_{A_{1} \otimes B_{1}} \otimes \Psi_{A_{2}, A_{3} ; B_{2}}^{L}\right) \\
& \left(\Psi^{L} \Psi^{R} 2\right) \quad\left(\Psi_{A_{1} \otimes A_{3}, A_{2} ; B_{1}}^{L} \otimes \mathbf{1}_{B_{2}}\right) \circ \Psi_{A_{1}, A_{3} ; B_{1} \otimes A_{2} \otimes B_{2}}^{R}= \\
& \left(E\left(\mathbf{1}_{A_{1}} \otimes c_{A_{2}, A_{3}}\right) \otimes \mathbf{1}_{B_{1} \otimes B_{2}}\right) \circ\left(\Psi_{A_{1}, A_{2} \otimes A_{3} ; B_{1}}^{R} \otimes \mathbf{1}_{B_{2}}\right) \circ\left(\mathbf{1}_{A_{1} \otimes B_{1}} \otimes \Psi_{A_{2}, A_{3} ; B_{2}}^{R}\right)
\end{aligned}
$$

The equation $\left(\Psi^{L} \Psi^{L}\right)$ is obtained from the equation ( $\Psi \Psi 1$ ) in the proof of Proposition 2 above by adding the superscripts $L$ to $\Psi$ and deleting both occurrences of $E$ in the subscripted indices. The equation $\left(\Psi^{L} \Psi^{R} 1\right)$ is obtained in a similar manner from $(\Psi \Psi 2)$ read from right to left. The equation $\left(\Psi^{L} \Psi^{R} 2\right)$, which is analogous to $\left(\Psi^{L} \Psi^{L}\right)$, could be obtained similarly from an equation of $\mathcal{M}_{c}^{s t r}$, which we did not need, and did not mention before.

We have moreover eight equations obtained from the equations $(\Psi c 1)-(\Psi c 4)$ by adding uniformly the superscripts $L$ or $R$ to $\Psi$ and deleting some occurrences of $E$ in the subscripted indices. These equations enable us to obtain a normal form that looks as follows.

A block $\vec{f}_{i}$ is a composition of $\Psi$-factors $f_{i_{k}} \circ \ldots \circ f_{i_{1}}$ all with the same $\tau$ value, such that $f_{i_{j+1}} \circ f_{i_{j}}$ is never of the form of the left-hand side of $\left(\Psi^{L} \Psi^{L}\right)$, and it is never the case that $f_{i_{j}}$ is a $\Psi^{R}$-factor while $f_{i_{j+1}}$ is a $\Psi^{L}$-factor. Our normal form is $g \circ \vec{f}_{n} \circ \ldots \circ \vec{f}_{1}$ such that, as before, $n \geq 0$ and the arrangement of the blocks is strictly increasing (see the proof of Proposition 1); the arrow term $g$ has no occurrence of $\psi^{L}$ and $\psi^{R}$ (but $c$ may occur). With this normal form we proceed as in the proofs of Propositions 1 and 2.

If we define $\Psi_{A_{1}, A_{2} ; B}^{R}$ as

$$
\left(\mathbf{1}_{B} \otimes \psi_{A_{1}, A_{2}}^{R}\right) \circ\left(c_{A_{1}, B} \otimes \mathbf{1}_{E A_{2}}\right): A_{1} \otimes B \otimes E A_{2} \rightarrow B \otimes E\left(A_{1} \otimes A_{2}\right)
$$

then $\Psi^{L}$ and $\Psi^{R}$ would be more symmetric, and we could use a modification of our normal form that would not favour pushing $\Psi^{L}$ to the right as in $\left(\Psi^{L} \Psi^{R} 1\right)$ and $\left(\Psi^{L} \Psi^{R} 2\right)$. In that case, however, our exposition would be somewhat less economical.

## 7 Coherence with conjunctive relevant endofunctors

A conjunctive relevant category is a symmetric monoidal category with a diagonal natural transformation, whose components are the arrows

$$
\Delta_{A}: A \rightarrow A \otimes A .
$$

For $\Delta$ we assume the following coherence equations:

$$
\begin{array}{ll}
(\Delta a) & a_{A, A, A} \circ\left(\Delta_{A} \otimes \mathbf{1}_{A}\right) \circ \Delta_{A}=\left(\mathbf{1}_{A} \otimes \Delta_{A}\right) \circ \Delta_{A}, \\
(\Delta l) & l_{I} \circ \Delta_{I}=\mathbf{1}_{I}, \\
(\Delta c) & c_{A, A} \circ \Delta_{A}=\Delta_{A}, \\
\text { with } c_{A, B, C, D}^{m}={ }_{d f} a_{A, C, B \otimes D}^{-1} \circ\left(\mathbf{1}_{A} \otimes\left(a_{C, B, D} \circ\left(c_{B, C} \otimes \mathbf{1}_{D}\right) \circ a_{B, C, D}^{-1}\right)\right) \circ a_{A, B, C \otimes D}: \\
(A \otimes B) \otimes(C \otimes D) \rightarrow(A \otimes C) \otimes(B \otimes D), \\
(\Delta a c) \quad & \Delta_{A \otimes B}=c_{A, A, B, B}^{m} \circ\left(\Delta_{A} \otimes \Delta_{B}\right),
\end{array}
$$

(see [2], Section 2, [17], Section 1, and [3], Sections 9.1-2; the denomination relevant comes from the connection with the structural fragment of relevant logic).

A conjunctive relevant endofunctor in a conjunctive relevant category is a linear endofunctor $E$ in this category that preserves $\Delta$ globally; i.e. we have the equation
$(\psi \Delta) \quad E \Delta_{A}=\psi_{A, A} \circ \Delta_{E A}$.
Let $\mathcal{R}$ be the free conjunctive relevant category with a family of conjunctive relevant endofunctors. We define the functor $G$ from $\mathcal{R}$ to the category Rel of relations between finite ordinals as the functor $G$ from $\mathcal{M}_{c}$ to Fun with an additional clause that corresponds to the following picture:
$G \Delta_{A}$


Let $\mathcal{R}^{-}$be the free conjunctive relevant category, and let $G$ from $\mathcal{R}^{-}$to Rel (as a matter of fact, $F u n^{o p}$ ) be defined by restricting $G$ from $\mathcal{R}$ to Rel. Then one can find in [17] (Section 5) a proof of $\mathcal{R}^{-}$-Coherence.

The category $\mathcal{R}$ is equivalent to its strictification $\mathcal{R}^{\text {str }}$, as $\mathcal{M}$ is equivalent to $\mathcal{M}^{\text {str }}$ (see Section 4), and, as before, our goal is to prove the following proposition, which entails $\mathcal{R}$-Coherence.

Proposition 4. The functor $G H$ from $\mathcal{R}^{\text {str }}$ to Rel is faithful.

We prove first the following auxiliary lemma concerning $\mathcal{M}_{c}$ (see Section 2 for the notions of diversification and scope).
$\mathcal{M}_{c}$-Theoremhood Lemma. For $A$ diversified on generating objects and $B$ diversified, there is an arrow $f: A \rightarrow B$ of $\mathcal{M}_{c}$ iff the generators of $A$ and $B$ coincide, and for every generating functor $E^{i}$ of $B$ the union of the scopes of the occurrences of $E^{i}$ in $A$ is equal to the scope of $E^{i}$ in $B$.

Proof. From left to right the lemma is trivially proved by induction on the length of $f$ in developed form. For the other direction, suppose $\left\{E^{1}, \ldots, E^{n}\right\}$ is the set of generating functors of $B$. We proceed by induction on $n$. If $n=0$, the set we just mentioned is empty, and we have trivially an arrow from $A$ to $B$ of symmetric monoidal categories.

For the induction step, let $E^{1}$ in $E^{1} B_{1}$ be the leftmost $E$ of $B$. Since $E^{1}$ is not in the scope of any other $E$ in $B$, by the assumptions of the lemma, it is not in the scope of any other $E$ in $A$ either. So we may assume that $A$ is of the form

$$
D_{1} \otimes E^{1} A_{1} \otimes \ldots \otimes D_{n} \otimes E^{1} A_{n} \otimes D_{n+1}
$$

with parentheses associated arbitrarily, and $D_{i}$ being $E^{1}$-free. It is clear that we have an arrow of $\mathcal{M}_{c}$ from $A$ to $E^{1}\left(A_{1} \otimes \ldots \otimes A_{n}\right) \otimes D_{1} \otimes \ldots \otimes D_{n+1}$. By the induction hypothesis there is an arrow of $\mathcal{M}_{c}$ from $A_{1} \otimes \ldots \otimes A_{n}$ to $B_{1}$, and hence an arrow of $\mathcal{M}_{c}$ from $E^{1}\left(A_{1} \otimes \ldots \otimes A_{n}\right)$ to $E^{1} B_{1}$. By appealing again to the induction hypothesis, we have an arrow of $\mathcal{M}_{c}$ from $p \otimes D_{1} \otimes \ldots \otimes D_{n}$ to $B$ in which $E^{1} B_{1}$ is replaced by $p$. From all that we obtain an arrow of $\mathcal{M}_{c}$ from $A$ to $B$.

Let $B$ be a part (proper or not) of an object of $\mathcal{M}_{c}^{\text {str }}$, denoted by $A[B]$, and let $A\left[B^{\prime}\right]$ be obtained from this object by replacing $B$ by $B^{\prime}$. (We replace a single part $B$ by a single $B^{\prime}$.) For $f: B \rightarrow B^{\prime}$, let $A[f]: A[B] \rightarrow A\left[B^{\prime}\right]$ be constructed out of $f$ with identity arrows, $\otimes$ and $E$ in the obvious way. We can then prove the following.

Lemma 1. For the arrow term

$$
f: A\left[E A_{1} \otimes D \otimes E A_{2}\right] \rightarrow B
$$

of $\mathcal{M}_{c}^{\text {str }}$ and $g: B \rightarrow C$ a $\psi$-factor such that the ordinals corresponding to the outermost occurrences of $E$ in $E A_{1}$ and $E A_{2}$ are respectively $i$ and $j$, and $(G H f)(i) \neq(G H f)(j)$, while $(G H(g \circ f))(i)=(G H(g \circ f))(j)$, there exists an arrow term

$$
f^{\prime}: A\left[E\left(A_{1} \otimes A_{2}\right) \otimes D\right] \rightarrow C
$$

of $\mathcal{M}_{c}^{\text {str }}$ such that $g \circ f=f^{\prime} \circ A\left[\Psi_{A_{1}, A_{2} ; D}\right]$.
Proof. Note first that every arrow term of $\mathcal{M}_{c}^{\text {str }}$ is a substitution instance of an arrow term of $\mathcal{M}_{c}^{s t r}$ with a diversified target. So we may assume that $C$ in the
lemma is diversified. That $f^{\prime}$ exists follows from the assumption that we have $g \circ f$ and from the $\mathcal{M}_{c^{\prime}}$-Theoremhood Lemma. That $g \circ f=f^{\prime} \circ A\left[\Psi_{A_{1}, A_{2} ; D}\right]$ follows from $\mathcal{M}_{c}$-Coherence.

Remark. Consider the arrow term $f: A[D \otimes D] \rightarrow B$ of $\mathcal{M}_{c}^{\text {str }}$ with $D$ atomic, and let $i$ and $j$, with $i<j$, be the ordinals corresponding respectively either to the outermost occurrences of $E$ in $D$, when $D$ is of the form $E D^{\prime}$, or otherwise to the two occurrences of $D$, when $D$ is a propositional letter. If $(G H f)(j)<$ $(G H f)(i)$, then for $h$ being $f \circ A\left[c_{D, D}\right]$ we have $(G H h)(i)<(G H h)(j)$ and $f=h \circ A\left[c_{D, D}\right]$.

A developed arrow term made only of $\Delta$-factors is called a $\Delta$-term. If $h$ is a $\Delta$-term, then for $G H h: G H A \rightarrow G H B$ the converse relation $(G H h)^{-1}$ : $G H B \rightarrow G H A$ is an onto function. A $\Delta$-term is atomized when for every $\Delta$-factor in it, in the head $\Delta_{A}$ of this $\Delta$-factor, $A$ is atomic.

Let $h$ be an atomized $\Delta$-term such that $E^{j}$ occurs exactly once in its source. By naturality and functorial equations $h$ is equal to an atomized $\Delta$ term $h_{3} \circ h_{2} \circ h_{1}$ such that, for every factor of $h_{1}$, its head is neither in the scope of $E^{j}$ nor is it of the form $\Delta_{E^{j} A^{\prime}}$; all the heads of the factors of $h_{2}$ are of the form $\Delta_{E^{j} A^{\prime}}$; and all the heads of the factors of $h_{3}$ are in the scope of $E^{j}$ (c.f. the proof of $\mathcal{E}$-Coherence in Section 2). Analogously, we can transform every atomized $\Delta$-term into the normal form $h_{3} \circ h_{2} \circ h_{1}$ relative to an occurrence $E^{i}$ in its source. (One has to replace this particular occurrence of $E^{i}$ by a genuinely new $E^{j}$, and then factor the newly obtained arrow term as above; at the end, one substitutes $E^{i}$ for $E^{j}$ everywhere in the term.)

A $\Delta$-capped arrow term is an arrow term $f \circ h$ of $\mathcal{R}^{\text {str }}$ such that $h: D \rightarrow A$ is a $\Delta$-term and $f: A \rightarrow C$ an arrow term of $\mathcal{M}_{c}^{\text {str }}$. A $\Delta$-capped arrow term $f \circ h$ is atomized when $h$ is an atomized $\Delta$-term.

A short circuit in an arrow term $f \circ h$, with $h: D \rightarrow A$, is a pair of ordinals $(i, j)$ such that $i, j \in G H A, i<j,(G H h)^{-1}(i)=(G H h)^{-1}(j)$ and $(G H f)(j)=$ $(G H f)(i)$.

A useless crossing is defined analogously to a short circuit save that we have $(G H f)(j)<(G H f)(i)$. For an example of a short circuit and a useless crossing see the picture after Lemma 5.

Lemma 2. Every arrow term of $\mathcal{R}^{\text {str }}$ is equal to an atomized $\Delta$-capped arrow term.

To prove this lemma we just apply naturality and functorial equations together with $\Delta_{I}=\mathbf{1}_{I}$, which is $(\Delta l)$ strictified, and $(\Delta a c)$.

Lemma 3. Every arrow term of $\mathcal{R}^{\text {str }}$ is equal to an atomized $\Delta$-capped arrow term without short circuits.

Proof. We apply first Lemma 2, and then we proceed by induction on the number $n$ of short circuits in an atomized $\Delta$-capped arrow term. If $n=0$, then we are done. If $n>0$, then our arrow term is of the form $k \circ g \circ f \circ h$, for $f \circ h$ an atomized $\Delta$-capped arrow term without short circuits, $g \circ f \circ h$ an atomized $\Delta$-capped arrow term with a single short circuit $(i, j)$ and $g$ a $\psi$-factor. Let $h_{3} \circ h_{2} \circ h_{1}$ be the normal form of the $\Delta$-term $h$ relative to the occurrence $E^{i}$ in the source of $h$ that corresponds to the ordinal $(G H h)^{-1}(i)$, which is equal to $(G H h)^{-1}(j)$, and let $g \circ f$ be transformed according to Lemma 1 ; here $h_{2}$ is not an identity arrow. Now we can apply naturality and functorial equations to "permute" $h_{3}$ with $A\left[\Psi_{A_{1}, A_{2} ; D}\right]$, and then the equations $(\Delta a)$ strictified, $(\Delta c)$ and $(\psi \Delta)$ in order to decrease $n$. After applying $(\psi \Delta)$, we may have to apply again $\Delta_{I}=\mathbf{1}_{I}$ and ( $\Delta a c$ ) in order to atomize the resulting $\Delta$-capped arrow term to which we apply the induction hypothesis.

Lemma 4. If for the $\Delta$-terms $f, g: A \rightarrow B$ we have $G H f=G H g$, then $f=g$.
Proof. We proceed essentially as in the proof of $\mathcal{E}$-Coherence in Section 2, by relying on $\mathcal{R}^{-}$-Coherence, instead of Mac Lane's monoidal coherence, in case $E$ does not occur in $A$. When $E$ occurs in $A$, a difference with the proof of $\mathcal{E}$-Coherence is that the interpolant $C$ is determined not only by $A^{\prime}$ and $B^{\prime}$, but we must take into account $G f$, which is equal to $G g$. From $G f=G g$, we may infer also $G f_{1}=G g_{1}$ and $G f_{2}=G g_{2}$.

Another difference with the proof of $\mathcal{E}$-Coherence is that the number of occurrences of $E$ in $A$ is not equal to this number in $B$. In $C$ we may have more than one occurrence of $E^{j}$. There will however be no essential difference with the previous proof, because we do not have to deal with $\Delta$-terms like $E^{j} \Delta_{E^{j} A}$, which have $\Delta$ in the scope of $E^{j}$ and $E^{j}$ in the index of $\Delta$. Between two occurrences of $E^{j}$ in $C$ there will always be a $\otimes$ in whose scope they are. Hence in $f_{1}$ and $g_{1}$ the subformula $E^{j} D$ will again amount to a generating object, and in $f_{2}$ and $g_{2}$ only what is within the scope of $E^{j}$ counts. Since there may be more than one occurrence of $E^{j}$ in $C$ we may need to apply the induction hypothesis more than once to establish that $f_{2}=g_{2}$.

Note that the normal form of $f$ relative to an occurrence of $E^{i}$ in $A$ is just a refinement of the factorization $f_{2} \circ f_{1}$ used in the proof of $\mathcal{E}$-Coherence in Section 2, and in the proof we have just finished. For $\mathcal{E}$-Coherence, we could take the factorization $f_{2} \circ f_{1}$ to be such that all the heads of the factors of $f_{1}$ are in the scope of $E^{j}$, and no head of the factors of $f_{2}$ is in the scope of $E^{j}$, while in the case of $\Delta$-terms, switching the roles of $f_{1}$ and $f_{2}$ is not possible.

An arrow term of $\mathcal{R}^{\text {str }}$ is in normal form when it is an atomized $\Delta$-capped arrow term without short circuits and without useless crossings. We can then prove the following.

Lemma 5. Every arrow term of $\mathcal{R}^{\text {str }}$ is equal to an arrow term in normal form.

We just apply Lemma 3, the Remark and the equation $(\Delta c)$. Note that we could not apply the Remark without previously applying Lemma 3. For example, we could have

where dotted lines are tied to a short circuit and bold lines to a useless crossing.
We can now prove Proposition 4.
Proof of Proposition 4. Note that for an arrow term $f \circ h$ of $\mathcal{R}^{s t r}$ in normal form, where $h: A \rightarrow B$ is a $\Delta$-term and $f: B \rightarrow C$ is an arrow term of $\mathcal{M}_{c}^{s t r}$, we have that $G(f \circ h)$ determines uniquely $G h, G f$ and $B$. This matter, which is not entirely trivial, is established along the lines of the more general Decomposition Proposition of [4] (Section 8). We conclude the proof of the proposition by using Lemma 4 and Proposition 2, i.e. $\mathcal{M}_{c}$-Coherence.

## 8 Coherence in cartesian categories with relevant endofunctors

A cartesian category is a conjunctive relevant category with the monoidal unit object $I$ being a terminal object. The unique arrow from $A$ to $I$ is $i_{A}: A \rightarrow I$. This notion of cartesian category is equivalent to the usual notion, where a cartesian category is a category with all finite products (see [3], Sections 9.1-2; some authors use the denomination cartesian for categories with different finite limits than just finite products; see [7], Vol. I, Section A1.2, and [11], Section 4.1).

In accordance with what we had before, a cartesian endofunctor in a cartesian category should preserve $i$, which would yield the equation

$$
E i_{A}=\psi_{0} \circ i_{E A}
$$

analogous to $(\psi \Delta)$. However, since $G I=\varnothing$, we see easily that no definition of $G i_{A}$ would enable us to obtain coherence with $(\psi i)$; even the functoriality of
$G$, i.e. $G(g \circ f)=G g \circ G f$, would fail (cf. the last picture in Section 3, and [4], beginning of Section 5).

We still obtain coherence however for conjunctive relevant endofunctors in cartesian categories, and we are going to prove this now. Let $\mathcal{C}$ be the free cartesian category with a family of conjunctive relevant functors. We define the functor $G$ from $\mathcal{C}$ to Rel as the functor $G$ from $\mathcal{R}$ to Rel with an additional clause that says that $G$ i is the empty relation between $G A$ and $\varnothing$, which is $G I$.

The category $\mathcal{C}$ is equivalent to its strictification $\mathcal{C}^{s t r}$, as $\mathcal{M}$ is equivalent to $\mathcal{M}^{\text {str }}$ (see Section 4), and, as before, our goal is to prove the following proposition, which entails $\mathcal{C}$-Coherence.

## Proposition 5. The functor $G H$ from $\mathcal{C}^{\text {str }}$ to Rel is faithful.

Proof. We proceed analogously to what we had for the proof of Proposition 4 of the preceding section. We modify the lemmata and the terminology given there in order to take into account the presence of $i$ in $\mathcal{C}^{s t r}$.

A developed arrow term made only of $\Delta$-factors and $i$-factors is called a $\Delta \mathrm{i}$-term. If $h$ is a $\Delta \mathrm{i}$-term, then for $G H h: G H A \rightarrow G H B$ the converse relation $(G H h)^{-1}: G H B \rightarrow G H A$ is a function. A $\Delta i$-term is atomized when for every $\Delta$-factor and every i -factor in it, in the heads $\Delta_{A}$ or $i_{A}$ of this $\Delta$-factor or i -factor, $A$ is atomic.

For every atomized $\Delta i$-term $h$, and every occurrence $E^{i}$ in its source, we define the normal form $h_{3} \circ h_{2} \circ h_{1}$ of $h$ relative to this occurrence of $E^{i}$ exactly as it is defined for atomized $\Delta$-terms in the preceding section. A $\Delta \mathrm{i}$-capped arrow term is an arrow term $f \circ h$ of $\mathcal{C}^{s t r}$ such that $h$ is a $\Delta i$-term and $f$ is an arrow term of $\mathcal{M}_{c}^{s t r}$. A $\Delta \mathrm{i}$-capped arrow term $f \circ h$ is atomized when $h$ is an atomized $\Delta i$-term. The notions of short circuit and useless crossing are defined exactly as in the preceding section.

An arrow term of $\mathcal{C}^{\text {str }}$ is in normal form when it is an atomized $\Delta \mathrm{i}$-capped arrow term without short circuits and without useless crossings. Lemmata 2-5 with $\mathcal{R}^{\text {str }}$ replaced by $\mathcal{C}^{s t r}$ and $\Delta$ by $\Delta \mathrm{i}$ can be proved along the lines of the proofs in the preceding section. For the proof of the modification of Lemma 4, where we relied before on $\mathcal{R}^{-}$-Coherence, we rely now on cartesian coherence, i.e. the faithfulness of $G$ from the free cartesian category into $\operatorname{Rel}$ (see [3], Section 9.2 , and references therein).

In contradistinction to what we had in the proof of Proposition 4 at the end of the preceding section, we do not have now any difficulty in obtaining that $G(f \circ h)$ determines uniquely $G h, G f$ and $B$. This is because we may assume that the target of $f \circ h$ is $\otimes$-free. To obtain that, we may compose with

$$
\begin{aligned}
& E^{n}\left(r_{a} \circ\left(\mathbf{1}_{a} \otimes \mathfrak{i}_{B}\right)\right): E^{n}(A \otimes B) \rightarrow E^{n} A \quad \text { and } \\
& E^{n}\left(l_{B} \circ\left(\mathfrak{i}_{A} \otimes \mathbf{1}_{B}\right)\right): E^{n}(A \otimes B) \rightarrow E^{n} B,
\end{aligned}
$$

for $E^{n}$ being the sequence of $n$ occurrences of $E$; then we use the following "extensionality" equation of $\mathcal{C}$ :

$$
\begin{array}{r}
E^{n-1} \psi_{A, B} \circ \ldots \circ \psi_{E^{n-1} A, E^{n-1} B \circ} \circ\left(E^{n}\left(r_{a} \circ\left(\mathbf{1}_{a} \otimes \mathfrak{i}_{B}\right)\right) \otimes E^{n}\left(l_{B} \circ\left(\mathfrak{i}_{A} \otimes \mathbf{1}_{B}\right)\right)\right) \circ \\
\Delta_{E^{n}(A \otimes B)}=\mathbf{1}_{E^{n}(A \otimes B)} .
\end{array}
$$

Since we may assume that the target of $f \circ h$ is $\otimes$-free, we may assume that $h$ is $\Delta$-free, and $G h, G f$ and $B$ are then determined uniquely out of $G(f \circ h)$ in a straightforward manner.

## 9 Coherence in cocartesian categories with endofunctors

A disjunctive relevant category is a symmetric monoidal category with a codiagonal natural transformation whose components are the arrows

$$
\nabla_{A}: A \otimes A \rightarrow A
$$

For $\nabla$ we assume coherence equations dual to $(\Delta a),(\Delta l),(\Delta c)$ and $(\Delta a c)$.
A disjunctive relevant endofunctor in a disjunctive relevant category is a linear endofunctor $E$ in this category that satisfies the equation
$(\psi \nabla) \quad E \nabla_{A} \circ \psi_{A, A}=\nabla_{E A}$.
This equation, together with others, enables us to reduce to a propositional letter all the indices of $\nabla$, and that together with the naturality of $\nabla$ is all we need essentially to push every occurrence of $\nabla$ to the left. This enables us to prove coherence for disjunctive relevant categories.

A cocartesian category is a disjunctive relevant category with the monoidal unit object $I$ being an initial object. The unique arrow from $I$ to $A$ is $!_{A}: I \rightarrow$ $A$. Equivalently, cocartesian categories are defined as categories with all finite coproducts.

It will follow from the coherence result below that any endofunctor $E$ in a cocartesian category is a disjunctive relevant endofunctor with the definitions

$$
\begin{aligned}
& \psi_{A, B}={ }_{d f} \nabla_{E(A \otimes B)} \circ\left(E\left(\left(\mathbf{1}_{A} \otimes!_{B}\right) \circ r_{A}^{-1}\right) \otimes E\left(\left(!_{A} \otimes \mathbf{1}_{B}\right) \circ l_{B}^{-1}\right)\right), \\
& \psi_{0}={ }_{d f}!_{E I} .
\end{aligned}
$$

Moreover, this endofunctor preserves !, in the sense that it satisfies the equation

$$
(\psi!) \quad E!_{A} \circ \psi_{0}=!_{E A} .
$$

Unlike $(\psi i)$, this equation is in accordance with coherence. Note that the equations of the definitions of $\psi_{0}$ and $\psi$ above follow from the initiality of $I$, from the requirement that $\otimes$ is a coproduct and from the equations $(\psi l)$ and $(\psi r)$.

Let $\mathcal{D}$ be the free cocartesian category with a family of endofunctors. We define the functor $G$ from $\mathcal{D}$ to Fun as the functor $G$ from $\mathcal{M}_{c}$ to Fun with the additional clause that corresponds to the following picture:

and the clause that says that $G!_{A}$ is the empty function from $\varnothing$, which is $G I$, to $G A$. Then we can prove $\mathcal{D}$-Coherence.

We proceed essentially as in the proof of $\mathcal{E}$-Coherence in Section 2, and as in the proofs of Lemma 4 and its modifications in the two preceding sections. We rely now on cocartesian coherence, instead of monoidal coherence, $\mathcal{R}^{-}$Coherence and cartesian coherence respectively. (The only difference is that now we work relative to an occurrence $E^{j}$ in the target $B$, and the factorization $f_{2} \circ f_{1}$ is such that all the heads of the factors of $f_{1}$ are in the scope of $E^{j}$ and no head of a factor of $f_{2}$ is in the scope of $E^{j}$.)

An alternative way to prove $\mathcal{D}$-Coherence is to rely on the factorization $f_{2} \circ f_{1}$ such that $f_{1}$ is an arrow term of $\mathcal{L}_{c}$, and in the developed arrow term $f_{2}$ every factor is either a $\nabla$-factor or a !-factor such that the index of its head is a propositional letter. For that we use the equations $(\psi \nabla)$ and $(\psi!)$ above.

Cocartesian coherence, i.e. the faithfulness of $G$ from the free cocartesian category into Fun, follows from cartesian coherence (see [3], Section 9.2, and references therein). Cocartesian coherence may be proved by relying on a normal form inspired by Gentzen's cut elimination (see [3], Sections 9.1-2), but we could rely alternatively on a developed strictified normal form $f_{3} \circ f_{2} \circ f_{1}$, where $f_{1}$ has atomized $c$-factors only, $f_{2}$ has atomized $\nabla$-factors only, and $f_{3}$ has atomized !-factors only.

The coherence results of this paper yield coherence results for categories with arrows oriented in the opposite direction. In these categories we do not have monoidal functors with $\psi$ and $\psi_{0}$, but comonoidal functors with arrows oriented oppositely to $\psi$ and $\psi_{0}$.

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