Coherence for Monoidal Monads and Comonads

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Abstract

The goal of this paper is to prove coherence results with respect to relational graphs for monoidal monads and comonads, i.e. monads and comonads in a monoidal category such that the endofunctor of the monad or comonad is a monoidal functor (this means that it preserves the monoidal structure up to a natural transformation that need not be an isomorphism). These results are proved first in the absence of symmetry in the monoidal structure, and then with this symmetry. The monoidal structure is also allowed to be given with finite products or finite coproducts. Monoidal comonads with finite products axiomatize a plausible notion of identity of deductions in a fragment of the modal logic S4.

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1 Introduction

A monoidal monad is a monad in a monoidal category such that the endofunctor of the monad is a monoidal functor, which means that it preserves the monoidal structure up to a natural transformation that need not be an isomorphism. (The notion of monoidal functor stems from [7], Section II.1, and the notion of monoidal monad from [8] and [9]; for historical remarks on the notions of monad and comonad see [11], notes at the end of Chapter VI.) This natural transformation has as components the arrows

$$\psi_{A,B}\colon TA\otimes TB\rightarrow T(A\otimes B),$$

and we also have the arrow $\psi_0\colon I\rightarrow TI$, which coincides with the unit η_I of the monad; these arrows satisfy the equations given in Section 3 below.

Our goal in this paper is to prove coherence for this and related notions of monad with respect to relational graphs; i.e. with respect to the category *Rel*, whose arrows are relations between finite ordinals. Sometimes it will be sufficient to have a subcategory of *Rel*, like the category *Fun* whose arrows are functional relations between finite ordinals, or the simplicial category Δ, whose arrows are order-preserving functional relations between finite ordinals. Coherence states that there is a faithful functor from a freely generated monoidal monad, or a related categorial structure for which we prove coherence, into *Rel* or a subcategory of it. We obtain thereby a characterization of the freely generated monoidal monad, or related categorial structure, in terms of graphs. Such coherence results give very useful procedures for deciding whether a diagram of canonical arrows commutes. (A general treatment of coherence in this spirit may be found in [3].)

Before we deal with monoidal monads, we consider coherence for notions of strong monad, which stem from [8], [9] and [14]. For monoidal monads we prove coherence first in the absence of symmetry in the monoidal structure, and then with this symmetry. After that we prove coherence for monoidal monads where the monoidal structure is cartesian (i.e. with finite products), or cocartesian (i.e. with finite coproducts). This makes the first part of the paper (Sections 2-6).

A monoidal comonad is a comonad in a monoidal category such that the endofunctor of the comonad is a monoidal functor. This notion is parallel, but not dual, to the notion of monoidal monad. A dual notion would be the notion of comonoidal comonad, where instead of ψ and ψ_0 we have arrows oriented in the opposite direction. More recent papers on the notion of monoidal comonad, or the dual notion of comonoidal monad, called also Hopf monad, opmonoidal monad or bimonad, are [13], [12], [1] and [15]. In [16], [2] and references therein, one finds for the canonical arrows of a comonoidal monad graphical interpretations different from ours and more involved; coherence is not proved however.

The second part of the paper (Sections 7-11) is parallel to the first part; instead of monads we have comonads in the same monoidal contexts, and we prove coherence for these notions. These results are not dual to those in the first part. By duality, we can obtain from the results of both parts of the paper coherence results for comonoidal monads and comonads.

We rely for this paper on [6], where basic coherence results for monoidal endofunctors are proved. Here these endofunctors become endofunctors of monads and comonads. We presuppose the reader is acquainted with the terminology and notation of this previous paper, but to make the exposition here more self-contained we will repeat some definitions.

2 Coherence for strong monads

Let a *left monoidal* endofunctor of a monoidal category $\langle \mathcal{A}, \otimes, I, a, l, r \rangle$ (in the notation of [7], Section II.1) be a functor T from $\mathcal{A}$ to $\mathcal{A}$ such that the monoidal structure of $\mathcal{A}$ is preserved *locally* by T up to a natural transformation whose components are the arrows

$$\psi^L_{A,B} \colon TA \otimes B \to T(A \otimes B);$$

this means that the following equations hold:

$$(\psi^L a) \qquad Ta_{A,B,C} \circ \psi^L_{A\otimes B,C} \circ (\psi^L_{A,B} \otimes \mathbf{1}_C) = \psi^L_{A,B\otimes C} \circ a_{TA,B,C},$$

$$(\psi^L r) \qquad Tr_A \circ \psi^L_{A,I} = r_{TA}.$$

A *right monoidal* endofunctor is defined analogously with respect to a natural transformation whose components are the arrows

$$\psi^R_{A,B} \colon A \otimes TB \to T(A \otimes B).$$

The equations corresponding to $(\psi^L a)$ and $(\psi^L r)$ are

$$(\psi^R a) \qquad Ta_{A,B,C} \circ \psi^R_{A\otimes B,C} = \psi^R_{A,B\otimes C} \circ (\mathbf{1}_A \otimes \psi^R_{B,C}) \circ a_{A,B,TC},$$

$$(\psi^R l) \qquad Tl_A \circ \psi^R_{I,A} = l_{TA}.$$

A *left strong* monad in a monoidal category $\mathcal{A}$ is a monad $\langle T, \eta, \mu \rangle$ (in the notation of [11], Section VI.1) in $\mathcal{A}$ such that T is a left monoidal functor, and we have moreover the equations

$$(\psi^L \eta) \qquad \psi^L_{A,B} \circ (\eta_A \otimes \mathbf{1}_B) = \eta_{A\otimes B},$$

$$(\psi^L \mu) \qquad \psi^L_{A,B} \circ (\mu_A \otimes \mathbf{1}_B) = \mu_{A\otimes B} \circ T\psi^L_{A,B} \circ \psi^L_{TA,B}.$$

(These equations might be interpreted as saying that η is a *left monoidal* natural transformation from the identity functor, which is left monoidal, to the left monoidal functor T, while μ is a left monoidal natural transformation from the left monoidal functor TT to T.)

A *right strong* monad is defined analogously with a right monoidal functor T. The equations corresponding to $(\psi^L \eta)$ and $(\psi^L \mu)$ are

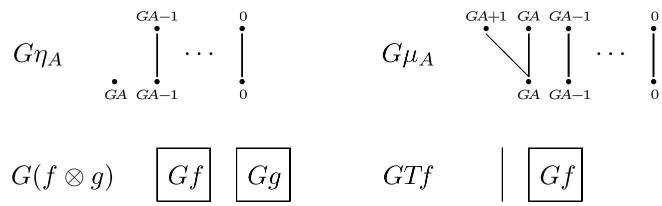
$$(\psi^R \eta) \qquad \psi^R_{A,B} \circ (\mathbf{1}_A \otimes \eta_B) = \eta_{A\otimes B},$$

$$(\psi^R \mu) \qquad \psi^R_{A,B} \circ (\mathbf{1}_A \otimes \mu_B) = \mu_{A\otimes B} \circ T\psi^R_{A,B} \circ \psi^R_{A,TB}.$$

The notions of left strong and right strong monad are derived from [8] and [9] in [14].

Let $\mathcal{L}^L\mathcal{S}$ be the category of the left strong monad freely generated by an arbitrary set of objects, and let Δ be the simplicial category (see [11], Section

VII.5); the arrows of Δ are order-preserving functions between finite ordinals. We define a functor G from $\mathcal{L}^L\mathcal{S}$ to Δ by stipulating that GA for A an object $\mathcal{L}^L\mathcal{S}$ is the number of occurrences of T in A, while Gf for an arrow term f of $\mathcal{L}^L\mathcal{S}$ is defined inductively on the complexity of f. If f is $a_{A,B,C}$, l_A, r_A, or $\psi^L_{A,B}$, then Gf is the identity function; next, we have clauses corresponding to the following pictures:

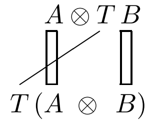


and $G(g \circ f)$ is the composition of the functions Gf and Gg. We can prove the following.

$\mathcal{L}^L\mathcal{S}$-Coherence. *The functor G from $\mathcal{L}^L\mathcal{S}$ to Δ is faithful.*

Proof. By naturality and functorial equations, and the equations $(\psi^L\eta)$ and $(\psi^L\mu)$, every arrow term of $f : A \to B$ of $\mathcal{L}^L\mathcal{S}$ is equal to an arrow term $f_2 \circ f_1 : A \to B$ such that η and μ do not occur in $f_1 : A \to C$, and a, l, r and ψ^L do not occur in $f_2 : C \to B$. We can uniquely determine Gf_2 from Gf (since Gf_1 is the identity function, $Gf_2 = Gf$), and from B and Gf_2 we obtain a unique C as a possible source of f_2. By the isomorphism of Δ with the monad freely generated by a single object (see the references in [4], Section 3), we can conclude that f_2 is uniquely determined by Gf, while f_1 is unique by $\mathcal{L}^L$-Coherence (see [6], Section 5). $\dashv$

Let $\mathcal{L}^R\mathcal{S}$ be the category of the right strong monad freely generated by an arbitrary set of objects. It is easy to show that this category is isomorphic to $\mathcal{L}^L\mathcal{S}$; it is a mirror image of $\mathcal{L}^L\mathcal{S}$. Hence $\mathcal{L}^R\mathcal{S}$-Coherence could be a result, exactly analogous to $\mathcal{L}^L\mathcal{S}$-Coherence, about a faithful functor from $\mathcal{L}^R\mathcal{S}$ to the simplicial category Δ. For future use, however, we need another coherence result for $\mathcal{L}^R\mathcal{S}$, which is given with respect to a functor G from $\mathcal{L}^R\mathcal{S}$ to the category *Fun* whose arrows are arbitrary functions between finite ordinals. This functor is defined as G from $\mathcal{L}^L\mathcal{S}$ to Δ except for the additional clause for $G\psi^R_{A,B}$ corresponding to the following picture:



In this picture we obtain a crossing if there is a T in A. Hence with this clause we abandon the category Δ, and must consider also functions that are not order-preserving. For this functor G we can prove the following.

$\mathcal{L}^R\mathcal{S}$-COHERENCE. *The functor G from $\mathcal{L}^R\mathcal{S}$ to Fun is faithful.*

The proof is as for $\mathcal{L}^L\mathcal{S}$-Coherence, except for the parenthetical remark about determining Gf_2 out of Gf. Now Gf_2 is not equal to Gf, but it is still uniquely determined by it. Note that we could prove $\mathcal{L}^L\mathcal{S}$-Coherence with respect to a functor from $\mathcal{L}^L\mathcal{S}$ to *Fun* analogous to the functor G from $\mathcal{L}^R\mathcal{S}$ to *Fun*.

3 Coherence for monoidal monads

Let a *locally* monoidal endofunctor of a monoidal category $\langle\mathcal{A},\otimes,I,a,l,r\rangle$ be a functor T that is both left monoidal and right monoidal, and we have moreover the equation

$$(\psi^L\psi^R a)\qquad Ta_{A,B,C}\circ\psi^L_{A\otimes B,C}\circ(\psi^R_{A,B}\otimes\mathbf{1}_C)=\psi^R_{A,B\otimes C}\circ(\mathbf{1}_A\otimes\psi^L_{B,C})\circ a_{A,TB,C}.$$

A *monoidal monad* in a monoidal category $\mathcal{A}$ is a monad $\langle T,\eta,\mu\rangle$ in $\mathcal{A}$ both left strong and right strong, such that T is a locally monoidal endofunctor, and we have moreover the equation

$$(\psi^L\psi^R\mu)\qquad \mu_{A\otimes B}\circ T\psi^L_{A,B}\circ\psi^R_{TA,B}=\mu_{A\otimes B}\circ T\psi^R_{A,B}\circ\psi^L_{A,TB}\colon\; TA\otimes TB\to T(A\otimes B).$$

An alternative definition of monoidal monad is obtained by stipulating that in a monoidal category $\mathcal{A}$ we have a monad $\langle T,\eta,\mu\rangle$ and a natural transformation whose components are the arrows

$$\psi_{A,B}\colon TA\otimes TB\to T(A\otimes B),$$

which satisfy the equations

$$(\psi a)\qquad Ta_{A,B,C}\circ\psi_{A\otimes B,C}\circ(\psi_{A,B}\otimes\mathbf{1}_{TC})=\psi_{A,B\otimes C}\circ(\mathbf{1}_{TA}\otimes\psi_{B,C})\circ a_{TA,TB,TC},$$

$$(\psi l)\qquad Tl_A\circ\psi_{I,A}\circ(\eta_I\otimes\mathbf{1}_{TA})=l_{TA},$$

$$(\psi r)\qquad Tr_A\circ\psi_{A,I}\circ(\mathbf{1}_{TA}\otimes\eta_I)=r_{TA},$$

$$(\psi\eta)\qquad \psi_{A,B}\circ(\eta_A\otimes\eta_B)=\eta_{A\otimes B},$$

$$(\psi\mu)\qquad \psi_{A,B}\circ(\mu_A\otimes\mu_B)=\mu_{A\otimes B}\circ T\psi_{A,B}\circ\psi_{TA,TB}.$$

The first three of these equations, together with $\psi_0=\eta_I$, say that T is a monoidal functor, while the last two equations say that η and μ are monoidal natural transformations, in the sense of [7] (Section II.1; see also [11], Section XI.2). Coherence for monoidal endofunctors is proved in [6] (Section 4).

With $\psi_{A,B}$ being defined as either of the two sides of the equation $(\psi^L\psi^R\mu)$, and with

$$\psi^L_{A,B} =_{df} \psi_{A,B} \circ (\mathbf{1}_{TA} \otimes \eta_B),$$

$$\psi^R_{A,B} =_{df} \psi_{A,B} \circ (\eta_A \otimes \mathbf{1}_{TB}),$$

we can show that the two definitions of monoidal monad amount to the same notion. Both of these definitions stem from [8] and [9].

Let $\mathcal{LS}$ be the category of the monoidal monad, with ψ^L and ψ^R primitive, freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{LS}$ to the category *Fun* by combining what we had in the preceding section for the functors G from $\mathcal{L}^L\mathcal{S}$ to Δ and $\mathcal{L}^R\mathcal{S}$ to *Fun*. We can prove the following.

$\mathcal{LS}$-Coherence. *The functor G from $\mathcal{LS}$ to Fun is faithful.*

Proof. By naturality and functorial equations, and the equations $(\psi^L\eta)$, $(\psi^L\mu)$, $(\psi^R\eta)$ and $(\psi^R\mu)$, every arrow term of $f : A \to B$ of $\mathcal{LS}$ is equal to an arrow term $f_2 \circ f_1 : A \to B$ such that η and μ do not occur in $f_1 : A \to C$, and a, l, r, ψ^L and ψ^R do not occur in $f_2 : C \to B$. We can uniquely determine Gf_2 from Gf, and as in the proof of $\mathcal{L}^L\mathcal{S}$-Coherence and $\mathcal{L}^R\mathcal{S}$-Coherence, from B and Gf_2 we obtain a unique C as a possible source of f_2. As in the previous proofs, we can conclude that f_2 is uniquely determined by Gf. However, Gf_1 is not thereby uniquely determined, as the two sides of $(\psi^L\psi^R\mu)$ show. Hence f_1 is not unique either.

Let the normal form of the proof of $\mathcal{L}$-Coherence of [6] (Section 5) for f_1 be $g_m \dots g_1$. If in this normal form for a ψ^R-factor g_i and a ψ^L-factor g_{i+1} we have $\tau(g_{i+1}) = \tau(g_i)+1$ and the function Gf_2 has the same value when it is applied to $\tau(g_i)$ and $\tau(g_{i+1})$, then we rely essentially on the equation $(\psi^L\psi^R\mu)$ to permute g_i with g_{i+1}. By proceeding in this manner, we obtain an arrow term f'_1 such that the permutation Gf'_1 has the least possible number of inversions. By $\mathcal{L}$-Coherence, f'_1 is unique. $\dashv$

4 Coherence for symmetric monoidal monads

A *locally linear* endofunctor is a locally monoidal endofunctor T in a symmetric monoidal category that satisfies the equation

$$(\psi^L\psi^R c) \quad Tc_{A,B} \circ \psi^L_{A,B} = \psi^R_{B,A} \circ c_{TA,B}.$$

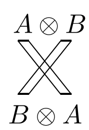
From this equation, which says that T preserves c locally up to ψ^L and ψ^R, we obtain immediately a definition of ψ^R in terms of ψ^L, and vice versa.

A *symmetric* monoidal monad is a monoidal monad in a symmetric monoidal category whose endofunctor is a locally linear monoidal endofunctor. In the language of monoidal monads with ψ primitive, the equation $(\psi^L\psi^R c)$ of symmetric monoidal monads is replaced by the equation

$$(\psi c) \qquad Tc_{A,B} \circ \psi_{A,B} = \psi_{B,A} \circ c_{TA,TB}$$

(see [6], Section 6).

Let $\mathcal{L}_c\mathcal{S}$ be the category of the symmetric monoidal monad, with ψ^L and ψ^R primitive, freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{L}_c\mathcal{S}$ to the category *Fun* by stipulating first that GA is the number of occurrences of generating objects, i.e. propositional letters, in A plus the number of occurrences of T in A. Up to now we took GA to be just the number of occurrences of T in A, but we could as well have counted also occurrences of generating objects; this was however superfluous up to now. The remainder of the definition of G is as the definition of the functor G from $\mathcal{LS}$ to *Fun*, with the additional standard clause for $Gc_{A,B}$ corresponding to the picture



We can prove the following.

$\mathcal{L}_c\mathcal{S}$-COHERENCE. *The functor G from $\mathcal{L}_c\mathcal{S}$ to Fun is faithful.*

For the proof we proceed as for $\mathcal{LS}$-Coherence, by relying on $\mathcal{L}_c$-Coherence of [6] (Section 6).

5 Coherence for cartesian monoidal monads

A *cartesian* monoidal monad (not to be confused with the cartesian monads [10], Section 4.1) is a symmetric monoidal monad in a cartesian category (by which is meant a monoidal category whose monoidal structure is given by finite products), which satisfies moreover the equation

$$(\psi\Delta) \qquad T\Delta_A = \psi_{A,A} \circ \Delta_{TA},$$

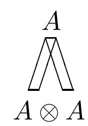
where $\Delta_A \colon A \to A \otimes A$ is a component of the diagonal natural transformation of the cartesian structure. The equation $(\psi\Delta)$ says intuitively that T preserves Δ up to ψ. In the terminology of [6] (Section 7), T is a conjunctive relevant endofunctor.

Note that in the definition of cartesian monoidal monad we do not assume the equation

$$(\psi¡) \qquad T¡_A = \eta_I \circ ¡_{TA},$$

where $¡_A : A \to I$ is the unique arrow from A to the terminal object (empty product) I. The equation $(\psi¡)$ says intuitively that T preserves $¡$ up to ψ, but we cannot assume that equation if we are guided by coherence, as the following pictures show:

We will now deal with coherence for the notion of cartesian monoidal monad. Let $\mathcal{CS}$ be the category of the cartesian monoidal monad freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{CS}$ to the category *Rel*, whose arrows are arbitrary relations between finite ordinals, by adding to the definition of G from $\mathcal{L}_c\mathcal{S}$ to *Fun* the clause for $G\Delta_A$ that corresponds to the following picture:



and the clause that says that $G¡_A$ is the empty relation between GA and $\emptyset$ (see [6], Sections 7-8).

As an auxiliary result for the proof of $\mathcal{CS}$-Coherence we establish first a lemma for which we need the notions of diversified object and scope of [6] (Section 2). An object that is a propositional formula is diversified when every generator (which means generating object, i.e. propositional letter, or generating functor) occurs in it at most once; for E^iC a subformula of D, the scope in D of the outermost occurrence of E^i in E^iC is the set of all the generators in C. The category $\mathcal{L}_c$ is the free symmetric monoidal category with a family of locally linear endofunctors (see the beginning of the preceding section and [6], Section 6). Here is our auxiliary lemma.

$\mathcal{L}_c$-THEOREMHOOD LEMMA. *For A and B diversified objects, there is an arrow $f : A \to B$ of $\mathcal{L}_c$ iff the generators of A and B coincide, and for every generating functor E^i of A the scope of E^i in A is a subset of the scope of E^i in B.*

PROOF. We proceed as in the proof of the $\mathcal{M}_c$-Theoremhood Lemma of [6] (Section 7), except for the assumption about the form of A in the induction step, which is now

$$D_1 \otimes A_1 \otimes \ldots \otimes D_{i-1} \otimes A_{i-1} \otimes D_i \otimes E^1 A_i \otimes D_{i+1} \otimes A_{i+1} \otimes \ldots \otimes D_n \otimes A_n \otimes D_{n+1},$$

with the union of the generators of $A_1, \ldots, A_n$ making the scope of E^1 in B. $\dashv$

Let $\mathcal{L}_c\mu$ be the category defined like $\mathcal{L}_c$, with ψ^l and ψ^R primitive, save that we have in addition the primitive arrow terms $\mu^i_A : E^iE^iA \to E^iA$ for which we assume the naturality and associativity equations of monads, and the equations $(\psi^L\mu)$ and $(\psi^L\psi^R\mu)$ with μ and T replaced respectively by μ^i and E^i. This category does not differ essentially from $\mathcal{L}_c\mathcal{S}$ in which η is absent. From the proof of $\mathcal{L}_c\mathcal{S}$-Coherence we may easily infer $\mathcal{L}_c\mu$-*Coherence* in the following form:

> *For all arrow terms $f, g\colon A \to B$ of $\mathcal{L}_c\mu$ with B diversified we have $f = g$ in $\mathcal{L}_c\mu$.*

We define ψ^i in $\mathcal{L}_c\mu$ as we did previously in Section 3.

We have introduced the category $\mathcal{L}_c\mu$ to formulate the following lemma without complications involving graphs.

$\mathcal{L}_c\mu$-THEOREMHOOD LEMMA. *For A diversified on generating objects and B diversified, there is an arrow $f : A \to B$ of $\mathcal{L}_c$ iff the generators of A and B coincide, and for every generating functor E^i of B the union of the scopes of the occurrences of E^i in A is a subset of the union of the scope of E^i in B with the set $\{E^i\}$.*

PROOF. The union with the set $\{E^i\}$ is mentioned above because an occurrence of E^i may be in the scope of another occurrence of E^i in A. Such an occurrence of E^i is called *nested*.

In the beginning we proceed for this proof as for the proof of the $\mathcal{M}_c$-Theoremhood Lemma of [6] (Section 7) until the assumption about the form of A in the induction step of the main induction. This form is now like the form mentioned in the proof of the $\mathcal{L}_c$-Theoremhood Lemma save that some of the A_j's for $j \neq i$ may be replaced by E^1A_j.

Then we have an auxiliary induction on the number m of nested occurrences of E^1 in A, in order to prove that there is an arrow f' of $\mathcal{L}_c\mu$ from A to the formula A' obtained from A by deleting all the nested occurrences of E^1. In the basis of this auxiliary induction, when $m = 0$, we have an identity arrow. In the induction step of this auxiliary induction, since E^1 is not in the scope of any E in B, we have a subformula of A like $E^1(C_1 \otimes E^1C_2 \otimes C_3)$. Let A'' be obtained from A by replacing this subformula with $E^1(C_1 \otimes C_2 \otimes C_3)$. It is clear that we have an arrow $g\colon A \to A''$ of $\mathcal{L}_c\mu$, and by the induction hypothesis of the auxiliary induction we have a desired arrow $f''\colon A'' \to A'$. So we have a desired arrow $f'\colon A \to A'$.

Then as in the $\mathcal{M}_c$-Theoremhood and $\mathcal{L}_c$-Theoremhood Lemmata we have a desired arrow from A' to

$$E^1(A_1 \otimes \ldots \otimes A_n) \otimes D_1 \otimes \ldots \otimes D_{n+1},$$

and we proceed as before for the remainder of the proof. $\dashv$

We can then prove the following lemma analogous to Lemma 1 of [6] (Section 7).

LEMMA 1. *For the arrow term*

$$f\colon A[EA_1 \otimes EA_2] \rightarrow B$$

of $\mathcal{L}_c\mu$ and $g : B \rightarrow C$ a μ-factor such that the ordinals corresponding to the outermost occurrences of E in EA_1 and EA_2 are respectively i and j, and $(GHf)(i) \neq (GHf)(j)$, while $(GH(g \circ f))(i) = (GH(g \circ f))(j)$, there exists an arrow term

$$f'\colon A[E(A_1 \otimes A_2)] \rightarrow C$$

of $\mathcal{L}_c\mu$ such that $g \circ f = f' \circ A[\psi_{A_1,A_2}]$.

PROOF. Note first that every arrow term of $\mathcal{L}_c\mu$ is a substitution instance of an arrow term of $\mathcal{L}_c\mu$ with a diversified target. So we may assume that C in the lemma is diversified. That f' exists follows from the assumption that we have $g \circ f$ and from the $\mathcal{L}_c\mu$-Theoremhood Lemma. That $g \circ f = f' \circ A[\psi_{A_1,A_2}]$ follows from $\mathcal{L}_c\mu$-Coherence. $\dashv$

We can now prove the following.

$\mathcal{CS}$-COHERENCE. *The functor G from $\mathcal{CS}$ to Rel is faithful.*

PROOF. We establish first that every arrow term f of $\mathcal{CS}$ is equal to an arrow term $f_3 \circ f_2 \circ f_1$ such that in the developed arrow term f_1 all the heads of factors are of the form Δ_A or $¡_A$ with A atomic (see [6], Section 2, for the notions of developed, head, factor and atomic), while f_2 is an arrow term of $\mathcal{L}_c\mathcal{S}$ without occurrences of η, and in the developed arrow term f_3 all the heads are of the form η_B. This is established as for Lemma 2 of [6] (Section 7) and for $\mathcal{C}$-Coherence of [6] (Section 8); we rely moreover on naturality and functorial equations to produce f_3.

The remainder of the proof is analogous to the proof of Lemma 5 of [6] (Section 7), which is based on Lemma 3 (ibid.). The notion of short circuit is the same, but in the proof of Lemma 3 we have that g is a μ-factor, and not a ψ-factor.

That Gf determines uniquely Gf_1, Gf_2, Gf_3, and the targets of f_1 and f_2 is established as at the end of of the proof of Proposition 5 of [6] (Section 8). We may assume that the target of f is $\otimes$-free, and so f_1 will be Δ-free. $\dashv$

6 Coherence for cocartesian monoidal monads

A *cocartesian* monoidal monad is a symmetric monoidal monad in a cocartesian category, by which we mean a monoidal category whose monoidal structure is given by finite coproducts; we have moreover the equation

$$(\psi\ def)\qquad \psi_{A,B} = \nabla_{T(A\otimes B)} \circ (T\iota^1_{A,B} \otimes T\iota^2_{A,B}),$$

where $\nabla_A : A\otimes A \to A$ is a component of the codiagonal natural transformation, while $\iota^1 : A \to A\otimes B$ and $\iota^2 : B \to A\otimes B$ are components of the injection natural transformations of the cocartesian structure.

More simply, we can define a cocartesian monoidal monad as a cocartesian category with a monad on it. The definition of ψ is given by (ψ *def*).

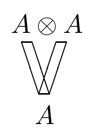
In every cocartesian monoidal monad T preserves ∇ up to ψ, in the sense that we have the equation

$$T\nabla_A \circ \psi_{A,A} = \nabla_{TA}.$$

If $!_A : I \to A$ is the unique arrow from the initial object I to A, then T preserves also ! up to ψ_0, which is defined as η_I, in the sense that we have the equation

$$T!_A \circ \eta_I = !_{TA}.$$

We will now deal with coherence for the notion of cocartesian monoidal monad. Let $\mathcal{DS}$ be the category of the cocartesian monoidal monad freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{DS}$ to the category *Fun*, whose arrows are arbitrary relations between finite ordinals, by adding to the definition of G from $\mathcal{L}_c\mathcal{S}$ to *Fun* the clause for $G\nabla_A$ that corresponds to the following picture:



and the clause that says that $G!_A$ is the empty function from $\emptyset$ to GA (see [6], Section 9). Then we can prove the following.

$\mathcal{DS}$-Coherence. *The functor G from $\mathcal{DS}$ to Fun is faithful.*

Proof. We establish first that every arrow term f of $\mathcal{DS}$ is equal to an arrow term $f_2 \circ f_1$ such that f_1 is an arrow term of $\mathcal{L}_c\mathcal{S}$, and in the developed arrow term f_2 every factor is either a ∇-factor or a !-factor such that the index of the head is a propositional letter.

Then we can ascertain that the target of f_1, which is the source of f_2, is uniquely determined by Gf. The graph Gf_2 is uniquely determined by Gf, and so is the graph Gf_1 if in f_1 we get rid of useless crossings (cf. [6], Section 7). Then we relay on $\mathcal{L}_c\mathcal{S}$-Coherence of Section 4 and $\mathcal{D}$-Coherence of [6] (Section 9) to obtain $\mathcal{DS}$-Coherence. (As a matter of fact, what we need is a rather trivial instance of $\mathcal{D}$-Coherence.) $\dashv$

7 Coherence for strong comonads

A *left strong* comonad in a monoidal category $\mathcal{A}$ is a comonad $\langle L, \varepsilon, \delta \rangle$ (in the notation of [11], Section VI.1) in $\mathcal{A}$ such that L is a left monoidal functor (see Section 2), and we have moreover the equations

$$(\psi^L\varepsilon) \qquad \varepsilon_{A\otimes B} \circ \psi^L_{A,B} = \varepsilon_A \otimes \mathbf{1}_B,$$

$$(\psi^L\delta) \qquad \delta_{A\otimes B} \circ \psi^L_{A,B} = L\psi^L_{A,B} \circ \psi^L_{LA,B} \circ (\delta_A \otimes \mathbf{1}_B).$$

(These equations might be interpreted as saying that ε is a left monoidal natural transformation from L to the identity functor, while δ is a left monoidal natural transformation from L to LL.)

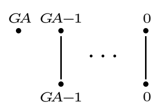
A *right strong* comonad is defined analogously with a right monoidal functor L. The equations corresponding to $(\psi^L\varepsilon)$ and $(\psi^L\delta)$ are

$$(\psi^R\varepsilon) \qquad \varepsilon_{A\otimes B} \circ \psi^R_{A,B} = \mathbf{1}_A \otimes \varepsilon_B,$$

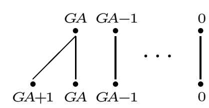
$$(\psi^R\delta) \qquad \delta_{A\otimes B} \circ \psi^R_{A,B} = L\psi^R_{A,B} \circ \psi^R_{A,LB} \circ (\mathbf{1}_A \otimes \delta_B).$$

Let $\mathcal{L}^L\mathcal{S}^{co}$ be the category of the left strong comonad freely generated by an arbitrary set of objects. We define a functor G from $\mathcal{L}^L\mathcal{S}^{co}$ to the category Δ^{op} as the functor G from $\mathcal{L}^L\mathcal{S}$ to the simplicial category Δ in Section 2, with the clauses for $G\eta_A$ and $G\mu_A$ replaced by dual clauses, which correspond to the following pictures:

$G\varepsilon_A$



$G\delta_A$



We can prove the following, by proceeding as for the proof of $\mathcal{L}^L\mathcal{S}$-Coherence in Section 2.

$\mathcal{L}^L\mathcal{S}^{co}$-COHERENCE. *The functor G from $\mathcal{L}^L\mathcal{S}^{co}$ to Δ^{op} is faithful.*

We can prove for right strong comonads coherence results analogous to the two versions of $\mathcal{L}^R\mathcal{S}$-Coherence in Section 2.

8 Coherence for monoidal comonads

A *monoidal* comonad in a monoidal category $\mathcal{A}$ is a comonad $\langle L, \varepsilon, \delta \rangle$ (in the notation of [11], Section VI.1) in $\mathcal{A}$ together with a natural transformation whose components are the arrows

$$\psi_{A,B} \colon LA \otimes LB \to L(A \otimes B),$$

and together with the arrow $\psi_0 : I \to LI$, such that L with ψ and ψ_0 is a monoidal functor (which means that we have the equations (ψa), (ψl) and (ψr) with T and η_I replaced respectively by L and ψ_0), and we have moreover the equations

$$(\psi\varepsilon) \qquad \varepsilon_{A\otimes B} \circ \psi_{A,B} = \varepsilon_A \otimes \varepsilon_B,$$

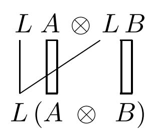
$$(\psi\delta) \qquad \delta_{A\otimes B} \circ \psi_{A,B} = L\psi_{A,B} \circ \psi_{LA,LB} \circ (\delta_A \otimes \delta_B),$$

$$(\psi_0\epsilon) \qquad \varepsilon_I \circ \psi_0 = \mathbf{1}_I,$$

$$(\psi_0\delta) \qquad \delta_I \circ \psi_0 = L\psi_0 \circ \psi_0,$$

which say that ε and δ are monoidal natural transformations. (References concerning the notion of monoidal comonad are given in Section 1.)

Let $\mathcal{MS}^{co}$ be the category of the monoidal comonad freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{MS}^{co}$ to *Rel* with the clause for $G\psi_{A,B}$ corresponding to the following picture:



We also have the clause that says that $G\psi_0$ is the empty relation from $\emptyset$ to $\{\emptyset\}$, the clauses for $G\varepsilon_A$ and $G\delta_A$ given above, and the remaining clauses as in Section 2.

For the proof of $\mathcal{MS}^{co}$-Coherence we need the following notion of normal form. An arrow term $f_2 \circ f_1$ of $\mathcal{MS}^{co}$ is in normal form when every factor of the developed arrow term $f_1 : A \to C$ is an ε-factor or a δ-factor, and $f_2 : C \to B$ is an arrow term of the category $\mathcal{M}$, which is the free monoidal category with a single monoidal endofunctor (see [6], Section 3; the family of monoidal endofunctors of $\mathcal{M}$ is here taken to be the singleton $\{L\}$). It is easy to see that the equations of $\mathcal{MS}^{co}$ yield that every arrow term is equal to an arrow term in normal form. To ascertain that $G(f_2 \circ f_1)$ determines uniquely Gf_1, Gf_2 and C we rely on a general proposition about decomposing an arbitrary binary relation between finite ordinals into three functions.

To formulate this proposition, let $<_l$ be the lexicographical order on $n \times m$; i.e. for $x_1, x_2 \in n$ and $y_1, y_2 \in m$ we have

$$(x_1, y_1) <_l (x_2, y_2) \text{ iff } (x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_1 < y_2)).$$

We call this the *left* lexicographical order, while the *right* lexicographical order $<_r$ is defined by

$$(x_1, y_1) <_r (x_2, y_2) \text{ iff } (y_1 < y_2 \text{ or } (y_1 = y_2 \text{ and } x_1 < x_2)).$$

Let $\langle \nu, \mu, \beta \rangle$ be a triple of functions $\nu\colon k \to n$, $\mu\colon k \to m$ and $\beta\colon k \to k$, for β a bijection, such that for $z \in k$ and

$$l_{\nu,\mu,\beta}(z) =_{df} (\nu(z), \mu(\beta(z))),$$
$$r_{\nu,\mu,\beta}(z) =_{df} (\nu(\beta^{-1}(z)), \mu(z)),$$

we have for every $u, v \in k$

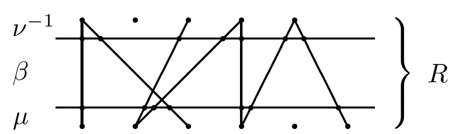
$$(*) \quad \text{if } u < v, \text{ then } (l_{\nu,\mu,\beta}(u) <_l l_{\nu,\mu,\beta}(v) \text{ and } r_{\nu,\mu,\beta}(u) <_r r_{\nu,\mu,\beta}(v)).$$

An alternative condition equivalent to $(*)$ is to say that $l_{\nu,\mu,\beta}(z)$ is the $(z{+}1)$-th pair in the $<_l$ ordering of the image of $l_{\nu,\mu,\beta}$, and analogously with r. The image of $l_{\nu,\mu,\beta}$ coincides with the image of $r_{\nu,\mu,\beta}$; it coincides also with the set of ordered pairs $\mu \circ \beta \circ \nu^{-1}$, whose cardinality is k.

We call triples of functions such as $\langle \nu, \mu, \beta \rangle$ above *coordinated*. That a triple of functions is coordinated amounts to saying that ν and μ are order-preserving and that in k there are no analogues of the short circuits and useless crossings of [6] (Section 7). We can now formulate our general proposition about decomposition.

Decomposition Proposition. *For every relation $R \subseteq n \times m$ there is a unique coordinated triple of functions $\langle \nu, \mu, \beta \rangle$ such that $R = \mu \circ \beta \circ \nu^{-1}$. The domain of ν, μ and β is the cardinality of R.*

Here $\circ$ on the right-hand side is composition of relations, and ν^{-1} is the relation converse to the function ν. We denote the cardinality of the set of ordered pairs R by $|R|$. The Decomposition Proposition is illustrated by the following example:



which also makes its truth pretty obvious.

We will however prove this proposition formally. For that, let the bijection $l_R\colon |R| \to R$ be defined by

$$l_R(z) \text{ is the } (z{+}1)\text{-th ordered pair of } R \text{ in the ordering } <_l.$$

We define analogously the bijection $r_R\colon |R| \to R$ via $<_r$.

Given R, consider the following functions:

$$\nu_R =_{df} \ p^1 \circ l_R : |R| \to n,$$
$$\mu_R =_{df} \ p^2 \circ r_R : |R| \to m,$$
$$\beta_R =_{df} r_R^{-1} \circ l_R : |R| \to |R|,$$

where p^1 and p^2 are respectively the first and second projection with domain $n \times m$.

To show that l_{ν_R,μ_R,β_R} and r_{ν_R,μ_R,β_R} satisfy $(*)$ it is enough to verify that the following lemma holds.

Lemma 1. *$l_{\nu_R,\mu_R,\beta_R} = l_R$ and $r_{\nu_R,\mu_R,\beta_R} = r_R$.*

Proof. For the first equation we have

$$\begin{aligned}(\nu_R(z), \mu_R(\beta_R(z))) &= (p^1(l_R(z)), p^2(r_R(r_R^{-1}(l_R(z)))))\\ &= (p^1(l_R(z)), p^2(l_R(z)))\\ &= l_R(z),\end{aligned}$$

and analogously for the second equation. ⊣

Since we also have $R = \mu_R \circ \beta_R \circ \nu_R^{-1}$, the triple $\tau(R) = \langle \nu_R, \mu_R, \beta_R \rangle$ is a coordinated triple of functions such as required by the Decomposition Proposition.

To show that $\tau(R)$ is unique we proceed as follows. It is enough to verify besides $R = \mu_R \circ \beta_R \circ \nu_R^{-1}$ that for every coordinated triple of functions $\langle \nu, \mu, \beta \rangle$ we have

$$\tau(\mu \circ \beta \circ \nu^{-1}) = \langle \nu, \mu, \beta \rangle.$$

For that we rely on the following lemmata.

Lemma 2. *$l_{\mu \circ \beta \circ \nu^{-1}} = l_{\nu,\mu,\beta}$ and $r_{\mu \circ \beta \circ \nu^{-1}} = r_{\nu,\mu,\beta}$.*

For the proof we rely on the comment after $(*)$.

Lemma 3. *$\nu_{\mu \circ \beta \circ \nu^{-1}} = \nu$, $\mu_{\mu \circ \beta \circ \nu^{-1}} = \mu$ and $\beta_{\mu \circ \beta \circ \nu^{-1}} = \beta$.*

Proof. For the first equation we have

$$\begin{aligned}\nu_{\mu \circ \beta \circ \nu^{-1}}(z) &= p^1(l_{\mu \circ \beta \circ \nu^{-1}}(z)),\\ &= p^1(\nu(z), \mu(\beta(z))), \quad \text{by Lemma 2}\\ &= \nu(z).\end{aligned}$$

The second equation is derived analogously, while for the third equation we have

$$\begin{aligned}\beta_{\mu \circ \beta \circ \nu^{-1}}(z) &= r^{-1}_{\mu \circ \beta \circ \nu^{-1}}(l_{\mu \circ \beta \circ \nu^{-1}}(z)),\\ &= r^{-1}_{\nu,\mu,\beta}(\nu(z), \mu(\beta(z))), \quad \text{by Lemma 2}\\ &= \beta(z).\end{aligned}$$

since we have

$$\begin{aligned}r_{\nu,\mu,\beta}(\beta(z)) &= \nu(\beta^{-1}(\beta(z)), \mu(\beta(z))),\\ &= (\nu(z), \mu(\beta(z))).\end{aligned}$$

⊣

This concludes the proof of the Decomposition Proposition. (The Decomposition Proposition could be used to obtain a normal form for arrow terms of the category *Rel*—a normal form alternative to the iota normal form of [5]; Section 13.)

We can now finish the proof of the following.

$\mathcal{MS}^{co}$-COHERENCE. *The functor G from $\mathcal{MS}^{co}$ to Rel is faithful.*

PROOF. We rely on the normal form $f_2 \circ f_1$, which we introduced before the Decomposition Proposition. In f_1 we find what corresponds to ν^{-1} in the Decomposition Proposition, and in f_2 what corresponds to $\mu \circ \beta$. We rely then on coherence for comonads (see [4], Section 3, and references therein) and on $\mathcal{M}$-Coherence of [6] (Section 4). $\dashv$

The normal form of this proof could be refined to $f_4 \circ f_3 \circ f_2 \circ f_1$ where in the developed arrow term f_1 every factor is an ε-factor, in the developed arrow term f_2 every factor is a δ-factor, in the developed arrow term f_4 every factor is a ψ_0-factor, and f_3 is an arrow term of $\mathcal{M}$ without occurrences of ψ_0.

9 Coherence for symmetric monoidal comonads

A *symmetric* monoidal comonad is a monoidal comonad in a symmetric monoidal category whose endofunctor L is a monoidal functor (see the beginning of the preceding section) that satisfies the equation (ψc) of Section 4 with T replaced by L; namely, this endofunctor is a linear endofunctor in the sense of [6] (Section 6).

Let $\mathcal{M}_c\mathcal{S}^{co}$ be the category of the symmetric monoidal comonad freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{M}_c\mathcal{S}^{co}$ to *Rel* as G from $\mathcal{MS}^{co}$ to *Rel*, save that now we have that GA is the number of occurrences of generating objects, i.e. propositional letters, in A plus the number of occurrences of L in A (see Section 4). We have moreover a clause for $Gc_{A,B}$ as in Section 4. We can prove the following.

$\mathcal{M}_c\mathcal{S}^{co}$-COHERENCE. *The functor G from $\mathcal{M}_c\mathcal{S}^{co}$ to Rel is faithful.*

For the proof we proceed as for $\mathcal{MS}^{co}$-Coherence, by relying on a normal form $f_2 \circ f_1$ where f_1 is as before while f_2 is an arrow term of the category $\mathcal{M}_c$, which is the free symmetric monoidal category with a single linear endofunctor; we appeal then to $\mathcal{M}_c$-Coherence of [6] (Section 6).

10 Coherence for cartesian monoidal comonads

A *cartesian* monoidal comonad is a symmetric monoidal comonad in a cartesian category, which satisfies moreover the equation $(\psi\Delta)$ of Section 5 with T

replaced by L.

Let $\mathcal{CS}^{co}$ be the category of the cartesian monoidal comonad freely generated by an arbitrary set of objects. This category may be taken as axiomatizing identity of deductions in the $\{\Box, \wedge, \top\}$ fragment of the modal logic S4 (cf. [4]). We define the functor G from $\mathcal{CS}^{co}$ to *Rel* as G from $\mathcal{M}_c\mathcal{S}^{co}$ to *Rel* with additional clauses for $G\Delta_A$ and $G¡_A$ as in Section 5. We can prove the following.

$\mathcal{CS}^{co}$-COHERENCE. *The functor G from $\mathcal{CS}^{co}$ to Rel is faithful.*

For the proof we proceed as for C-Coherence in [6] (Section 8). We rely again on the possibility to assume that the targets are $\otimes$-free.

11 Coherence for cocartesian monoidal comonads

A *cocartesian* monoidal comonad is a symmetric monoidal comonad in a cocartesian category; we have moreover the equation $(\psi\ def)$ of Section 6. The equation

$$(\psi_0\ def) \qquad \psi_0 = !_{LI}$$

follows from the assumption that I is an initial object, which comes with the assumption that we are in a cocartesian category.

More simply, we can define a cocartesian monoidal comonad as a cocartesian category with a comonad in it. The definitions of ψ and ψ_0 are then given by $(\psi\ def)$ and $(\psi_0\ def)$.

Let $\mathcal{DS}^{co}$ be the category of the cocartesian monoidal comonad freely generated by an arbitrary set of objects. We define the functor G from $\mathcal{DS}^{co}$ to *Rel* as G from $\mathcal{M}_c\mathcal{S}^{co}$ to *Rel* with additional clauses for $G\nabla_A$ and $G!_A$ as in Section 6. We can prove the following.

$\mathcal{DS}^{co}$-COHERENCE. *The functor G from $\mathcal{DS}^{co}$ to Rel is faithful.*

PROOF. We rely on a normal form $f_2 \circ f_1$ where every factor of the developed arrow term $f_1 : A \rightarrow C$ is an ε-factor or a δ-factor, and $f_2 : C \rightarrow B$ is an arrow term of the category $\mathcal{D}$, which is the free cocartesian category with a single endofunctor (see [6], Section 9). To ascertain that $G(f_2 \circ f_1)$ determines uniquely Gf_1, Gf_2 and C we use the Decomposition Proposition of Section 8. We rely then on coherence for comonads, as in the proof of $\mathcal{MS}^{co}$-Coherence in Section 8, and on $\mathcal{D}$-Coherence of [6] (Section 9). $\dashv$

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