The Small-Community Phenomenon in Networks

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Abstract

We investigate several geometric models of network which simultaneously have some nice global properties, that the small diameter property, the small-community phenomenon, which is defined to capture the common experience that (almost) every one in our society belongs to some meaningful small communities by the authors (2011), and that under certain conditions on the parameters, the power law degree distribution, which significantly strengths the results given by van den Esker (2008), and Jordan (2010). The results above, together with our previous progress in Li and Peng (2011), build a mathematical foundation for the study of communities and the small-community phenomenon in various networks.

In the proof of the power law degree distribution, we develop the method of *alternating concentration analysis* to build concentration inequality by alternatively and iteratively applying both the sub- and super-martingale inequalities, which seems powerful, and which may have more potential applications.

1 Introduction

With the availability of massive datasets of many real world networks, we are able to observe and study the underlying dynamic mechanisms and many interesting phenomena in large-scale networks in a quantitative way. Some properties such as sparse, high-clustering, hierarchical structure, the power law degree distribution and small diameter appear in a wide range of networks, ranging from Internet graphs, collaboration graphs to PPI (Protein-Protein Interaction) networks. Modeling these interesting properties and phenomena not only provides us a good way to better understand how these networks evolve and why these global phenomena occur through local growing rules, but also gives us insights on the development of new technologies or even cancer drugs.

A typical network always simultaneously exhibits several properties. For example, in a Web graph, the nodes are web-pages and directed edges are hyperlinks between the pages, the number of nodes with indegree k is proportional to $k^{-\beta}$ for some constant β , i.e., the in-degree sequence obeys the power law degree distribution ([AJB99, KKR⁺99]). It has also been observed that the Web graphs have a small average distance [AJB99, BKM⁺00]. In this paper, when it is not confused, "small" means that the quantity is a polylogarithmic function of the number of graph nodes. Furthermore, the most community-like subgraphs in the large Web graphs turn out to have size about 100, which seems to be a general property in many large real networks [LLDM08, LLM10]. The above mentioned three properties are by no means particular in the technological networks, and they are shared also by a wide range of social networks, such as the friendship network of LiveJournal.

The first two properties, i.e., the power law degree distribution and the small diameter property, have been explored extensively in the past decades. However, to our knowledge, the third property that good communities in large-scale networks have small sizes is still widely open due to the reason that there were no

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mathematical definition of good communities in network, which motivates us to mathematically study the common experiences, or observations, or small experiments of the small-community phenomenon in networks.

The authors proposed a mathematical definition for communities based on the concept of conductance, defined the *small-community phenomenon* in networks, and conjectured that small communities are ubiquitous in various networks (referred to our earlier work [LP11]). Intuitively, a given network is said to have the small-community phenomenon if almost every node in the network is contained in some good community of small size (referred to Section 2 for the formal definition). We found theoretical evidence for our conjecture that some classical network models (e.g., Kleinberg's small world model [Kle00] and Ravasz-Barabási Hierarchical model [RB03]) do have the small-community phenomenon, though models without this phenomenon exist.

There are also other reasons for us to make such a conjecture. Firstly, we all have the common experience that everyone in the society belong to some small meaningful groups which may correspond to classmates, friends or relatives. Secondly, the existing empirical studies provide us evidence that large communities are rare in large networks and good communities are of small size. Besides the direct evidence given by [LLDM08, LLM10, GR09], there are also some implicit evidence. For example, [Lan05] has shown that spectral graph partitioning fails to generate highly unbalanced cuts over many large scale social networks and [KB10] pointed out that this failure may be caused by the abundance of small dense communities. In summary, we have reasons to conjecture that the small communities are ubiquitous, at least in many large social networks, which arises a number of new problems in both theory and applications of the small-community phenomenon in networks.

We are interested in *evolving* models that simultaneously have these "good" properties: the power law degree distribution, the small diameter and the small-community phenomenon, which are shared in typical Web graphs and large social networks. Models with one or two properties are easily constructed in some natural way. In particular, the power law degree distribution arises from the preferential attachment scheme; the small diameter originates from a broad class of graph processes [Bol01]; the small community may be caused by the notion of *homophily* that, similar or close individuals have great tendency to associate with each other, which is a common reason for two people establish a relationship with each other in our society.

However, when trying to define a model that unify all the three properties, we usually come across conflicts that are hard to reconcile. Not strictly speaking, the first two properties usually result from some expander like graphs while the small community corresponds to highly structured graphs which seem anti-expander like graphs to some extent [LP11]. Still, Ravasz-Barabási Hierarchical model [RB03] satisfies all these requirements as shown in [LP11]. However, the Ravasz-Barabási Hierarchical model has a very unnatural growing rule, which can only capture very special networks.

Another good candidate may be the Geometric Preferential Attachment (henceforth GPA) model introduced by Flaxman, Frieze and Verta [FFV07a, FFV07b], whose motivation was to model networks with power law degree distribution and small expansion. This model is defined on a unit-area spherical surface S, on which distance can be naturally introduced. The authors of [FFV07a] combined the rich-get-richer effect and the concept of homophily in a simple way that every new comer chooses neighbors only from those exiting vertices that are not far from them using the preferential attachment scheme, and proved that the power law distribution occurs under some conditions of the parameters in the model. In [LP11], we have shown that good communities exist for every node in a model under these conditions. However, the found communities are of relatively large size and the diameter is also large.

In the present paper, we will first study a base model that is a GPA model with additive fitness. We generalize the result of [Jor10] and show that under some appropriate conditions, the base model have both the power law degree distribution and the small-community phenomenon. However, in this situation the diameter of the model is large. To resolve this problem, we try to incorporate a simple growth rule into our base model that leads to small diameter and does not change too much the degree sequence. The rule we try to use is the uniform recursive tree, i.e., each time a new vertex chooses a neighbor uniformly at random from exiting vertices. It has been well known that such a simple process results in a graph of diameter and maximum degree of order $\Theta(\ln n)$, where n is the number of generated vertices [SM95]. We give two alternate ways to incorporate this rule. Though the resulted two models are similar, their structures are

different. The first one is a hybrid model, which can be regarded as a composition of two independent parts: a local graph, which has the power law degree distribution, and a global graph, which may connect vertices that are far away. The hybrid model as a whole has the small diameter and the small community structure. The second is a self-loop model, in which we treat the additive fitness in our base model as the number of self-loops attached with the new vertex. This gives a new interpretation for the use of fitness in the preferential attachment schemes. With some further operation, the self-loop model is shown to have all the three good properties.

The methodology we use to show the power law degree distribution may be of independent interest. The proof technique is inspired by the work [Jor10], who investigated the asymptotic behavior of the degree sequence of the base model (see Section 2). In our proof of the concentration inequalities, there are subtle restrictions on parameters for which deeper mathematics is needed. Rather than using the coupling techniques as that in [FFV07a, vdE08], we recursively utilize the submartingale and supermartingale concentration inequalities [CL06] to give a better bound at each step, which will result in a sharp bound of the desired quantity.

Further related works Avin studied a random distance graph that incorporates both the Erdös-Rényi graph and the random geometric graph [Avi08]. This graph is shown to have several good properties, e.g., the small diameter and high clustering coefficient et al. Hybrid model composed of a power law graph and a grid-like local graph is studied by several groups of researchers, see [CL04, KB10, FG09]. Clusters or communities based on the concept of conductance was studied in [KVV04] and [LLDM08, LLM10], in which the spectral algorithms and other approximation algorithms were used to detect good clusters or communities.

In Section 2, we will introduce the definition of the small-community phenomenon as well as our models, and then state the main results of the paper. In the next three sections, we show that the models have the desired properties. In Section 7, we discuss the effect of the choice of a parameter on the properties of our proposed models. Finally, we give a brief conclusion in Section 8.

2 basic definitions, the model and main results

2.1 The small-community Phenomenon

In a graph G = (V, E), the degree of a node $v \in V$ is denoted as $\deg_G(v)$. The volume of a subset of $S \subseteq V$ is defined to be the sum of degrees of vertices in it, namely, $\operatorname{vol}(S) = \sum_{v \in S} \deg_G(v)$.

Our definition of communities is inspired by the work of Leskovec et al. [LLDM08], who used the conductance to measure the goodness of a community. We introduced the concept of (α, β, γ) -community based on the conductance and the size of a set of nodes [LP11]. The conductance $\Phi(S)$ of S is the ratio between the number of edges coming out of S and the volume of it or its complement \bar{S} , whichever is smaller, i.e.,

$$\Phi(S) = \frac{|e(S, \bar{S})|}{\min\{\operatorname{vol}(S), \operatorname{vol}(\bar{S})\}} \ ,$$

where e(S,T) denotes the set of edges with one endpoint in S and the other in T. Now we formulate the (α, β, γ) -community as follows:

Definition 1. Given a graph G = (V, E) with |V| = n, a connected set $S \subset V$ with $|S| = \omega(1)$ is a strong (α, β) -community if

$$\Phi(S) \le \frac{\alpha}{|S|^{\beta}} \ . \tag{1}$$

Moreover, if $|S| = O((\ln n)^{\gamma})$, then we say that S is a strong (α, β, γ) -community.

Note that in the above definition we require that the size of a community is not too small (i.e., $|S| = \omega(1)$). This requirement helps us to avoid the trivial case in our definition (when |S| is constant, it can always be

treated as a proper community by choosing large α). In fact, a meaningful community in society always can not be too small because of lack of requisite variety or other group function [All04].

To characterize the feature that almost every one in the network belongs to some small community, we give the following definition.

Definition 2. A network (model) G is said to exhibit the small-community phenomenon, if almost every node belongs to some (α, β, γ) -community, where $\alpha, \beta, \gamma > 0$ are some global constants.

2.2 The Geometric Model

The base model we will use is a geometric preferential attachment model with additive fitness. Such a model has been studied in [vdE08, Jor10] (see also [FFV07a, FFV07b]). Assume that a self-loop counts as degree 1. The model is defined on a unit-area spherical surface S (i.e., the radius of the sphere is $\frac{1}{2\sqrt{\pi}}$). Let n be the number of vertices we are going to generate. Let $\xi > 0$ be an arbitrary constant and $m, r, \delta = \xi m$ be some parameters which may depend on n (Note that this is the essential difference from the cases studied in [Jor10]). Intuitively speaking, m is the number of edges we are going to add in each step; r is the distance restriction on the two endpoints of an edge; δ is the additive fitness. Let $B_R(v)$ denote the spherical cap of radius R around v in S, i.e., $B_R(v) = \{u \in S : ||u - v|| \le R\}$, where $||\cdot||$ denotes the angular distance on S. Let $A_R = \text{area}(B_R(v))$ be the area of the spherical cap of radius R, which is independent of v.

The base model: We start the process from a graph G_1 , which is composed of a uniformly generated (from S) node x_1 with 2m self-loops. At each time t+1 for t>0, if $G_t=(V_t,E_t)$, we first generate a new node x_{t+1} uniformly at random from S and then connect it to some existing vertices or itself. Specifically, if there is no node in $B_r(x_{t+1})$, then we add 2m self-loops to x_{t+1} ; if $B_r(x_{t+1}) \cap V_t \neq \emptyset$, then we choose independently m contacts (with replacement) from $B_r(x_{t+1})$ for the new comer such that for any i with $1 \leq i \leq m$, the probability that some vertex $v \in B_r(x_{t+1})$ is chosen as the ith contact is defined by

$$\Pr[y_i^{t+1} = v] = \frac{\deg_t(v) + \delta}{\sum_{w \in B_r(x_{t+1}) \cap V_t} (\deg_t(w) + \delta)} . \tag{2}$$

Remark: in [vdE08] (also in [FFV07a, FFV07b]), a self-loop parameter $\alpha > 2$ was introduced to avoid a technical problem when proving the power law degree distribution. In their settings, a node $v \in B_r(x_{t+1})$ is chosen as the contact with probability

$$\frac{\deg_t(v) + \delta}{\max\{\sum_{w \in B_r(x_{t+1}) \cap V_t} (\deg_t(w) + \delta), \alpha(m + \delta/2) A_r t\}} , \tag{3}$$

where $\delta > -m$. The case of $\alpha = 0$ is left open in these papers. Jordan [Jor10] investigated the asymptotic behavior of the degree sequence in the case of $\alpha = 0$. In his study, $m, r, \delta > 0$ are constants that not depend on n, which converges to infinity. However, in our situation, we need a strong concentration result such that the parameters may depend on n. We will give such a result when $\alpha = 0$ and $\delta > 0$, which strengths the results in [vdE08, Jor10] and partially answers the open question in [FFV07a, FFV07b].

We can show that when $\delta = \xi m > 0$ and $r = r_0 = n^{-\frac{1}{2}}(\ln n)^{c_0}$, where $c_0 = c_0(\xi)$ is large and may depend on ξ , the base model has the power law degree distribution and the small-community phenomenon but does not have the small diameter. To incorporate the missing property while not changing the other two properties too much, we introduce some operations that essentially generate a uniform recursive tree. We give two different operations such that the resulted two variants of the base model both have the three properties to some extent.

1. **The hybrid model**: In this model, every edge has an attribute that indicates whether it is a *local-edge* or a *long-edge*, which indicates that the two endpoints of the edge are local- or long-contacts of each other. A local- (or long-) edge contributes to the local- (or long-) degree of both of its endpoints. We

¹ almost every means $1 - o_n(1)$, where n is the number of vertices in G

start from the $G_1^{\rm H}$ the same as G_1 in the above and let the self-loops of x_1 be local-edges. At each step t+1 for $t\geq 1$, to form $G_{t+1}^{\rm H}$ from $G_t^{\rm H}$, a new vertex x_{t+1} is chosen uniformly at random from S. First we choose for the new comer m local-contacts $y_i, 1\leq i\leq m$, independently at random as in the base model with $\deg_t(v)$ in Eq. (2) denoting the local-degree of v at time t. Then we choose for x_{t+1} one other long contact z uniformly from x_1, \dots, x_t .

This model can be seen as composed of two parts: a local power law graph and a global uniform recursive tree, which can be generated in two phases: firstly, we can generate the local power law graph following the rules used in the base geometric model and then generate a recursive tree as follows: sequentially for $t \ge 1$, x_{t+1} connects a long-contact which is chosen uniformly at random from x_1, \dots, x_t .

The independence of the local part and the global part of the hybrid model conforms to our intuition that local contacts and long contacts are formed by different mechanisms. Previous studies on such a model usually has a global power law graph and a local grid-like graph (see eg. [CL04]), which is comparable with ours.

2. The self-loop model: In this model, every new node is born with δ flexible self-loops which may be eliminated in later steps. Now we generate x_1 uniformly at random from S and add $2m + \delta$ self-loops to it with $\delta \geq 2$ loops marked flexible. This is the start graph G_1^S . At each step t+1 for $t \geq 1$, to form G_{t+1}^S from G_t^S , a new vertex x_{t+1} is chosen uniformly at random from S and δ flexible self-loops are added to it. We first choose m contacts $y_i, 1 \leq i \leq m$ independently at random as in the base model with $\deg_t(v)$ in Eq. (2) denoting the number of non-flexible edges incident to v at time t. Then we choose for x_{t+1} one other contact z uniformly from the set of existing nodes containing flexible self-loop(s)(such a set cannot be empty because x_t is a member of it) and delete one flexible self-loop from both x_{t+1} and z. The newly added edge (x_{t+1}, z) is marked flexible. Note that the edge-rewiring keeps the degree of vertices unchanged, which facilitates the analysis of its degree distribution.

This model can be seen as composed of two parts: a flexible part and a non-flexible part, which can be generated in several phases: we first generate the non-flexible part following the growth rules of the base model. We then add δ flexible self-loops to each vertex. Then sequentially for each $t \geq 1$, x_{t+1} connects a contact z which is chosen uniformly at random from x_1, \dots, x_t , containing flexible self-loop(s), a flexible self-loop of x_{t+1} and z is deleted and a new flexible edge (x_{t+1}, z) is added.

We give a plausible explanation of the self-loops emerging in this model. It is widely studied in social sciences that people in our society have not only evident relationships with others, but some implicitly one-sided "parasocial" interactions with the celebrities, virtual characters and so on, in which relationship only one part knows a great deal about the other, but the other does not [HW56]. Such a relationship can barely be reflected by the usually used friendship networks, which mainly coins the two-sided friendship. Our model incorporates the parasocial relationships as self-loops and the edge-rewiring may be roughly interpreted as that the long-distance relationship is established at the expense of its parasocial connections.

2.3 Main Results

Our main results are that the two models have rather good properties. Assume that $\delta = m\xi$, where $\xi > 0$ is some constant and $r_0 = n^{-1/2} (\ln n)^{c_0}$ for some large constant c_0 which may depend on ξ .

For $r \ge r_0$, it is obvious that the diameter of the base model is $\Omega(1/r) = \Omega(n^{1/2}(\ln n)^{-c_0})$ (see Section 4), which is large, while the short diameters of the uniform recursive trees imply the small diameter results in our two generalized models.

Theorem 1. (Small Diameter Property)

- 1. For any $m \ge 1, r > 0$, with high probability, the diameter of G_n^H is $O(\ln n)$.
- 2. For $m \geq K_1(\xi) \ln n$ and r > 0, with high probability, the diameter of G_n^S is $O(\ln n)$, where $K_1(\xi)$ is some constant depending on ξ .

By the geometric structure of the models, it is natural to think of that a group of vertices close to each other behaves like a good community. We will make this intuition rigorous by considering the R-neighborhood $C_R(v)$ of a vertex v, which is the set of all vertices within distance at most R from v in G_n and show that for some appropriate r and R, $C_R(v)$ is a good community for every v. We give the following result:

Theorem 2. (Small-Community Phenomenon) If $r = r_0$ and $m \ge K_2(\xi) \ln n$, where $K_2(\xi)$ is some constant depending on ξ , both G_n^H and G_n^S have the small-community phenomenon, i.e., in each model, with high probability, for every node $v \in V_n$, there exists some (α, β, γ) -community containing v, where α, β, γ are some constants independent of n.

A simple corollary of the above theorem is that the base model G_n also has the small-community phenomenon, which indicates that the community structure is mainly determined by the geometric structure of our model and that the effect of long edges is little for the reason that every new node can establish $m \gg \ln n$ local edges while only 1 long edges.

The power law degree distribution stems from the preferential attachment scheme used in our base model, for which we have:

Theorem 3. (Degree Distribution of the Base Model) In the base model, if $r \ge r_0$, $m = O(\ln^2 n)$ and $\delta = m\xi$ for any constant $\xi > 0$, there exist some constants C_k and μ , such that for all $k = k(n) \ge m$,

$$E[d_k(t)] = C_k \frac{n}{k^{3+\xi}} + O(\frac{n}{(nr^2)^{\mu}}) , \qquad (4)$$

where $d_k(t)$ denotes the number of vertices with degree k in the base model G_t , $C_k = C_k(m, \xi)$ tends to a limit $C_{\infty}(m, \delta)$ which only depends on m, δ as $k \to \infty$, and μ is some constant depending on ξ and strictly less than 1.

Theorem 3 has already significantly strengthened the results in both van den Esker [vdE08] and Jordan [Jor10]. The proof of this theorem requires the new technique of recursively bounding the concentration inequalities as we will build in Section 6.

Based on Theorem 3, we are able to show that in our generalized models, the networks satisfy a nice power law degree distribution.

Theorem 4. (Power Law Degree Distribution) For $r \ge r_0$ and $m = O(\ln^2 n)$, the expected degree sequences of the local graph of the hybrid model G_n^H and the whole graph of the self-loop model G_n^S both follow a power law distribution with exponent $3 + \xi$. More specifically, there exist some constants C_k^H , C_k^S and μ , such that for all $k = k(n) \ge m$,

- 1. in the hybrid model, $E[d_k(n)] = C_k^H \frac{n}{k^{3+\xi}} + O(\frac{n}{(nr^2)^{\mu}})$, where $d_k(t)$ denotes the number of vertices with local-degree k in G_t^H ;
- 2. in the self-loop model, $E[d_k(n)] = C_k^S \frac{n}{k^{3+\xi}} + O(\frac{n}{(nr^2)^{\mu}})$, where $d_k(t)$ denotes the number of vertices with total degree k in G_t^S .

In the above statements, both C_k^H and C_k^S tend to some limits that depend on m, δ only as $k \to \infty$, and μ is some constant depending on ξ and strictly less than 1.

From the above theorems, we know that when $r = r_0 = n^{-1/2}(\ln n)^{c_0}$, the two generalized models simultaneously have all the three properties to some extent (as in the hybrid model, only the local part has the power law degree distribution). What about the cases when r is too large or too small? We give some evidence that at least one of the three properties disappears in such cases. In particular, when r is large, we have the following new phenomenon.

Theorem 5. (Large Community and Small Expander) In the base model G_n , let $r = n^{-1/2+\epsilon}$, where $\epsilon > 0$ and $m \ge K \ln n$, for some sufficiently large constant K.

- 1. If $R = n^{-1/2+\rho}$, for any $\rho > \epsilon$, then $|C_R(v)| = \Theta(n^{2\rho})$ and $\Phi(C_R(v)) = O(\frac{1}{n^{\rho-\epsilon}})$, with high probability.
- 2. With high probability, for all R = o(r), $\Phi(C_R(v)) = \Omega(1)$.

Theorem 5 indicates that when $r = n^{-1/2+\epsilon}$, there exists some large community for every node, which may not belong to any small community for the reason that the most natural candidate, i.e., the small neighborhood is not a good community. We remark that Theorem 5 may imply a new phenomenon in networks. It would be interesting to find some real world networks, in which there is a large fraction of nodes each of which is contained in both a good but large community and a small expander. We also note that the two generalized models have the same phenomenon for such a large r.

In the remaining sections of the paper, we are devoted to proving our main results, Theorems 1, 2, 3, 4, and 5. We will organize the paper as follows. In Section 3, we introduce some basic tools for our proof, and basic properties of our network models. In Sections 4 and 5, we prove Theorems 1 and 2, respectively. In Section 6, we prove Theorems 3 and 4. In Section 7, we prove Theorem 5. Finally in Section 8, we discuss some further issues following the results in this paper.

3 Useful tools and basic facts

Before proving the main results, we first give several basic facts which will be useful in our proofs of the main results.

We will use the following form of the Chernoff bound (see eg. Theorem 1.1 in [DP09]).

Lemma 1. If X_1, \dots, X_t are independently distributed in [0,1] and $X = \sum_{i=1}^t X_i$, then for $0 < \zeta \le 1$,

$$\Pr[|X - E[X]| \ge \zeta E[X]] \le 2e^{-\frac{\zeta^2 E[X]}{3}}.$$
 (5)

The following submartingale concentration inequality will be used extensively in our proofs (referred to Theorems 2.38 and 2.41 in [CL06]).

Lemma 2. Suppose that $\{X_0, \dots, X_t\}$ is a sequence of random variables associated with a filter $\{\mathcal{F}_0, \dots, \mathcal{F}_t\}$ and G is some event on the probability space. If for $1 \leq i \leq t$,

$$E[X_i|\mathcal{F}_{i-1},\mathcal{G}] \leq X_{i-1},$$

$$Var[X_i|\mathcal{F}_{i-1},\mathcal{G}] \leq \sigma_i^2,$$

$$X_i - E[X_i|\mathcal{F}_{i-1},\mathcal{G}] \leq M,$$

where σ_i^2, M are non-negative constants. Then we have

$$\Pr[X_t \ge X_0 + \lambda] \le e^{-\frac{\lambda^2}{2\sum_{i=1}^t \sigma_i^2 + M\lambda/3}} + \Pr[\neg \mathcal{G}]. \tag{6}$$

The supermartingale concentration inequality is similar and we omit it here.

In the following sections, we will use constants c_0, c_1 and c_2 which may depend on ξ to characterize some bounds. We state here the conditions that the three constants should satisfy.

$$(c_0 - c_1 - 1)(1 - 1/(\xi + 2)) < c_1 < 2(c_0 - c_1 - 1)(1 - 2/(2 + \xi))$$

$$(7)$$

$$(c_0 - c_1 - 1)(1 - 1/(\xi + 2)) < c_1 < 2(c_0 - c_1 - 1)(1 - 2/(2 + \xi))$$

$$c_2 = c_1 \frac{\ln(\xi(1 + \xi/2) + 1)}{\ln((7 + 400/\xi)^2(\xi(1 + \xi/2) + 1))}$$
(8)

Note that for fixed ξ we can always choose c_0 to be large enough to guarantee that c_2 is also large, which will ensure that the bounds we obtain in the proof are good.

In the definition of our base model, a new vertex will create 2m self-loops if there is no existing vertex within distance at most r from it. This rule is made to guarantee that at each step the degree of the graph grows by 2m, which facilitates further analysis. Moreover, in most interesting cases when $r = r_0 = n^{-1/2}(\ln n)^{c_0}$, if t grows as large as $\tau = O(\frac{n}{(\ln n)^{2c_0-1}})$, then with high probability for any vertex that comes after time τ , there will be many existing nodes within distance at most r from it. Therefore we will focus on the processes that all the *later* comers will choose existing nodes as neighbors other than creating 2m self-loops.

In analyzing the degree sequence of our base model, it is convenient to compare the chosen probability given in Eq. (2) with the traditional case (eg. [Bol03]), in which at each step t+1 an existing vertex v with degree k is chosen with probability $\frac{k}{2t}$, where 2t is the total degree of all existing vertices. Thus it is natural to consider of using a good estimation of (2) for further analysis. In particular, we would like to have some good bound on the normalized quantity of the denominator of (2). Let $T_t(u)$ denote this quantity, namely, $T_t(u) = \sum_{v \in B_r(u) \cap V_t} (\deg_t(v) + \delta)$. A closely related quantity is $Z_t(u) = \sum_{v \in B_r(u) \cap V_t} 1$, which is the number of vertices in $B_r(u)$ at time t. We have several simple facts on these two quantities.

Lemma 3. If $u \in S$ and t > 0, then the expectation of $T_t(u)$ is $A_r(2m + \delta)t$.

Proof. Note that

$$E[T_t(u)] = E[\sum_{v \in B_r(u) \cap V_t} (\deg_t(v) + \delta)] = E[\sum_{v \in V_t} (\deg_t(v) + \delta) 1_{v \in B_r(u)}]$$

$$= E[\sum_{v \in V_t} \deg_t(v) 1_{v \in B_r(u)}] + \delta A_r t .$$

$$(9)$$

The first part of (9) is $2A_rmt$ as given in Lemma 1 and 2 in [FFV07a], which completes the proof.

Let A_r denote the area of $B_r(v)$. Then $A_r = \text{area}(B_r(v)) \sim r^2/4$, for r = o(1). Let $t_r = \frac{12(\ln n)^2 n^{c_1/c_0}}{r^{2(1-c_1/c_0)}}$ and thus $A_r t_r \sim 3(\ln n)^2 (nr^2)^{c_1/c_0}$. We will consider that $r \geq r_0 = n^{-1/2} (\ln n)^{c_0}$ and let $t_0 := t_{r_0} = \frac{12n}{(\ln n)^{2c_0-2c_1-2}}$. We first give an estimation of the quantity $Z_t(u)$.

Lemma 4. If $r \ge r_0$, then for any $t \ge t_r$, with probability at least $1 - 2n^{-\ln n}$, we have that

$$|Z_t(u) - A_r t| \le \frac{1}{(nr^2)^{c_1/2c_0}} A_r t.$$

Proof. Noticing that $Z_t(u) = \sum_{i=1}^t 1_{x_i \in B_r(u)}$ and that $\Pr[1_{x_i \in B_r(u)} = 1] = A_r$, we can obtain the result by simply applying the Chernoff bound.

From the above lemma, we can give a rough bound on $T_t(u)$.

Lemma 5. If $r \ge r_0$, then for any $t \ge t_r$, with probability at least $1 - 4n^{-\ln n}$, we have that

$$\left(1 - \frac{1}{(nr^2)^{c_1/2c_0}}\right)\left(1 + \xi\right)mA_rt \le T_t(u) \le 4\left(1 + \frac{1}{(nr^2)^{c_1/2c_0}}\right)\left(2 + \xi\right)mA_r. \tag{10}$$

Proof. The left inequality is obvious by using the trivial relation that $T_t(u) \ge m(1+\xi)Z_t$ and the bound on Z_t given in Lemma 4.

To see the right inequality, we note that the sum of the degrees of vertices in $B_r(u)$ is equal to the sum of out-degrees of all vertices in $B_r(u)$, which is equal to mZ_t , plus the sum of the in-degrees of vertices in $B_r(u)$, which is at most the sum of out-degrees of all vertices in $B_{2r}(u)$. Therefore, $T_t(u) \leq (m+\delta)Z_t + m\sum_{v\in V_t\cap B_{2r}(u)}1 \leq (2m+\delta)\sum_{v\in V_t\cap B_{2r}(u)}1 \leq (2m+\delta)A_{2r}t(1+\frac{1}{(nr^2)^{c_1/2c_0}}) = 4(2+\xi)mA_rt(1+\frac{1}{(nr^2)^{c_1/2c_0}})$, with probability $1-2n^{-\ln n}$.

4 Small Diameter

It is obvious that the diameter in the base model is at least $\Omega(1/r) = \Omega(n^{1/2}(\ln n)^{-c_0})$ for all $r \geq r_0$, since any vertex can connect nodes that within distance at most r from it and the maximum distance of two vertices is $\Omega(1)$. However, with the addition of the ability to choose uniformly from the subset of previous vertices, the diameter can be reduced to $O(\ln n)$, with high probability. We will use the following classic result on the diameter and the maximum degree of a uniform recursive tree.

Lemma 6. With high probability, the diameter and the maximum degree in a uniform recursive tree is $\Theta(\ln n)$.

Proof. This is a classic result, for which a proof is referred to such as [Pit94, DL95].

Now the diameter of the two generalized models can be bounded as follows.

Proof of Theorem 1. We consider the two models separately.

- 1. In the hybrid model, no matter how the local graph grows, the global graph is the same as the uniform recursive tree, which gives an upper bound $O(\ln n)$ on the diameter of the whole graph.
- 2. For the self-loop model, the constructed tree in the flexible part are restricted to having degree at most δ and thus may be different from a uniform recursive tree. However, by Lemma 6, the maximum degree of a uniform recursive tree is $L \ln n$, where L is the hidden constant in $\Theta(\ln n)$, from which we know that if $\delta \geq L \ln n$, then with high probability, the constructed tree in the flexible part is the same as the uniform recursive tree. Therefore, the diameter of the self-loop model is again upper bounded by $O(\ln n)$. Finally, we note that $\delta = m\xi \geq L \ln n$ is equivalent to $m \geq L \ln n/\xi$, which completes the proof.

This completes the proof of Theorem 1.

5 The Small-Community Phenomenon

In this section, we consider the community structure and we will require that $r=r_0$. We start from the intuition that a group of people close to each other form a good community, which can be thought of geographical communities ([KB10]). In particular, for a node v, we define the R-neighborhood $C_R(v)$ of v to be the set of vertices within distance at most R from v in G_n , i.e., $C_R(v) = B_R(v) \cap V_n$. Let $R_0 = n^{-1/2} (\ln n)^{2c_0}$, we will show that $C_{R_0}(v)$ is a good community. In this section, we will assume that $m \geq K_2(\xi) \ln n$, where $K_2(\xi)$ is some large constant depending on ξ .

Note that given v, the probability that a node generated uniformly at random from S will land in $B_{R_0}(v)$ is $A_{R_0} \sim R_0^2/4 = \frac{(\ln n)^{4c_0}}{4n}$. Using the Chernoff bound, it is easy to show that with high probability, the number of nodes in C_{R_0} is $\Theta((\ln n)^{4c_0})$, which means that the size of such R-neighborhood is small. Now we consider the connectivity of the subgraph induced by $C_{R_0}(v)$.

Lemma 7. In the base model, if $r = r_0 = n^{-1/2} (\ln n)^{c_0}$, then for any $v \in V_n$, the R_0 -neighborhood $C_{R_0}(v)$ induces a connected subgraph in G_n with high probability.

Proof. We will first show that for every v, $C_{r/2}(v)$ induces a connected subgraph in G_n with high probability. The lemma then follows from the fact that any two vertices u, u' in $C_{R_0}(v)$ can be connected by a set of paths between vertices $u = v_1, v_2, \dots, v_k = u'$ such that each vertex pair (v_i, v_{i+1}) is within distance r/2.

Now we consider the connectivity of $C_{r/2}(v)$.

Let $A_rT=12\ln n$, and thus $T=\frac{12\dot{n}}{(\ln n)^{2c_0-1}}$. Let H_0 be the subgraph induced by nodes within distance at most r/2 from v at time T. Now let x_{t_1}, \dots, x_{t_k} be the nodes that land in $B_{r/2}(v)$ after time T and let H_s be the corresponding subgraph when vertex x_{t_s} is added in $B_{r/2}(v)$. Since every vertex x_j will land in $B_{r/2}(v)$ with probability $A_{r/2}$, we know that with high probability, for $t \geq T$, the number of nodes

in $B_{r/2}(v)$ will be in the range $[\kappa_1 A_{r/2}t, \kappa_2 A_{r/2}t]$ for some constants κ_1, κ_2 . In particular, we have that $|H_0| \leq \kappa_2 A_{r/2}T = 3\kappa_2 \ln n$ and $\kappa_1 A_{r/2}t_s \leq |H_s| \leq \kappa_2 A_{r/2}t_s$.

Now let X_s be the number of connected components of H_s and let Y_s be the number of connected components of H_s connected to $x_{t_{s+1}}$. Then we have

$$X_{s+1} = X_s - Y_s + 1, X_0 \le 3\kappa_2 \ln n .$$

We show that if $s \leq 6\kappa_2 \ln n$, X_s decreases by at least 1 for every $s \geq 1$ with probability at least $\frac{7}{10}$, from which we know that the probability that $H_{6\kappa_2 \ln n}$ is not connected is bounded by $O(n^{-3})$ and then the Lemma follows from the fact that each later coming vertex $x_{t_{s+1}}$ such that $s \geq 6\kappa_2 \ln n$ will connect the H_s with probability at least $1 - O(n^{-10})$.

Let \mathcal{E} denote the event that for any $u \in V_n$ and for each $t \geq T$, $T_t(u) \leq 32(2+\xi)mA_{r/2}t$, then as in the proof Lemma 5, the probability that \mathcal{E} holds is $1 - O(n^{-4})$. Now Conditioned on \mathcal{E} , for each $1 \leq s \leq 6\kappa_2 \ln n$, since x_{t_s} is in $B_{r/2}(v)$, we have that $|x_{t_s} - u| \leq r$ for every vertex $u \in H_{s-1}$ and thus x_{t_s} will connect u with probability at least

$$\frac{m+\delta}{T_{t_s-1}(x_{t_s})} \ge \frac{1}{32A_{r/2}t_s} .$$

Therefore, the probability that x_{t_s} will not connect any vertex in $B_{r/2}(v)$ is

$$\Pr[Y_s = 0] \le (1 - \frac{|H_s|}{32A_{r/2}t_s})^m \le n^{-10}$$
,

where the last inequality follows from the fact that $m \geq K_2(\xi) \ln n$.

Now we consider the case that H_s has at least two connected components, namely, $X_s \geq 2$. The probability that $x_{t_{s+1}}$ will connect at most one component is that

$$\Pr[Y_s = 1 | X_s \ge 2] \le 2(1 - \frac{1}{32A_{r/2}t_s})^m \le 1/10$$
,

where we used the fact that $32A_{r/2}t_s \ge 96 \ln n$ and that $m \ge K_2(\xi) \ln n$.

Therefore, X_s decreases by at least 1 for every $1 \le s \le 6\kappa_2 \ln n$ with probability at least $\frac{7}{10}$, which completes our proof.

Now we show that the conductance of $C_{R_0}(v)$ in each model is small.

Lemma 8. In both the hybrid model and the self-loop model, with high probability, for any $v \in V_n$, we have that

$$\Phi(C_{R_0}(v)) = O\left(\frac{1}{|C_{R_0}(v)|^{1/4c_0}}\right) . \tag{11}$$

Proof. We first consider the hybrid model. For convenience, we abbreviate $C_{R_0}(v)$ as C. Let $e(C, \bar{C})$ denote the set of edges that connecting C and its complement. Let $e_1(C, \bar{C})$ and $e_2(C, \bar{C})$ denote edges in $e(C, \bar{C})$ that are local and long, respectively. Then we have: $e(C, \bar{C}) = e_1(C, \bar{C}) \cup e_2(C, \bar{C})$.

Local edges connecting C and \bar{C} must lie between the two spherical segments separated by the boundary of $C_{R_0}(v)$. More specifically, if $e = (u, w) \in e_1(C, \bar{C})$, then one of u, w lies on the strip $str_1 = B_{R_0+r}(v) \setminus B_{R_0}(v)$ and the other point lies on the strip $str_2 = B_{R_0-r}(v) \setminus B_{R_0}(v)$. With high probability, the total number of vertices in str_1 is at most $n(2rR_0+r^2)$ and the total number of vertices in str_2 is at most $n(2rR_0-r^2)$. Hence the number of local edges that lies between the two strips is at most $4mnrR_0$, namely, $|e_1(C,\bar{C})| \leq 4mnrR_0$.

Now we consider the long edges that connects C and the rest of the graph. We will show that the number of such edges is relatively small compared with the local edges therein. More precisely, we have the following lemma.

Lemma 9. In the hybrid model, let Y_t denote the sum of the long-degrees of vertices in $B_{R_0}(v) \cap V_t$. Then $Y_n \leq cA_{R_0}n$ for some constant c, with high probability.

Proof. By definition, we have the following recurrence for Y_t .

$$E[Y_{t+1}|Y_t] = Y_t + A_{R_0} + \frac{|B_{R_0}(v) \cap V_t|}{t} . \tag{12}$$

Let $A_{R_0}T = 12 \ln n$, and thus $T = \frac{12n}{(\ln n)^{4c_0-1}}$. Let \mathcal{F} denote the event that for all $t \geq T$, the relation $|B_{R_0}(v) \cap V_t| \in [\kappa_1 A_{R_0} t, \kappa_2 A_{R_0} t]$ holds for some constants κ_1 and κ_2 and that the maximum long-degree of vertices x_1, \dots, x_T is $L \ln n$. By Lemma 6 and the Chernoff bound, we know that $\Pr[\mathcal{F}] \geq 1 - O(n^{-3})$.

Now we know that for $t \geq T$,

$$E[Y_{t+1}|Y_t, \mathcal{F}] \le Y_t + A_{R_0} + \frac{\kappa_2 A_{R_0} t}{t}$$
,

from which we have

$$E[Y_{t+1}|Y_t, \mathcal{F}] - (1+\kappa_2)A_{R_0}(t+1) \le Y_t - (1+\kappa_2)A_{R_0}t . \tag{13}$$

Conditioned on \mathcal{F} , we know that the number of vertices in $B_{R_0}(v) \cap V_t$ is $\kappa_2 A_{R_0} T \leq 12\kappa_2 \ln n$ and every vertex in this set has degree at most $L \ln n$, from which we know that $Y_T \leq 12\kappa_2 L(\ln n)^2$. Now define

$$X_{\tau} = \left\{ \begin{array}{ll} Y_{\tau} - (1 + \kappa_2) A_{R_0} \tau & \text{for } \tau \geq T + 1 \\ 12 \kappa_2 L (\ln n)^2 & \text{for } \tau = T \end{array} \right.,$$

By inequality (13), X_T, \dots, X_t forms a submartingale with error $O(n^{-3})$. We also have that for $\tau > T$,

$$X_{\tau} - \mathrm{E}[X_{\tau}|X_{\tau-1}] \le 1 ,$$

and

$$\begin{aligned}
\operatorname{Var}[X_{\tau}|X_{\tau-1}] &= \operatorname{Var}[Y_{\tau}|X_{\tau-1}] \\
&\leq \operatorname{E}[(Y_{\tau} - Y_{\tau-1})^{2}|X_{\tau-1}] \\
&\leq (1 + \kappa_{2})A_{R_{0}}.
\end{aligned}$$

Now we apply the submartingale concentration inequality as in Lemma 2, we have that

$$\Pr[X_t \ge X_T + \lambda] \le e^{-\frac{\lambda^2}{2(\sum_{\tau=T+1}^t (1+C_2)A_{R_0} + \lambda/3)}} + O(n^{-3})$$

$$\le e^{-\frac{\lambda^2}{2t(1+C_2)A_{R_0} + 2\lambda/3}} + O(n^{-3}).$$

Let $\lambda = c' \sqrt{\ln n A_{R_0} t}$ for some constant c'. Then

$$\Pr[X_t \ge X_T + c'\sqrt{\ln n A_{R_0} t}] \le O(n^{-3})$$
.

Finally, using $X_t = Y_t - (1 + \kappa_2) A_{R_0} t$, we have

$$\Pr[Y_t \ge (1 + \kappa_2) A_{R_0} t + c' \sqrt{\ln n A_{R_0} t} + 12\kappa_2 L(\ln n)^2] \le O(n^{-3}).$$

In particular, with high probability $Y_n \leq cA_{R_0}n$ for some constant c, which completes the proof.

By Lemma 9, we know that $|e_2(C, \bar{C})| \leq cA_{R_0}n$. Thus, the total number of edges between C and \bar{C} is

$$|e(C,\bar{C})| = O(mrR_0n + R_0^2n)$$
 (14)

The volume of C is at least $m|C| \sim mR_0^2 n$, which means that

$$\Phi(C) = O\left(\frac{m4rR_0n + R_0^2n}{mR_0^2n}\right) = O((\ln n)^{-1}) = O\left(\frac{1}{|C|^{1/(4c_0)}}\right) . \tag{15}$$

Finally, we briefly discuss the case in the self-loop model. Let $\delta \geq K_1(\xi) \ln n$, then with high probability, the constructed tree in the flexible part of the model is a uniform recursive tree as the same as that in the proof of Theorem 1. Therefore, the edges that connect an R-neighborhood and its complement can be also bounded by the same argument as that in the case of the hybrid model, which then gives the same result as (15).

Now we can show that the two models have the small-community phenomenon.

Proof of Theorem 2. For each $v \in V_n$, the R_0 -neighborhood $C_{R_0}(v)$ is of size $\Theta((\ln n)^{4c_0})$. By Lemmas 7 and 8, we know that $C_{R_0}(v)$ is an (α, β, γ) -community of v, where α is the hidden constant in term $O\left(\frac{1}{|C_{R_0}(v)|^{1/4c_0}}\right)$ in Eq. (15), $\beta = 1/4c_0$, and $\gamma = 4c_0$. This completes the proof of Theorem 2.

Note that the proof of Theorem 2 also implies that the base model G_t has the small-community phenomenon. In fact, in this case, we do not need to consider the effect of the edges generated in the uniform recursive tree, which simplifies the analysis. We can easily show that the R_0 -neighborhood $C_{R_0}(v)$ has small size, induces a connected subgraph and has conductance $\Phi(C_{R_0}(v)) = O(\frac{m4rR_0n}{mR_0^2n}) = O((\ln n)^{-c_0}) = O(\frac{1}{|C_{R_0}(v)|^{1/4}}) \le \frac{\alpha'}{|C_{R_0}(v)|^{1/4}}$, i.e., every node in the base model is contained in a $(\alpha', 1/4, 4c_0)$ -community.

6 The Power Law Degree Distribution

In this section we prove Theorems 3 and 4. In Subsection 6.1, we prove Theorem 3 by assuming a concentration inequality of the degree sequence, in Subsection 6.2, we develop an *alternating concentration method* to prove the concentration inequality desired, and in Subsection 6.3, we prove Theorem 4.

6.1 The degree sequence on the base model

To prove Theorem 3, we analyze a recurrence on $E[d_k(t)]$ as usual. Recall that $T_t(u) = \sum_{v \in B_r(u) \cap V_t} (\deg_t(v) + \delta)$. As mentioned above, we will first give a good estimation of $T_t(u)$ and show that $T_t(u)$ concentrates around its expectation, building on which we can derive the degree sequence from the recurrence on $E[d_k(t)]$.

its expectation, building on which we can derive the degree sequence from the recurrence on $E[d_k(t)]$. Recall that $t_r = \frac{12(\ln n)^2 n^{c_1/c_0}}{r^{2(1-c_1/c_0)}}$ for any $r \ge r_0$. We have the following concentration inequality of $T_t(u)$:

Lemma 10. (Alternating Concentration Theorem) If $r \geq r_0$, then for all $t \geq t_r$, we have that

$$\Pr[|T_t(u) - (2+\xi)mA_r t| \ge \frac{1}{(nr^2)^{c_2/2c_0}} mA_r t] = O(n^{-2}) , \qquad (16)$$

where c_1, c_2 are some constants satisfying the conditions in Eq. (7) and (8).

Lemma 10 is one of our key technical contributions in this paper which is interesting by its own. To prove it, we will need to develop an *alternating concentration method*, allowing us to alternatively and iteratively apply both the submartingale and supermartingale inequalities to prove a desired concentration result. The full proof of Lemma 10 is given in Subsection 6.2.

The role of Lemma 10 is to give a good estimation of $\mathrm{E}\left[\frac{1_{|x_{t+1}-v|\leq r}}{T_t(x_{t+1})}|G_t\right]$ to analyze the recurrence of $\mathrm{E}[d_k(t)]$. In this subsection, we prove Theorem 3 by assuming Lemma 10 as follows.

Proof of Theorem 3. Define $D_k(t) := \{v \in V(G_t) | \deg_{G_t}(v) = k\}$. Then $d_k(t) = |D_k(t)|$. The recurrence for the expectation of $d_k(t)$ can be written as follows.

$$E[d_{k}(t+1)|G_{t}] = d_{k}(t) + \sum_{v \in D_{k-1}(t)} \left(mE\left[\frac{(k-1+\delta)1_{|x_{t+1}-v| \le r}}{T_{t}(x_{t+1})}|G_{t}\right] \right) - \sum_{v \in D_{k}(t)} \left(mE\left[\frac{(k+\delta)1_{|x_{t+1}-v| \le r}}{T_{t}(x_{t+1})}|G_{t}\right] \right) + O(mE[\eta_{k}(G_{t}, x_{t+1})|G_{t}]) ,$$

$$(17)$$

where $\eta_k(G_t, x_{t+1})$ denotes the probability that a parallel edge from the new vertex x_{t+1} to a vertex of degree no more than k is created, which is at most

$$\binom{m}{2} \sum_{i=m}^{k} \sum_{v \in D_{i}(t)} (i+\delta)^{2} \left(\frac{1_{|v-x_{t+1}| \le r}}{T_{t}(x_{t+1})}\right)^{2}.$$

Now for $t \ge t_r$, let \mathcal{A}_t denote the event that $|T_t(u) - (2+\xi)mA_rt| \le \frac{1}{(nr^2)^{c_2/2c_0}} mA_rt$. By Lemma 10, we have

$$\Pr[\mathcal{A}_t] = 1 - O(n^{-2})$$

Therefore for $t \geq t_r$,

$$E\left[\sum_{v \in D_{k}(t)} \frac{(k+\delta)1_{|x_{t+1}-v| \leq r}}{T_{t}(x_{t+1})}\right]$$

$$= E\left[\sum_{v \in D_{k}(t)} \frac{(k+\delta)1_{|x_{t+1}-v| \leq r}}{(2+\xi)mA_{r}t} \left(1 + O(\frac{1}{(nr^{2})^{c_{2}/2c_{0}}})\right) |\mathcal{A}_{t}\right] \Pr[\mathcal{A}_{t}] + O(n^{-2})$$

$$= \frac{(k+\delta)}{(2+\xi)mt} \left(1 + O(\frac{1}{(nr^{2})^{c_{2}/2c_{0}}})\right) \operatorname{E}[d_{k}(t)|\mathcal{A}] \Pr[\mathcal{A}] + O(n^{-2})$$

$$= \frac{(k+\delta)}{(2+\xi)mt} \left(1 + O(\frac{1}{(nr^{2})^{c_{2}/2c_{0}}})\right) \left(\operatorname{E}[d_{k}(t)] - \operatorname{E}[d_{k}(t)|\neg\mathcal{A}] \Pr[\neg\mathcal{A}]\right) + O(n^{-2})$$

$$= \frac{(k+\delta)\operatorname{E}[d_{k}(t)]}{(2+\xi)mt} + O(\frac{1}{(nr^{2})^{c_{2}/2c_{0}}}).$$

Similarly, we have

$$E\left[\sum_{v \in D_{k-1}(t)} \frac{(k-1+\delta)1_{|x_{t+1}-v| \le r}}{T_t(x_{t+1})}\right] \\
= \frac{(k-1+\delta)E[d_{k-1}(t)]}{(2+\xi)mt} + O(\frac{1}{(nr^2)^{c_2/2c_0}}).$$

The error term can be bounded as follows.

$$E[\eta_k(G_t, x_{t+1})] \le {m \choose 2} E\left[\sum_{i=m}^k \sum_{v \in D_i(t)} (k+\delta)^2 \left(\frac{1_{|v-x_{t+1}| \le r}}{T_t(x_{t+1})}\right)^2\right].$$

$$\leq \binom{m}{2} E \left[\sum_{i=m}^{k} \sum_{v \in D_i(t)} (k+\delta)^2 \frac{1}{m^2 A_r t^2} \left(1 + O(\frac{1}{(nr^2)^{c_2/2c_0}}) \right) \right] + O(n^{-2}) \\
\leq O\left(\frac{(k+\delta)^2}{A_r t} \right) + O(n^{-2}) .$$

If $k + \delta \le k_0(t) = (nr^2)^{c_1/2c_0 - c_2/4c_0}$, then $\mathbb{E}[\eta_k(G_t, x_{t+1})] = O(\frac{1}{(\ln n)^2(nr^2)^{c_2/2c_0}})$ and $\mathbb{E}[m\eta_k(G_t, x_{t+1})] = O(\frac{1}{(nr^2)^{c_2/2c_0}})$ given the fact that $m = O(\ln^2 n)$.

Let $\bar{d}_k(t) := E[d_k(t)]$. Now the recurrence can be simplified as

$$\bar{d}_{k}(t+1) = \bar{d}_{k}(t) - \frac{(k+\delta)\bar{d}_{k}(t)}{(2+\xi)t} + \frac{(k-1+\delta)\bar{d}_{k-1}(t)}{(2+\xi)t} + 1_{k=m} + O(\frac{1}{(nr^{2})^{c_{2}/2c_{0}}}) .$$
(18)

We define a new recurrence related to (18). For j < m, let $f_j = 0$ and for $j \ge m$, let

$$f_k = \frac{k-1+\delta}{2+\xi} f_{k-1} - \frac{k+\delta}{2+\xi} f_k + 1_{d=m} , \qquad (19)$$

which has solution that $f_m = \frac{2+\xi}{2+\xi+m+\delta}$ and for $k \geq m+1$,

$$f_k = \prod_{j=m+1}^k \frac{j-1+\delta}{2+\xi+j+\delta} f_m$$

$$= \frac{\Gamma(k+\delta)\Gamma(m+4+\xi+\delta)}{\Gamma(3+\xi+k+\delta)\Gamma(m+1+\delta)} \frac{2+\xi}{2+\xi+m+\delta}$$

$$= \frac{\phi_k(m,\delta)}{k^{3+\xi}} ,$$

where $\phi_k(m, \delta)$ tends to a limit $\phi_{\infty}(m, \delta)$ which depends on m, δ only as $k \to \infty$. Now we show that

$$|\bar{d}_k(t) - f_k t| \le M \left(t_r + \frac{n + Lt}{(nr^2)^{c_2/2c_0}} \right)$$
 (20)

where M is some large constant and L is the hidden constant in term $O(\frac{1}{(nr^2)^{c_2/2c_0}})$ in Eq. (18). We prove (20) by induction.

- 1. First note that for $t \leq t_r$, the above relation holds trivially; for $t \geq t_r$ and $k \geq k_0(t)$, the inequality follows from the fact that $\bar{d}_k(t) \leq 2mt/k$.
- 2. Now assume that $t \geq t_r$ and $k \leq k_0(t)$, we have

$$|\bar{d}(k+1) - f_k(t+1)|$$

$$= |\bar{d}_k(t) - f_k(t+1) - \frac{(k+\delta)\bar{d}_k(t)}{(2+\xi)t} + \frac{(k-1+\delta)\bar{d}_{k-1}(t)}{(2+\xi)t} + O(\frac{1}{(nr^2)^{c_2/2c_0}})|$$

$$= |\bar{d}_k(t) - f_k t - \left(\frac{k-1+\delta}{2+\xi}f_{k-1} - \frac{k+\delta}{2+\xi}f_k\right)$$

$$- \frac{(k+\delta)\bar{d}_k(t)}{(2+\xi)t} + \frac{(k-1+\delta)\bar{d}_{k-1}(t)}{(2+\xi)t} + O(\frac{1}{(nr^2)^{c_2/2c_0}})|$$

$$\leq \left(1 - \frac{k+\delta}{(2+\xi)t}\right) |\bar{d}_k(t) - f_k t| + \frac{k-1+\delta}{(2+\xi)t} |d_{k-1}(t) - f_{k-1}t| + O(\frac{1}{(nr^2)^{c_2/2c_0}}) \\
\leq M \left(t_r + \frac{n+Lt}{(nr^2)^{c_2/2c_0}}\right) + L \frac{1}{(nr^2)^{c_2/2c_0}} \\
\leq M \left(t_r + \frac{n+L(t+1)}{(nr^2)^{c_2/2c_0}}\right) .$$

This completes the induction and the proof of Theorem 3.

6.2 Estimation of $T_t(u)$ - Alternating Concentration Analysis

In this subsection, we prove the Alternating Concentration Theorem, that is, Lemma 10.

As mentioned above, Flaxman et al. [FFV07a, FFV07b] and van den Esker [vdE08] introduced a new parameter $\alpha > 2$ to facilitate the analysis and they used the traditional coupling technique to bound $T_t(u)$. In our settings, we do not use the additional parameter α and we can still get a nice bound. Our idea is to develop a refined method based on the recurrence directly implied in the definition of $T_t(u)$. By using this recurrence, we can start from the weak bound as given in Lemma 5, and iteratively improve both the upper bound and the lower bound of $T_t(u)$. This improvement can be done by using the submartingale and supermartingale concentration inequalities as in the proof of Lemma 9. This allows us to show that the accumulated error in the whole process is small and therefore guarantees the desired bound.

At first, we show that a lower bound can be achieved from a rough lower bound on $T_t(u)$.

Lemma 11. Fix $r \geq r_0$. If for any $t \geq t_r$,

$$\Pr[T_t(u) \le (b_l - \frac{r_l}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t] \le \epsilon_l , \qquad (21)$$

for some $b_l \in [1/2, 1)$ and $r_l = o((nr^2)^{c_1/2c_0})$, then for any $t \ge t_r$,

$$\Pr[T_t(u) \ge (b_u + \frac{r_u}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t] \le \epsilon_u , \qquad (22)$$

where $b_u = \frac{\xi+1}{2+\xi-\frac{1}{b_l}} \in (1,\infty)$, $r_u = 7 + 40r_l/\xi$, $\epsilon_u = n\epsilon_l + 5n^{-\ln n+1}$ and c_1 is some constant satisfying the condition given in Eq. (7).

Proof of Lemma 11. We will mainly use the following recurrence.

$$\begin{split} \mathbf{E}[T_{t+1}(u)|G_t] &= T_t(u) + m(1+\xi)\mathbf{E}[\mathbf{1}_{|x_{t+1}-u| \le r}|G_t] \\ &+ \sum_{v \in V} m \Pr[y_i^{t+1} = v|G_t] \mathbf{1}_{|u-v| \le r} \ , \end{split}$$

where

$$\Pr[y_i^{t+1} = v | G_t] = \operatorname{E}\left[\frac{(\deg_t(v) + \delta) 1_{|x_{t+1} - v| \le r}}{T_t(x_{t+1})} | G_t\right].$$

Let \mathcal{G} denote the event that for all $t \geq t_r$, the following inequalities hold: $T_t(u) \geq (b_l - \frac{r_l}{(nr^2)^{c_1/2c_0}})(2 + \xi)mA_rt$ and $(1 - \frac{1}{(nr^2)^{c_1/2c_0}})(1 + \xi)mA_rt \leq T_t(u) \leq 4(2 + \xi)mA_r(1 + \frac{1}{(nr^2)^{c_1/2c_0}})$. Then by Lemma 5 and the bound given in (21), $\Pr[\neg \mathcal{G}] \leq n\epsilon_l + 4n^{-\ln n+1}$. Conditioned on \mathcal{G} , for $t \geq t_r$, we have

$$\Pr[y_i^{t+1} = v | G_t, \mathcal{G}] \leq \operatorname{E}\left[\frac{(\deg_t(v) + \delta) 1_{v \in B_r(x_{t+1})}}{(b_l - \frac{r_l}{(nr^2)^{c_1/2c_0}})(2 + \xi) m A_r t} | G_t, \mathcal{G}\right]$$

$$= \frac{\deg_t(v) + \delta}{(b_l - \frac{r_l}{(nr^2)^{c_1/2c_0}})(2 + \xi) m t}$$

$$\leq \frac{\deg_t(v) + \delta}{b_l(2 + \xi) m t} (1 + \frac{4r_l}{(nr^2)^{c_1/2c_0}}).$$

Therefore,

$$E[T_{t+1}(u)|G_t,\mathcal{G}] \leq T_t(u) + m(1+\xi)A_r + \frac{1}{b_l(2+\xi)t}(1 + \frac{4r_l}{(nr^2)^{c_1/2c_0}})T_t(u)$$

$$\leq (1 + \frac{1}{b_l(2+\xi)t})T_t(u) + (\xi + 1 + \frac{40r_l}{(nr^2)^{c_1/2c_0}})mA_r,$$

where the second inequality uses the rough upper bound on $T_t(u)$ in Lemma 5.

Let $b_u = \frac{\xi+1}{2+\xi-\frac{1}{h_l}}$, and $s = 40r_l/\xi$, then

$$E[T_{t+1}(u)|G_t,\mathcal{G}] - (b_u + \frac{s}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r(t+1)$$

$$\leq (1 + \frac{1}{b_l(2+\xi)t}) \Big(T_t(u) - (b_u + \frac{s}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t \Big)$$

$$+ \Big(\frac{b_u}{b_l} + \xi + 1 - b_u(2+\xi) + (2s + 40r_l - s(2+\xi)) \frac{1}{(nr^2)^{c_1/2c_0}} \Big) mA_r$$

$$\leq (1 + \frac{1}{b_l(2+\xi)t}) \Big(T_t(u) - (b_u + \frac{s}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t \Big) .$$
(23)

Now define

$$X_i = \begin{cases} \frac{T_i(u) - (b_u + \frac{s}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_ri}{\prod_{j=t_r}^{i-1}(1 + \frac{1}{b_l(2+\xi)j})} & \text{for } i > t_r \\ T_{t_r}(u) - (b_u + \frac{s}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_rt_r & \text{for } i = t_r \end{cases}.$$

From inequality (27), we know that $E[X_i|G_{i-1},\mathcal{G}] \leq X_{i-1}$ for $t_r < i \leq t$. Let $\Delta_i = \prod_{j=t_r}^i (1 + \frac{1}{b_l(2+\xi)j}) \sim (\frac{i}{t_r})^{1/b_l(2+\xi)}$. We have that

$$X_i - \mathrm{E}[X_i|G_{i-1},\mathcal{G}] = \frac{T_i(u) - \mathrm{E}[T_i(u)|G_{i-1},\mathcal{G}]}{\Delta_{i-1}} \le (2+\xi)m$$
,

and

$$\operatorname{Var}[X_{i}|G_{i-1},\mathcal{G}] = \frac{\operatorname{Var}[T_{i}(u)|G_{i-1},\mathcal{G}]}{\Delta_{i-1}^{2}} \\
\leq \frac{\operatorname{E}[(T_{i}(u) - T_{i-1}(u))^{2}|G_{i-1},\mathcal{G}]}{\Delta_{i-1}^{2}} \\
\leq (2 + \xi)m \frac{\frac{T_{i-1}(u)}{b_{l}(2+\xi)(i-1)} + (\xi + 1 + \frac{40r_{l}}{(nr^{2})^{c_{1}/2c_{0}}})mA_{r}}{\Delta_{i-1}^{2}} \\
\leq \frac{(\xi + 3)^{2}m^{2}A_{r}}{\Delta_{i-1}^{2}} .$$

Therefore, the sequence X_{t_r}, \dots, X_t satisfies the conditions in Lemma 2 with $\Pr[\neg \mathcal{G}] \leq n\epsilon + 4n^{-\ln n + 1}$ and

$$\sum_{i=t_r+1}^{t} \operatorname{Var}[X_i|G_{i-1}, \mathcal{G}]$$

$$\leq \sum_{i=t_r+1}^{t} \frac{(\xi+3)^2 m^2 A_r}{\Delta_{i-1}^2}$$

$$\leq \sum_{i=t_r+1}^{t} \frac{(\xi+3)^2 m^2 A_r t_r^{2/b_l(2+\xi)}}{i^{2/b_l(2+\xi)}}$$

$$\leq (\frac{mA_r t_r}{\ln n})^2.$$
(24)

The last inequality can be seen by using the fact that $A_r t_r \sim 3(\ln n)^2 (nr^2)^{c_1/c_0}$, the assumption that $(c_0 - c_1 - 1)(1 - 1/(\xi + 2)) < c_1$. Specifically,

1. if $2/b_l(\xi + 2) = 1$, then

$$\sum_{i=t_{+}+1}^{t} \frac{(\xi+3)^{2} m^{2} A_{r} t_{r}^{2/b_{l}(2+\xi)}}{i^{2/b_{l}(2+\xi)}} \leq O(m^{2} A_{r} t_{r} \ln(t/t_{r})) = O(\frac{m^{2} A_{r}^{2} t_{r}^{2}}{A_{r} t_{r} / \ln \ln n}) \leq (\frac{m A_{r} t_{r}}{\ln n})^{2}.$$

2. if $2/b_l(\xi+2) > 1$, then

$$\sum_{i=t_r+1}^t \frac{(\xi+3)^2 m^2 A_r t_r^{2/b_l(2+\xi)}}{i^{2/b_l(2+\xi)}} \le O(m^2 A_r t_r) = O(\frac{m^2 A_r^2 t_r^2}{A_r t_r}) \le (\frac{m A_r t_r}{\ln n})^2.$$

3. if $2/b_l(\xi + 2) < 1$, then

$$\begin{split} \sum_{i=t_r+1}^t \frac{(\xi+3)^2 m^2 A_r t_r^{2/b_l(2+\xi)}}{i^{2/b_l(2+\xi)}} &\leq O(m^2 A_r t (\frac{t_r}{t})^{2/b_l(\xi+2)}) = O(\frac{m^2 A_r^2 t_r^2}{A_r t_r (\frac{t_r}{t})^{1-2/b_l(\xi+2)}}) \\ &\leq \frac{m^2 A_r^2 t_r^2}{3(\ln n)^{2+2(1-\frac{2}{b_l(\xi+2)})} (nr^2)^{c_1/c_0 + (c_1/c_0-1)(1-\frac{2}{b_l(\xi+2)})}} \\ &\leq \frac{m^2 A_r^2 t_r^2}{3(\ln n)^{2+2(1-\frac{2}{b_l(\xi+2)})} (nr^2)^{c_1/c_0 + (c_1/c_0-1)(1-\frac{2}{(\xi+2)})}} \leq (\frac{m A_r t_r}{\ln n})^2 \ . \end{split}$$

If we let $\lambda = 2mA_rt_r$, then using the submartingale concentration inequality, we have

$$\begin{aligned} & \Pr[X_t \geq X_{t_r} + \lambda] \\ & \leq & e^{-\frac{\lambda^2}{2\sum_{j=t_r+1}^t \operatorname{Var}[X_i|G_{i-1},\mathcal{G}] + 2(2+\xi)m\lambda/3}} + \Pr[\neg \mathcal{G}] \\ & \leq & n\epsilon_l + 5n^{-\ln n + 1} \end{aligned}$$

On the other hand, we have that $X_{t_r} \leq 5mA_rt_r$ conditioned on \mathcal{G} . Thus, $\Delta_{t-1}(X_{t_r}+\lambda) \leq 7(\frac{t}{t_r})^{1/b_l(2+\xi)}mA_rt_r = 7(\frac{t_r}{t})^{1-1/b_l(2+\xi)}mA_rt \leq \frac{7}{(nr^2)^{c_1/2c_0}}mA_rt$, where the last inequality follows from the assumption that $(2c_0 - 2c_1 - 2)(1 - 2/(2 + \xi)) > c_1$. Therefore,

$$\Pr[T_{t}(u) \geq (b_{u} + \frac{s+7}{(nr^{2})^{c_{1}/2c_{0}}})(2+\xi)mA_{r}t]$$

$$\leq \Pr[\frac{T_{t}(u) - (b_{u} + \frac{s}{(nr^{2})^{c_{1}/2c_{0}}})(2+\xi)mA_{r}t}{\Delta_{t-1}} \geq X_{t_{r}} + \lambda]$$

$$\leq \Pr[X_{t} \geq X_{t_{r}} + \lambda]$$

$$\leq n\epsilon_{l} + 5n^{-\ln n+1}.$$

The proof completes by letting $r_u = 7 + 40r_l/\xi$ and $\epsilon_u = n\epsilon_l + 5n^{-\ln n + 1}$.

Similarly, from a rough upper bound, we can obtain an upper bound on $T_t(u)$.

Lemma 12. Fix $r \ge r_0$. If for any $t \ge t_r$,

$$\Pr[T_t(u) \ge (b_u + \frac{r_u}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t] \le \epsilon_u, \tag{25}$$

for some $b_u \in (1,4)$ and $r_u = o((nr^2)^{c_1/2c_0})$, then for any $t \ge t_0$,

$$\Pr[T_t(u) \le (b_l - \frac{r_l}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t] \le \epsilon_l, \tag{26}$$

where $b_l = \frac{\xi+1}{2+\xi-\frac{1}{b_u}} \in (1/2,1)$, $r_l = 7 + 40r_u/\xi$, $\epsilon_l = n\epsilon_u + 5n^{-\ln n+1}$, and c_1 is some constant satisfying the condition given in Eq. (7).

Proof. The proof here is similar to proof of Lemma 11. Note that we should instead use the supermartingale concentration inequality and let \mathcal{G}' denote the good event defined similar to \mathcal{G} in the above proof, which will lead to the following recurrence.

$$E[T_{t+1}(u)|G_t, \mathcal{G}'] \geq T_t(u) + m(1+\xi)A_r + \frac{1}{b_u(2+\xi)t}(1 - \frac{r_u}{(nr^2)^{c_1/2c_0}})T_t(u)$$

$$\geq (1 + \frac{1}{b_u(2+\xi)t})T_t(u) + (\xi + 1 - \frac{5r_u}{(nr^2)^{c_1/2c_0}})mA_r.$$

Let $b_l = \frac{\xi + 1}{2 + \xi - \frac{1}{h}}$, and $s' = 40r_u/\xi$, then

$$E[T_{t+1}(u)|G_t, \mathcal{G}'] - (b_l - \frac{s'}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r(t+1)$$

$$\geq (1 + \frac{1}{b_u(2+\xi)t}) \Big(T_t(u) - (b_l - \frac{s'}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t \Big)$$

$$+ \Big(\frac{b_l}{b_u} + \xi + 1 - b_l(2+\xi) + (-s' - 5r_u + s'(2+\xi)) \frac{1}{(nr^2)^{c_1/2c_0}} \Big) mA_r$$

$$\geq (1 + \frac{1}{b_u(2+\xi)t}) \left(T_t(u) - (b_l - \frac{s'}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_r t \right) . \tag{27}$$

We then define the corresponding supermartingale X'_{t_r}, \dots, X'_t using the above inequality. In this case, we will also use the conditions (7) on the constants c_0 and c_1 . Then by setting $\lambda' = \frac{29}{4} m A_r t$, where λ' corresponds to the parameter λ in Lemma 11, and using $X'_{t_r} \geq \frac{1}{4} m A_r t_r$, we will get

$$\Pr[T_{t}(u) \leq (b_{l} - \frac{s' + 7}{(nr^{2})^{c_{1}/2c_{0}}})(2 + \xi)mA_{r}t]$$

$$\leq \Pr[\frac{T_{t}(u) - (b_{l} - \frac{s'}{(nr^{2})^{c_{1}/2c_{0}}})(2 + \xi)mA_{r}t}{\Delta_{t-1}} \leq X'_{t_{r}} - \lambda']$$

$$\leq \Pr[X'_{t} \leq X'_{t_{r}} - \lambda']$$

$$\leq n\epsilon_{u} + 5n^{-\ln n + 1}.$$

and complete the proof by letting $r_l = 7 + 40r_u/\xi$ and $\epsilon_l = n\epsilon_u + 5n^{-\ln n + 1}$.

Now we are ready to prove Lemma 10. Intuitively, we will iteratively apply the above two lemmas and show that if we start with a rough lower bound l_1 , then by Lemma 11, we can get an upper bound u, from which we can again get a new lower bound l_2 by Lemma 12. We prove that $l_2 > l_1$, which means that we get a better lower bound in every iteration. The same holds for the upper bound.

Proof of Lemma 10. If $\frac{\xi+1}{2+\xi-\frac{2+\xi}{1+\xi}} > 4$, then we start our iterative process from the rough upper bound in Lemma 5. Otherwise, we start the process from the rough lower bound.

Assume we start from the rough lower bound, and the case of starting from the rough upper bound is similar. By Lemma 5, we know that for all $t \ge t_r$, $T_t(u) \ge (1 - \frac{1}{(nr^2)^{c_1/2c_0}})(1 + \xi)mA_rt$ with probability at

least $1-4n^{-\ln n}$. We define the start point of our iterative process by letting $b_l^{(1)}=\frac{1+\xi}{2+\xi}\in(1/2,1), r_l^{(1)}=1,$ $\epsilon_l^{(1)}=5n^{-\ln n}$.

For $i \geq 1$, assume that we have that $T_t(u) \geq (b_l^{(i)} - \frac{r_l^{(i)}}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_rt$ with error probability $\epsilon_l^{(i)}$ for any $t \geq t_r$. Now we substitute the corresponding parameters in Lemma 11 to give an upper bound that $T_t(u) \leq (b_u^{(i)} + \frac{r_u^{(i)}}{(nr^2)^{c_1/2c_0}})(2+\xi)mA_rt$ for all $t \geq t_r$ with error probability $\epsilon_u^{(i)}$, where $b_u^{(i)} = \frac{1+\xi}{2+\xi-\frac{1}{b_l^{(i)}}} \in (1,4]$, $r_s^{(i)} = (7+40/\xi)r_s^{(i)} > 7+40r_s^{(i)}/\xi$, $\epsilon_s^{(i)} = r_s^{(i)} + 5r_s^{-\ln n+1}$

$$\begin{split} r_u^{(i)} &= (7+40/\xi) r_l^{(i)} \geq 7+40 r_l^{(i)}/\xi, \, \epsilon_u^{(i)} = n \epsilon_l^{(i)} + 5 n^{-\ln n + 1}. \\ & \text{Again we substitute the corresponding parameters in Lemma 12 to give an improved lower bound that} \\ T_t(u) &\geq (b_l^{(i+1)} - \frac{r_l^{(i+1)}}{(nr^2)^{c_1/2c_0}})(2+\xi) m A_r t \text{ for all } t \geq t_r \text{ with error } \epsilon_l^{(i+1)}, \text{ where } b_l^{(i+1)} = \frac{1+\xi}{2+\xi-\frac{1}{b_u^{(i)}}} \in (1/2,1), \\ r_l^{(i+1)} &= (7+40/\xi) r_u^{(i)} \geq 7+40 r_u^{(i)}/\xi, \, \epsilon_l^{(i+1)} = n \epsilon_u^{(i)} + 5 n^{-\ln n + 1}. \\ \text{Let } C(\xi) &= 7+40/\xi. \text{ Then } r_l^{(i+1)} = C(\xi)^2 r_l^{(i)} \text{ and } \epsilon_l^{(i+1)} \leq n^2 \epsilon_l^{(i)} + 10 n^{-\ln n + 2}. \end{split}$$

Now we show that for every $i, b_l^{(i+1)}$ is strictly greater than $b_l^{(i)}$, i.e., the process gives better lower bound after every two consecutive steps. Then by the fact that $b_l^{(i)} < 1$, we have that $\{b_l^{(i)}\}_{i \geq 1}$ converges to 1. Similarly, it can be shown that the procedure gives better upper bound; namely, $\{b_u^{(i)}\}_{i \geq 1}$ is a decreasing sequence which converges to 1. In the following, we actually prove a stronger result that after each iteration, the distance between $b_l^{(i)}$ and 1 decreases by a multiple factor, which guarantees that the $\{b_l^{(i)}\}_{i \geq 1}$ converges quickly to 1.

We calculate the distance between $b_l^{(i+1)}$ and 1, which gives that

$$\begin{array}{rcl} 1-b_l^{(i+1)} & = & 1-\frac{1+\xi}{2+\xi-\frac{1}{b_u^{(i)}}} \\ & = & 1-\frac{1+\xi}{2+\xi-\frac{1}{\frac{1+\xi}{2+\xi-\frac{1}{b_l^{(i)}}}}} \\ & = & \frac{1-b_l^{(i)}}{\xi(2+\xi)b_l^{(i)}+1} \\ & \leq & \frac{1-b_l^{(i)}}{\xi(1+\xi/2)+1}. \end{array}$$

Therefore, the sequence $\{1-b_l^{(i)}\}_{i\geq 1}$ decreases by a multiple factor at least $\frac{1}{\xi(1+\xi/2)+1}$ at each step. On the other hand, since $T_t(u)\geq [1-(1-b_l^{(i)})-\frac{r_l^{(i)}}{(\ln n)^{c_1}}](2+\xi)mA_rt$, the best bound is determined by the maximum of $\frac{r_l^{(i)}}{(\ln n)^{c_1}}$ and $1-b_l^{(i)}$, which is at most $\frac{1/2}{(\xi(1+\xi/2)+1)^i}$. We terminate the iteration at the step $k_0=\lceil\frac{(c_1/2c_0)\ln(nr^2)}{\ln(C(\xi)^2(\xi(1+\xi/2)+1))}\rceil\leq \frac{\ln n}{4}$, in which case $\frac{1/2}{(\xi(1+\xi/2)+1)^{k_0}}\leq \frac{r_l^{(k_0)}}{(nr^2)^{c_1/2c_0}}=\frac{C(\xi)^{2k_0}}{(nr^2)^{c_1/2c_0}}$, and

$$\Pr[T_t(u) \le (1 - \frac{1}{(nr^2)^{c_2/2c_0}})(2 + \xi)mA_r t]$$

$$\le \Pr[T_t(u) \le (1 - \frac{2C(\xi)^{2k_0}}{(nr^2)^{c_1/2c_0}})(2 + \xi)mA_r t]$$

$$\le \epsilon_l^{(k_0)} \le 2n^{2k_0 - \ln n + 2} \le n^{-\ln n/2 + 2},$$

where we used the assumption that $c_2 = c_1 \frac{\ln(\xi(1+\xi/2)+1)}{\ln(C(\xi)^2(\xi(1+\xi/2)+1))}$.

The upper bound can be obtained similarly by noting that the sequence $\{b_u^{(i)}-1\}_{i\geq 1}$ decreases by a

multiple factor at least $\frac{1}{\xi(2+\xi)+1} \le \frac{1}{\xi(1+\xi/2)+1}$ at each step. Hence, we have that

$$\Pr[|T_t(u) - (2+\xi)mA_r t| \ge \frac{1}{(nr^2)^{c_2/2c_0}} mA_r t] \le n^{-2}.$$
(28)

6.3 Power Law Distribution of the Generalized Models

In this subsection, we prove Theorem 4, based on both the result and the proof of Theorem 3.

Proof of Theorem 4. Since the local-degree sequences in the hybrid model is exactly the same as the degree sequences in the base model, by Theorem 3, the local graph of G_n^H has the power law degree distribution. Now for the self-loop model, in which the degree of a node v can be expressed as $\deg_t(v) + \delta$, where

Now for the self-loop model, in which the degree of a node v can be expressed as $\deg_t(v) + \delta$, where $\deg_t(v)$ is the number of non-flexible edges incident to v at time t. Now we can write the recurrence as follows.

$$E[d_{k+\delta}(t+1)|G_t] = d_{k+\delta}(t) + \sum_{v \in D_{k-1+\delta}(t)} \left(m E\left[\frac{(k-1+\delta)1_{|x_{t+1}-v| \le r}}{T_t(x_{t+1})}|G_t\right] \right) - \sum_{v \in D_{k+\delta}(t)} \left(m E\left[\frac{(k+\delta)1_{|x_{t+1}-v| \le r}}{T_t(x_{t+1})}|G_t\right] \right) + O(m E[\eta_k(G_t, x_{t+1})|G_t]), \tag{29}$$

Solving the recurrence, we can also arrive at (19), which gives the solution of the form $\frac{\phi_k'(m,\delta)}{(k+\delta)^{3+\xi}}$, where $\phi_k'(m,\delta)$ tends to a limit $\phi_\infty'(m,\delta)$ which depends only on m,δ as $k\to\infty$. This finishes the proof that the degree sequence of the self-loop model follows a power law distribution.

7 Large Community and Small Expander

In this section, we will prove Theorem 5.

Before proving the result, we give a brief discussion on the choice of r. In the previous sections, we considered the case when $r = n^{-1/2} (\ln n)^{c_0}$ for some sufficiently large constant c_0 . The base model as well as the two generalized models has the small-community phenomenon and the power law degree distribution. Now we consider other choices of r and show that if r is too small or too large, then there is a strong evidence indicating that the model does not have the power law degree distribution or the small-community phenomenon, respectively.

When r is as small as $r = n^{-1/2-\epsilon}$, for any $\epsilon > 0$, then every node connects only a very small fraction of neighbors and the whole graph is almost surely disconnected ([Pen03]). Furthermore, there are many isolated vertices in the base model in this range of r, which indicates that the base model is very unlikely to have the power law degree distribution.

When r is as large as $r=n^{-1/2+\epsilon}$, for any $\epsilon>0$, we have shown that the models have the power law degree distribution. However, the small-community phenomenon does not seem to exist in this situation. In particular, there exists an interesting division of the structure of the R-neighborhood when R varies. Specifically, we have shown in [LP11] that under this range of r, if $R=n^{-1/2+\rho}$ for any $\rho>\epsilon$, then with high probability, for any v, $C_R(v)$ is an (α,β) -community for some constants α,β of size $\Theta(n^{2\rho})$, which indicates that every node belongs to some large community. Here we show that with high probability, for all R=o(r), and for any $v\in V_n$, the conductance $\Phi(C_R(v))$ of $C_R(v)$ is larger than some constant, which indicates that the R-neighborhood is not a good community.

Now we give the proof of Theorem 5.

Proof of Theorem 5. The first part of the theorem is given in [LP11]. Here we prove the second part.

For some fixed R = o(r), we let $C = C_R(v)$ and $C' = C_{r-R}(v)$ for convenience. Then for any vertex $u \in C$ and $u' \in C'$, the distance between u and u' is at most r. The areas of $B_R(v)$ and $B_{r-R}(v)$ are

$$area(B_R(v)) \sim R^2/4$$

 $area(B_{r-R}(v)) \sim (r-R)^2/4 \sim r^2/4$,

respectively, which means that a uniformly generated point will land in $B_R(v)$ and $B_{r-R}(v)$ with probability $R^2/4$ and $r^2/4$, respectively.

We will show that there are many edges between $C'\setminus C$ and C. To be more specific, let C_1 (or C'_1) be the vertices in C (or C') that were born before or at time n/2 and C_2 (or C'_2) be the set of vertices in C (or (C') that were born after time n/2. We show that the sum of the number of edges $e(C_1, C'_2)$ between C_1 and C'_2 , and the number of edges $e(C_2, C'_1)$ between C_2 and C'_1 are large.

Let \mathcal{E} denote the event that for any $u \in V_n$ and for each $t \geq t_0$, $T_t(u) \leq 8(2+\xi)mA_rt$, then by Lemma 5, the probability that \mathcal{E} holds is $1 - O(n^{-\ln n})$. Now Conditioned on \mathcal{E} , for any vertex $x_j \in C_2'$, the probability that the *i*-th contact of x_j lies in C_1 is at least $\frac{(m+\delta)|C_1|}{T_{j-1}(x_j)} \geq \frac{|C_1|}{4(2+\xi)A_rn} \geq \frac{|C_1|}{8A_rn}$. Thus, $|e(C_1, C_2')|$ dominates $Bi(m|C_2'|, \frac{|C_1|}{8A_rn})$, where Bi(N, p) denotes the binomial distribution with parameters N and p.

Similarly, for any vertex $x_j \in C_2$, the probability that the *i*-th contact of x_j lies in C_1' is thus at least $\frac{(m+\delta)|C_1'|}{T_{j-1}(x_j)} \geq \frac{(1+\xi)|C_1'|}{4(2+\xi)A_rn} \geq \frac{|C_1'|}{8A_rn}$. Thus, $|e(C_2, C_1')|$ dominates $Bi(m|C_2|, \frac{|C_1'|}{8A_rn})$. Totally, the expected number of edges between the C and $C' \setminus C$ is

$$E[|e(C, C' \setminus C)|] \ge \frac{m|C_2'||C_1|}{8A_r n} + \frac{m|C_2||C_1'|}{8A_r n},$$

which is at least m|C|/16 conditioned on the event A that C'_1 and C'_2 are both of size at least $A_r n/4$. Therefore, by Hoeffdings inequality and the fact that $\Pr[\neg A] = O(n^{-3})$, we see that $|e(C, \bar{C})| \ge |e(C, C' \setminus C)| \ge |e(C, C' \setminus C)|$ m|C|/32 with probability at least $1 - e^{-m|C|/32}$.

On the other hand, $|C| = o(A_r n)$ with high probability. Therefore,

$$\Pr[\exists R = o(r), \exists v, |e(C_R(v), \bar{C}_R(v))| \le m|C_R(v)|/32] \le \sum_{k=1}^{o(A_r n)} \binom{n}{k} e^{-mk/32} = o(1),$$

where the last inequality follows from the assumption that $m > K \ln n$, for some large constant K.

Finally we note that $\operatorname{vol}(C_R(v)) \leq m|C_R(v)| + |e(C_R(v), \bar{C}_R(v))|$ and then we have

$$\Phi(C_R(v)) \ge \frac{m|C_R(v)|/32}{m|C_R(v)| + m|C_R(v)|/32} = \Omega(1), \tag{30}$$

with high probability. This proves Theorem 5.

Finally, we remark that the above proof can be adapted to the two generalized models $G_n^{\rm H}$ and $G_n^{\rm S}$. Since the number of long edges is relatively small compared with the number of long edges, the effect of long edges do not change the community structure too much. Specifically, to show that for R = o(r), $C_R(v)$ is an expander in G_n^H and G_n^S , we just need to use that $\operatorname{vol}(C_R(v)) \leq (m+1)|C_R(v)| + |e(C_R(v), \bar{C}_R(v))|$, and $|e(C_R(v), \bar{C}_R(v))| \geq m|C|/32$, which follows exactly the same as above.

Conclusion 8

We investigate the small-community phenomenon in networks and give two models that unify the three typical properties of large-scale networks: the power law degree distribution, the small-community phenomenon and the small diameter property. The proposed network models provide us insights of how real networks evolve and may have potential applications in, e.g., wireless ad-hoc model and sensor networks.

We have shown that the choice of parameters is subtle if one wants all the three properties to coexist. The fundamental conflicts is discussed, i.e., the power law degree distribution generated by the preferential attachment scheme and the small diameter always lead to an expander like graph, while the small-community phenomenon corresponds naturally to anti-expander in some sense, which means that the conductance of many subsets of small size is of order o(1). Other reasons for such conflicts worth further investigation.

Finally, our proof technique for the power law degree distribution is of its own interest and it partially solves the open problems in [FFV07a] et al. It is interesting to find other applications of this method, in particular, in the analysis of randomized algorithms and network modeling.

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