A Coding Theoretic Study on MLL Proof Nets

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Abstract

Coding theory is useful for real world applications. A notable example is digital television. Basically, coding theory is to study a way of detecting and/or correcting data that may be true or false. Moreover coding theory is an area of mathematics, in which there is an interplay between many branches of mathematics, e.g., abstract algebra, combinatorics, discrete geometry, information theory, etc. In this paper we propose a novel approach for analyzing proof nets of Multiplicative Linear Logic (MLL) by coding theory. We define families of proof structures and introduce a metric space for each family. In each family,

1. an MLL proof net is a true code element, and
2. a proof structure that is not an MLL proof net is a false (or corrupted) code element.

The definition of our metrics reflects the duality of the multiplicative connectives elegantly. In this paper we show that in the framework one error-detecting is possible but one error-correcting not. Our proof of the impossibility of one error-correcting is interesting in the sense that a proof theoretic property is proved using a graph theoretic argument. In addition, we show that affine logic and MLL + MIX are not appropriate for this framework. That explains why MLL is better than such similar logics.

Keywords: Linear Logic, proof nets, error-correcting codes, graph isomorphisms, combinatorics

1 Introduction

The study of the multiplicative fragment of Linear Logic without multiplicative constants (for short MLL) [Gir87] is successful from both semantical and syntactical point of view. In semantical point of view there are good semantical models including coherent spaces. In syntactical point of view the theory of MLL proof nets has obtained a firm status without doubt. On the other hand the intuitionistic multiplicative fragment of Linear Logic without multiplicative constants (for short IMLL) is also studied, for example, in [Mat07]. IMLL can be seen as a subsystem of MLL. IMLL is easier to be studied more deeply than MLL, because we can use intuitions inspired from the conventional lambda-calculus theory as well as graph-theoretic intuitions from the MLL proof nets theory. We exploited both benefits in [Mat07].
In order to study MLL more deeply, how should we do? One approach is to interpret MLL intuitionistically by using Gödel's double negation interpretation. One example is [Has05]. However in such an approach multiplicative constants must be introduced. Definitely introducing multiplicative constants makes things complicated. Another approach we propose in this paper is to adopt *coding theoretic* framework.
Basically, coding theory [Bay98, MS93] is to study a way of detecting and/or correcting data that may be true or false. Moreover coding theory is an area of mathematics, in which there is an interplay between many branches of mathematics, e.g., abstract algebra, combinatorics, discrete geometry, information theory, etc. In this paper we propose a novel approach for analyzing proof nets of Multiplicative Linear Logic (MLL) by coding theory. We define families of proof structures and introduce a metric space for each family. In each family,

1. an MLL proof net is a true code element, which is usually called a *codeword* in the literature of coding theory;
2. a proof structure that is not an MLL proof net is a false (or corrupted) code element.

Figure 1 shows an explanatory example. All three examples in Figure 1 are MLL proof nets in a standard notation of [Gir87]. In our framework the left and the middle proof nets belong to the same family, because when we forget $\otimes$ and $\invamp$ symbols, these are the same (although in fact, these are equal without forgetting those symbols. We will discuss the matter later). But the right proof net does not belong to the family, because when we forget $\otimes$ and $\invamp$ symbols from the right one, we can not identify this one with the previous one by the mismatch of the literals p and $p^{\perp}$. The subtle point will be discussed later in a more precise way (see Subsection 3.1). The definition of our metrics reflects the duality of the multiplicative connectives

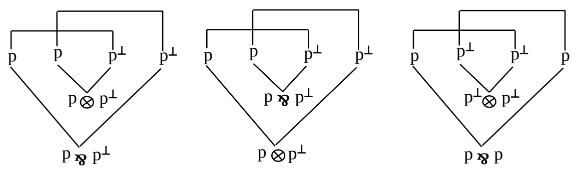

Figure 1: An explanatory example

elegantly. Moreover introducing the framework makes it possible to apply different results and techniques of other branches of mathematics to the study of MLL proof nets. In particular, our concern is closely related to the following question: given a condition about proof nets (for example, that of the number of ID-links), how many proof nets do we have such that they satisfy the condition? As far as we know, in the literature, there are only a few discussions about such a counting problem on proof theory.
So far, most of the study of MLL proof nets have focused on individual proof nets (e.g., sequentialization theorem [Gir87]) or the relationship between identifiable proof nets (e.g., cut-elimination and η-expansion). On the other hand, our approach focuses on a relationship between similar, but different proof nets. In particular, our notion of similarity of proof nets seems to be unable to be understood by conventional type theory.
The main technical achievement of this paper is Theorem 3, which says that in our framework one error-detecting is possible but one error-correcting not. Our proof of the theorem is interesting in the sense that a proof-theoretic property is proved by a graph-theoretic argument.

The Structure of the Paper: Section 2 introduces basic properties of MLL proof nets. MLL proof nets are defined and sequentialization theorem on them is described. Moreover, the notion of empires, which are needed in order to prove the main theorems, is introduced. Section 3 introduces the notion of PS-families (families of proof structures) and distances on them. It is shown that they are metric spaces. Then other basic properties w.r.t PS-families and the main theorems are stated. Most of details of the proofs of the main theorems are put into Appendices. An example is also given (Example 1). Finally, future research directions about PS-families and elementary results on them are stated.

2 The MLL System

2.1 The Basic Theory of MLL Proof Nets

In this section, we present multiplicative proof nets. We also call these *MLL proof nets* (or simply, *proof nets*). First we define MLL formulas. In this paper, we only consider MLL formulas with the only one propositional variable p because the restriction does not give any essential differences w.r.t our main results. By the same reason we restrict ID-links to them with literal conclusions. Moreover we do not consider Cut-links and Cut-elimination because our main results do not concern them.

Definition 1 (Literals) *A literal is p or $p^{\perp}$. The positive literal is p and the negative literal is $p^{\perp}$.*

Definition 2 (MLL Formulas) *MLL formulas (or simply formulas) F is any of the followings:*

- *F is a literal;*
- *F is $F_1 \otimes F_2$, where F_1 and F_2 are MLL formulas. Then F is called $\otimes$-formula.*
- *F is $F_1 \parr F_2$, where F_1 and F_2 are MLL formulas. Then F is called $\parr$-formula.*

We denote the set of all the MLL formulas by MLLFml.

Definition 3 (Negations of MLL Formulas) *Let F be an MLL formula. The negation $F^\perp$ of F is defined as follows according to the form of F:*

- *if F is p, then $F^\perp \equiv_{\mathrm{def}} p^\perp$;*
- *if F is $p^\perp$, then $F^\perp \equiv_{\mathrm{def}} p$;*
- *if F is $F_1 \otimes F_2$, then $F^\perp \equiv_{\mathrm{def}} F_1^\perp \parr F_2^\perp$;*
- *if F is $F_1 \parr F_2$, then $F^\perp \equiv_{\mathrm{def}} F_1^\perp \otimes F_2^\perp$.*

So, $F^\perp$ is actually an MLL formula.

Definition 4 (Indexed MLL Formulas) *An indexed MLL formula is a pair $\langle F, i\rangle$, where F is an MLL formula and i is a natural number.*

Figure 2 shows the links we use in this paper. We call each link in Figure 2 *an MLL link* (or simply *link*). In Figure 2,

1. In ID-link, $\langle A, i\rangle$ and $\langle A^\perp, j\rangle$ are called conclusions of the link.
2. In $\otimes$-link (resp. $\parr$-link) $\langle A, i\rangle$ is called the left premise, $\langle B, j\rangle$ the right premise and $\langle A \otimes B, k\rangle$ (resp. $\langle A \parr B, k\rangle$) the conclusion of the link.

Moreover we call links except ID-links *multiplicative links*.

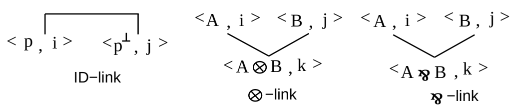

Figure 2: MLL links

Definition 5 (MLL Proof Structures) *Let $\mathbb{F}$ be a finite set of MLL formula occurrences, i.e., a finite set of indexed MLL formulas and $\mathbb{L}$ be a finite set of MLL link occurrences such that for each $L \in \mathbb{L}$, the conclusions and the premises of L belong to $\mathbb{F}$. The pair $\Theta = \langle \mathbb{F}, \mathbb{L}\rangle$ is an MLL proof structure (or simply, a proof structure) if Θ satisfies the following conditions:*

1. *for any $\langle F_0, i\rangle$ and $\langle F_0', j\rangle$ in $\mathbb{F}$, if $i = j$, then $F_0 = F_0'$ (i.e., in $\mathbb{F}$, each element has a different index number).*
2. *for each formula occurrence $F \in \mathbb{F}$ and for each link occurrence $L \in \mathbb{L}$, if F is a premise of L then L is unique, i.e., F is not a premise of any other link $L' \in \mathbb{L}$.*
3. *for each formula occurrence $F \in \mathbb{F}$, there is a unique link occurrence $L \in \mathbb{L}$ such that F is a conclusion of L.*

Remark. In the following, when we discuss proof structures or proof nets, in many cases, we conveniently forget indices for them, because such information is superfluous in many cases. Moreover, when we draw a proof structure or a proof net, we also forget such an index, because locative information in such drawings plays an index.

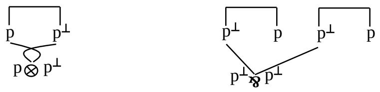

Figure 3: Two examples of MLL proof structures

We say that in $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$, a formula occurrence $F \in \mathbb{F}$ is a conclusion of Θ if for any $L \in \mathbb{L}$, F is not a premise of L.
It is well-known that a proof structure does not necessarily correspond to a sequent calculus proof. For example, two MLL proof structures in Figure 3 do not the corresponding sequent calculus proofs. The following sequentializability is a judgement on the correspondence.

Definition 6 (Sequentializability) *A MLL proof structure $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$ is sequentializable if any of the following conditions holds:*

1. *$\mathbb{L} = \{L\}$ and L is an ID-link;*
2. *There is a ⅋-link $L \in \mathbb{L}$ such that the conclusion $A ⅋ B$ of L is a conclusion of Θ and $\langle \mathbb{F} - \{A ⅋ B\}, \mathbb{L} - \{L\} \rangle$ is sequentializable.*
3. *There is a $\otimes$-link $L \in \mathbb{L}$ and there are two subsets $\mathbb{F}_1$ and $\mathbb{F}_2$ of $\mathbb{F}$ and two subsets $\mathbb{L}_1$ and $\mathbb{L}_2$ of $\mathbb{L}$ such that (a) the conclusion $A \otimes B$ of L is a conclusion of Θ, (b) $\mathbb{F} = \mathbb{F}_1 \uplus \mathbb{F}_2 \uplus \{A \otimes B\}$, (c) $\mathbb{L} = \mathbb{L}_1 \uplus \mathbb{L}_2 \uplus \{L\}$, and (d) $\langle \mathbb{F}_1, \mathbb{L}_1 \rangle$ (respectively $\langle \mathbb{F}_2, \mathbb{L}_2 \rangle$) is an MLL proof structure and sequentializable, where $\uplus$ denotes the disjoint union operator.*

Definition 7 (MLL Proof Nets) *An MLL proof structure Θ is an MLL proof net if Θ is sequentializable.*

Next we give a graph-theoretic characterization of MLL proof nets, following [Gir96]. The characterization was firstly proved in [Gir87] and then an improvement was given in [DR89]. In order to characterize MLL proof nets among MLL proof structures, we introduce *Danos-Regnier graphs (for short, DR-graphs)*. Let Θ be an MLL proof structure. We assume that we are given a function S from the set of the occurrences of ⅋-links in Θ to $\{0, 1\}$. Such a function is called a *DR-switching* for Θ. Then the Danos-Regnier graph Θ_S for Θ and S is a undirected graph such that

1. the nodes are all the formula occurrences in Θ, and
2. the edges are generated by the rules of Figure 4.

In the following we also use the alternative notation $S(\Theta)$ for the Danos-Regnier graph Θ_S.
The following theorem by Girard, Danos, and Regnier [Gir87, DR89], which is called *sequentialization theorem*, is the most important theorem in the theory of MLL proof nets.

Theorem 1 *An MLL proof structure Θ is an MLL proof net iff for each switching function S for Θ, the Danos-Regnier graph Θ_S is acyclic and connected.*

2.2 Empires

In this subsection, we introduce *empires* following [Gir06]. The notion is needed to establish our main results. First we fix a proof structure $\Theta = \langle \mathbb{F}_\Theta, \mathbb{L}_\Theta \rangle$. Moreover we introduce the notations $\mathrm{fml}(\Theta) \equiv_{\mathrm{def}} \mathbb{F}_\Theta$ and $\mathrm{lnk}(\Theta) \equiv_{\mathrm{def}} \mathbb{L}_\Theta$.

Definition 8 (Empires) *The empire of a formula A in a proof net $\Theta = \langle \mathbb{F}, \mathbb{L} \rangle$ (denoted by $e_\Theta(A)$) is defined in the following manner: let S be a DR-switching for Θ. Then an undirected maximal connected graph $(\Theta_S)^A$ (or simply ${\Theta_S}^A$) is defined as follows:*

1. *If there is a link $L \in E$ such that A is a premise of L and there is the edge e from A to the conclusion of L in Θ_S, then $(\Theta_S)^A$ is the maximal connected graph including A obtained from Θ_S by deleting e;*

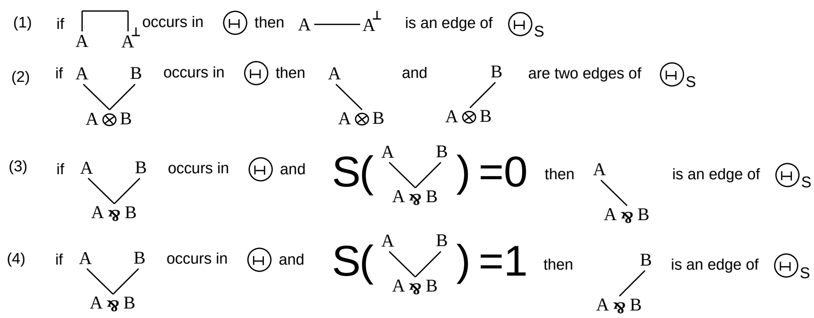

Figure 4: The rules for the generation of the edges of a Danos-Regnier graph Θ_S

2. *otherwise,* $(\Theta_S)^A = \Theta_S$.

Then the empire A in Θ *(denoted by* $e_\Theta(A)$*) is defined as follows:*

$$e_\Theta(A) \equiv_{\mathrm{def}} \bigcap_{S \text{ is a DR-switching for } \Theta} \mathrm{fml}({\Theta_S}^A)$$

From the definition it is obvious that $A \in e_\Theta(A)$. Although the empire $e_\Theta(A)$ is defined as a set of formula occurrences, by considering the set $\mathbb{L}_{e_\Theta(A)}$of links whose conclusions and premises are all included in $e_\Theta(A)$, the empire $e_\Theta(A)$ can be considered as the pair $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)}\rangle$.
Appendix B gives basic properties on empires. Many of them are used in Section 3.

3 Families of Proof Structures

3.1 Our Framework

Firstly we define families of proof-structures. Informally two proof structures Θ_1 and Θ_2 that belong to the same family means that Θ_2 is obtained from Θ_1 by replacing several $\otimes$-links (resp. $⅋$-links) by $⅋$-links (resp. $\otimes$-links). We define such families using graph isomorphisms on directed graphs in a mathematically rigorous way. The reader might feel that the following definitions in this subsection are too cumbersome. But there is a subtle point of the definitions. That is the reason why we insist on a rigorous style. We will discuss this matter at the end of the subsection.

Definition 9 (Strip Function) *A function* $\mathrm{strp}_{\otimes ⅋} : \mathrm{MLLFml} \to \{p, p^\perp, \otimes, ⅋\}$ *is defined as follows:*

1. $\mathrm{strp}_{\otimes ⅋}(p) = p$ *and* $\mathrm{strp}_{\otimes ⅋}(p^\perp) = p^\perp$*;*

2. $\mathrm{strp}_{\otimes ⅋}(A \otimes B) = A \otimes B$ *and* $\mathrm{strp}_{\otimes ⅋}(A ⅋ B) = ⅋$.

Definition 10 (Labelled Directed Graphs) *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be sets. A labelled directed graph with labels* $\mathbb{B}$ *(resp.* $\mathbb{A}$ *and* $\mathbb{B}$*) is a tuple* $\langle V, E, \ell_E : E \to \mathbb{B}\rangle$ *(resp.* $\langle V, E, \ell_V : V \to \mathbb{A}, \ell_E : E \to \mathbb{B}\rangle$*) satisfying the following conditions:*

1. V *is a set;*

2. E *is a set with two functions* $\mathrm{src} : E \to V$ *and* $\mathrm{tgt} : E \to V$.

In the following, we suppose $\mathbb{A} = \{p, p^\perp, \otimes, ⅋\}$ and $\mathbb{B} = \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}$.

Next we define a translation from proof structures to labelled directed graphs and that with a function $f : \mathrm{MLLFml} \to \mathbb{A}$ as a parameter.

Definition 11 (Labelled Directed Graphs Induced by Proof Structures) *Let* $\Theta = \langle \mathbb{F}, \mathbb{L}\rangle$ *be a proof structure and* $f : \mathrm{MLLFml} \to \mathbb{A}$. *A labelled directed graph* $G(\Theta) = \langle V, E, \ell_E : E \to \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}\rangle$ *and* $G^f(\Theta) = \langle V, E, \ell_V^f : V \to \mathbb{A}, \ell_E : E \to \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}\rangle$ *is defined from* Θ *in the following way:*

1. $V = \{i \mid \langle A,i\rangle \in \mathbb{F}\}$ *and* $\ell_V^f = \{\langle i, f(A)\rangle \mid \langle A,i\rangle \in \mathbb{F}\}$*;*
   Since in Θ*, each formula occurrence has a unique index, we can easily see that* V *is bijective to* $\mathbb{F}$*.*

2. E *and* ℓ_E *is the least set satisfying the following conditions:*

   - *If* $L \in \mathbb{L}$ *is an ID-link occurrence with conclusions* $\langle p,i\rangle$ *and* $\langle p^\perp, j\rangle$*, then there is an edge* $e \in E$ *such that* $\mathrm{src}(e) = i$ *and* $\mathrm{tgt}(e) = j$ *and* $\langle e, \mathbf{ID}\rangle \in \ell_E$*;*
   - *If* $L \in \mathbb{L}$ *is a* $\otimes$*-link occurrence with the form* $\frac{\langle A,i\rangle \quad \langle B,j\rangle}{\langle A\otimes B,k\rangle}$ *, then there are two edges* $e_1 \in E$ *and* $e_2 \in E$ *such that* $\mathrm{src}(e_1) = i$*,* $\mathrm{tgt}(e_1) = k$*,* $\mathrm{src}(e_2) = j$*,* $\mathrm{tgt}(e_2) = k$*,* $\langle e_1, \mathbf{L}\rangle \in \ell_E$*, and* $\langle e_2, \mathbf{R}\rangle \in \ell_E$*;*
   - *If* $L \in \mathbb{L}$ *is a* $⅋$*-link occurrence with the form* $\frac{\langle A,i\rangle \quad \langle B,j\rangle}{\langle A⅋B,k\rangle}$ *, then there are two edges* $e_1 \in E$ *and* $e_2 \in E$ *such that* $\mathrm{src}(e_1) = i$*,* $\mathrm{tgt}(e_1) = k$*,* $\mathrm{src}(e_2) = j$*,* $\mathrm{tgt}(e_2) = k$*,* $\langle e_1, \mathbf{L}\rangle \in \ell_E$*, and* $\langle e_2, \mathbf{R}\rangle \in \ell_E$*.*

The next definition is a slight extension of the standard definition of graph isomorphisms.

Definition 12 (Graph Isomorphisms on Labelled Directed Graphs) *Let* $G_1 = \langle V_1, E_1, \ell_{E_1}\rangle$ *(resp.* $G_1 = \langle V_1, E_1, \ell_{V_1}, \ell_{E_1}\rangle$*) and* $G_2 = \langle V_2, E_2, \ell_{V_2}, \ell_{E_2}\rangle$ *(resp.* $G_2 = \langle V_2, E_2, \ell_{E_2}\rangle$*) be labelled directed graphs. Then a graph homomorphism from* G_1 *to* G_2 *is a pair* $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ *satisfying the following conditions:*

1. *for any* $e \in E_1$*,* $h_V(\mathrm{src}(e)) = \mathrm{src}(h_E(e))$ *and* $h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_E(e))$*;*

2. *(only the case where* ℓ_{V_1} *and* ℓ_{V_2} *are specified) for any* $v \in V_1$*,* $\ell_{V_1}(v) = \ell_{V_2}(h_V(v))$*;*

3. *for any* $e \in E_1$*,* $\ell_{E_1}(e) = \ell_{E_2}(h_E(e))$*.*

The graph homomorphism $\langle h_V, h_E\rangle$ *is a graph isomorphism if* $h_V : V_1 \to V_2$ *and* $h_E : E_1 \to E_2$ *are both bijections (then, we write* $\langle h_V, h_E\rangle : G_1 \simeq G_2$*).*

Definition 13 (PS-families) *Let* Θ_1 *and* Θ_2 *be proof structures. Then* $\Theta_1 \sim \Theta_2$ *if there is a graph isomorphism* $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ *from* $G(\Theta_1) = \langle V_1, E_1, \ell_{E_1}\rangle$ *to* $G(\Theta_2) = \langle V_2, E_2, \ell_{E_2}\rangle$*. It is obvious that* $\sim$ *is an equivalence relation. Therefore for a given proof structure* Θ*, we can define the equivalence class* $[\Theta]$ *such that* $\Theta' \in [\Theta]$ *iff* $\Theta \sim \Theta'$*. Then we say* $[\Theta]$ *is a* **PS-family** *of* Θ*. We also say* Θ *belongs to the PS-family* $[\Theta]$*.*

Remark. We define a PS-family as an equivalence class generated by the relation $\sim$. Of course, we can define a PS-family as an MLL proof structure in which all the occurrences of multiplicative links are of $\frac{A \quad B}{A@B}$ instead of $\otimes$- and $⅋$-links, where @ is a new symbol. The reader might prefer to this form. But it seems a matter of taste.

We denote a PS-family by $\mathscr{F}$.
Next, given a PS-family $\mathscr{F}$, we introduce a metric $d_{\mathscr{F}}$ on $\mathscr{F}$.

Definition 14 *Let* $\mathscr{F}$ *be a PS-family. We assume that two MLL proof structures* Θ_1 *and* Θ_2 *belong to* $\mathscr{F}$*. So, by definition we have at least one graph isomorphism* $\langle h_V, h_E\rangle$ *from* $G(\Theta_1)$ *to* $G(\Theta_2)$*. Moreover let* $G^{\mathrm{strp}_{\otimes⅋}}(\Theta_1) = \langle V_1, E_1, \ell_{V_1}^{\mathrm{strp}_{\otimes⅋}}, \ell_{E_1}\rangle$ *and* $G^{\mathrm{strp}_{\otimes⅋}}(\Theta_2) = \langle V_2, E_2, \ell_{V_2}^{\mathrm{strp}_{\otimes⅋}}, \ell_{E_2}\rangle$*. Then* $d_{\mathscr{F}}(\Theta_1, \Theta_2) \in \mathbb{N}$ *is defined as follows:*

$$d_{\mathscr{F}}(\Theta_1,\Theta_2) = \min\{|\{v_1 \in V_1 \mid \ell_{V_2}^{\mathrm{strp}_{\otimes⅋}}(h_V(v_1)) \neq \ell_{V_1}^{\mathrm{strp}_{\otimes⅋}}(v_1)\}| \mid \langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)\}$$

Before proving that $\langle \mathscr{F}, d_{\mathscr{F}}\rangle$ is a metric space, we must define an equality between two MLL proof structures, because the statement concerns the equality on $\mathscr{F}$. In order to define the equality, we use Definition 11 with the parameter $\mathrm{strp}_{\otimes⅋}$.

Definition 15 (Equality on MLL Proof Structures) *Let* Θ_1 *and* Θ_2 *be proof structures. Then* $\Theta_1 = \Theta_2$ *if there is a graph isomorphism* $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ *from* $G^{\mathrm{strp}_{\otimes⅋}}(\Theta_1) = \langle V_1, E_1, \ell_{V_1}^{\mathrm{strp}_{\otimes⅋}}, \ell_{E_1}\rangle$ *to* $G^{\mathrm{strp}_{\otimes⅋}}(\Theta_2) = \langle V_2, E_2, \ell_{V_2}^{\mathrm{strp}_{\otimes⅋}}, \ell_{E_2}\rangle$*.*

It is obvious that $=$ is an equivalence relation.

Proposition 1 *The pair $\langle \mathscr{F}, d_{\mathscr{F}} : \mathscr{F} \to \mathbb{N}\rangle$ is a metric space.*

Proof. The non-negativity of $d_{\mathscr{F}}$ is also obvious. It is obvious that $d_{\mathscr{F}}$ is symmetry.
The formula $\Theta_1 = \Theta_2 \Rightarrow d_{\mathscr{F}}(\Theta_1,\Theta_2) = 0$ is obvious. Next we prove that $d_{\mathscr{F}}(\Theta_1,\Theta_2) = 0 \Rightarrow \Theta_1 = \Theta_2$. Let $G(\Theta_1) = \langle V_1, E_1, \ell_{E_1}\rangle$ and $G(\Theta_2) = \langle V_2, E_2, \ell_{E_2}\rangle$. Since Θ_1 and Θ_2 belong to the same PS-family $\mathscr{F}$, there is a graph isomorphism $\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle$ from $G(\Theta_1)$ to $G(\Theta_2)$. By Definition 12, this means that both $h_V : V_1 \to V_2$ and $h_E : E_1 \to E_2$ are bijections and

1. for any $e \in E_1$, $h_V(\mathrm{src}(e)) = \mathrm{src}(h_E(e))$ and $h_V(\mathrm{tgt}(e)) = \mathrm{tgt}(h_E(e))$;
2. for any $e \in E_1$, $\ell_{E_1}(e) = \ell_{E_2}(h_E(e))$.

On the other hand, since $d_{\mathscr{F}}(\Theta_1,\Theta_2) = 0$, we find a graph isomorphism
$\langle h_V : V_1 \to V_2, h_E : E_1 \to E_2\rangle : G(\Theta_1) \to G(\Theta_2)$ with the following additional property: for any $v \in V_1$, $\ell^{\mathrm{strp}\otimes\parr}_{V_1}(v) = \ell^{\mathrm{strp}\otimes\parr}_{V_2}(h_V(v))$. So, we have a graph isomorphism from $G^{\mathrm{strp}\otimes\parr}(\Theta_1)$ to $G^{\mathrm{strp}\otimes\parr}(\Theta_2)$. By Definition 15, we obtain $\Theta_1 = \Theta_2$.
In order to prove the triangle equality on $d_{\mathscr{F}}$, we need the following claim.

Claim 1 *Let Θ_1 and Θ_2 be two proof structures belonging to the same PS-family $\mathbb{F}$. Moreover let $\langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)$, $V_h = \{v_1 \in V_1 \,|\, \ell^{\mathrm{strp}\otimes\parr}_{V_2}(h_V(v_1)) \neq \ell^{\mathrm{strp}\otimes\parr}_{V_1}(v_1)\}$, and $d_{\mathscr{F}}(\Theta_1,\Theta_2) = |V_h|$. In addition let $V' \subseteq V_h$ and Θ_0 be the proof structure obtained from Θ_1 by replacing the $\otimes$-link (resp. the $\parr$-link) corresponding to v by the $\parr$-link (resp. the $\otimes$-link) for each $v \in V'$. Then $d_{\mathscr{F}}(\Theta_1,\Theta_0) = |V'|$.*

proof of Claim 1: We assume that $d_{\mathscr{F}}(\Theta_1,\Theta_0) < |V'|$. Then we have $\langle h^0_V, h^0_E\rangle : G(\Theta_1) \simeq G(\Theta_0)$ such that $d_{\mathscr{F}}(\Theta_1,\Theta_0) = |\{v_1 \in V_1 \,|\, \ell^{\mathrm{strp}\otimes\parr}_{V_0}(h^0_V(v_1)) \neq \ell^{\mathrm{strp}\otimes\parr}_{V_1}(v_1)\}|$.
On the other hand, $\langle h_V, h_E\rangle : G(\Theta_1) \simeq G(\Theta_2)$ can be decomposed into $\langle h^{10}_V, h^{10}_E\rangle : G(\Theta_1) \simeq G(\Theta_0)$ and $\langle h^{02}_V, h^{02}_E\rangle : G(\Theta_0) \simeq G(\Theta_2)$ (i.e., $\langle h_V, h_E\rangle = \langle h^{02}_V, h^{02}_E\rangle \circ \langle h^{10}_V, h^{10}_E\rangle$) such that $|V_{10}| + |V_{02}| = |V_h|$, where $V_{10} = \{v_1 \in V_1 \,|\, \ell^{\mathrm{strp}\otimes\parr}_{V_0}(h^{10}_V(v_1)) \neq \ell^{\mathrm{strp}\otimes\parr}_{V_1}(v_1)\}$ and $V_{02} = \{v_0 \in V_0 \,|\, \ell^{\mathrm{strp}\otimes\parr}_{V_2}(h^{02}_V(v_0)) \neq \ell^{\mathrm{strp}\otimes\parr}_{V_0}(v_0)\}$. We note $|V'| = |V_{10}|$. Then $\langle h^{02}_V \circ h^0_V, h^{02}_E \circ h^0_E\rangle : G(\Theta_0) \simeq G(\Theta_2)$ and

$$
\begin{aligned}
& |\{v_1 \in V_1 \,|\, \ell^{\mathrm{strp}\otimes\parr}_{V_2}((h^{02}_V \circ h^0_V)(v_1)) \neq \ell^{\mathrm{strp}\otimes\parr}_{V_1}(v_1)\}| \\
= \quad & d_{\mathscr{F}}(\Theta_1,\Theta_0) + |V_{20}| < |V'| + |V_{02}| = |V_{10}| + |V_{02}| = |V_h| = d_{\mathscr{F}}(\Theta_1,\Theta_2)\,.
\end{aligned}
$$

This is a contradiction. *the end of the proof of Claim 1*
Using the claim, we can prove the triangle equality on $d_{\mathscr{F}}$ similar to that of the set of all the binary words with a fixed length. □

We give a justification of the definitions above using Figure 1. Let Θ_1, Θ_2, and Θ_3 be the left proof net, the middle proof net, and the right proof net of Figure 1 respectively. Then $G(\Theta_1) \sim G(\Theta_2)$, since $G(\Theta_1)$ and $G(\Theta_2)$ are graph-isomorphic to the left directed graph of Figure 5. But note that there are two graph isomorphisms $\{\otimes \mapsto \otimes, \parr \mapsto \parr\}$ and $\{\otimes \mapsto \parr, \parr \mapsto \otimes\}$ between $G(\Theta_1)$ and $G(\Theta_2)$. By the former one, we can identify Θ_1 with Θ_2, while in the latter one, there are two differences w.r.t multiplicative nodes. Therefore $d_{\mathbb{F}}(\Theta_1,\Theta_2) = 0$. That's why we need the min operator for the definition of $d_{\mathbb{F}}(\Theta_1,\Theta_2)$. So, Θ_1 and Θ_2 belong to the same PS-family. But $\neg(G(\Theta_1) \sim G(\Theta_3))$ (and also $\neg(G(\Theta_2) \sim G(\Theta_3))$), since $G(\Theta_3)$ is graph-isomorphic to the right directed graph of Figure 5 and the left one of Figure 5 are not graph-isomorphic to the right one. So, Θ_3 does not belong to the same PS-family as Θ_1 and Θ_2.
Note that direction of edges labelled with **ID** are indispensable, because if we eliminated the information, then the two graphs of Figure 5 would be isomorphic. However, direction of edges labelled with **L** or **R** is redundant, because we can always identify the conclusions of the graph without the information by looking for the nodes without an outgoing edge. But we prefer to the conventional definition of directed graphs.
In order to avoid the min operator for the definition of $d_{\mathbb{F}}(\Theta_1,\Theta_2)$, we need to consider only PS-families in which there is the unique graph isomorphism between $G(\Theta_1)$ and $G(\Theta_2)$ for each two members Θ_1 and Θ_2. In order to do that, we restrict PS-families to them with exactly one conclusion, because each multiplicative link in an element in such a PS-family is given an absolute position from the root of the proof structure. We call such a PS-family *closed PS-family*. A closed PS-family is PS-connected in the sense of Definition 17 (Subsection 3.4). For example, two proof structures in Figure 6 belonging to the same closed PS-family has the unique graph isomorphism between them. The restriction is similar to that of closed loops in knot theory (see [Ada94]).
On the other hand, for any MLL proof net without closedness condition, the following proposition holds.

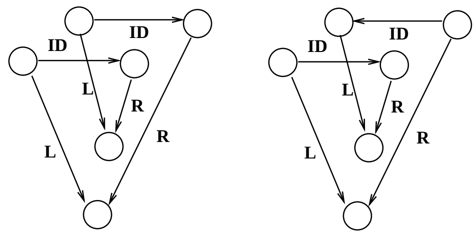

Figure 5: The induced directed graphs from Θ_1 and Θ_2, and that of Θ_3

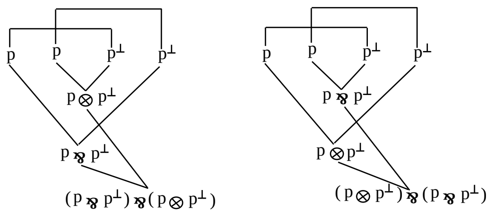

Figure 6: Two elements of a closed PS-family

Proposition 2 *Let Θ be an MLL proof net. Then the identity map $\langle \mathrm{id}_V, \mathrm{id}_E \rangle$ is the only one graph automorphism on $G^{\mathrm{strp}_{\otimes⅋}}(\Theta) = \langle V, E, \ell_V^{\mathrm{strp}_{\otimes⅋}}, \ell_E \rangle$.*

Our proof of Proposition 2 is given in Appendix C.

3.2 Basic Results

Our proposal in this paper starts from the following trivial proposition. We note that this proposition is stated in Subsection 11.3.3 of [Gir06].

Proposition 3 *Let Θ be an MLL proof net.*

1. *Let $L_\otimes : \frac{A \quad B}{A \otimes B}$ be a $\otimes$-link in Θ. Let Θ' be the proof structure Θ except that $L_\otimes$ is replaced by $L'_⅋ : \frac{A \quad B}{A ⅋ B}$. Then Θ' is not an MLL proof net.*

2. *Let $L_⅋ : \frac{C \quad D}{C ⅋ D}$ be a ⅋-link in Θ. Let Θ'' be the proof structure Θ except that $L_⅋$ is replaced by $L'_\otimes : \frac{C \quad D}{C \otimes D}$. Then Θ' is not an MLL proof net.*

Proof.

1. It is obvious that there is a formula X (resp. Y) in $\mathrm{fml}(\Theta)$ such that $X \neq A$ (resp. $Y \neq B$) and $X \in e_\Theta(A)$ (resp. $Y \in e_\Theta(B)$) since if A (resp. B) is a literal, then we just take X (resp. Y) as the other conclusion of the ID-link whose conclusion is A (resp. B), and otherwise, we just take X (resp. Y) as the formula immediately above A (resp. B). On the other hand since $e_\Theta(A) \cap e_\Theta(B) = \emptyset$ by Proposition 14, when we pick up a DR-switching S for Θ arbitrarily, the unique path X from Y in $S(\Theta)$ always passes $A, A \otimes B, B$. Then let S' be a DR-switching for Θ' obtained from S by adding a selection for $L'_⅋$. Then it is obvious that X and Y is disconnected in $S'(\Theta')$.

2. Let S be a DR-switching for Θ. Then by Proposition 15 there is the unique path θ from C to D in $S(\Theta)$ such that θ does not include $C⅋D$. Then Let S'' be the DR-switching for Θ'' obtained form S by deleting the ⅋-switch for $L_⅋$. It is obvious that $S''(\Theta'')$ has a cycle including θ and $C \otimes D$. □

Remark. Proposition 3 does not hold in neither MLL+MIX [Gir87] nor Affine Logic [Bla92]. For example $(p⅋p^\perp) \otimes (p⅋p^\perp)$ is provable in MLL, MLL+MIX, and Affine Logic. The formula $(p⅋p^\perp)⅋(p⅋p^\perp)$ is not provable in MLL, but provable in both MLL+MIX and Affine Logic,

The following corollary is obvious.

Corollary 1 *Let Θ_1 and Θ_2 be MLL proof nets belonging to the same PS-family $\mathscr{F}$. Then $d_{\mathscr{F}}(\Theta_1,\Theta_2) \geq 2$.*

This corollary says that if a PS-family $\mathscr{F}$ has n MLL proof nets, then $\mathscr{F}$ can be used as a **one error-detecting code system** with n different code elements(see Appendix A). But since neither MLL+MIX nor Affine Logic has the property, these can not be used as such a system.
The following proposition is basically a slight extension of Corollary 17.1 of Subsection 11.A.2 of [Gir06]. The extension is by a suggestion of an anonymous referee of the previous version of this paper.

Proposition 4 *Let $\Theta = \langle \mathbb{F}_\Theta, \mathbb{L}_\Theta \rangle$ be an MLL proof net. Let $\mathbb{L}^{\mathrm{ID}}_\Theta$, $\mathbb{L}^{\otimes}_\Theta$, and $\mathbb{L}^{\parr}_\Theta$ be the set of the ID-links, the $\otimes$-links, and the $\parr$-links in $\mathbb{L}$ respectively and con_Θ be the set of the conclusions in $\mathbb{F}_\Theta$. Then $|\mathrm{con}_\Theta| + |\mathbb{L}^{\parr}_\Theta| = |\mathbb{L}^{\mathrm{ID}}_\Theta| + 1$ and $|\mathbb{L}^{\mathrm{ID}}_\Theta| - |\mathbb{L}^{\otimes}_\Theta| = 1$.*

Proof. We prove this by induction on $|\mathbb{L}_\Theta|$.

1. The case where $|\mathbb{L}_\Theta| = 1$:
   Then $|\mathbb{L}^{\mathrm{ID}}_\Theta| = |\mathbb{L}| = 1$, $|\mathrm{con}_\Theta| = 2$, and $|\mathbb{L}^{\parr}_\Theta| = |\mathbb{L}^{\otimes}_\Theta| = 0$. The statements holds obviously.

2. The case where $|\mathbb{L}_\Theta| > 1$:

   (a) The case where Θ includes a $\parr$-formula as a conclusion:
   We choose one $\parr$-link $L_\parr$ among such $\parr$-links. Let $\Theta_0 = \langle \mathbb{F}_{\Theta_0}, \mathbb{L}_{\Theta_0} \rangle$ be Θ except that $L_\parr$ is removed. Since Θ_0 is also an MLL proof net (otherwise, Θ is not an MLL proof net), by inductive hypothesis $|\mathrm{con}_{\Theta_0}| + |\mathbb{L}^{\parr}_{\Theta_0}| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_0}| + 1$ and $|\mathbb{L}^{\mathrm{ID}}_{\Theta_0}| - |\mathbb{L}^{\otimes}_{\Theta_0}| = 1$. But since $|\mathbb{L}^{\mathrm{ID}}_\Theta| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_0}|$, $\mathrm{con}_\Theta = \mathrm{con}_{\Theta_0} - 1$, and $|\mathbb{L}^{\parr}_\Theta| = |\mathbb{L}^{\parr}_{\Theta_0}| + 1$, $|\mathbb{L}^{\otimes}_\Theta| = |\mathbb{L}^{\otimes}_{\Theta_0}|$, the statements hold.

   (b) The case where the conclusions of Θ do not have any $\parr$-formula:
   In this case, $|\mathbb{L}^{\otimes}_\Theta|$ must be greater than 0. Then by Splitting lemma (Lemma 2), we have a $\otimes$-conclusion $A \otimes B$ and its $\otimes$-link $L_{A\otimes B}$ in Θ such that Θ is decomposed into $\Theta_1 = e_\Theta(A)$, $\Theta_2 = e_\Theta(B)$, and $\otimes$-link $L_{A\otimes B}$ By inductive hypothesis $|\mathrm{con}_{\Theta_1}| + |\mathbb{L}^{\parr}_{\Theta_1}| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_1}| + 1$, $|\mathbb{L}^{\mathrm{ID}}_{\Theta_1}| - |\mathbb{L}^{\otimes}_{\Theta_1}| = 1$, $|\mathrm{con}_{\Theta_2}| + |\mathbb{L}^{\parr}_{\Theta_2}| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_2}| + 1$, and $|\mathbb{L}^{\mathrm{ID}}_{\Theta_2}| - |\mathbb{L}^{\otimes}_{\Theta_2}| = 1$ hold. Moreover since $|\mathbb{L}^{\mathrm{ID}}_\Theta| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_1}| + |\mathbb{L}^{\mathrm{ID}}_{\Theta_2}|$, $|\mathrm{con}_\Theta| = |\mathrm{con}_{\Theta_1}| + |\mathrm{con}_{\Theta_2}| - 1$, $|\mathbb{L}^{\parr}_\Theta| = |\mathbb{L}^{\parr}_{\Theta_1}| + |\mathbb{L}^{\parr}_{\Theta_2}|$, and $|\mathbb{L}^{\otimes}_\Theta| = |\mathbb{L}^{\otimes}_{\Theta_1}| + |\mathbb{L}^{\otimes}_{\Theta_2}| + 1$, the statements holds. □

Remark. Proposition 4 does not hold in MLL+MIX. A counterexample in MLL+MIX is again $(p \parr p^\perp) \parr (p \parr p^\perp)$.

Corollary 2 *Let $\mathscr{F}$ be a PS-family. Let Θ_1 and Θ_2 be MLL proof nets belonging to $\mathscr{F}$. Then the number of $\otimes$-links (resp. $\parr$-links) occurring in Θ_1 is the same as that of Θ_2.*

Proof. Since Θ_1 and Θ_2 are members of $\mathscr{F}$, $|\mathrm{con}_{\Theta_1}| = |\mathrm{con}_{\Theta_2}|$ and $|\mathbb{L}^{\mathrm{ID}}_{\Theta_1}| = |\mathbb{L}^{\mathrm{ID}}_{\Theta_2}|$. Therefore by Proposition 4, $|\mathbb{L}^{\otimes}_{\Theta_1}| = |\mathbb{L}^{\otimes}_{\Theta_2}|$ and $|\mathbb{L}^{\parr}_{\Theta_1}| = |\mathbb{L}^{\parr}_{\Theta_2}|$. □

Next, we define an important notion in the next subsection.

Definition 16 ($\otimes$-$\parr$-exchange) *Let Θ be a proof structure. Moreover let $L_\otimes : \frac{A \quad B}{A\otimes B}$ and $L_\parr : \frac{C \quad D}{C \parr D}$ be a $\otimes$-link and a $\parr$-link in Θ respectively. Then $\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr)$ be a proof structure obtained from Θ replacing $L_\otimes$ by $L'_\parr : \frac{A \quad B}{A \parr B}$ and $L_\parr$ by $L'_\otimes : \frac{C \quad D}{C\otimes D}$ simultaneously. Then $\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr)$ is called a $\otimes$-$\parr$-exchange of Θ by $L_\otimes$ and $L_\parr$.*
More generally, when $\langle L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}} \rangle$ is a list of $\otimes$-links and $\langle L_{\parr_1}, \ldots, L_{\parr_{\ell_2}} \rangle$ a list of $\parr$-links, then $\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}} \rangle, \langle L_{\parr_1}, \ldots, L_{\parr_{\ell_2}} \rangle)$ is defined to be a proof structure obtained from Θ by replacing $L_{\otimes_1}, \ldots, L_{\otimes_{\ell_1}}$ by the list of $\parr$-links $L'_{\parr_1}, \ldots, L'_{\parr_{\ell_1}}$ and $L_{\parr_1}, \ldots, L_{\parr_{\ell_2}}$ by the list of $\otimes$-links $L'_{\otimes_1}, \ldots, L'_{\otimes_{\ell_2}}$ simultaneously.

It is obvious that Θ and $\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr)$ belong to the same PS-family. Moreover, $\mathrm{ex}_{\otimes\parr}(\mathrm{ex}_{\otimes\parr}(\Theta, L_\otimes, L_\parr), L'_\otimes, L'_\parr)$ is Θ. Then for each two proof structures Θ_1 and Θ_2, we define a relation $\Theta_1 \Leftrightarrow \Theta_2$ if there are $\otimes$-link $L_\otimes$ and $\parr$-link $L_\parr$ in Θ_1 such that Θ_2 is $\mathrm{ex}_{\otimes\parr}(\Theta_1, L_\otimes, L_\parr)$. Then $\Leftrightarrow$ is a symmetric relation from the observation above. On the other hand, if Θ is an MLL proof net and $\Theta \Leftrightarrow \Theta'$, then Θ' is not always an MLL proof net. Figure 7 shows such an example. Theorem 2 below describes a necessary and sufficient condition that Θ' is an MLL proof net.

As to general $\otimes$-$\parr$-exchange $\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, L_{\otimes_{\ell_1}}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr_{\ell_2}}\rangle)$, note that we do not assume that each element of $\langle L_{\otimes 1}, \ldots, L_{\otimes_{\ell_1}}\rangle$ (resp. $\langle L_{\parr 1}, \ldots, L_{\parr_{\ell_2}}\rangle$) does not appear in Θ like substitution of λ-calculus, because of convenience. In addition, note that Proposition 3 states when Θ is an MLL proof net and $L_\otimes : \frac{A \quad B}{A\otimes B}$ (resp. $L_\parr : \frac{C \quad D}{C\parr D}$) appears in Θ, then $\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_\otimes\rangle, \langle\rangle)$ (resp. $\mathrm{ex}_{\otimes\parr}(\Theta, \langle\rangle, \langle L_\parr\rangle)$) is not an MLL proof net (although these two belong to the same PS-family as Θ).

Moreover from Corollary 2, we can easily see that if Θ_1 and Θ_2 are MLL proof nets that belong to the

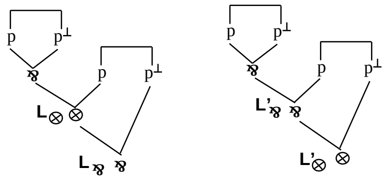

Figure 7: A counterexample

same PS-family, then there is a sequence of proof structures $\Theta'_1, \ldots, \Theta'_k\,(k \geq 0)$ such that $\Theta_1 \Leftrightarrow \Theta'_1 \Leftrightarrow \cdots \Leftrightarrow \Theta'_k \Leftrightarrow \Theta_2$. Theorem 3 below says that we can always find such a sequence $\Theta'_1, \ldots, \Theta'_k$ such that each element $\Theta'_i\,(1 \leq i \leq k)$ is an **MLL proof net**. This does not seem trivial.

3.3 Main Theorems

In this section, we answer the following question: "in our framework is error-correcting possible?" Our answer is negative. Corollary 3 says that this is impossible even for one error-correcting.

Before that, we state a characterization of the condition $d_{\mathscr{F}}(\Theta_1, \Theta_2) = 2$, where $\mathscr{F}$ is a PS-family and Θ_1 and Θ_2 are MLL proof nets belonging to $\mathscr{F}$. The characterization is used in the proof of Lemma 1 of Appendix E, which is needed to prove Theorem 3.

Theorem 2 *Let Θ be an MLL proof net. Moreover let $L_{1\otimes} : \frac{A \quad B}{A\otimes B}$ and $L_{2\parr} : \frac{C \quad D}{C\parr D}$ be a $\otimes$-link and a $\parr$-link in Θ respectively. Then $\mathrm{ex}_{\otimes\parr}(\Theta, L_{1\otimes}, L_{\parr 2})$ is an MLL proof net iff one of the followings holds in Θ:*

(1) C is a conclusion of $e_\Theta(A)$ and D is a conclusion of $e_\Theta(B)$;

(2) D is a conclusion of $e_\Theta(A)$ and C is a conclusion of $e_\Theta(B)$.

Our proof of Theorem 2 is given in Appendix D.

Theorem 3 *Let Θ and Θ' be two MLL proof nets belonging to the same PS-family $\mathscr{F}$. Then there is $n \in \mathbb{N}$ and a sequence of MLL proof nets $\Theta_1, \ldots, \Theta_n$ such that*

$$\Theta \Leftrightarrow \Theta_1 \Leftrightarrow \cdots \Leftrightarrow \Theta_n \Leftrightarrow \Theta'.$$

Proof. We assume that Θ and Θ' are MLL proof nets, but we do not have such a sequence of MLL proof nets for any $n \in \mathbb{N}$. Moreover we can choose two MLL proof nets Θ and Θ' in $\mathscr{F}$ such that there is no MLL proof net Θ'' such that $d(\Theta, \Theta'') < d(\Theta, \Theta')$ and $d(\Theta'', \Theta') < d(\Theta, \Theta')$ since it is sufficient to prove the theorem. Then from Corollary 2, we can easily deduce that $d_{\mathscr{F}}(\Theta, \Theta')$ is even, i.e., $d_{\mathscr{F}}(\Theta, \Theta') = 2m$. In addition there are m $\otimes$-links $L_{\otimes 1} : \frac{A_1 \quad B_1}{A_1\otimes B_1}, \ldots, L_{\otimes m} : \frac{A_m \quad B_m}{A_m\otimes B_m}$ in Θ and m $\parr$-links $L_{\parr 1} : \frac{C_1 \quad D_1}{C_1\parr D_1}, \ldots, L_{\parr m} : \frac{C_m \quad D_m}{C_m\parr D_m}$ in Θ such that Θ' is $\mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m}\rangle)$. Let $\Theta_{i,j}\,(1 \leq i, j \leq m)$ be $\mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes i}, L_{\parr j})$. Then our assumption means that $\Theta_{i,j}$ is not an MLL proof net for any $i, j\,(1 \leq i, j \leq m)$ (The assumption is used in the proof of Lemma 1 of Appendix E). Then we derive a contradiction from these settings by induction on lexicographic order $\langle m, |\mathbb{L}_\Theta|\rangle$, where $|\mathbb{L}_\Theta|$ is the number of link occurrences in Θ.

(1) The case where $m = 0$ and $m = 1$:
It is obvious.

(2) The case were $m > 1$:

(a) The case where Θ consists of exactly one ID-link:
In this case there is neither a $\otimes$-link nor a $\parr$-link in Θ. This is a contradiction to $m > 1$.

(b) The case where Θ includes a $\parr$-formula $C \parr D$ as a conclusion:
We choose such a $\parr$-link $L_{\parr} : \frac{C \quad D}{C \otimes D}$.

(i) The case where $C \parr D$ is not $C_j \parr D_j$ for any $j\,(1 \leq j \leq m)$:
Let Θ_0 be Θ except that $L_{\parr}$ is eliminated. Then we can apply inductive hypothesis to Θ_0 and a subproof net of Θ', $\mathrm{ex}_{\otimes\parr}(\Theta_0, \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m}\rangle)$. We derive a contradiction.

(ii) The case where $C \parr D$ is $C_{j_0} \parr D_{j_0}$ for some $j_0\,(1 \leq j_0 \leq m)$:
In this case, by Lemma 1, Θ' is not an MLL proof net. This is a contradiction.

(c) The case where the conclusions of Θ do not have any $\parr$-formula:
In this case, by Splitting lemma (Lemma 2), we have a $\otimes$-conclusion $A \otimes B$ and its $\otimes$-link $L_{A \otimes B}$ in Θ such that Θ is decomposed into $e^{PN}_{\Theta}(A)$, $e^{PN}_{\Theta}(B)$, and $\otimes$-link $L_{A \otimes B}$

(i) The case where $A \otimes B$ is not $A_i \otimes B_i$ for any $i\,(1 \leq i \leq m)$:
In this case if the number of $\parr$-links from $L_{\parr 1}, \ldots, L_{\parr m}$ in $e_{\Theta}(A)$ is the same as the number of $\otimes$-links from $L_{\otimes 1}, \ldots, L_{\otimes m}$ in $e_{\Theta}(A)$, then we can apply inductive hypothesis to $e_{\Theta}(A)$ and a subproof net of Θ', $\mathrm{ex}_{\otimes\parr}(e_{\Theta}(A), \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m}\rangle)$. Then we derive a contradiction. Otherwise, let Θ'_A be
$\mathrm{ex}_{\otimes\parr}(e_{\Theta}(A), \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m}\rangle)$. Then by Corollary 2, Θ'_A is not an MLL proof net. Therefore Θ' is not an MLL proof net. This is a contradiction.

(ii) The case where $A \otimes B$ is $A_i \otimes B_i$ for some $i\,(1 \leq i \leq m)$:
Then we can find a DR-switching S' for Θ' such that $S'(\Theta')$ is disconnected since The $\otimes$-link $L_{\otimes i}$ is replaced by a $\parr$-link $L_{\otimes i}$. Therefore Θ' is not an MLL proof net. This is a contradiction.

Therefore, for some $i_0, j_0\,(1 \leq i_0, j_0 \leq m)$, $\Theta_{i_0, j_0} (= \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes i_0}, L_{\parr j_0}))$ is an MLL proof net. We have done. □

Lemma 1 *The assumptions are inherited from the case (2-b-ii) of the proof above of Theorem 3. Then,* $\Theta' = \mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, L_{\otimes m}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr j_0}, \ldots, L_{\parr m}\rangle)$ *is not an MLL proof net.*

A proof of the lemma is given in Appendix E.
When a PS-family $\mathscr{F}$ has at least two MLL proof nets, we define the distance $d(\mathscr{F})$ of $\mathscr{F}$ itself in the usual manner:

$$d(\mathscr{F}) = \min\{d_{\mathscr{F}}(\Theta_1, \Theta_2) \,|\, \Theta_1, \Theta_2 \in \mathscr{F} \wedge (\Theta_1 \,\text{and}\, \Theta_2 \,\text{are MLL proof nets}) \wedge \Theta_1 \neq \Theta_2\}$$

Then from Theorem 3 the following corollary is easily derived.

Corollary 3 *For any PS-family $\mathscr{F}$, if the number of the MLL proof nets in $\mathscr{F}$ is equal to or greater than 2, then $d(\mathscr{F}) = 2$.*

Corollary 3 means that one error-correcting is impossible for any PS-family of MLL.

Example 1 *Our proof of Theorem 3 states that when Θ and Θ' are MLL proof nets belonging to the same PS-family $\mathscr{F}$ and $d_{\mathscr{F}}(\Theta, \Theta') \geq 2$, we can always find an MLL proof net Θ'' such that $d_{\mathscr{F}}(\Theta, \Theta'') = 2$ and $d_{\mathscr{F}}(\Theta'', \Theta') = d_{\mathscr{F}}(\Theta, \Theta') - 2$. We show an example in the following.*
For two MLL proof nets Θ of the left side of Figure 8 and Θ' of the right side of Figure 8 belonging to the same PS-family, $d(\Theta, \Theta') = 4$ holds. Then when we let the left side of Figure 9 be Θ_1, then $\Theta_1 = \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes 1}, L_{\parr 2})$ (and $\Theta = \mathrm{ex}_{\otimes\parr}(\Theta_1, L'_{\otimes 2}, L'_{\parr 1})$). Moreover we find $d(\Theta, \Theta_1) = 2$ and $d(\Theta_1, \Theta') = 2$. But such a Θ_1 is not unique. In fact when we let Θ_2 be the right side of Figure 9, then $\Theta_2 = \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes 2}, L_{\parr 2})$ (and $\Theta = \mathrm{ex}_{\otimes\parr}(\Theta_2, L'_{\otimes 2}, L'_{\parr 2})$). By the way, the PS-family has nine MLL proof nets.
Warning: *This example is not a substitute for Corollary 3. The statement of Corollary 3 is a universal one. Therefore one example is not enough to prove the statement.*

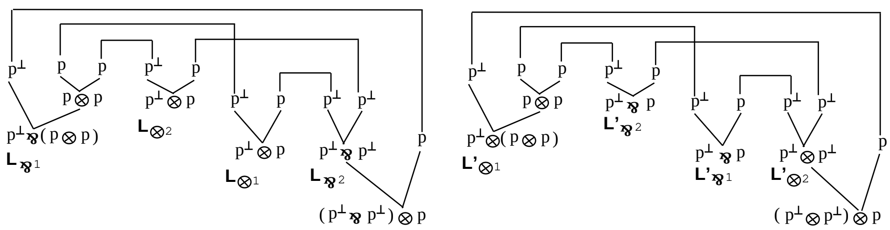

Figure 8: MLL proof nets Θ and Θ'

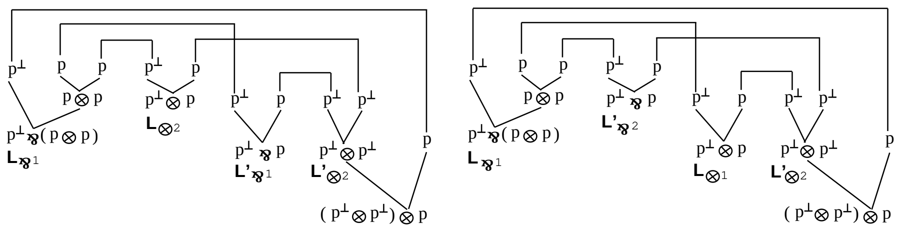

Figure 9: MLL proof nets Θ_1 and Θ_2

3.4 Other Topics

In this section we discuss ongoing research directions in our framework.

3.4.1 The Number of MLL Proof Nets in a PS-family

It is interesting to consider how many MLL proof nets a given PS-family has. We have a characterization of the PS-families without any MLL proof nets as an elementary result.
Firstly we note that the number of the multiplicative links in an element of a given PS family $\mathscr{F}$ is always the same.

Definition 17 (PS-connected) *Let $\mathscr{F}$ be a PS-family. Then $\mathscr{F}$ has the element $\Theta_\otimes$ that has only $\otimes$-links as its multiplicative links (if any). Then there is exactly one DR-switching S for $\Theta_\otimes$ that is empty set. $\mathscr{F}$ is PS-connected if the unique DR-graph $S(\Theta_\otimes)$ is connected.*

Proposition 5 *Let $\mathscr{F}$ be a PS-family. Then $\mathscr{F}$ does not have any MLL proof nets iff $\mathscr{F}$ is not PS-connected.*

Proof.

1. If part:
   We assume that that $\mathscr{F}$ is not PS-connected. We can easily see that for each element Θ of $\mathscr{F}$ and each DR-switching S for Θ, the DR-graph Θ_S is disconnected. Therefore, there is no MLL proof nets in $\mathscr{F}$.

2. Only-if part:
   We prove that if $\mathscr{F}$ is PS-connected, then $\mathscr{F}$ has at least one MLL proof nets by induction on the number n of the multiplicative links in $\mathscr{F}$.

   (a) The case where $n = 0$:
   $\mathscr{F}$ is PS-connected, $\mathscr{F}$ must be the singleton consisting of exactly one ID-link. Therefore $\mathscr{F}$ has exactly one MLL proof net.

   (b) The case where $n > 0$:

      i. The case where there is an element Θ of $\mathscr{F}$ such that by removing one multiplicative link $L : \frac{A \quad B}{A@B}$ of Θ and its conclusion $A@B$, two disjoint proof structures Θ_1 with a conclusion A and Θ_2 with a conclusion B is obtained:

Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be the PS-families that Θ_1 and Θ_2 belong to respectively. Both $\mathscr{F}_1$ and $\mathscr{F}_2$ are PS-connected. Therefore by inductive hypothesis $\mathscr{F}_1$ and $\mathscr{F}_2$ have MLL proof nets Θ'_1 and Θ'_2 respectively. Then let Θ' be the proof structure obtained from Θ'_1 and Θ'_2 by connecting A and B via $\otimes$-link $L' : \frac{A \quad B}{A \otimes B}$. Then it is obvious that Θ' is an MLL proof net and Θ' is an element of $\mathscr{F}$.

ii. Otherwise:
Then there is an element Θ of $\mathscr{F}$ such that by removing one multiplicative link $L : \frac{A \quad B}{A @ B}$ of Θ and its conclusion $A@B$, one proof structure Θ_0 with conclusions A and B is obtained. Let $\mathscr{F}_0$ be the PS-family that Θ_0 belongs to. $\mathscr{F}_0$ is PS-connected. Therefore by inductive hypothesis $\mathscr{F}_0$ has an MLL proof net Θ'_0. Then let Θ' be the proof structure obtained from Θ'_0 by connecting A and B via $⅋$-link $L' : \frac{A \quad B}{A ⅋ B}$. Then it is obvious that Θ' is an MLL proof net and Θ' is an element of $\mathscr{F}$. $\Box$

But it is not so easy to give a similar characterization of PS-families with exactly m MLL proof nets for a given $m (\geq 1)$. At this moment we just obtain the following elementary result.

Proposition 6 *For any positive integer m, there are denumerable PS-families with exactly m MLL proof nets.*

Proof. If $m = 1$, then it is enough to see the left side of Figure 10 in order to confirm that the statement is correct. Similarly if $m > 1$, it is enough to see the right side of Figure 10 for the same purpose. $\Box$

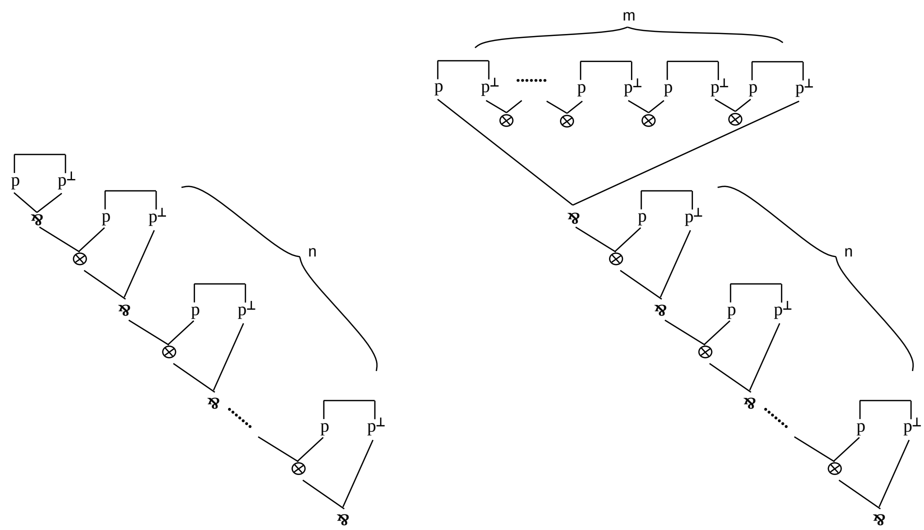

Figure 10: Witnesses for Proposition 6

But it seems difficult to obtain a characterization of the PS-families even with exactly one MLL proof net. The reason is as follows:

1. There are primitive patterns of such PS-families.
2. Moreover by combining such primitive patterns appropriately, we can get compound PS-families with exactly one MLL proof net.

In order to get such a characterization, it seems that an appropriate language that describes (denumerable) sets of PS-families is needed like the regular language for describing sets of words. But since the purpose of this paper is to introduce the new notion of PS-families and metric spaces associated with them, the question is left open as an interesting one.

3.4.2 The Composition of PS-families

MLL proof nets are composable: we get a MLL proof net by connecting two MLL proof nets via Cut-link. But this is not the case about MLL proof structures: we may obtain a vicious circle by connecting two MLL proof structures via Cut-link (see Section 11.2.6 of [Gir06]). Therefore we need a care about the composition of PS-families because a PS-family always includes MLL proof structures that are not MLL proof nets. Moreover this issue is closely related to recent works of Samson Abramsky and his colleagues about compact closed categories (For example, see [Abr07]). But since the paper is already long, the issue will be treated elsewhere.

4 Concluding Remarks

In this paper, we introduced the notion of PS-families over MLL proof structures and metric spaces with associated with them. Moreover we proved that in the case where A PS-family has more than two MLL proof nets, the distance of the PS-family is 2.

Although our main result is the impossibility of one error-correcting in MLL, the remedy is possible. By introducing general $\otimes_n$-links and $\parr_n$-links [DR89], where $n \geq 3$ and these general links have n premises instead of exactly two premises, we can construct a PS-family $\mathscr{F}$ such that $d(\mathscr{F}) = n$. For example, when let Θ_1 (resp. Θ_2 be the general MLL proof net of the left (resp right) side, $d_{\mathscr{F}}(\Theta_1, \Theta_2) = 4$, where $\mathscr{F}$ is the PS-family belonging to Θ_1 and Θ_2. Moreover it is obvious that $d(\mathscr{F}) = 4$. But at this moment we are not sure whether such an easy modification makes good codes (although our main purpose is not to find good codes from PS-families). Nevertheless, we believe that Theorem 3 is a fundamental theorem in this direction of study, because a general version of Theorem 3 seems to be derived in the extended framework.

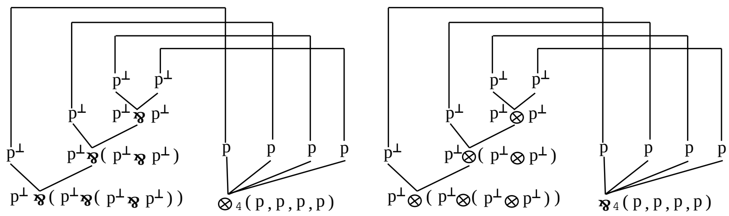

Figure 11: General MLL proof nets Θ_1 and Θ_2

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A Codes over Binary Words

In this appendix, we present basic knowledge about codes over binary finite words. The contents are elementary. The reader can find these materials in any coding theory's textbooks, for example [Bay98, MS93]. The purpose of the appendix is to help the reader understand this paper easily by comparing with the standard theory. If the reader knows these things already, please ignore the appendix.

Definition 18 (Binary Finite Words) *A binary word w with length $n\,(\in \mathbb{N})$ is an element of $\{0,1\}^n$. For each $i\,(1 \leq i \leq n)$, $w[i]$ ($\in \{0,1\}$) denotes i-th element of w.*

Definition 19 (Distance of Binary Words with the Same Length) *Let w_1 and w_2 be binary words with the same length n. The the distance of w_1 and w_2, $d(w_1,w_2)$ is defined as follows:*

$$d(w_1,w_2) = |\{w_1[i] \in \{0,1\} \,|\, 1 \leq i \leq n \wedge w_1[i] \neq w_2[i]\}|$$

For example, $d(00110,10011) = 3$.

Definition 20 (Code over Words with Length n) *A code C over words with length n is a subset of $\{0,1\}^n$. An element of C is called* **codeword**. *The distance of C is defined as follows:*

$$d(C) = \min\{d(w_1,w_2) \,|\, w_1,w_2 \in C \wedge w_1 \neq w_2\}$$

Example 2 (Hamming $\langle 7,4\rangle$ code) *The Hamming $\langle 7,4\rangle$ code C is the subset of $\{0,1\}^7$ such that $w \in \{0,1\}^7$ is an element of C iff w satisfies the following three equations(where $\oplus$ denotes 'exclusive or'):*

$$w[1] \oplus w[2] \oplus w[4] \oplus w[5] = 0$$
$$w[2] \oplus w[3] \oplus w[4] \oplus w[6] = 0$$
$$w[1] \oplus w[3] \oplus w[4] \oplus w[7] = 0$$

Then we can easily see $|C| = 16$ and $d(C) = 3$ by easy calculation. As a result the Hamming $\langle 7,4\rangle$ code is **one error-correcting** *because when a given $w \in \{0,1\}^7$, if $d(w,w') = 1$ for some $w' \in C$, then for any $w''(\neq w') \in C$, $d(w,w'') > 1$. Therefore we can judge that w is w' with one error. Moreover the Hamming $\langle 7,4\rangle$ code is* **two error-detecting** *because when a given $w \in \{0,1\}^7$, if $d(w,w') = 2$ for some $w' \in C$, then for any $w''(\neq w') \in C$, $d(w,w'') \geq 2$. Therefore we can judge that w has exactly two errors. But since there may be a different codeword $w''' \in C$ from w' such that $d(w,w''') = 2$, we can not judge that w is w' with two errors.*

On the other hand, in the Hamming $\langle 7,4\rangle$ code C, we can not do one error-correcting and two error-detecting at the same time, because there are $w_1,w_2 \in C$ and $w' \in \{0,1\}^7 - C$ such that $d(w_1,w') = 2$ and $d(w',w_2) = 1$. Therefore we can not decide whether w' is w_1 with two errors or w_2 with one error. We have to decide whether we adopt the one error-correcting interpretation or the two error-detecting interpretation. If we adopt the one error-correcting interpretation, then w' is w_2 with one error. If we adopt the two error-detecting interpretation, then w' has two errors, but we can not say w' is w_1 with two errors by the reason of the paragraph above.

B Basic Properties of Empires

In this section we prove basic properties of empires. These properties are well-known in the literature, for example [Gir87, BW95, Gir96, Gir06]. Before presenting results, we fix terminology about paths of indexed formulas in a DR-graph.

Definition 21 *Let Θ be an MLL proof net, S be a DR-switching for Θ, and $A, B \in \mathrm{fml}(\Theta)$. Then there is a unique path θ from A to B in Θ_S. We say that θ* **passes immediately above or adjacent to** *A (resp. B) if θ includes a formula C such that there is the link L whose conclusion is A (resp. B) and C is a premise or another conclusion of L. We say that θ* **passes immediately below** *A (resp. B) if θ includes a formula C such that there is the link L whose premise is A (resp. B) and C is the conclusion of L.*

Proposition 7 *Let $B \in e_\Theta(A)$ and $L \in \mathbb{L}_{e_\Theta(A)}$ such that the conclusion of L is B. Then if B' is a premise or a conclusion of L, then $B' \in e_\Theta(A)$.*

Proof. We prove this by case analysis. If $B' = B$, then it is obvious. So we assume $B' \neq B$ in the following.

1. The case where L is an ID-link:
   Then B and B' are literals which are dual each other. Since $B \in e_\Theta(A)$, for each DR-switching S, $B \in \mathrm{fml}({\Theta_S}^A)$. Then it is obvious that $B' \in \mathrm{fml}({\Theta_S}^A)$. So, $B' \in e_\Theta(A)$.

2. The case where L is a $\otimes$-link:
   Then B' is a premise of L. The rest of the proof of this case is similar to the case above.

3. The case where L is a $⅋$-link:
   Then B' is a premise of L. Without loss of generality, we can assume that B' is the left premise of L. We assume $B' \notin e_\Theta(A)$. Then there a DR-switching S such that $B' \notin \mathrm{fml}({\Theta_S}^A)$. By the assumption S selects the right premise B'' in L. Since Θ_S is acyclic and connected, there is a unique path θ from B to B' in Θ_S. If θ does not include A, then by the definition of $\mathrm{fml}({\Theta_S}^A)$ and by $B \in \mathrm{fml}({\Theta_S}^A)$, we derive $B' \in \mathrm{fml}({\Theta_S}^A)$, which is a contradiction. So, θ includes A and θ has two subpaths θ_1 from B to A that passes immediately above or adjacent to A and θ_2 from A to B' that passes immediately below A. Then if θ includes B'', then θ_1 includes B'' and letting S' be S except S' selects B', we obtain $B \notin \mathrm{fml}({\Theta_{S'}}^A)$ and then $B \notin e_\Theta(A)$, which is a contradiction. Therefore θ does not include B''. Then let S' be the DR-switching such that S' is S except that S' selects the left premise B' in L. Then $\Theta_{S'}$ has a cycle. This is a contradiction. □

The following corollary is easily derived from the proposition above.

Corollary 4 *The pair $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)} \rangle$ is an MLL proof structure.*

Proposition 8 *If $B_1 \in e_\Theta(A)$, $B_2 \notin e_\Theta(A)$, and L is a $⅋$-link such that B_1 and B_2 are the premise of L, then the conclusion B of L does not belongs to $e_\Theta(A)$.*

Proof. We assume that $B \in e_\Theta(A)$. Then by Proposition 7, $B_2 \in e_\Theta(A)$. This is a contradiction. □

Proposition 9 *If $B \in e_\Theta(A)$ such that $B \neq A$ and L is a $\otimes$-link such that B is a premise of L, then the premises and the conclusion of L belong to $e_\Theta(A)$.*

Proof. Similar to the case 2 of the proof of Proposition 7. □

Proposition 10 *If $B_1, B_2 \in e_\Theta(A)$ such that $B_1 \neq B_2, B_1 \neq A, B_2 \neq A$ and L is a $⅋$-link such that B_1 and B_2 are the premises of L, then the conclusion B of L belongs to $e_\Theta(A)$.*

Proof. From the assumption for each DR-switching S for Θ, $B_1, B_2 \in \mathrm{fml}({\Theta_S}^A)$. If S selects B_1 in L, then there is an edge between B_1 and B in ${\Theta_S}^A$. That is $B \in \mathrm{fml}({\Theta_S}^A)$. On the other hand, if S selects B_2 in L, then there is an edge between B_2 and B in ${\Theta_S}^A$. That is $B \in \mathrm{fml}({\Theta_S}^A)$. Hence $B \in e_\Theta(A)$. □

Next, we prove that there is a DR-switching S such that $\mathrm{fml}({\Theta_S}^A) = e_\Theta(A)$.

Definition 22 *Let S be a DR-switching for an MLL proof net Θ including A. we say that S is a principal DR-switching (or simply principal switching) for A in Θ if S satisfies the following conditions:*

1. *if there is a $\invamp$-link L such that a premise of L is A, then S selects A, not the other premise of L in L and*

2. *if there is a $\invamp$-link L such that one premise B_1 of L belongs to $e_\Theta(A)$ and the other premise B_2 of L does not belong to $e_\Theta(A)$, then S selects B_2 in L.*

When a given MLL proof net Θ and a formula A in Θ, we can easily see that we can always find a principal DR-switching for A in Θ from the definition above, since if we find a $\invamp$-link satisfying any of the assumptions of the conditions, then we can always choose the switch for the $\invamp$-link that satisfies the conditions.

Proposition 11 *Let S be a DR-switching for an MLL proof net Θ. Then S is a principal DR-switching for a formula A in Θ iff* $\mathrm{fml}(\Theta_S{}^A) = e_\Theta(A)$.

Proof. The if-part is obvious. Hence we concentrate on the only-if part in the following.
Let S be a principal DR-switching. It is obvious that $e_\Theta(A) \subseteq \mathrm{fml}(\Theta_S{}^A)$ from the definition of empires. In order to prove $\mathrm{fml}(\Theta_S{}^A) \subseteq e_\Theta(A)$, we need the following claim.

Claim 2 *Let $B \in \mathrm{fml}(\Theta_S{}^A)$. If the unique path θ from A to B in $\Theta_S{}^A$ includes a $\invamp$-formula $C \invamp D$, then C and D must belong to $e_\Theta(A)$.*

Proof of Claim 2. We prove the claim by induction on the number of $\invamp$-formulas in θ.
If θ does not include any $\invamp$-formula, then the claim is obvious.
Let $C \invamp D$ be the nearest $\invamp$-formula to B in θ and E be the formula immediately before $C \invamp D$ in θ. Then we consider the subpath θ' of θ from A to E. Then the number of $\invamp$-formulas in θ' is less than that of θ. So by inductive hypothesis, the premises of each $\invamp$-formula in θ' belong to $e_\Theta(A)$. Then from Proposition 7, Proposition 9, and Proposition 10, the formulas in θ' must belong to $e_\Theta(A)$. So $E \in e_\Theta(A)$. Then the following two cases are considered:

1. The case where E is either C or D:
   Without loss of generality, we can assume that E is C. Then we assume that $C \invamp D \notin e_\Theta(A)$. But this contradicts that S is a principal DR-switching.

2. The case where E is neither C nor D:
   Since $E \in e_\Theta(A)$, from Proposition 7 we can derive $C \invamp D \in e_\Theta(A)$. Then again by Proposition 7, C and D must belong to $e_\Theta(A)$. *the end of proof of Claim 2.*

the end of proof of Claim 2
Hence using the claim, from Proposition 7, Proposition 9, and Proposition 10, we can derive $B \in e_\Theta(A)$. □

Corollary 5 $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)} \rangle$ *is an MLL proof net.*

Proof. Since $\Theta' = \langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)} \rangle$ is a proof structure by Corollary 4, we concentrate on the correctness criterion. Let S' be a DR-switching for $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)} \rangle$. Then there is a principal DR-switching S for A in Θ which is an extension of S'. Then by Proposition 11, $\mathrm{fml}(\Theta_S{}^A) = e_\Theta(A) = \mathrm{fml}(\Theta'_{S'})$. Therefore $\Theta_S{}^A = \Theta'_{S'}$. This means that $\Theta'_{S'}$ is acyclic and connected. □

Corollary 6 $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)} \rangle$ *is the greatest MLL sub-proof net of Θ among the MLL sub-proof nets of Θ with a conclusion A.*

Proof. Let Θ' be an MLL sub-proof net of Θ with conclusion A such that $e_\Theta(A) \subsetneq \mathrm{fml}(\Theta')$. Then, if S is a principal switching for A in Θ, then by Proposition 11, $\mathrm{fml}(\Theta_S{}^A) = e_\Theta(A)$. So there is a formula $B \in \mathrm{fml}(\Theta')$ such that $B \notin \mathrm{fml}(\Theta_S{}^A)$. Next we consider the MLL proof net Θ' as the *root* proof net instead of Θ. Note that for any DR-switching S'_0 for Θ', there is no path θ' in $\Theta'_{S'_0}$ such that θ' passes immediately below A. Moreover since $e_\Theta(A) \subsetneq \mathrm{fml}(\Theta')$, by extending a principal switching S_0 for A in $e_\Theta(A)$, we can obtain a DR-switching S'_0 for Θ'. But then A and B are disconnected in $\Theta'_{S'_0}$ by the note above. This is a contradiction. □

Corollary 7 *If A is a conclusion of an MLL proof net Θ, then $\langle e_\Theta(A), \mathbb{L}_{e_\Theta(A)}\rangle = \Theta$.*

Corollary 8 *If B is a conclusion of $e_\Theta(A)$, then $A \in e_\Theta(B)$ (but A is not necessarily a conclusion of $e_\Theta(B)$).*

Proposition 12 *If $B \notin e_\Theta(A)$ and $A \notin e_\Theta(B)$, then $e_\Theta(A) \cap e_\Theta(B) = \emptyset$.*

Proof. We derive a contradiction from assumptions $B \notin e_\Theta(A)$, $e_\Theta(A) \cap e_\Theta(B) \neq \emptyset$, and $A \notin e_\Theta(B)$. We assume that $C \in e_\Theta(A) \cap e_\Theta(B)$. We claim the following.

Claim 3 *There is a principal switching S_B^f for B such that there is no path from A to B in $(\Theta_{S_B^f})^A$.*

Proof of Claim 3 Let S_B be a principal switching for B. Then by Proposition 11, $\mathrm{fml}((\Theta_{S_B})^B) = e_\Theta(B)$. Since $A \notin e_\Theta(B) = \mathrm{fml}((\Theta_{S_B})^B)$, in Θ_{S_B} there is a unique path θ from A to B in Θ_{S_B} such that θ passes immediately below B. Then if each formula in θ except A is not included in $(\Theta_{S_B})^A$, then we have done. We just let S_B^f be S_B. Next we assume that θ includes a formula in $(\Theta_{S_B})^A$ except A. Then, since Θ_{S_B} is acyclic and connected and by the definition of $(\Theta_{S_B})^A$, θ from A to B must be included in $(\Theta_{S_B})^A$. On the other hand, since $B \notin e_\Theta(A)$, there is a $\parr$-link $L : \frac{E \quad F}{E \parr F}$ such that exactly one premise of L (i.e., E or F) and $E \parr F$ are not included in $e_\Theta(A)$. Without loss of generality we can assume that (i) $E \in e_\Theta(A)$, (ii) $F \notin e_\Theta(A)$, and (iii) θ includes the subpath $E, E \parr F$ by picking up the first $\parr$-link in θ among such $\parr$-links. Moreover we can show that such a $\parr$-link is unique in θ (otherwise, we have a $L_0 : \frac{E_0 \quad F_0}{E_0 \parr F_0}$ in Θ such that (i') $E_0 \in e_\Theta(A)$, (ii')$F_0 \notin e_\Theta(A)$, and (iii') θ includes the subpath $E_0 \parr F_0, E_0$ without loss of generality. Then $S_B(\Theta)$ has a cycle because $S_B(\Theta)$ has a path from E to E_0 other than the subpath of θ from E to E_0. This is a contradiction).

Subclaim 1 *Let S'_B be the DR-switching S_B except that S'_B chooses the other formula, i.e., F in L. Then, $S'{}_B$ is a principal switching for B.*

Proof of Subclaim 1 We suppose not. On the other hand, since S_B is principal for B, θ in $(\Theta_{S_B})^A$ passes immediately above or adjacent to A and immediately below B (Otherwise, θ passes immediately above or adjacent to B. This means that $A \in \mathrm{fml}(\Theta_{S_B}) = e_\Theta(B)$). Since θ includes E and $E \parr F$, we have $E \notin e_\Theta(B)$ and $E \parr F \notin e_\Theta(B)$. Therefore we must have $F \in e_\Theta(B)$, because otherwise (i.e. $F \notin e_\Theta(B)$), it is obvious that S'_B is a principal switching for B. Since $F \in e_\Theta(B)$, we have a unique path θ' from B to $E \parr F$ through F in $(\Theta_{S'_B})^B$. On the other hand, the subpath θ_0 of θ from $E \parr F$ to B in Θ_{S_B} survives in $\Theta_{S'_B}$. Therefore θ' and θ_0 make a cycle in $\Theta_{S'_B}$. This is a contradiction. *the end of proof of Subclaim 1*
Then the following two cases can be considered:

1. The case where there is no path from A to F in $(\Theta_{S'_B})^A$:
   We suppose that there is a unique path θ' from A to B in $(\Theta_{S'_B})^A$. Then θ' does not pass E, because if θ' includes E, then θ' from A to B in $(\Theta_{S'_B})^A$ survives in $(\Theta_{S_B})^A$ and therefore θ and θ' makes a cycle including $B, E \parr F, E$ in $(\Theta_{S_B})^A$. Moreover θ' does not pass $E \parr F$ because if θ' includes $E \parr F$, then θ' also includes F, which contradicts the assumption. Therefore θ' survives in $(\Theta_{S_B})^A$. Then θ and θ' make a cycle including A and B in $(\Theta_{S_B})^A$. This is a contradiction. Therefore since there is no path θ' from A to B in $(\Theta_{S'_B})^A$, we have done. We just let $\Theta_{S_B}^f$ be $\Theta_{S'_B}$.

2. The case where there is a unique path θ' from A to F in $(\Theta_{S'_B})^A$:
   Since $F \notin e_\Theta(A)$, there is a $\parr$-link L' in θ' such that exactly one premise and the conclusion of the link are not included in $e_\Theta(A)$. Moreover it is obvious that such a $\parr$-link is unique in θ'. Let $L' : \frac{E' \quad F'}{E' \parr F'}$ be the unique $\parr$-link. Without loss of generality we assume that θ' passes E', $E' \in e_\Theta(A)$, and $F' \notin e_\Theta(A)$. Let S''_B be the DR-switching S'_B except S''_B chooses the other formula, i.e., F' in L'. Moreover by the similar discussion to that of S'_B, S''_B is a principal switching for B. Then if S''_B does not satisfy the condition for S_B^f, then we repeat the discussions above to S''_B. Since the number of $\parr$-links in Θ is finite, we can eventually find a principal switching S_B^f for B such that there is no path from A to B in $(\Theta_{S_B^f})^A$.

the end of proof of Claim 3
Then since S_B^f is a principal switching for B in Θ and $A \notin e_\Theta(B)$, for any formula $C \in e_\Theta(B)$, there is no path from A to C in $(\Theta_{S_B^f})^A$. This means that $C \notin e_\Theta(A)$. This contradicts the assumption $C \in e_\Theta(A) \cap e_\Theta(B)$. □

The following proposition is given in a stronger form than Lemma 5 of [Gir96] slightly.

Proposition 13 *If $B \notin e_\Theta(A)$ and $A \in e_\Theta(B)$, then A is not a conclusion of $e_\Theta(B)$ and $e_\Theta(A) \subsetneq e_\Theta(B)$.*

Proof.

1. The proof that A is not a conclusion of $e_\Theta(B)$:
   We suppose that A is a conclusion of $e_\Theta(B)$. Let S be a DR-switching. Then we claim that $B \in \mathrm{fml}({\Theta_S}^A)$. We prove this using case analysis.

   (a) The case where S is a principal switching for B:
   By Proposition 11 $\mathrm{fml}({\Theta_S}^B) = e_\Theta(B)$. From assumptions we can easily see that A and B are a leaf or the root in the tree ${\Theta_S}^B$. Moreover since A is a conclusion of $e_\Theta(B)$, the unique path θ from A to B in ${\Theta_S}^B$ immediately above or adjacent to A. This means that $B \in \mathrm{fml}({\Theta_S}^A)$.

   (b) The case where S is not a principal switching for B:
   Then $A \in e_\Theta(B) \subsetneq \mathrm{fml}({\Theta_S}^B)$. Then there is a unique path θ from A to B in ${\Theta_S}^B$. We suppose that θ passes immediately below A. Then there is the link L' whose premise is A and the link L' must be a $\invamp$-link, since A is a conclusion of $e_\Theta(B)$. Moreover S chooses the premise A in L'. This means that a formula that is not included in $e_\Theta(B)$ is included in θ. On the other hand let S_B be a principal switching for B obtained from S with the minimal effort. Then for any $\invamp$-link $L_0 \in \mathbb{L}_{e_\Theta(B)}$, $S_B(L_0) = S(L_0)$ because of the minimal assumption. Therefore there is no path θ' from A to B in ${\Theta_{S_B}}^B$ such that θ' passes immediately above or adjacent to A, because there is no such path in ${\Theta_S}^B$. Moreover S_B chooses another premise other than A because S_B is a principal switching for B. Hence there is no path θ' from A to B in Θ_{S_B} such that θ' passes immediately below A because S_B selects the other premise other than A in L'. This means that there is a path θ'' from B to A in Θ_{S_B} such that θ'' passes immediately below B and immediately above or adjacent to A. This contradicts that $A \in e_\Theta(B)$. Therefore θ passes immediately above or adjacent to A. This means that $B \in \mathrm{fml}({\Theta_S}^A)$.

   Therefore,
   $$B \in \bigcap_{S \text{ is a DR-switching for } \Theta} \mathrm{fml}({\Theta_S}^A) = e_\Theta(A).$$
   This contradicts the assumption $B \notin e_\Theta(A)$.

2. The proof of $e_\Theta(A) \subsetneq e_\Theta(B)$:
   Let S_A be a principal switching for A. By Proposition 11 $B \notin e_\Theta(A) = \mathrm{fml}((\Theta_{S_A})^A)$. Let S_B be a principal switching for B obtained from S_A by changing $\invamp$-switches with the minimal effort.

   Claim 4 *Then still $B \notin \mathrm{fml}((\Theta_{S_B})^A)$.*

   Proof of Claim 4. We assume that $B \in \mathrm{fml}((\Theta_{S_B})^A)$. Then there is a unique path θ from A to B in $(\Theta_{S_B})^A$ such that θ passes immediately above or adjacent to A. Since $B \notin \mathrm{fml}((\Theta_{S_A})^A)$ and $B \in \mathrm{fml}((\Theta_{S_B})^A)$, the path θ must include the conclusion of a $\invamp$-link L_0 such that $S_A(L_0) \neq S_B(L_0)$. On the other hand, by the minimal assumption about the change from S_A to S_B, the conclusion of L_0 is not included in $e_\Theta(B)$. Moreover since $A \in e_\Theta(B) = (\Theta_{S_B})^B$, there is a path θ' from A to B in $(\Theta_{S_B})^B$ such that all the $\invamp$-formulas in θ' are included in $e_\Theta(B)$. Therefore since these two paths θ and θ' from A to B in Θ_{S_B} are different, θ and θ' make a cycle in Θ_{S_B}. This is a contradiction. *the end of proof of Claim 4*
   Then we can prove the following.

   Claim 5 $\mathrm{fml}((\Theta_{S_B})^A) \subseteq \mathrm{fml}((\Theta_{S_B})^B)$

   Proof of Claim 5. We assume that there is a formula $C \in \mathrm{fml}((\Theta_{S_B})^A)$, but $C \notin \mathrm{fml}((\Theta_{S_B})^B)$. Since $A \in e_\Theta(B) = \mathrm{fml}((\Theta_{S_B})^B)$ and S_B is a principal switching for B, the unique path π' from A to C in Θ_{S_B} must include B in order to go out from $e_\Theta(B) = \mathrm{fml}((\Theta_{S_B})^B)$. On the other hand, since $C \in \mathrm{fml}((\Theta_{S_B})^A)$, there is the unique path π'' from A to C in $(\Theta_{S_B})^A$ such that π'' passes immediately

above or adjacent to A. By uniqueness π' and π'' coincide in Θ_{S_B}. Therefore there is a subpath π'_0 of π' from A to B such that π'_0 passes immediately above or adjacent to both A and B. Hence we can derive $B \in \mathrm{fml}((\Theta_{S_B})^A)$. This contradicts $B \notin \mathrm{fml}((\Theta_{S_B})^A)$. *the end of proof of Claim 5*
Therefore $e_\Theta(A) \subseteq \mathrm{fml}((\Theta_{S_B})^A) \subseteq \mathrm{fml}((\Theta_{S_B})^B) = e_\Theta(B)$. □

Proposition 14 *Let Θ be an MLL proof net including $\otimes$-link $L : \frac{A \quad B}{A \otimes B}$. Then $e_\Theta(A) \cap e_\Theta(B) = \emptyset$.*

Proof. We assume $e_\Theta(A) \cap e_\Theta(B) \neq \emptyset$. Then $A \otimes B \notin e_\Theta(A) \cap e_\Theta(B)$. Otherwise, there is a DR-switching S for Θ such that Θ_S has a cycle including A and $A \otimes B$. Therefore there is a formula C such that $C \in e_\Theta(A) \cap e_\Theta(B)$ and $k \neq \ell$. Then when we consider $e_\Theta(A \otimes B)$, we can easily see that there is an arbitrary DR-switching S for Θ such that Θ_S has a cycle including C and $A ⅋ B$, since there is a unique path from A to C in Θ^A_S and there is also the unique path from B to C in $\Theta_S{}^B$. This is a contradiction. □

Proposition 15 *Let Θ be an MLL proof net including ⅋-link $L : \frac{A \quad B}{A ⅋ B}$. Then $e_\Theta(A) = e_\Theta(B)$.*

Proof.

Claim 6 $e_\Theta(A) \cap e_\Theta(B) \neq \emptyset$

Proof of Claim 6. We assume that $e_\Theta(A) \cap e_\Theta(B) = \emptyset$. We take a principal switching S_B for B. Then there is no path from A to B in Θ_{S_B}. In order to prove this, we assume that there is a path θ from A to B in Θ_{S_B}. The path θ does not pass immediately below B. If so, since S_B is a principal switching for B, S_B selects B in the ⅋-link L. Therefore, θ passes immediately above or adjacent to A. Moreover by the assumption, θ includes the subpath $A ⅋ B, B$. Then let S_A be S_B except that S_A chooses A in L. Then $S_A(\Theta)$ has a cycle including the subpath of θ from A to $A ⅋ B$ and the path $A ⅋ B, A$. Therefore the path θ does not pass immediately below B. On the other hand, the path θ does not pass immediately above or adjacent to B because $A \notin e_\Theta(B)$ (since $e_\Theta(A) \cap e_\Theta(B) = \emptyset$) and $\mathrm{fml}((\Theta_{S_B})^B) = e_\Theta(B)$. Therefore Θ_{S_B} is disconnected. This is a contradiction. *the end of proof of Claim 6*
Then by Proposition 12, $B \in e_\Theta(A)$ or $A \in e_\Theta(B)$.

1. The case where $B \in e_\Theta(A)$ and $A \in e_\Theta(B)$:
   It is obvious that $A ⅋ B \notin e_\Theta(A)$, since otherwise we can easily find a DR-switching S such that Θ_S has a cycle including A and $A ⅋ B$. Similarly $A ⅋ B \notin e_\Theta(B)$. So B is a conclusion of $e_\Theta(A)$ and A is a conclusion of $e_\Theta(B)$.
   Let S_B be a principal switching for B. In addition, let S_A be a principal switching for A obtained from S_B by changing ⅋-switches with the minimal effort. Then the following claim holds.

   Claim 7 *Let $C \in \mathrm{fml}((\Theta_{S_B})^B)$ and θ be a unique path from A to C in $(\Theta_{S_B})^B$. Then each formula in θ is included in $(\Theta_{S_A})^A$.*

   Proof of Claim 7. At first we note that θ passes immediately above or adjacent to A because S_B selects B in the ⅋-link L. We assume that the statement does not hold. Then without loss of generality, there is a subpath $E, E ⅋ F$ in θ such that the subpath of θ from A to E in $(\Theta_{S_B})^B$ survives in $(\Theta_{S_A})^A$ and $E \in \mathrm{fml}((\Theta_{S_A})^A)$, but $E ⅋ F \notin \mathrm{fml}((\Theta_{S_A})^A)$. Moreover, since S_A is principal for A, there is a path π in Θ_{S_A} from A to F such that π passes immediately below A in Θ_{S_A}. Then each formula in π except A does not belong to $\mathrm{fml}((\Theta_{S_B})^B)$. In fact, let G be the first formula in π except A such that $G \in \mathrm{fml}((\Theta_{S_B})^B)$. Then the subpath π' of π from $A ⅋ B$ to G in Θ_{S_A} survives in Θ_{S_B}. On the other hand, since $G \in \mathrm{fml}((\Theta_{S_B})^B)$, there is a unique path ξ from B to G in $(\Theta_{S_B})^B$ such that ξ passes immediately above or adjacent to B. Then since S_B selects B in the ⅋-link L, π' and ξ makes a cycle in Θ_{S_B}. This is a contradiction. Therefore, each formula in π except A does not belong to $\mathrm{fml}((\Theta_{S_B})^B)$. But $F \in \mathrm{fml}((\Theta_{S_B})^B)$ because $E ⅋ F$ belongs to θ and θ is included in $(\Theta_{S_B})^B$. This is a contradiction. *the end of proof of Claim 7*
   Since S_B (resp. S_A) is a principal switching for B (resp. A), Claim 7 means $e_\Theta(B) \subseteq e_\Theta(A)$. Similarly we can prove $e_\Theta(A) \subseteq e_\Theta(B)$. So $e_\Theta(A) = e_\Theta(B)$.

2. The case where $B \notin e_\Theta(A)$ and $A \in e_\Theta(B)$:
   Then by Proposition 13, $e_\Theta(A) \subsetneq e_\Theta(B)$ and A is not a conclusion of $e_\Theta(B)$. But this implies $A ⅋ B \in e_\Theta(B)$, which contradicts the definition of empires. Therefore this case never happens.

3. The case where $B \in e_\Theta(A)$ and $A \notin e_\Theta(B)$:
   Similar to the case immediately above. $\Box$

The next goal is to prove Splitting lemma (Lemma 2). In order to do that, we introduce a strict partial order on $\otimes$-formulas in a MLL proof net.

Definition 23 *Let Θ be an MLL proof net. Let $L : \frac{A \quad B}{A \otimes B}$ and $L' : \frac{A' \quad B'}{A' \otimes B'}$ be $\otimes$-links in Θ. Then,*

$$A \otimes B < A' \otimes B' \text{ iff } e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A') \vee e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(B')$$

Proposition 16 *$<$ is a strict partial order.*

Proof.

- transitivity:
  We assume that $A \otimes B < A' \otimes B'$ and $A' \otimes B' < A'' \otimes B''$. By definition,
  $(e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A') \vee e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(B')) \wedge (e_{\Theta_0}(A' \otimes B') \subseteq e_{\Theta_0}(A'') \vee e_{\Theta_0}(A' \otimes B') \subseteq e_{\Theta_0}(B''))$.
  We only consider the case where $e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A') \wedge e_{\Theta_0}(A' \otimes B') \subseteq e_{\Theta_0}(B'')$ because the other three cases are similar. Since $e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A')$ and $e_{\Theta_0}(A') \subseteq e_{\Theta_0}(A' \otimes B')$, we obtain $e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A' \otimes B')$. Therefore from $e_{\Theta_0}(A' \otimes B') \subseteq e_{\Theta_0}(B'')$, we obtain $e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(B'')$. So, $A \otimes B < A'' \otimes B''$.

- irreflexivity:
  We assume that $A \otimes B < A \otimes B$. Then by definition $e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A) \vee e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(B)$. We only consider the case where $e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A)$, because the other case is similar. Then $B \in e_{\Theta_0}(A \otimes B) \subseteq e_{\Theta_0}(A)$ and $B \in e_{\Theta_0}(B)$. So $e_{\Theta_0}(A) \cap e_{\Theta_0}(B) \neq \emptyset$. From Proposition 14 We derive a contradiction. $\Box$

Lemma 2 (Splitting Lemma) *Let Θ be an MLL proof net whose conclusions does not include any $\invamp$-formulas. Then there is a conclusion $L : \frac{A \quad B}{A \otimes B}$ in Θ such that $\mathrm{fml}(\Theta) = \{A \otimes B\} \uplus e_\Theta(A) \uplus e_\Theta(B)$.*

Proof. Let $T = \{A_1 \otimes B_1, \ldots, A_\ell \otimes B_\ell\}$ be the conclusions in Θ that are a $\otimes$-formula. Then let $\ell_0\,(1 \leq \ell_0 \leq \ell)$ be an index such that $A_{\ell_0} \otimes B_{\ell_0}$ is a maximal element in T w.r.t the strict partial order $<$. We can always find the index by the finiteness of Θ. We claim that $A_{\ell_0} \otimes B_{\ell_0}$ is $A \otimes B$ of the the statement. We assume that ℓ_0 is not. Then without loss of generality, there is a conclusion C of $e_\Theta(A_{\ell_0})$ such that C is not a conclusion of Θ. Then without loss of generality there is an index $\ell'\,(1 \leq \ell' \leq \ell)$ such that C is hereditarily above $B_{\ell'}$. Hence by Proposition 7, $C \in e_\Theta(B_{\ell'})$. Moreover, from the definition of empires, $B_{\ell'} \notin e_\Theta(A_{\ell_0})$. Then, by Proposition 12, $A_{\ell_0} \in e_\Theta(B_{\ell'})$. Hence by Proposition 13, $e_\Theta(A_{\ell_0}) \subsetneq e_\Theta(B_{\ell'})$. So, since $e_\Theta(A_{\ell_0} \otimes B_{\ell_0}) \subsetneq e_\Theta(B_{\ell'})$. Hence $A_{\ell_0} \otimes B_{\ell_0} < A_{\ell'} \otimes B_{\ell'}$. This contradicts the maximality of $A_{\ell_0} \otimes B_{\ell_0}$ w.r.t $<$ over T. $\Box$

C Proof of Proposition 2

Proof of Proposition 2 We prove this proposition by induction on the number of the links in Θ. Before that, we prove the following claim.

Claim 8 *Let $\langle h_V, h_E \rangle$ be an other graph automorphism on $G^{\mathrm{strp}_{\otimes\invamp}}(\Theta)$ than $\langle \mathrm{id}_V, \mathrm{id}_E \rangle$. Then $\forall v \in V. h_V(v) \neq v$.*

proof of Claim 8: We assume $h_V = \mathrm{id}_V$. Since $\langle h_V, h_E \rangle \neq \langle \mathrm{id}_V, \mathrm{id}_E \rangle$, there is $e_0 \in E$ such that $e_0 = \mathrm{id}_E(e_0) \neq h_E(e_0)$. On the other hand, since $\langle h_V, h_E \rangle$ is a graph automorphism, $\ell_E(e_0) = \ell_E(h_E(e_0)) \in \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}$. Therefore the link L that induces e_0 is different from the link L' that induces $h_E(e_0)$. Then since (a) two different links does not share the same formula except that the formula is one premise of the one link and one conclusion of the other link, but (b) $\mathrm{src}(e_0)$ is a conclusion (resp. premise) of L iff $\mathrm{src}(h_E(e_0))$ is a conclusion (resp. premise) of L', hence, $h_V(\mathrm{src}(e_0)) = \mathrm{src}(h_E(e_0)) \neq \mathrm{src}(e_0)$. Therefore $h_V \neq \mathrm{id}_V$.
So, there is $v_0 \in V$ such that $v_0 = \mathrm{id}_V(v_0) \neq h_V(v_0)$. The the following subclaim holds.

Subclaim 2 *For any $e \in E$ and $v \in V$, if* $\mathrm{src}(e) = v_0$ *and* $\mathrm{tgt}(e) = v$*, or* $\mathrm{src}(e) = v$ *and* $\mathrm{tgt}(e) = v_0$*, then* $v \neq h_V(v)$.

proof of Subclaim 2: We only consider the case where $\mathrm{src}(e) = v_0$ and $\mathrm{tgt}(e) = v$, because the other case is similar. Since $v_0 \neq h(v_0)$ and $\ell_V^{\mathrm{strp}_{\otimes\parr}}(v_0) = \ell_V^{\mathrm{strp}_{\otimes\parr}}(h_V(v_0))$, hence, $e \neq h_E(e)$. Then since $\ell_E(e) = \ell_E(h_E(e)) \in \{\mathbf{L}, \mathbf{R}, \mathbf{ID}\}$, by the same discussion above, we can derive $v = \mathrm{tgt}(e) \neq h_V(\mathrm{tgt}(e)) = h_V(v)$. *the end of the proof of Subclaim 2*
Since Θ is an MLL proof net, starting from $v_0 \in V$, we can reach any $v \in V$ by moving from a node $v_1 \in V$ to another node $v_2 \in V$ repeatedly such that v_1 and v_2 are a premise or a conclusion of the same link. Then through the travelling, by applying the subclaim, we can derive the claim. *the end of the proof of Claim 8*
Then we prove the proposition using the claim above.

1. The case where Θ consists of exactly one ID-link $\overline{p \quad p^\perp}$:
   It is obvious that the identity map is the only graph automorphism on $G^{\mathrm{strp}_{\otimes\parr}}(\Theta)$.

2. The case where there is a $\parr$-formula $\langle A \parr B, k_1\rangle$ among the conclusions in Θ:
   Let $\langle h_V, h_E\rangle$ be an other graph automorphism on $G^{\mathrm{strp}_{\otimes\parr}}(\Theta)$ than $\langle \mathrm{id}_V, \mathrm{id}_E\rangle$. By Claim 8, Θ must have a conclusion $\langle A \parr B, k_2\rangle$ such that $k_1 \neq k_2$, $h_V(k_1) = k_2$, and $h_V(k_2) = k_1$. Let Θ_0 be the proof net obtained from Θ deleting the two $\parr$-links associated with $\langle A \parr B, k_1\rangle$ and $\langle A \parr B, k_2\rangle$ (let the two $\parr$-links be $L_{\parr 1} : \frac{\langle A, i_1\rangle \quad \langle B, j_1\rangle}{\langle A \parr B, k_1\rangle}$ and $L_{\parr 2} : \frac{\langle A, i_2\rangle \quad \langle B, j_2\rangle}{\langle A \parr B, k_2\rangle}$ respectively). We apply inductive hypothesis to Θ_0. Then the only graph automorphism on $G^{\mathrm{strp}_{\otimes\parr}}(\Theta_0)(= \langle V_0, E_0, \ell_{V_0}^{\mathrm{strp}_{\otimes\parr}}, \ell_{E_0}\rangle)$ is $\langle \mathrm{id}_{V_0}, \mathrm{id}_{E_0}\rangle$. Therefore $\langle h_V, h_E\rangle$ must be an extension of $\langle \mathrm{id}_{V_0}, \mathrm{id}_{E_0}\rangle$. But it is impossible, because since $h_V(k_1) = k_2$, and $h_V(k_2) = k_1$, we must have $h_V(i_1) = i_2$, $h_V(i_2) = i_1$, $h_V(j_1) = j_2$, and $h_V(j_2) = j_1$.

3. The case where there is no $\parr$-formula among the conclusions in Θ:
   In this case, by applying Lemma 2 (Appendix B) to Θ we can find $\langle A_1 \otimes B_1, k_1\rangle$ such that $\mathrm{fml}(\Theta) = \{\langle A_1 \otimes B_1, k_1\rangle\} \uplus e_\Theta(\langle A, i_1\rangle) \uplus e_\Theta(\langle B, j_1\rangle)$. Let $\langle h_V, h_E\rangle$ be an other graph automorphism on $G^{\mathrm{strp}_{\otimes\parr}}(\Theta)$ than $\langle \mathrm{id}_V, \mathrm{id}_E\rangle$. By Claim 8, Θ must have a conclusion $\langle A \otimes B, k_2\rangle$ such that $k_1 \neq k_2$, $h_V(k_1) = k_2$, and $h_V(k_2) = k_1$. Moreover by symmetry, we must have $\mathrm{fml}(\Theta) = \{\langle A \otimes B, k_2\rangle\} \uplus e_\Theta(\langle A, i_2\rangle) \uplus e_\Theta(\langle B, j_2\rangle)$. Moreover by symmetry, it is enough to consider the following two cases.

   (a) The case where
   $\mathrm{fml}(\Theta) = \{\langle A \otimes B, k_1\rangle, \langle A \otimes B, k_2\rangle\} \uplus e_\Theta(\langle A, i_1\rangle) \uplus e_\Theta(\langle A, i_2\rangle) \uplus \big(e_\Theta(\langle B, j_1\rangle) \cap e_\Theta(\langle B, j_2\rangle)\big)$

   (b) The case where
   $\mathrm{fml}(\Theta) = \{\langle A \otimes B, k_1\rangle, \langle A \otimes B, k_2\rangle\} \uplus e_\Theta(\langle B, j_1\rangle) \uplus e_\Theta(\langle B, j_2\rangle) \uplus \big(e_\Theta(\langle A, i_1\rangle) \cap e_\Theta(\langle A, i_2\rangle)\big)$

   We only consider the case (a) because the case (b) is similar. Then let Θ_0 be the proof net whose formulas are $e_\Theta(\langle B, j_1\rangle) \cap e_\Theta(\langle B, j_2\rangle)$. We apply inductive hypothesis to Θ_0. Then the only graph automorphism on $G^{\mathrm{strp}_{\otimes\parr}}(\Theta_0)(= \langle V_0, E_0, \ell_{V_0}^{\mathrm{strp}_{\otimes\parr}}, \ell_{E_0}\rangle)$ is $\langle \mathrm{id}_{V_0}, \mathrm{id}_{E_0}\rangle$. Therefore $\langle h_V, h_E\rangle$ must be an extension of $\langle \mathrm{id}_{V_0}, \mathrm{id}_{E_0}\rangle$. But it is impossible, because since $h_V(k_1) = k_2$, and $h_V(k_2) = k_1$, we must have $h_V(i_1) = i_2$, $h_V(i_2) = i_1$, $h_V(j_1) = j_2$, and $h_V(j_2) = j_1$. $\square$

D Proof of Theorem 2

Proof of Theorem 2. At first we fix our notation. Let Θ' be $\mathrm{ex}_{\otimes\parr}(\Theta, L_{1\otimes}, L_{\parr 2})$ and $L'_{1\parr}$ and $L'_{2\otimes}$ be $\frac{A \quad B}{A \parr B}$ and $\frac{C \quad D}{C \otimes D}$ respectively.
• If part

1. The case where C is a conclusion of $e_\Theta(A)$ and D is a conclusion of $e_\Theta(B)$:
   Let S' be a DR-switching for Θ'. We assume that $\Theta'_{S'}$ has a cycle or is disconnected.

   (a) The case where S' selects A in $L'_{1\parr}$:
   By the assumption on $\Theta'_{S'}$, (i) there is a cycle including $C \otimes D$ in $\Theta'_{S'}$ or (ii) A and B are disconnected in $\Theta'_{S'}$. Then let S be the DR-switching for Θ such that S is S' except that S chooses the left or the right premise of $L_{2\parr}$ and the domain of S does not include $L'_{1\parr}$. Then there are two unique paths θ_1 and θ_2 in Θ_S from A to C and from B to D respectively. From our

assumption about C and D, we can easily see that all the indexed formulas in θ_1 and θ_2 are included in $e_\Theta(A)$ and $e_\Theta(B)$ respectively. In particular,

- θ_1 passes immediately above or adjacent to both A and C, and
- θ_2 passes immediately above or adjacent to both B and D.

Moreover, by our assumption and Proposition 14 we obtain $e_\Theta(A) \cap e_\Theta(B) = \emptyset$. Therefore if we consider θ_1 and θ_2 as two sets of indexed formulas, θ_1 and θ_2 are disjoint. Moreover, two paths θ_1 and θ_2 in Θ_S are preserved in $\Theta'_{S'}$ because θ_1 (resp. θ_2) includes neither $A \otimes B$ nor $C \parr D$. Hence if we let $(\theta_2)^r$ be the reverse of θ_2, then $\theta_1, C \otimes D, (\theta_2)^r$ is the unique path from A to B in $\Theta'_{S'}$. Hence the case (ii) is impossible. So the case (i) holds.
If $\Theta'_{S'}$ has a cycle π, then one of the following conditions must be satisfied:

(a-1) The case where the cycle π in $\Theta'_{S'}$ includes $C, C \otimes D, D$:
Since $C \in e_\Theta(A)$, $D \in e_\Theta(B)$ and $e_\Theta(A) \cap e_\Theta(B) = \emptyset$, π must include at least one indexed formula from each of the following three types of indexed formulas except $C, C \otimes D, D$: (I) indexed formulas from $e_\Theta(A)$ different from A, (II) indexed formulas from $e_\Theta(B)$ different from B, and (III) indexed formulas that are not included in $e_\Theta(A) \cup e_\Theta(B)$. Let E be an indexed formula of the type (I) that is included in π and F be an indexed formula of the type (II) that is included in π. Then there is a path τ_1 from A to E in $\Theta'_{S'}$ such that all the indexed formulas in τ_1 are included in $e_\Theta(A)$ and τ_1 passes immediately above or adjacent to A. Similarly, there is a path τ_2 from B to F in $\Theta'_{S'}$ such that all the indexed formulas in τ_2 are included in $e_\Theta(B)$ and τ_2 passes immediately above or adjacent to B. On the other hand since π has indexed formulas of type (III), there is the subpath π' of π from E to F such that π' includes at least one indexed formula that is not included in $e_\Theta(A) \cup e_\Theta(B)$. Since Θ is an MLL proof net, Θ_S must be acyclic and connected. But there is the cycle $A \otimes B, \tau_1, \pi', (\tau_2)^r, A \otimes B$ in Θ_S. This is a contradiction.

(a-2) The case where the cycle π in $\Theta'_{S'}$ includes C and $C \otimes D$, but does not include D:
In this case there is the subpath π_0 of π from C to $C \otimes D$ in $\Theta'_{S'}$ such that π_0 passes immediately above or adjacent to C and immediately below $C \otimes D$. We let the DR-switching S for Θ select C in $L_{2\parr}$. Since Θ is an MLL proof net, Θ_S must be acyclic and connected. But since π_0 in $\Theta'_{S'}$ survives in Θ_S, Θ_S has a cycle. This is a contradiction.

(a-3) The case where the cycle π in $\Theta'_{S'}$ includes D and $C \otimes D$, but does not include C:
Similar to the case immediately above except that we let the DR-switching S for Θ select D in $L_{2\parr}$.

(b) The case where S' selects B in $L'_{1\parr}$:
Similar to the case above.

2. The case where D is a conclusion of $e_\Theta(A)$ and C is a conclusion of $e_\Theta(B)$:
Similar to the case above.

• Only-if part
We suppose that Θ and $\Theta' (= \mathrm{ex}_{\otimes\parr}(\Theta, L_{1\otimes}, L_{\parr 2}))$ are proof nets, but neither (1) nor (2) of the statement of the theorem holds. Then we derive a contradiction. Basically we find a DR-switching S' for Θ' such that $S'(\Theta')$ has a cycle. We prove this by case analysis.

1. The case where $C \parr D \in e_\Theta(A)$:
By Proposition 7, $C \in e_\Theta(A)$ and $D \in e_\Theta(A)$. Let S be a principal DR-switching for A in Θ. Without loss of generality we assume that S selects C in $L_{2\parr}$. Since Θ_S is acyclic and connected, there are two unique paths θ_1 from A to C and θ_2 from A to D in $(\Theta_S)^A$ such that both θ_1 and θ_2 pass immediately above or adjacent to A. Moreover, since $e_\Theta(A) \cap e_\Theta(B) = \emptyset$ and all the formulas in θ_1 and θ_2 are included in $e_\Theta(A)$, neither θ_1 nor θ_2 includes B. We have two cases.

   (a) The case where both θ_1 and θ_2 pass $C \parr D$:
   In this case, both θ_1 and θ_2 pass immediately below $C \parr D$. Otherwise, let S_0 be S except S_0 selects D in $L_{2\parr}$. Then $S_0(\Theta)$ has a cycle including the subpath of θ_1 from A to $C \parr D$, the path $C \parr D, D$, and $(\theta_2)^r$ from D to A. This is a contradiction. Therefore since S selects C in $L_{2\parr}$, θ_1 from A to C is a subpath of θ_2 from A to D in $S(\Theta)^A$. Hence θ_2 has the subpath θ_{21} from $C \parr D$ to D such that θ_{21} passes immediately above or adjacent to $C \parr D$. Let S' be S except that the

$\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A or B. Then S' is a DR-switching for Θ' and $S'(\Theta')$ includes a cycle $\theta_{21}, C \otimes D$.

(b) Otherwise:
In this case, neither θ_1 nor θ_2 includes $C \parr D$ (otherwise, we have a cycle including $C, C \parr D$ in Θ_S or when we let $S_{\mathbf{R}}$ be the DR-switching obtained from S by selecting D in $L_{2\parr}$, we have a cycle including $D, C \parr D$ in $\Theta_{S_{\mathbf{R}}}$). Therefore θ_1 (resp. θ_2) passes immediately above or adjacent to C (resp. D). Then let S' be S except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A or B. Since $e_\Theta(A) \cap e_\Theta(B) = \emptyset$ and all the formulas in θ_1 and θ_2 are included in $e_\Theta(A)$, both θ_1 and θ_2 in $S(\Theta)$ survive in $S'(\Theta')$. Then we find a cycle $\theta_1, C \otimes D, (\theta_2)^r$ in $S'(\Theta')$.

2. The case where $C \parr D \in e_\Theta(B)$:
Similar to the case above.

3. The case where $C \parr D \notin e_\Theta(A)$ and $C \parr D \notin e_\Theta(B)$:
Moreover we divide the case into two cases.

(a) The case where $e_\Theta(A \otimes B) \cap e_\Theta(C \parr D) = \emptyset$:
Let $S_{\mathbf{L}}$ be a DR-switching for Θ selecting C in $L_{2\parr}$. Then there is the unique path θ from D to $C \parr D$ in $\Theta_{S_{\mathbf{L}}}$ with length > 1. Let $S'_{\mathbf{L}}$ be $S_{\mathbf{L}}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A (or B). Then $\Theta'_{S'_{\mathbf{L}}}$ has a cycle θ, D. This is a contradiction.

(b) The case where $e_\Theta(A \otimes B) \cap e_\Theta(C \parr D) \neq \emptyset$:
Then by Proposition 12, $C \parr D \in e_\Theta(A \otimes B)$ or $A \otimes B \in e_\Theta(C \parr D)$.

(b-1) The case where $C \parr D \in e_\Theta(A \otimes B)$:
Since neither (1) nor (2) of the statement of the theorem holds, one of the following four cases must hold.

(b-1-1) The case where neither C nor D is a conclusion of $e_\Theta(A)$:
In this case, since $C \parr D \notin e_\Theta(A)$, $C \notin e_\Theta(A)$ and $D \notin e_\Theta(A)$. Let $S_{\mathbf{L}A}$ be a principal switching for A and Θ such that $S_{\mathbf{L}A}$ selects C in $L_{2\parr}$. Then there are two unique paths θ_1 from A to C and θ_2 from A to D in $\Theta_{S_{\mathbf{L}A}}$ such that both θ_1 and θ_2 pass immediately below A. Let S' be $S_{\mathbf{L}A}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A (or B). Then $\Theta'_{S'}$ has a cycle $\theta_1, C \otimes D, (\theta_2)^r$. This is a contradiction.

(b-1-2) The case where C is neither a conclusion of $e_\Theta(A)$ nor a conclusion of $e_\Theta(B)$:
Since $C \parr D \notin e_\Theta(A)$ (resp. $C \parr D \notin e_\Theta(B)$), We can easily see that $C \notin e_\Theta(A)$ (resp $C \notin e_\Theta(B)$), since if $C \in e_\Theta(A)$ (resp. $C \in e_\Theta(B)$), then C is a conclusion of $e_\Theta(A)$ (resp. $e_\Theta(B)$). Let S_B be a principal switching for B in Θ. Since $C \notin e_\Theta(B)$, there is the unique path θ_1 from B to C in Θ_{S_B} such that θ_1 passes immediately below B. Then we have two cases:
(b-1-2-1) The case where θ_1 includes A:
There is the unique path θ_2 from A from D in Θ_{S_B}. Let θ'_1 be the subpath of θ_1 from A to C. Let S' be S_B except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A. Then $\Theta'_{S'}$ is a cycle $\theta'_1, C \otimes D, (\theta_2)^r$ since θ'_1 and θ_2 are preserved when moving to $\Theta'_{S'}$ from Θ_{S_B}.
(b-1-2-2) The case where θ_1 does not include A:
There is the unique path θ_2 from A from D in Θ_{S_B}. Let θ'_1 be the subpath of θ_1 from $A \otimes B$ to C. Let S' be S_B except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A. Then $\Theta'_{S'}$ is a cycle $\theta'_1, C \otimes D, (\theta_2)^r, A \otimes B$ since θ'_1 and θ_2 are preserved when moving to $\Theta'_{S'}$ from Θ_{S_B} except $A \otimes B$ is replaced by $A \parr B$.

(b-1-3) The case where neither C nor D is a conclusion of $e_\Theta(B)$:
Similar to the case (b-1-1) above.

(b-1-4) The case where D is neither a conclusion of $e_\Theta(A)$ nor a conclusion of $e_\Theta(B)$:
Similar to the case (b-1-2) above.

(b-2) The case where $C \parr D \notin e_\Theta(A \otimes B)$ and $A \otimes B \in e_\Theta(C \parr D)$
By Proposition 13, $A \otimes B$ is not a conclusion of $e_\Theta(C \parr D)$ and $e_\Theta(A \otimes B) \subsetneq e_\Theta(C \parr D)$. In this case we easily find a DR-switching S' for Θ' such that $\Theta'_{S'}$ has a cycle including

$C \otimes D$. In the following we prove the claim. Let $S_{A\otimes B}$ be a principal switching for $A \otimes B$ in $e_\Theta(C \parr D)$. Then we can obtain a principal switching $S_{C\parr D}$ for $C \parr D$ in Θ by extending $S_{A\otimes B}$. Then the unique path θ from C to D in $S_{C\parr D}(\Theta)$ includes neither $A, A\otimes B, A$ nor $B, A\otimes B, A$, because in order that θ includes $A, A\otimes B, A$ or $B, A\otimes B, A$, θ must enter $e_\Theta(A\otimes B)$ from a conclusion of $e_\Theta(A\otimes B)$ other than $A\otimes B$. But this is impossible because $S_{C\parr D}$ is an extension of $S_{A\otimes B}$ that is a principal switching for $A\otimes B$. Then we have three cases about θ from C to D.

(b-2-1) The case where θ includes neither A, B, nor $A\otimes B$:
Let S' be $S_{C\parr D}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A or B. Since θ from C to D in $S_{C\parr D}(\Theta)$ survives in $S'(\Theta')$, $S'(\Theta')$ has a cycle $\theta, C\otimes D, C$.

(b-2-1) The case where θ includes $A, A\otimes B$ or $A\otimes B, A$:
Let S' be $S_{C\parr D}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects A. Since θ from C to D in $S_{C\parr D}(\Theta)$ survives in $S'(\Theta')$, $S'(\Theta')$ has a cycle $\theta, C\otimes D, C$.

(b-2-2) The case where θ includes $B, A\otimes B$ or $A\otimes B, B$:
Let S' be $S_{C\parr D}$ except that the $\parr$-switch for $L_{2\parr}$ is deleted and the $\parr$-switch for $L'_{1\parr}$ selects B. Since θ from C to D in $S_{C\parr D}(\Theta)$ survives in $S'(\Theta')$, $S'(\Theta')$ has a cycle $\theta, C\otimes D, C$. $\square$

E Proof of Lemma 1

In this section, we prove Lemma 1 by proving the following generalized main lemma by induction.

Lemma 3 (Generalized Main Lemma) *Let Θ be an MLL proof net with a conclusion $C_0 \parr D_0$ with the $\parr$-link $L_{\parr 0} : \frac{C_0 \quad D_0}{C_0 \parr D_0}$. We assume that m_1 $\otimes$-links $L_{\otimes 1} : \frac{A_1 \quad B_1}{A_1\otimes B_1}, \ldots, L_{\otimes m_1} : \frac{A_{m_1} \quad B_{m_1}}{A_{m_1}\otimes B_{m_1}}$ and m_2 $\parr$-links $L_{\parr 1} : \frac{C_1 \quad D_1}{C_1 \parr D_1}, \ldots, L_{\parr m_2} : \frac{C_{m_2} \quad D_{m_2}}{C_{m_2}\parr D_{m_2}}$ occur in Θ, where $m_1, m_2 \in \mathbb{N}$. Moreover we assume that (a) $\Theta_{i,j} = \mathrm{ex}_{\otimes\parr}(\Theta, L_{\otimes i}, L_{\parr j})$ is not an MLL proof net for each i, j $(1 \le i \le m_1, 0 \le j \le m_2)$. Moreover we define Θ' as follows:*

$$\Theta' \equiv_{\mathrm{def}} \mathrm{ex}_{\otimes\parr}(\Theta, \langle L_{\otimes 1}, \ldots, L_{\otimes m_1}\rangle, \langle L_{\parr 0}, L_{\parr 1}, \ldots, L_{\parr m_2}\rangle)$$

Then Θ' is not an MLL proof net.

Proof of Lemma 3 Let Θ_0 be the MLL proof net obtained from Θ by deleting $L_{\parr 0} : \frac{C_0 \quad D_0}{C_0 \parr D_0}$. Moreover let Θ'_0 be

$$\mathrm{ex}_{\otimes\parr}(\Theta_0, \langle L_{\otimes 1}, \ldots, L_{\otimes m_1}\rangle, \langle L_{\parr 1}, \ldots, L_{\parr m_2}\rangle).$$

We prove the lemma by induction on lexicographic order $\langle m_1, |\mathbb{L}_\Theta|\rangle$, $|\mathbb{L}_\Theta|$ is the number of link occurrences in Θ. If $m_1 = 0$, then we can easily see that there is a DR-switching S'_0 for Θ'_0 such that there is a path θ' from C_0 to D_0 in $S'_0(\Theta'_0)$. Therefore $S'_0(\Theta')$ has a cycle.
In the following, we prove the induction step: we assume $m_1 > 0$.

- The case where $m_2 = 0$:
Since $m_1 > 0$, it is obvious that there is a DR-switching S' for Θ'_0 such that $S'(\Theta'_0)$ is disconnected. If $S'(\Theta'_0)$ has more than two maximally connected components, then we have done. If $S'(\Theta'_0)$ has exactly two maximally components, then $m_1 = 1$. Therefore by condition (a), Θ' is not an MLL proof net.

- The case where $m_2 > 0$:
By inductive hypothesis, Θ'_0 is not an MLL proof net. Therefore, there is a DR-switching S' for Θ'_0 such that $S'(\Theta'_0)$ has a cycle or is disconnected. If $S'(\Theta'_0)$ has a cycle, then we have done: $S'(\Theta')$ also has a cycle. If $S'(\Theta'_0)$ has more than two maximally connected components, then we have done: $S'(\Theta')$ is disconnected. Therefore we can assume that $S'(\Theta'_0)$ has exactly two maximally connected components in which each component does not have any cycle for any DR-switching S' for Θ'_0 (note that the number of the edges of $S'(\Theta'_0)$ is always the same for any DR-switching S' for Θ'_0). Hence there is i_0 $(1 \le i_0 \le m_1)$ such that one connected component has A_{i_0} and the other has B_{i_0}. Moreover,

since $\mathrm{ex}_{\otimes\wp}(e_{\Theta_0}(A_{i_0}),\langle L_{\otimes 1},\ldots,L_{\otimes m_1}\rangle,\langle L_{\wp 1},\ldots,L_{\wp m_2}\rangle)$ is a subproof structure of Θ_0', $S'(\mathrm{ex}_{\otimes\wp}(e_{\Theta_0}(A_{i_0}),\langle L_{\otimes 1},\ldots,L_{\otimes m_1}\rangle,\langle L_{\wp 1},\ldots,L_{\wp m_2}\rangle))$ must be acyclic and connected for any DR-switching S' for Θ_0'. Therefore $\mathrm{ex}_{\otimes\wp}(e_{\Theta_0}(A_{i_0}),\langle L_{\otimes 1},\ldots,L_{\otimes m_1}\rangle,\langle L_{\wp 1},\ldots,L_{\wp m_2}\rangle)$ is an MLL proof net. Therefore by inductive hypothesis, $\mathrm{ex}_{\otimes\wp}(e_{\Theta_0}(A_{i_0}),\langle L_{\otimes 1},\ldots,L_{\otimes m_1}\rangle,\langle L_{\wp 1},\ldots,L_{\wp m_2}\rangle)=e_{\Theta_0}(A_{i_0})$ (this means $e_{\Theta_0}(A_{i_0})$ has neither $\otimes$-link nor $\wp$-link to be exchanged). Moreover since by the condition (a) and $S'(\Theta_0')$ has exactly two maximally connected components for any S', for any $j\,(1\leq j\leq m_2)$, $e_{\Theta_0}(A_{i_0})$ has neither C_j nor D_j as a conclusion (otherwise, the condition (a) is violated, i.e., $e_{\Theta_0}(A_{i_0})$ has C_j (resp. D_j) as a conclusion and $e_{\Theta_0}(B_{i_0})$ has D_j (resp. C_j) as a conclusion). For the same reason,i.e., the condition (a), if $C_0(\mathrm{resp.}D_0)\in e_{\Theta_0}(A_{i_0})$, then $D_0(\mathrm{resp.}C_0)\in e_{\Theta_0}(A_{i_0})$ (see Figure 12). Then when let $S'_{A_{i_0}}$ be a principal switching for $e_{\Theta_0}(A_{i_0})$ in Θ', $S'_{A_{i_0}}(\Theta')$ is disconnected or has a cycle including $C_0, C_0\otimes D_0, D_0$. □

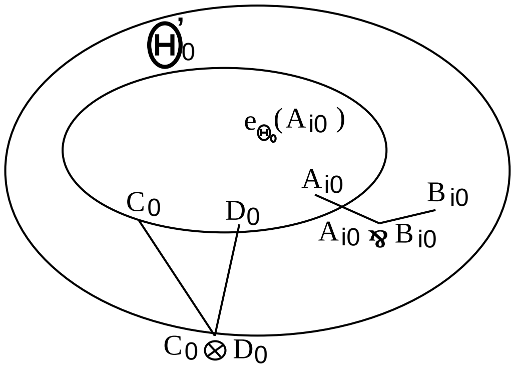

Figure 12: Θ'